Journal of Applied Mathematics and Computing (2024) 70:1441–1469
https://doi.org/10.1007/s12190-024-02008-4

ORIGINAL RESEARCH

Self-adaptive relaxed CQ algorithms for solving split feasibility problem with multiple output sets

Xiaolei Tong[1] · Tong Ling[1] · Luoyi Shi[2]

Received: 18 July 2023 / Revised: 24 January 2024 / Accepted: 29 January 2024 /
Published online: 26 February 2024

Abstract

In this paper, we propose two new self-adaptive relaxed CQ algorithms to solve the split feasibility problem with multiple output sets, which involve the computation of projections onto half-spaces instead of the computation onto the closed convex sets. Our proposed algorithms with selection technique reduce the computation of projections. And then, as a generalization, we construct two new algorithms to solve the variational inequalities over the solution set of split feasibility problem with multiple output sets. More importantly, strong convergence of all proposed algorithms is proved under suitable conditions. Finally, we conduct numerical experiments to show the efficiency and accuracy of our algorithms compared to some existing results.

Keywords Split feasibility problem · Multiple output sets · Strong convergence · Relaxed CQ algorithm · Self-adaptive algorithm

Mathematics Subject Classification 47H09 · 47H10 · 49J53 · 90C25

Tong Ling and Luoyi Shi have contributed equally to this work.

✉ Luoyi Shi
shiluoyi@tiangong.edu.cn

Xiaolei Tong
tongxiaoleimath@163.com

Tong Ling
lingtongmath@163.com

1 School of Mathematical Sciences, Tiangong University, Tianjin 300387, China

2 School of Software, Tiangong University, Tianjin 300387, China

1 Introduction

Let C and Q be nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint operator A^*. The split feasibility problem (SFP) is formulated as finding a point $x^* \in H_1$ satisfying

$$x^* \in C \text{ and } Ax^* \in Q. \tag{1}$$

The SFP was first proposed by Censor and Elfving [6] in 1994 for solving modeling inverse problems which arise from the phase retrieval problems and medical image reconstruction [4]. It has been found that the SFP can also be used to model problems with applications in different domains, for instance, intensity-modulated radiation therapy and gene regulatory network inference [5, 7, 8, 17].

For solving the SFP, Byrne [4] introduced the applicable and best-known CQ algorithm, which involves the computations of the projections P_C and P_Q onto C and Q. In addition, the step size of CQ algorithm depends on the operator norm, which is not easy to compute (or at least estimate). In 2004, Yang [19] generalized the CQ method to the so-called relaxed CQ algorithm, needing computation of the metric projection onto (relaxed sets) half-spaces C^n and Q^n. Since P_{C^n} and P_{Q^n} are easily calculated, this method appears to be very practical. To overcome the criterion for computing the norm of A (which is both complicated and costly), in 2012, López et al. [11] introduced a relaxed CQ algorithm for solving the SFP with a new adaptive way of determining the sequence of steps τ_n, defined as follows:

$$\tau_n := \frac{\rho_n f_n(x_n)}{\|\nabla f_n(x_n)\|^2},$$

where $\rho_n \in (0, 4)$, $\forall n \geq 1$ such that $\liminf_{n\to\infty} \rho_n(4 - \rho_n) > 0$. It was proved that the sequence $\{x_n\}$ with τ_n converges weakly to a solution of the SFP.

Some generalizations of the SFP have been studied by many authors. Recently, Reich and Tuyen [12] considered and studied the split feasibility problem with multiple output sets (SFPMOS). Let H, H_i, $i = 1, 2, ..., N$, be real Hilbert spaces and let $A_i : H \to H_i$, $i = 1, 2, ..., N$ be bounded linear operators. Let C and Q_i, $i = 1, 2, ..., N$ be nonempty, closed, and convex subsets of H and H_i, $i = 1, 2, ..., N$, respectively. The SFPMOS is formulated to find a point x^* such that

$$x^* \in \Gamma := C \cap (\cap_{i=1}^N A_i^{-1}(Q_i)) \neq \emptyset. \tag{2}$$

The solution set of problem (SFPMOS) is denoted by Γ.

Furthermore, Reich and Tuyen [12] proposed the following two algorithms for solving the SFPMOS by extending the CQ algorithm:

$$x_{n+1} = P_C(x_n - \lambda_n \sum_{i=1}^N A_i^*(I - P_{Q_i})A_i x_n), \tag{3}$$

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) P_C(x_n - \lambda_n \sum_{i=1}^{N} A_i^*(I - P_{Q_i}) A_i x_n), \tag{4}$$

where $f : C \to C$ is a strict contraction mapping, $\{\lambda_n\} \subset (0, \infty)$ and $\{\alpha_n\} \subset (0, 1)$. It is proved that if the sequence λ_n satisfies the condition:

$$0 < a \leq \lambda_n \leq b < \frac{2}{N \max_{i=1,2,\ldots N}\{\|A_i\|^2\}},$$

for all $n \geq 1$, then the sequence $\{x_n\}$ obtained weak and strong convergence results for (3) and (4), respectively.

However, we note that each iteration step of Algorithm (3) and Algorithm (4) requires the computation of metric projections onto sets C and Q_i and have to calculate or estimate the operator norm $\|A_i\|$. In general, this is not an easy task in practice.

Next, we consider several self-adaptive projection methods that would avoid the above case. In 2019, Yao et al. [14] presented two self-adaptive iterative algorithms for solving the multiple-sets split feasibility problem (MSSFP) and proved the weak and strong convergence results. MSSFP is to find a point x^* such that

$$x^* \in \bigcap_{i=1}^{s} C_i \text{ and } Ax^* \in \bigcap_{j=1}^{t} Q_j. \tag{5}$$

where $\{C_i\}_{i=1}^{s}$ and $\{Q_i\}_{j=1}^{t}$ are two finite families of closed convex subsets of H_1 and H_2.

Yao et al. [14] proposed the following self-adaptive method with selection technique:

$$\begin{cases} z_n = P_{C_{i_n}} x_n, \\ y_n = A^*(I - P_{Q_{j_n}}) A x_n, \\ x_{n+1} = x_n - \tau_n (x_n + y_n - z_n), \end{cases} \tag{6}$$

where

$$i_n \in \{i \,|\, \max_{i \in I_1} \|x_n - P_{C_i} x_n\|, I_1 = \{1, 2, ..., s\}\},$$

$$j_n \in \{j \,|\, \max_{j \in I_2} \|Ax_n - P_{Q_j} A x_n\|, I_2 = \{1, 2, ..., t\}\},$$

$$\tau_n = \lambda_n \frac{\|x_n - z_n\|^2 + \|y_n\|^2}{2\|x_n + y_n - z_n\|^2},$$

in which $\lambda_n > 0$.

In 2022, Reich et al. [14] introduced a split inverse problem called split common fixed point problem with multiple output sets. To solve this problem, they proposed a self-adaptive algorithm which is based on the viscosity approximation method and prove a strong convergence theorem. More details in [14].

Very recently, Taddele et al. [16] propose two new self-adaptive relaxed CQ algorithms for solving SFPMOS. The algorithms they proposed are as follows:

$$x_{n+1} = x_n - \rho_1^n (I - P_{C^n}) x_n - \tau_n \sum_{i=1}^{N} \vartheta_i A_i^* (I - P_{Q_i^n}) A_i x_n,$$
$$x_{n+1} = \alpha_n u + (1 - \alpha_n)(x_n - \rho_1^n (I - P_{C^n}) x_n - \tau_n \sum_{i=1}^{N} \vartheta_i A_i^* (I - P_{Q_i^n}) A_i x_n),$$

where $\tau_n := \frac{\rho_2^n \sum_{i=1}^N \vartheta_i \|(I-P_{Q_i^n})A_i x_n\|^2}{\bar{\tau_n}^2}$, $\bar{\tau_n} := \max\{\| \sum_{i=1}^N \vartheta_i A_i^* (I - P_{Q_i^n}) A_i x_n\|, \beta\}$. They establish a weak and a strong convergence theorems for the proposed algorithms. We note that although the step size of the algorithms of Taddele are adaptive, the computational effort required to compute the sum of the first N terms with respect to the projection $P_{Q_i^n}$ is undoubtedly enormous.

In addition, variational inequalities are crucial for both optimization theory and its applications. More importantly, we also turn our attention to the variational inequality problem over the solution set of the SFPMOS (in short, P2). Let $F : H \to H$ be a given operator. Suppose that the solution set Γ of SFPMOS is nonempty, P2 is described as follows:

$$\text{Find a point } x^* \in \Gamma \text{ such that } \langle Fx^*, x - x^* \rangle \geq 0, \ \forall\, x \in \Gamma. \tag{7}$$

The solution set of problem (P2) is denoted by VIP(F, Γ).

Motivated and inspired by [14] and [20], we further improve Reich's algorithms (4) for solving the SFPMOS (2). Especially, we develop and improve some previously discussed results in the following ways:

1. In contrast to Reich's algorithms (4), we propose two new relaxed CQ algorithms with adaptive step size to solve SFPMOS. It is worth noting that the parameters of these two algorithms are chosen without the prior knowledge of the norm of the transfer mappings. Our proposed algorithms employ selection techniques that reduce the computational effort of projection.

2. Based on these two algorithms in [14] and [20], we present compute the projections onto half-spaces instead of onto a closed convex set. Strong convergence results of the proposed Halpern-Type algorithms are proved.

3. As a generalisation of our proposed algorithms, we propose two new adaptive algorithms for solving the variational inequality problem over the solution set of the SFPMOS (7), which may not have been solved by others until now.

The paper is organized as follows: In Sect. 2, we recall some existing results, lemmas, and definitions for subsequent use. In Sect. 3, we present algorithms for solving SFPMOS and as a generalisation problem P2 in turn, and analyse the convergence of the proposed algorithms. To demonstrate that our proposed approach is implementable, several numerical examples are provided in Sect. 4.

2 Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$. Let the symbols "$\rightharpoonup$" and "$\to$" denote the weak and strong convergence, respectively. For any sequence $\{x_n\} \subset H$, $\omega_w(x_n) = \{x \in H : \exists \{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \rightharpoonup x\}$ denotes the weak w-limit set of $\{x_n\}$.

Definition 1 [2] Let C be a nonempty, closed and convex set of H. For every element $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$ such that

$$\|x - P_C x\| = \min\{\|x - y\| : y \in C\}.$$

The operator P_C is called a metric projection from H onto C.

It has the following well-known properties which are used in the subsequent convergence analysis.

Lemma 2 *[10] Let $C \subset H$ be a nonempty, closed and convex set. Then, the following assertions hold for any $x, y \in H$ and $z \in C$:*

(i) $\langle x - P_C x, z - P_C x \rangle \le 0$;
(ii) $\|P_C x - P_C y\| \le \|x - y\|$;
(iii) $\|P_C x - P_C y\|^2 \le \langle P_C x - P_C y, x - y \rangle$;
(iv) $\|P_C x - z\|^2 \le \|x - z\|^2 - \|x - P_C x\|^2$.

Next, we give some inequalities required for proving convergence analysis. For each $x, y \in H$, we have

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y \rangle. \tag{8}$$

Lemma 3 *(Peter-Paul Inequality) If a and b are non-negative real numbers, then*

$$ab \le \frac{a^2}{2\epsilon} + \frac{\epsilon b^2}{2}, \ \forall \epsilon > 0.$$

Definition 4 [2] Let $f : H \to (-\infty, +\infty]$ be a proper function, and let $x \in H$. Then f is called

(i) Lower semicontinuous at x if $x_n \to x$ implies $f(x) \le \liminf_{n\to\infty} f(x_n)$;
(ii) Weakly lower semicontinuous at x if $x_n \rightharpoonup x$ implies $f(x) \le \liminf_{n\to\infty} f(x_n)$.

Definition 5 [2] Let $f : H \to (-\infty, +\infty]$ be proper. The subdifferential of f is the set-valued operator

$$\partial f : H \to 2^H : x \mapsto \{u \in H | \langle y - x, u \rangle + f(x) \le f(y), \forall y \in H\}.$$

Let $x \in H$. Then f is subdifferentiable at x if $\partial f(x) \neq \emptyset$; the elements of $\partial f(x)$ are the subgradients of f at x.

Lemma 6 *[11] A mapping $T : H \to H$ is said to be:*

(i) L-Lipschitz continuous if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|, \ \forall\, x,\ y \in H;$$

(ii) κ-strongly monotone if there exists a constant $\kappa > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \kappa\|x - y\|^2, \ \forall\, x,\ y \in H.$$

Lemma 7 *[15] Let $\{\Upsilon_n\}$ be a positive sequence, $\{b_n\}$ be a sequence of real numbers, and $\{\alpha_n\}$ be a sequence in the open interval $(0, 1)$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$. Assume that*

$$\Upsilon_{n+1} \leq (1 - \alpha_n)\Upsilon_n + \alpha_n b_n, \ \ \forall\, n\ \geq\ 1.$$

If $\limsup_{k\to\infty} b_{n_k} \leq 0$ for every subsequence $\{\Upsilon_{n_k}\}$ of $\{\Upsilon_n\}$ satisfying $\liminf_{k\to\infty} (\Upsilon_{n_{k+1}} - \Upsilon_{n_k}) \leq 0$, then $\lim_{n\to\infty} \Upsilon_n = 0$.

3 Results

In this section, we consider a general case of the SFPMOS, in which C and Q_i are level sets of convex and subdifferential functions $c : H_1 \to \mathbb{R}$ and $q_i : H_i \to \mathbb{R}$ defined as follows:

$$C = \{x \in H : c(x) \leq 0\} \ \text{ and } \ Q_i = \{y \in H_i : q_i(y) \leq 0\}.$$

Let ∂c and ∂q denote the subdifferential of c and q. Then c and q are also weakly lower semicontinuous. We define the half-spaces C^n and Q_i^n $(i = 1, 2, ..., N)$ of C and Q_i, respectively:

$$\begin{aligned} C^n &:= \{x \in H : c(x_n) + \langle \xi_n, x - x_n\rangle \leq 0\}, \quad \xi_n \in \partial c(x_n), \\ Q_i^n &:= \{y \in H_i : q_i(A_i x_n) + \langle \eta_i^n, y - A_i x_n\rangle \leq 0\}, \quad \eta_i^n \in \partial q_i(A_i x_n). \end{aligned} \tag{9}$$

By the definition of the subgradient, it is easy to see that $C \subseteq C^n$ and $Q_i \subset Q_i^n$ ([9]).

Now, we introduce two self-adaptive relaxed CQ algorithms for solving the SFPMOS (2) in the case where the SFPMOS is consistent (i.e. $\Gamma \neq \emptyset$).

3.1 Two iterative algorithms for solving SFPMOS

Theorem 8 *Suppose the sequences $\{\lambda_n\}$, $\{\rho_n\}$ and $\{\alpha_n\}$ are in (0, 1) satisfying the following conditions:*

(i) $0 < a \leq \lambda_n \leq b < 1$, $0 < c \leq \rho_n \leq d < 1$;
(ii) $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Algorithm 1 Self-adaptive relaxed CQ algorithm I for SFPMOS

Step 0. Choose initial points x_0 arbitrary. Set $n = 1$.
Step 1. Set

$$\hat{d_n} = \|x_n - P_{C^n} x_n\|,$$
$$\bar{d_n} = \max\{\|A_i x_n - P_{Q_i^n} A_i x_n\|, i = 1, 2, ..., N\},$$
$$i_n \in \{i = 1, 2, ..., N : \|A_i x_n - P_{Q_i^n} A_i x_n\| = \bar{d_n}\}.$$

Step 2. Let $\Lambda_n = \max\{\hat{d_n}, \bar{d_n}\}$.
Case 1: $\hat{d_n} = \Lambda_n$. If $x_n = P_{C^n} x_n$, then stop; else compute

$$y_n = x_n - \lambda_n (x_n - P_{C^n} x_n). \tag{10}$$

Case 2: $\bar{d_n} = \Lambda_n$. If $A_{i_n} x_n = P_{Q_{i_n}^n} A_{i_n} x_n$, then stop; else compute

$$y_n = x_n - \tau_n A_{i_n}^* (A_{i_n} x_n - P_{Q_{i_n}^n} A_{i_n} x_n),$$

where

$$\tau_n = \rho_n \frac{\|A_{i_n} x_n - P_{Q_{i_n}^n} A_{i_n} x_n\|^2}{\|A_{i_n}^* (A_{i_n} x_n - P_{Q_{i_n}^n} A_{i_n} x_n)\|^2}. \tag{11}$$

Step 3. Compute

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) y_n. \tag{12}$$

Set $n = n + 1$ and go back to step 1.

Then $\{x_n\}$ generated by Algorithm 1 converges strongly to a solution x^*, where $x^* = P_\Gamma u$.

Proof Let $x^* \in \Gamma$. The proof is divided into four steps as follows.
Step 1. The sequence $\{x_n\}$ is bounded. We consider the following two cases:

Case 1: $\hat{d_n} = \Lambda_n$.
From the definition of $\{y_n\}$, we have

$$\begin{aligned}
\|y_n - x^*\|^2 &= \|x_n - \lambda_n (x_n - P_{C^n} x_n) - x^*\|^2 \\
&= \|x_n - x^*\|^2 + \lambda_n^2 \|x_n - P_{C^n} x_n\|^2 - 2\lambda_n \langle x_n - x^*, x_n - P_{C^n} x_n \rangle \\
&= \|x_n - x^*\|^2 + \lambda_n^2 \|x_n - P_{C^n} x_n\|^2 - 2\lambda_n \langle x_n - P_{C^n} x_n, x_n - P_{C^n} x_n \rangle \\
&\quad - 2\lambda_n \langle P_{C^n} x_n - x^*, x_n - P_{C^n} x_n \rangle \\
&= \|x_n - x^*\|^2 + \lambda_n^2 \|x_n - P_{C^n} x_n\|^2 - 2\lambda_n \|x_n - P_{C^n} x_n\|^2 \\
&\quad - 2\lambda_n \langle P_{C^n} x_n - x^*, x_n - P_{C^n} x_n \rangle.
\end{aligned} \tag{13}$$

Since $x^* \in C \subset C^n$, it follows from lemma 2 (i) that

$$\langle x_n - P_{C^n} x_n, P_{C^n} x_n - x^* \rangle \geq 0.$$

This implies that

$$\|y_n - x^*\|^2 \leq \|x_n - x^*\|^2 - 2\lambda_n(1-\lambda_n)\|x_n - P_{C^n} x_n\|^2. \tag{14}$$

Case 2: $\bar{d}_n = \Lambda_n$.
Using (11) and lemma 2 (i), we have

$$\begin{aligned}
\|y_n - x^*\|^2 &= \|x_n - \tau_n A_{i_n}^*(A_{i_n}x_n - P_{Q_{i_n}^n} A_{i_n}x_n) - x^*\|^2 \\
&= \|x_n - x^*\|^2 + \tau_n^2 \|A_{i_n}^*(A_{i_n}x_n - P_{Q_{i_n}^n} A_{i_n}x_n)\|^2 \\
&\quad - 2\tau_n \langle x_n - x^*, A_{i_n}^*(A_{i_n}x_n - P_{Q_{i_n}^n} A_{i_n}x_n)\rangle \\
&= \|x_n - x^*\|^2 + \tau_n^2 \|A_{i_n}^*(A_{i_n}x_n - P_{Q_{i_n}^n} A_{i_n}x_n)\|^2 \\
&\quad - 2\tau_n \langle A_{i_n}x_n - A_{i_n}x^*, A_{i_n}x_n - P_{Q_{i_n}^n} A_{i_n}x_n\rangle \\
&= \|x_n - x^*\|^2 + \tau_n^2 \|A_{i_n}^*(A_{i_n}x_n - P_{Q_{i_n}^n} A_{i_n}x_n)\|^2 \\
&\quad - 2\tau_n \langle A_{i_n}x_n - P_{Q_{i_n}^n} A_{i_n}x_n, A_{i_n}x_n - P_{Q_{i_n}^n} A_{i_n}x_n\rangle \\
&\quad - 2\tau_n \langle P_{Q_{i_n}^n} A_{i_n}x_n - A_{i_n}x^*, A_i x_n - P_{Q_{i_n}^n} A_{i_n}x_n\rangle \\
&\leq \|x_n - x^*\|^2 + \tau_n^2 \|A_{i_n}^*(A_{i_n}x_n - P_{Q_{i_n}^n} A_{i_n}x_n)\|^2 - 2\tau_n \|A_{i_n}x_n - P_{Q_{i_n}^n} A_{i_n}x_n\|^2 \\
&= \|x_n - x^*\|^2 + \left(\frac{\rho_n \|A_{i_n}x_n - P_{Q_{i_n}^n} A_{i_n}x_n\|^2}{\|A_{i_n}^*(A_{i_n}x_n - P_{Q_{i_n}^n} A_{i_n}x_n\|^2}\right)^2 \|A_{i_n}^*(A_i x_n - P_{Q_{i_n}^n} A_i x_n)\|^2 \\
&\quad - 2\left(\frac{\rho_n \|A_{i_n}x_n - P_{Q_{i_n}^n} A_{i_n}x_n\|^2}{\|A_{i_n}^*(A_{i_n}x_n - P_{Q_{i_n}^n} A_{i_n}x_n\|^2}\right)\|A_{i_n}x_n - P_{Q_{i_n}^n} A_{i_n}x_n\|^2 \\
&\leq \|x_n - x^*\|^2 - 2\rho_n(1-\rho_n)\frac{\|A_{i_n}x_n - P_{Q_{i_n}^n} A_{i_n}x_n\|^4}{\|A_{i_n}^*(A_{i_n}x_n - P_{Q_{i_n}^n} A_{i_n}x_n)\|^2}.
\end{aligned} \tag{15}$$

By (13) and (15), we obtain

$$\|y_n - x^*\| \leq \|x_n - x^*\|. \tag{16}$$

We next demonstrate that the sequence $\{x_n\}$ is bounded. Indeed, using (8) and (16), we get

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|\alpha_n u + (1-\alpha_n)y_n - x^*\|^2 \\
&= \|\alpha_n(u - x^*) + (1-\alpha_n)(y_n - x^*)\|^2 \\
&\leq \|(1-\alpha_n)(y_n - x^*)\|^2 + 2\alpha_n\langle u - x^*, x_{n+1} - x^*\rangle
\end{aligned}$$

$$\begin{aligned}&\le (1-\alpha_n)\|y_n-x^*\|^2+2\alpha_n\langle u-x^*, x_{n+1}-x^*\rangle\\&\le (1-\alpha_n)\|x_n-x^*\|^2+2\alpha_n\langle u-x^*, x_{n+1}-x^*\rangle.\end{aligned} \tag{17}$$

It follows from lemma 3 and (17) that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 \le\ &(1-\alpha_n)\|x_n-x^*\|^2+2\alpha_n\langle u-x^*, x_{n+1}-x^*\rangle\\ \le\ &(1-\alpha_n)\|x_n-x^*\|^2+2\alpha_n\|u-x^*\|\|x_{n+1}-x^*\|\\ \le\ &(1-\alpha_n)\|x_n-x^*\|^2+4\alpha_n\|u-x^*\|^2+\frac{1}{4}\alpha_n\|x_{n+1}-x^*\|^2\\ \le\ &\frac{1-\alpha_n}{1-\frac{1}{4}\alpha_n}\|x_n-x^*\|^2+\frac{\frac{3}{4}\alpha_n}{1-\frac{1}{4}\alpha_n}\frac{16}{3}\|u-x^*\|^2\\ \le\ &\max\{\|x_n-x^*\|^2, \frac{16}{3}\|u-x^*\|^2\} \qquad (18)\\ &\vdots\\ \le\ &\max\{\|x_0-x^*\|^2, \frac{16}{3}\|u-x^*\|^2\}. \qquad (19)\end{aligned}$$

This implies that the sequence $\{x_n\}$ is bounded, and $\{y_n\}$ and $\{A_i x_n\}$, $i = 1, ..., N$ as well.

Step 2. $\|x_n - P_{C^n} x_n\| \to 0$ and $\|A_{i_n} x_n - P_{Q^n_{i_n}} A_{i_n} x_n\| \to 0$ are hold as $n \to \infty$.

By (17), we obtain

$$\|x_{n+1}-x^*\|^2 \le (1-\alpha_n)\|x_n-x^*\|^2+2\alpha_n\langle u-x^*, x_{n+1}-x^*\rangle. \tag{20}$$

Setting $\Upsilon_n = \|x_n - x^*\|^2$, we obtain

$$\Upsilon_{n+1} \le (1-\alpha_n)\Upsilon_n+\alpha_n b_n, \tag{21}$$

where $b_n = 2\langle u - x^*, x_{n+1} - x^*\rangle$.

In view of lemma 7, it suffices to demonstrate that $\limsup_{k\to\infty} b_{n_k} \le 0$ for every subsequence Υ_{n_k} is an arbitrary subsequence of Υ_n satisfying

$$\liminf_{k\to\infty}(\Upsilon_{n_{k+1}}-\Upsilon_{n_k}) = 0.$$

From (17), we also obtain

$$\|x_{n+1}-x^*\|^2 \le \|y_n-x^*\|^2+2\alpha_n\langle u-x^*, x_{n+1}-x^*\rangle. \tag{22}$$

If we put (14) in (22), we obtain that

$$\begin{aligned}&\|x_{n+1}-x^*\|^2 \le \|x_n-x^*\|^2+2\alpha_n\langle u-x^*, x_{n+1}-x^*\rangle\\&-2\lambda_n(1-\lambda_n)\|x_n-P_{C^n}x_n\|^2.\end{aligned}$$

Using the hypothesis $\lambda_n \in [a, b] \subset (0, 1)$, we get from the inequality above that

$$\begin{aligned} \|x_n - P_{C^n} x_n\|^2 &\leq \frac{1}{2\lambda_n(1-\lambda_n)}(\Upsilon_n - \Upsilon_{n+1} + \alpha_n M) \\ &\leq \frac{1}{2a(1-b)}(\Upsilon_n - \Upsilon_{n+1} + \alpha_n M), \end{aligned} \tag{23}$$

where $M > 0$ is a constant such that $2\|u - x^*\|\|x_{n+1} - x^*\| \leq M$, for all $n \in N$.

Since $x^* \in \Gamma$, we note $A_i^*(I - P_{Q_i^n})A_i x^* = 0$. Hence, we can get

$$\|A_i^*(I - P_{Q_i^n})A_i x_n - A_i^*(I - P_{Q_i^n})A_i x^*\| \leq (\max_{1\leq i\leq N} \|A_i\|^2)\|x_n - x^*\|.$$

Since $\{x_n\}$ is also bounded, we have the sequence $\{\|A_i^*(I - P_{Q_i^n})A_i x_n\|\}_{n=1}^{\infty}$ is also bounded. Similarly, using (15), (22) and $\rho_n \in [c, d] \subset (0, 1)$, we see that

$$\|A_{i_n} x_n - P_{Q_{i_n}^n} A_{i_n} x_n\|^4 \leq \frac{K^2}{2c(1-d)}(\Upsilon_n - \Upsilon_{n+1} + \alpha_n M), \tag{24}$$

where $K > 0$ is a constant such that $\|A_{i_n}^*(I - P_{Q_{i_n}^n})A_{i_n} x_n\| \leq K$, for all $n \in N$.

Since $\Upsilon_n - \Upsilon_{n+1} \to 0$ and $\alpha_n \to 0$ $(n \to \infty)$, it follows from (23) and (24) that

$$\lim_{n\to\infty} \|x_n - P_{C^n} x_n\| = 0 \ \text{ and } \lim_{n\to\infty} \|A_{i_n} x_n - P_{Q_{i_n}^n} A_{i_n} x_n\| = 0. \tag{25}$$

Step 3. We show that $\omega_w(x_n) \subset \Gamma$.

Let $\hat{x} \in \omega_w(x_n)$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \hat{x}$ as $k \to \infty$. For each $i = 1, 2, ..., N$, since ∂c is bounded on bounded sets, there exists a constant $\xi > 0$ such that $\|\xi_n\| \leq \xi$ for all $n \geq 0$. Then, from the fact that $P_{C^n} x_n \in C^n$ and (9), it follows that

$$c(x_{n_k}) \leq \langle \xi_{n_k}, x_{n_k} - P_{C^{n_k}} x_{n_k}\rangle \leq \|\xi_{n_k}\|\|x_{n_k} - P_{C^{n_k}} x_{n_k}\| \leq \xi\|(I - P_{C^{n_k}})x_{n_k}\|.$$

By applying (25) and the weakly lower semicontinuity of c, we have

$$c(\hat{x}) \leq \liminf_{k\to\infty} c(x_{n_k}) \leq 0.$$

Consequently, $\hat{x} \in C$. In the same way, there exists a constant $\eta > 0$ such that $\|\eta_{i_n}^n\| \leq \eta$ for all $n \geq 0$. Since $P_{Q_{i_n}^n} A_{i_n} x_n \in Q_{i_n}^n$, it follows that

$$\begin{aligned} q_{i_n}(A_{i_n} x_{n_k}) &\leq \langle \eta_{i_n}^{n_k}, A_{i_n} x_{n_k} - P_{Q_{i_n}^n} A_{i_n} x_{n_k}\rangle \\ &\leq \|\eta_{i_n}^{n_k}\|\|A_{i_n} x_{n_k} - P_{Q_{i_n}^n} A_{i_n} x_{n_k}\| \\ &\leq \eta\|(I - P_{Q_{i_n}^{n_k}})A_{i_n} x_{n_k}\|. \end{aligned} \tag{26}$$

By applying (25) and the weakly lower semicontinuity of q_i, we have

$$q_{i_n}(A_{i_n}\hat{x}) \leq \liminf_{k\to\infty} q_{i_n}(A_{i_n}x_{n_k}) \leq 0.$$

According to the definition of i_n, it turns out that $A_i\hat{x} \in Q_i$ for $i = 1, 2, ..., N$. Consequently, we have $\hat{x} \in \Gamma$, which implies that $\omega_w(x_n) \subset \Gamma$.

Step 4. We prove that $\Upsilon_n \to 0$.

We now show that $\|x_{n_k+1} - x_{n_k}\| \to 0$ as $k \to \infty$. Indeed, it follows from the boundedness of the sequence $\{y_n\}$ and the assumption of $\alpha_n \to 0$ that

$$\|x_{n_k+1} - y_{n_k}\| = \alpha_{n_k}\|u - y_{n_k}\| \to 0, \ k \to \infty. \tag{27}$$

In Case 1, from the definition of y_n, λ_n, ρ_n and (25), we obtain that

$$\|y_{n_k} - x_{n_k}\| = \lambda_{n_k}\|x_{n_k} - P_{C^{n_k}}x_{n_k}\| \to 0, \ k \to \infty. \tag{28}$$

In Case 2, from the definition of y_n, τ_n, ρ_n and (25), we have

$$\begin{aligned} \|y_{n_k} - x_{n_k}\| &= \tau_{n_k}\|A_{i_n}^*(A_{i_n}x_{n_k} - P_{Q_{i_n}^{n_k}}A_{i_n}x_{n_k})\| \\ &= \rho_{n_k}\frac{\|A_{i_n}x_n - P_{Q_{i_n}^n}A_{i_n}x_n\|^2}{\|A_{i_n}^*(A_{i_n}x_n - P_{Q_{i_n}^n}A_{i_n}x_n)\|} \to 0, \ k \to \infty. \end{aligned} \tag{29}$$

Thus, from (28) and (29), we get $\|y_{n_k} - x_{n_k}\| \to 0$. Consequently, we have

$$\lim_{k\to\infty} \|x_{n_k+1} - x_{n_k}\| = 0. \tag{30}$$

Assume that $\{x_{n_{k_j}}\}$ is a subsequence of $\{x_{n_k}\}$, we also have

$$x_{n_{k_j}} \rightharpoonup \hat{x}, \ j \to \infty.$$

Furthermore, according to (30), $x^* = P_\Gamma u$ and lemma 2 (i), we have

$$\begin{aligned} \limsup_{k\to\infty} b_{n_k} &= \limsup_{k\to\infty} 2\langle u - x^*, x_{n_k+1} - x^*\rangle \\ &= \lim_{j\to\infty} 2\langle u - x^*, x_{n_{k_j}+1} - x^*\rangle \\ &= 2\langle u - x^*, \hat{x} - x^*\rangle \\ &\leq 0. \end{aligned} \tag{31}$$

Finally, applying Lemma 7 to (21), we conclude that $\Upsilon_n \to 0$, that is $x_n \to x^*$. This completes the proof. □

Assume that the sequence $\{x_n\}$ generated by Algorithm 2 is infinite. In other words, Algorithm 2 does not terminate in a finite number of iterations.

Algorithm 2 Self-adaptive relaxed CQ algorithm II for SFPMOS

Step 0. Choose initial points x_0 arbitrary. Set $n = 1$.
Step 1. Let

$$z_n = P_{C^n} x_n,$$

$$y_n^i = A_i^*(A_i x_n - P_{Q_i^n} A_i x_n),$$

$$d_n = \max\{\|x_n + y_i^n - z_n\|, i = 1, 2, ..., N\},$$

$$i_n \in \{i \in 1, 2, ..., N : \|x_n + y_n^i - z_n\| = d_n\}.$$

Step 2. If $d_n = 0$, then stop; else compute

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)[x_n - t_n(x_n + y_n^{i_n} - z_n)], \tag{32}$$

where

$$t_n = \rho_n \frac{\|x_n - P_{C^n} x_n\|^2 + \|A_{i_n} x_n - P_{Q_{i_n}^n} A_{i_n} x_n\|^2}{2\|x_n + y_n^{i_n} - z_n\|^2}. \tag{33}$$

Step 3. Set $n = n + 1$ and go back to step 1.

Lemma 9 *Let $\{x_n\}$ be a bounded sequence. If $\|x_n + y_{i_n}^n - z_n\| = 0$ holds if and only if x_n is a solution of Problem 2.*

Proof Assume $\|x_n + y_n^{i_n} - z_n\| = 0$ holds. For any $x^* \in \Gamma$, by lemma 2 (iii), we have

$$\begin{aligned}
&\|x_n - P_{C^n} x_n\|^2 + \|(I - P_{Q_{i_n}^n}) A_{i_n} x_n\|^2 \\
&\leq \langle x_n - P_{C^n} x_n, x_n - x^* \rangle + \langle (I - P_{Q_{i_n}^n}) A_{i_n} x_n, A_{i_n} x_n - A_{i_n} x^* \rangle \\
&= \langle x_n - P_{C^n} x_n + A_{i_n}^* (I - P_{Q_{i_n}^n}) A_{i_n} x_n, x_n - x^* \rangle \\
&= \langle x_n + y_n^{i_n} - z_n, x_n - x^* \rangle \\
&\leq \|x_n + y_n^{i_n} - z_n\| \|x_n - x^*\|.
\end{aligned} \tag{34}$$

Since $\{x_n\}$ is bounded and together with $d_n = 0$, we get

$$\|x_n - P_{C^n} x_n\| = 0 \quad \text{and} \quad \|(I - P_{Q_{i_n}^n}) A_{i_n} x_n\| = 0.$$

According to the definition of i_n, it follows from the above equation that

$$\|(I - P_{Q_i}^n) A_i x_n\| = 0, \text{ for all } i \in I.$$

Hence $x_n \in C^n$ and $A_i x_n \in Q_i^n$. From (9), we have that $x_n \in C$ and $A_i x_n \in \bigcap_{i=1}^N Q_i$. Therefore, $x_n \in \Gamma$. □

Theorem 10 *Suppose the sequences $\{\alpha_n\}$ and $\{\rho_n\}$ satisfying the following conditions:*
(C1) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C2) $0 < \liminf_{n\to\infty} \rho_n \leq \limsup_{n\to\infty} \rho_n < 4$.
Then the sequence $\{x_n\}$ generated by Algorithm 2 converges strongly to $x^* = P_\Gamma u$.

Proof The proof is divided into four steps as follows.

Step 1. The sequence $\{x_n\}$ is bounded.

Set $w_n = x_n - t_n(x_n + y_n^{i_n} - z_n)$. It follows from (33) and (34) that

$$\begin{aligned}
\|w_n - x^*\|^2 &= \|x_n - t_n(x_n + y_n^{i_n} - z_n) - x^*\|^2 \\
&= \|x_n - x^*\|^2 + t_n^2\|x_n + y_n^{i_n} - z_n\|^2 - 2t_n\langle x_n - x^*, x_n + y_n^{i_n} - z_n\rangle \\
&\le \|x_n - x^*\|^2 + t_n^2\|x_n + y_n^{i_n} - z_n\|^2 - 2t_n(\|x_n - P_{C^n}x_n\|^2 \\
&\quad + \|(I - P_{Q_{i_n}^n})A_{i_n}x_n\|^2) \\
&= \|x_n - x^*\|^2 - 4\rho_n(1 - \frac{1}{4}\rho_n)\frac{(\|x_n - P_{C^n}x_n\|^2 + \|A_{i_n}x_n - P_{Q_{i_n}^n}A_{i_n}x_n\|^2)^2}{\|x_n + y_n^{i_n} - z_n\|^2} \\
&\le \|x_n - x^*\|^2 - \rho_n(1 - \frac{1}{4}\rho_n)\frac{(\|x_n - P_{C^n}x_n\|^2 + \|A_{i_n}x_n - P_{Q_{i_n}^n}A_{i_n}x_n\|^2)^2}{\|x_n + y_n^{i_n} - z_n\|^2}.
\end{aligned} \tag{35}$$

It implies that

$$\|w_n - x^*\| \le \|x_n - x^*\|. \tag{36}$$

From (32) and (36), we get

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|\alpha_n u + (1 - \alpha_n)w_n - x^*\| \\
&= \|\alpha_n(u - x^*) + (1 - \alpha_n)(w_n - x^*)\| \\
&\le \alpha_n\|u - x^*\| + (1 - \alpha_n)\|w_n - x^*\| \\
&\le \alpha_n\|u - x^*\| + (1 - \alpha_n)\|x_n - x^*\| \\
&\le \max\{\|u - x^*\|, \|x_n - x^*\|\} \\
&\ \ \vdots \\
&\le \max\{\|u - x^*\|, \|x_0 - x^*\|\}.
\end{aligned} \tag{37}$$

Hence, $\{x_n\}$ is bounded and so are the sequences $\{Ax_n\}$, $\{P_C x_n\}$ and $\{P_{Q_i}x_n\}(i \in I)$.

Step 2. $\|x_n - P_{C^n}x_n\| \to 0$ and $\|A_{i_n}x_n - P_{Q_{i_n}^n}A_{i_n}x_n\| \to 0$ are hold, $n \to \infty$.

From (8) and (35), we deduce

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|\alpha_n u + (1 - \alpha_n)w_n - x^*\|^2 \\
&= \|\alpha_n(u - x^*) + (1 - \alpha_n)(w_n - x^*)\|^2 \\
&\le (1 - \alpha_n)\|w_n - x^*\|^2 + 2\alpha_n\langle u - x^*, x_{n+1} - x^*\rangle \\
&\le (1 - \alpha_n)\|x_n - x^*\|^2 + 2\alpha_n\langle u - x^*, x_{n+1} - x^*\rangle \\
&\quad -(1 - \alpha_n)\rho_n(1 - \frac{1}{4}\rho_n)\frac{(\|x_n - P_{C^n}x_n\|^2 + \|A_{i_n}x_n - P_{Q_{i_n}^n}A_{i_n}x_n\|^2)^2}{\|x_n + y_n^{i_n} - z_n\|^2}
\end{aligned}$$

$$\leq (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n\Big[2\langle u - x^*, x_{n+1} - x^*\rangle$$
$$-\frac{1-\alpha_n}{\alpha_n}\rho_n(1-\frac{1}{4}\rho_n)\frac{(\|x_n - P_{C^n}x_n\|^2 + \|A_{i_n}x_n - P_{Q^n_{i_n}}A_{i_n}x_n\|^2)^2}{\|x_n + y_n^{i_n} - z_n\|^2}\Big]. \quad (38)$$

Set $\Upsilon_n = \|x_n - x^*\|^2$, we have

$$\Upsilon_{n+1} \leq (1-\alpha_n)\Upsilon_n + \alpha_n b_n,\ n \geq 1, \quad (39)$$

where

$$b_n = 2\langle u - x^*, x_{n+1} - x^*\rangle - \frac{1-\alpha_n}{\alpha_n}\rho_n(1-\frac{1}{4}\rho_n)$$
$$\times\frac{(\|x_n - P_{C^n}x_n\|^2 + \|A_{i_n}x_n - P_{Q^n_{i_n}}A_{i_n}x_n\|^2)^2}{\|x_n + y_n^{i_n} - z_n\|^2}.$$

Now, we prove that $\Upsilon_n \to 0$ by lemma 7. Suppose that $\{\Upsilon_{n_k}\}$ is an arbitrary subsequence of satisfying

$$\liminf_{k\to\infty}(\Upsilon_{n_{k+1}} - \Upsilon_{n_k}) \geq 0. \quad (40)$$

Let $M > 0$ be a constant such that $2\|u - x^*\|\|x_{n+1} - x^*\| \leq M$, for all $n \in N$. By (38), we have

$$\begin{aligned}
&(1-\alpha_{n_k})\rho_{n_k}(1-\frac{1}{4}\rho_{n_k})\frac{(\|x_{n_k} - P_{C^{n_k}}x_{n_k}\|^2 + \|A_{i_n}x_{n_k} - P_{Q^{n_k}_{i_n}}A_{i_n}x_{n_k}\|^2)^2}{\|x_{n_k} + y_{n_k}^{i_n} - z_{n_k}\|^2}\\
&\leq (1-\alpha_{n_k})\|x_{n_k} - x^*\|^2 - \|x_{n_k+1} - x^*\|^2 + 2\alpha_{n_k}\langle u - x^*, x_{n_k+1} - x^*\rangle\\
&\leq \|x_{n_k} - x^*\|^2 - \|x_{n_k+1} - x^*\|^2 + 2\alpha_{n_k}\|u - x^*\|\|x_{n_k+1} - x^*\|\\
&\leq \Upsilon_{n_k} - \Upsilon_{n_{k+1}} + \alpha_{n_k}M.
\end{aligned} \quad (41)$$

From (40) and (41) together with conditions (C1) and (C2), we have that

$$\begin{aligned}
&\limsup_{k\to\infty}(1-\alpha_{n_k})\rho_{n_k}(1-\frac{1}{4}\rho_{n_k})\\
&\quad\frac{(\|x_{n_k} - P_{C^{n_k}}x_{n_k}\|^2 + \|A_{i_n}x_{n_k} - P_{Q^{n_k}_{i_n}}A_{i_n}x_{n_k}\|^2)^2}{\|x_{n_k} + y_{n_k}^{i_n} - z_{n_k}\|^2}\\
&\quad\leq \limsup_{k\to\infty}(\Upsilon_{n_k} - \Upsilon_{n_{k+1}} + \alpha_{n_k}M)\\
&\quad= -\liminf_{k\to\infty}(\Upsilon_{n_{k+1}} - \Upsilon_{n_k}) + \limsup_{k\to\infty}\alpha_{n_k}M\\
&\quad\leq 0.
\end{aligned} \quad (42)$$

Thus we see that

$$\lim_{k\to\infty} \frac{(\|x_{n_k} - P_{C^{n_k}} x_{n_k}\|^2 + \|A_{i_n} x_{n_k} - P_{Q_{i_n}^{n_k}} A_{i_n} x_{n_k}\|^2)^2}{\|x_{n_k} + y_{n_k}^{i_n} - z_{n_k}\|^2} = 0,$$

which yields

$$\|x_{n_k} - P_{C^{n_k}} x_{n_k}\| = 0, \quad \|A_{i_n} x_{n_k} - P_{Q_{i_n}^{n_k}} A_{i_n} x_{n_k}\| = 0. \tag{43}$$

Step 3. We show that $\omega_w(x_n) \subset \Gamma$.

Assume that $\hat{x} \in \omega_w(x_n)$ and $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ which converges weakly to $\hat{x}$. Since $P_{C^n} x_n \in C^n$ and (9), it follows that

$$c(x_{n_k}) \leq \langle \xi_{n_k}, x_{n_k} - P_{C^{n_k}} x_{n_k} \rangle \leq \|\xi_{n_k}\| \|x_{n_k} - P_{C^{n_k}} x_{n_k}\| \leq \xi \|(I - P_{C^{n_k}}) x_{n_k}\|, \tag{44}$$

where ξ satisfies $\|\xi_{n_k}\| \leq \xi$. By applying (43) and the weakly lower semicontinuity of c, we have

$$c(\hat{x}) \leq \liminf_{k\to\infty} c(x_{n_k}) \leq 0.$$

Consequently, $\hat{x} \in C$. In the same way, there exists a constant $\eta > 0$ such that $\|\eta_{i_n}^{n_k}\| \leq \eta$ for all $n \geq 0$.

Since $P_{Q_{i_n}^{n_k}}(A_{i_n} x_{n_k}) \in Q_{i_n}^{n_k}$, it follows that

$$\begin{aligned} q_{i_n}(A_{i_n} x_{n_k}) &\leq \langle \eta_{i_n}^{n_k}, A_{i_n} x_{n_k} - P_{Q_{i_n}^{n_k}} A_{i_n} x_{n_k} \rangle \\ &\leq \|\eta_{i_n}^{n_k}\| \|A_{i_n} x_{n_k} - P_{Q_{i_n}^{n_k}} A_{i_n} x_{n_k}\| \\ &\leq \eta \|(I - P_{Q_{i_n}^{n_k}}) A_{i_n} x_{n_k}\|. \end{aligned} \tag{45}$$

By applying (43) and the weakly lower semicontinuity of q_i, we have

$$q_{i_n}(A_{i_n} \hat{x}) \leq \liminf_{k\to\infty} q_{i_n}(A_{i_n} x_{n_k}) \leq 0. \tag{46}$$

It turns out that $A_i \hat{x} \in Q_i$ for $i = 1, 2, ..., N$. Consequently, we have $\hat{x} \in \Gamma$, which implies that $\omega_w(x_n) \subset \Gamma$.

Step 4. We prove that $x_n \to x^*$.

Moreover, we have the following estimation

$$\begin{aligned}\|x_{n_k+1}-x_{n_k}\| &= \|\alpha_{n_k}u+(1-\alpha_{n_k})(x_{n_k}-t_{n_k}(x_{n_k}+y_{n_k}^{i_n}-z_{n_k}))-x_{n_k}\|\\ &\le \alpha_n\|u-x_{n_k}\|+(1-\alpha_{n_k})t_{n_k}\|x_{n_k}+y_{n_k}^{i_n}-z_{n_k}\|\\ &\le \alpha_n\|u-x_{n_k}\|+t_{n_k}\|x_{n_k}+y_{n_k}^{i_n}-z_{n_k}\|\\ &\le \alpha_n\|u-x_{n_k}\|+\rho_{n_k}\\ &\quad\times\frac{\|x_{n_k}-P_{C^{n_k}}x_{n_k}\|^2+\|A_{i_{n_k}}x_{n_k}-P_{Q_{i_n}^{n_k}}A_{i_{n_k}}x_{n_k}\|^2}{2\|x_{n_k}+y_{n_k}^{i_n}-z_{n_k}\|}.\end{aligned} \tag{47}$$

Since $\{x_n\}$ is bounded, (43) together with (C1) and (C2), we have that

$$\lim_{k\to\infty}\|x_{n_k}-x_{n_k+1}\|=0. \tag{48}$$

Assume that $\{x_{n_{k_j}}\}$ is a subsequence of $\{x_{n_k}\}$, we also have $x_{n_{k_j}}\rightharpoonup\hat{x}$, as $j\to\infty$.
Furthermore, due to (48), $x^*=P_\Gamma u$ and lemma 2 (i), we infer that

$$\begin{aligned}\limsup_{k\to\infty} b_{n_k} &= \lim_{j\to\infty} b_{n_{k_j}}\\ &= \lim_{j\to\infty}\Big[2\langle u-x^*, x_{n_{k_j}+1}-x^*\rangle\\ -\frac{1-\alpha_{n_{k_j}}}{\alpha_{n_{k_j}}}&\rho_{n_{k_j}}(1-\frac{1}{4}\rho_{n_{k_j}})\frac{\left(\|x_{n_{k_j}}-P_{C^{n_{k_j}}}x_{n_{k_j}}\|^2+\|A_i x_{n_{k_j}}-P_{Q_{i_n}^{n_{k_j}}}A_i x_{n_{k_j}}\|^2\right)^2}{\|x_{n_{k_j}}+y_{n_{k_j}}^{i_n}-z_{n_{k_j}}\|^2}\Big]\\ &\le \lim_{j\to\infty} 2\langle u-x^*, x_{n_{k_j}+1}-x^*\rangle\\ &\le 2\langle u-x^*,\hat{x}-x^*\rangle\\ &\le 0.\end{aligned} \tag{49}$$

Finally, applying Lemma 7 to (39), we conclude that $\Gamma_n\to 0$, that is $x_n\to x^*$. This completes the proof. □

3.2 Two iterative algorithms for variational inequalities over the solution set of SFPMOS

Now, we introduce two self-adaptive algorithms for solving variational inequalities over the solution set of SFPMOS (7). Before convergence analysis of our algorithms, the following conditions are assumed.

Assumption 1 The control sequences satisfy the following conditions:

(i) $0<a\le\lambda_n\le b<1$, $0<c\le\rho_n\le d<1$;
(ii) $\{\beta_n\}\subset(0,1)$, $\lim_{n\to\infty}\beta_n=0$ and $\sum_{n=0}^{\infty}\beta_n=\infty$;
(iii) γ is a positive real number in the open interval $(0, 2\kappa/L^2)$.

Algorithm 3 Self-adaptive algorithm I for P2

Step 0. Choose initial points x_0 arbitrary. Set $n = 1$.
Step 1. Set

$$\hat{d_n} = \|x_n - P_{C^n} x_n\|,$$
$$\bar{d_n} = \max\{\|A_i x_n - P_{Q_i^n} A_i x_n\|, i = 1, 2, ..., N\},$$
$$i_n \in \{i = 1, 2, ..., N : \|A_i x_n - P_{Q_i^n} A_i x_n\| = \bar{d_n}\}.$$

Step 2. Let $\Lambda_n = \max\{\hat{d_n}, \bar{d_n}\}$.
Case 1: $\hat{d_n} = \Lambda_n$. If $x_n = P_{C^n} x_n$, then stop; else compute

$$y_n = x_n - \lambda_n (x_n - P_{C^n} x_n). \tag{50}$$

Case 2: $\bar{d_n} = \Lambda_n$. If $A_{i_n} x_n = P_{Q_{i_n}^n} A_{i_n} x_n$, then stop; else compute

$$y_n = x_n - \tau_n A_{i_n}^* (A_{i_n} x_n - P_{Q_{i_n}^n} A_{i_n} x_n),$$

where

$$\tau_n = \rho_n \frac{\|A_{i_n} x_n - P_{Q_{i_n}^n} A_{i_n} x_n\|^2}{\|A_{i_n}^* (A_{i_n} x_n - P_{Q_{i_n}^n} A_{i_n} x_n)\|^2}. \tag{51}$$

Step 3. Compute

$$x_{n+1} = (I - \beta_n \gamma F) y_n. \tag{52}$$

Set $n = n + 1$ and go back to step 1.

Lemma 11 *Suppose that $F : H \to H$ is L-Lipschitz and κ-strongly monotone, then (i)*

$$\|(I - \beta_n \gamma F) y_n - (I - \beta_n \gamma F) x^*\| \leq (1 - \beta_n \varphi) \|y_n - x^*\|,$$

where

$$\varphi = 1 - \sqrt{1 - \gamma(2\kappa - \gamma L^2)};$$

(ii)

$$\|x_{n+1} - x^*\| \leq \|x_n - x^*\| + \beta_n \gamma (\beta_n \gamma \|F y_n\|^2 - 2\langle y_n - x^*, F y_n \rangle)$$
$$-2\lambda_n (1 - \lambda_n) \|x_n - P_{C^n} x_n\|^2;$$

(iii)

$$\|x_{n+1} - x^*\| \leq \|x_n - x^*\| + \beta_n \gamma (\beta_n \gamma \|F y_n\|^2 - 2\langle y_n - x^*, F y_n \rangle)$$
$$- 2\rho_n (1 - \rho_n) \frac{\|A_i x_n - P_{Q_i^n} A_i x_n\|^4}{\|A_i^* (A_i x_n - P_{Q_i^n})\|^2}. \tag{53}$$

Proof (i) Taking any $x^* \in \Gamma$, it follows from Lemma 3.1 in [18] that

$$\|(I-\gamma F)y_n - (I-\gamma F)x^*\|^2 \leq (1-\gamma(2\kappa - \gamma L^2))\|y_n - x^*\|^2. \tag{54}$$

From (54), we have

$$\begin{aligned}
&\|(I-\beta_n\gamma F)y_n - (I-\beta_n\gamma F)x^*\| \\
&= \|(I-\beta_n)(y_n - x^*) + \beta_n(y_n - x^*) - \beta_n\gamma F y_n + \beta_n\gamma F x^*\| \\
&\leq (I-\beta_n)\|y_n - x^*\| + \beta_n\|(I-\gamma F)y_n - (I-\gamma F)x^*\| \\
&\leq (I-\beta_n)\|y_n - x^*\| + \beta_n\sqrt{1-\gamma(2\kappa-\gamma L^2)}\|y_n - x^*\|.
\end{aligned} \tag{55}$$

(ii) Using (14), we get

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|(I-\beta_n\gamma F)y_n - x^*\|^2 \\
&\leq \|y_n - x^* - \beta_n\gamma F y_n\|^2 \\
&\leq \|y_n - x^*\|^2 + \beta_n^2\gamma^2\|F y_n\|^2 - 2\beta_n\gamma\langle y_n - x^*, F y_n\rangle \\
&\leq \|x_n - x^*\|^2 + \beta_n\gamma(\beta_n\gamma\|F y_n\|^2 \\
&\quad - 2\langle y_n - x^*, F y_n\rangle) - 2\lambda_n(1-\lambda_n)\|x_n - P_{C^n}x_n\|^2.
\end{aligned} \tag{56}$$

(iii) Similar to (ii), through (15), we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 \leq \|x_n - x^*\|^2 &+ \beta_n\gamma(\beta_n\gamma\|F y_n\|^2 - 2\langle y_n - x^*, F y_n\rangle) \\
&- 2\rho_n(1-\rho_n)\frac{\|A_i x_n - P_{Q_i{}^n}A_i x_n\|^4}{\|A_i^*(A_i x_n - P_{Q_i^n})\|^2}.
\end{aligned} \tag{57}$$

□

Theorem 12 *Let $F : H \to H$ be an L-Lipschitz and κ-strongly monotone operator and under Assumption 1. Then the sequence $\{x_n\}$ generated by Algorithm 3 converges strongly to the unique solution of $VIP(F, \Gamma)$.*

Proof Let x^* be the unique solution of the variational inequality $VIP(F, \Gamma)$, that is,

$$\langle F x^*, z - x^*\rangle \geq 0,\ \forall z \in \Gamma. \tag{58}$$

From (16) in Theorem 8, (52) and lemma 11 (i), we have

$$\begin{aligned}
\|x_{n+1}-x^*\| &= \|(I-\beta_n\gamma F)y_n - x^*\| \\
&= \|(I-\beta_n\gamma F)y_n - (I-\beta_n\gamma F)x^* - \beta_n\gamma Fx^*\| \\
&\leq \|(I-\beta_n\gamma F)y_n - (I-\beta_n\gamma F)x^*\| + \beta_n\gamma\|Fx^*\| \\
&= (1-\beta_n\varphi)\|y_n - x^*\| + \beta_n\gamma\|Fx^*\| \\
&\leq (1-\beta_n\varphi)\|x_n - x^*\| + \beta_n\varphi\frac{\gamma}{\varphi}\|Fx^*\| \\
&\leq \max\{\|x_n - x^*\|, \frac{\gamma}{\varphi}\|Fx^*\|\} \\
&\vdots \\
&\leq \max\{\|x_0 - x^*\|, \frac{\gamma}{\varphi}\|Fx^*\|\}.
\end{aligned} \tag{59}$$

This implies that the sequence $\{x_n\}$ is bounded, and $\{y_n\}$ and $\{A_i x_n\}$, $i = 1, ..., N$ as well. Using lemma 11 (i), we get that

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &= \langle (I-\beta_n\gamma F)y_n - x^*, x_{n+1}-x^*\rangle \\
&= \langle (I-\beta_n\gamma F)y_n - (I-\beta_n\gamma F)x^*, x_{n+1}-x^*\rangle - \beta_n\gamma\langle Fx^*, x_{n+1}-x^*\rangle \\
&\leq \frac{\|(I-\beta_n\gamma F)y_n - (I-\beta_n\gamma F)x^*\|^2 + \|x_{n+1}-x^*\|^2}{2} \\
&\quad - \beta_n\gamma\langle Fx^*, x_{n+1}-x^*\rangle \\
&\leq \frac{(I-\beta_n\varphi)\|y_n - x^*\|^2 + \|x_{n+1}-x^*\|^2}{2} - \beta_n\gamma\langle Fx^*, x_{n+1}-x^*\rangle \\
&\leq \frac{(I-\beta_n\varphi)\|x_n - x^*\|^2 + \|x_{n+1}-x^*\|^2}{2} - \beta_n\gamma\langle Fx^*, x_{n+1}-x^*\rangle.
\end{aligned} \tag{60}$$

Hence, we infer that

$$\|x_{n+1}-x^*\|^2 \leq (I-\beta_n\varphi)\|x_n - x^*\|^2 - 2\beta_n\gamma\langle Fx^*, x_{n+1}-x^*\rangle.$$

Set $\Upsilon_n = \|x_n - x^*\|^2$, we have

$$\Upsilon_{n+1} \leq (1-\alpha_n)\Upsilon_n + \alpha_n b_n, \tag{61}$$

where $\alpha_n = \beta_n\varphi$ and $b_n = -\frac{2\gamma}{\varphi}\langle Fx^*, x_{n+1}-x^*\rangle$.

Now, we prove that $\Upsilon_n \to 0$ by using lemma 7. To this end, suppose that Υ_{n_k} is an arbitrary subsequence of Υ_n satisfying

$$\liminf_{k\to\infty}(\Upsilon_{n_{k+1}} - \Upsilon_{n_k}) = 0. \tag{62}$$

Since $\{x_n\}$ is bounded, $\{y_n\}$ is bounded too. Hence, there exists a positive real number $\bar{M}$ such that $\max\{\sup_n \|y_n\|, \sup_n \|Fy_n\|\} \leq \bar{M}$.

In Case 1, from lemma 11 (ii) and $\{\lambda_n\} \subset [a, b] \subset (0, 1)$, we get

$$2a(1-b)\|x_{n_k} - P_{C^{n_k}} x_{n_k}\|^2 \leq \Upsilon_{n_k} - \Upsilon_{n_{k+1}} + \beta_n\gamma(\beta_n\gamma\bar{M} + 2\bar{M} + 2\|x^*\|)\bar{M}.$$

Thus, by (62) and $\beta_n \to 0$, we obtain that

$$\begin{aligned}
&\limsup_{k\to\infty} \|x_{n_k} - P_{C^{n_k}} x_{n_k}\|^2 \\
&\le \frac{1}{2a(1-b)} \limsup_{k\to\infty} [\Upsilon_{n_k} - \Upsilon_{n_k+1} + \beta_n\gamma(\beta_n\gamma\bar{M} + 2\bar{M} + 2\|x^*\|)\bar{M}] \\
&= \frac{1}{2a(1-b)}\Big[\limsup_{k\to\infty}(\Upsilon_{n_k} - \Upsilon_{n_k+1}) + \limsup_{k\to\infty} \beta_n\gamma(\beta_n\gamma\bar{M} + 2\bar{M} + 2\|x^*\|)\bar{M}\Big] \\
&= \frac{1}{2a(1-b)}\Big[-\liminf_{k\to\infty}(\Upsilon_{n_k} - \Upsilon_{n_k+1}) + \limsup_{k\to\infty} \beta_n\gamma(\beta_n\gamma\bar{M} + 2\bar{M} + 2\|x^*\|)\bar{M}\Big] \\
&\le 0.
\end{aligned} \tag{63}$$

In Case 2, the same process as above, by lemma 11 (ii) and the condition $\{\rho_n\} \subset [c, d] \subset (0, 1)$, we infer that

$$\begin{aligned}
&2c(1-d)\|A_{i_n}x_{n_k} - P^{n_k}_{Q_{i_n}} A_{i_n}x_{n_k}\|^4 \\
&\quad \le K^2(\Upsilon_{n_k} - \Upsilon_{n_{k+1}} + \beta_n\gamma(\beta_n\gamma\bar{M} + 2\bar{M} + 2\|x^*\|)\bar{M}),
\end{aligned}$$

where $K > 0$ is a constant such that $\|A^*_{i_n}(I - P_{Q^n_{i_n}})A_{i_n}x_n\| \le K$, for all $n \in N$. Thus, we obtain that

$$\begin{aligned}
&\limsup_{k\to\infty} \|A_{i_n}x_{n_k} - P^{n_k}_{Q_{i_n}} A_{i_n}x_{n_k}\|^4 \\
&\le \frac{K^2}{c(1-d)} \limsup_{k\to\infty} [\Upsilon_{n_k} - \Upsilon_{n_k+1} + \beta_n\gamma(\beta_n\gamma\bar{M} + 2\bar{M} + 2\|x^*\|)\bar{M}] \\
&\le 0.
\end{aligned} \tag{64}$$

This implies that

$$\|x_{n_k} - P_{C^{n_k}} x_{n_k}\| = 0, \quad \|A_{i_n}x_{n_k} - P_{Q^{n_k}_{i_n}} A_{i_n}x_{n_k}\| = 0.$$

By the same process as Step 3 of Theorem 1, we obtain $\omega_w(x_n) \subset \Gamma$. Let $\hat{x} \in \omega_w(x_n)$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \hat{x}$ as $k \to \infty$.

Next, we show that $\limsup_{k\to\infty} b_{n_k} \le 0$. Assume that $\{x_{n_{k_j}}\}$ is a subsequence of $\{x_{n_k}\}$, we also have $x_{n_{k_j}} \rightharpoonup \hat{x}$. Thus, using the above mentioned and (58), we have

$$\begin{aligned}
\liminf_{k\to\infty} \langle Fx^*, x_{n_k} - x^*\rangle &= \lim_{j\to\infty} \langle Fx^*, x_{n_{k_j}} - x^*\rangle \\
&= \langle Fx^*, \hat{x} - x^*\rangle \\
&\ge 0.
\end{aligned} \tag{65}$$

From (27)–(30) in Theorem 8, we have

$$\lim_{k\to\infty} \|x_{n_k+1} - x_{n_k}\| = 0. \tag{66}$$

From (61) and (66), we have

$$\begin{aligned}\limsup_{k\to\infty} b_{n_k} &= \limsup_{k\to\infty} -\frac{2\gamma}{\varphi}\langle Fx^*, x_{n_k+1} - x^*\rangle \\ &= -\frac{2\gamma}{\varphi}\liminf_{k\to\infty}\langle Fx^*, x_{n_k+1} - x^*\rangle \\ &= -\frac{2\gamma}{\varphi}\langle Fx^*, \hat{x} - x^*\rangle \\ &\leq 0.\end{aligned} \tag{67}$$

Therefore we can immediately conclude that $\Upsilon_n \to 0$, that is, $x_n \to x^*$ as $n \to \infty$. This completes the proof of the theorem. □

Algorithm 4 Self-adaptive algorithm II for P2

Step 0. Choose initial points x_0 arbitrary. Set $n = 1$.
Step 1. Let

$$\begin{aligned} z_n &= P_{C^n} x_n, \\ y_n^i &= A_i^*(A_i x_n - P_{Q_i^n} A_i x_n), \\ d_n &= \max\{\|x_n + y_i^n - z_n\|, i = 1, 2, ..., N\}, \\ i_n &\in \{i \in 1, 2, ..., N : \|x_n + y_n^i - z_n\| = d_n\}. \end{aligned}$$

Step 2. If $d_n = 0$, then stop; else compute

$$x_{n+1} = (I - \beta_n \gamma F)(x_n - t_n(x_n + y_n^{i_n} - z_n)), \tag{68}$$

where

$$t_n = \rho_n \frac{\|x_n - P_{C^n} x_n\|^2 + \|A_{i_n} x_n - P_{Q_{i_n}^n} A_{i_n} x_n\|^2}{2\|x_n + y_n^{i_n} - z_n\|^2}.$$

Step 3. Set $n = n + 1$ and go back to step 1.

Theorem 13 *Let $F : H \to H$ be an L-Lipschitz and κ-strongly monotone operator and under assumption 1. Then the sequence $\{x_n\}$ generated by Algorithm 4 converges strongly to the unique solution of $VIP(F, \Gamma)$.*

Proof Set $w_n = x_n - t_n(x_n + y_n^{i_n} - z_n)$. It follows from (35) that

$$\|w_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \rho_n(1 - \frac{1}{4}\rho_n)$$

$$\times \frac{(\|x_n - P_{C^n} x_n\|^2 + \|A_{i_n} x_n - P_{Q^n_{i_n}} A_{i_n} x_n\|^2)^2}{\|x_n + y_n^{i_n} - z_n\|^2},$$

which implies

$$\|w_n - x^*\| \le \|x_n - x^*\|.$$

Using the above inequalities, we get

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|(I - \beta_n \gamma F) w_n - x^*\|^2 \\
&\le \|w_n - x^* - \beta_n \gamma F w_n\|^2 \\
&\le \|w_n - x^*\|^2 + \beta_n^2 \gamma^2 \|F w_n\|^2 - 2\beta_n \gamma \langle w_n - x^*, F w_n\rangle \\
&\le \|x_n - x^*\|^2 + \beta_n \gamma (\beta_n \gamma \|F y_n\|^2 - 2\langle y_n - x^*, F y_n\rangle) \\
&\quad - \rho_n (1 - \frac{1}{4}\rho_n) \frac{(\|x_n - P_{C^n} x_n\|^2 + \|A_{i_n} x_n - P_{Q^n_{i_n}} A_{i_n} x_n\|^2)^2}{\|x_n + y_n^{i_n} - z_n\|^2}.
\end{aligned} \tag{69}$$

According to (59), replacing w_n by y_n, we get that $\{x_n\}$ is bounded.

From (60)–(61), the following result obviously holds. Set $\Upsilon_n = \|x_n - x^*\|^2$, we have

$$\Upsilon_{n+1} \le (1 - \alpha_n)\Upsilon_n + \alpha_n b_n, \tag{70}$$

where $\alpha_n = \beta_n \varphi$ and $b_n = -\frac{2\gamma}{\varphi}\langle F x^*, x_{n+1} - x^*\rangle$.

Now, we prove that $\Upsilon_n \to 0$ by lemma 7. Suppose that $\{\Upsilon_{n_k}\}$ is an arbitrary subsequence of satisfying

$$\liminf_{k\to\infty} (\Upsilon_{n_{k+1}} - \Upsilon_{n_k}) \ge 0. \tag{71}$$

From (69), we get

$$\begin{aligned}
&\rho_n (1 - \frac{1}{4}\rho_n) \frac{(\|x_n - P_{C^n} x_n\|^2 + \|A_{i_n} x_n - P_{Q^n_{i_n}} A_{i_n} x_n\|^2)^2}{\|x_n + y_n^{i_n} - z_n\|^2} \le \Upsilon_n - \Upsilon_{n+1} \\
&+ \beta_n \gamma (\beta_n \gamma \|F y_n\|^2 - 2\langle y_n - x^*, F y_n\rangle).
\end{aligned} \tag{72}$$

Thus, from Assumption 1 (ii) and (62), we obtain

$$\begin{aligned}
&\limsup_{k\to\infty} \rho_{n_k} (1 - \frac{1}{4}\rho_{n_k}) \frac{(\|x_{n_k} - P_{C^{n_k}} x_{n_k}\|^2 + \|A_{i_n} x_{n_k} - P_{Q^{n_k}_{i_n}} A_{i_n} x_{n_k}\|^2)^2}{\|x_{n_k} + y_{n_k}^{i_n} - z_{n_k}\|^2} \\
&\le \limsup_{k\to\infty} [\Upsilon_{n_k} - \Upsilon_{n_{k+1}} + \beta_{n_k} \gamma (\beta_{n_k} \gamma \bar{M} + 2\bar{M} + 2\|x^*\|)\bar{M}]
\end{aligned} \tag{73}$$

$$= -\liminf_{k\to\infty}(\Upsilon_{n_{k+1}} - \Upsilon_{n_k}) + \limsup_{k\to\infty}\beta_{n_k}\gamma(\beta_{n_k}\gamma\bar{M} + 2\bar{M} + 2\|x^*\|)\bar{M} \tag{74}$$
$$\le 0,$$

where $\bar{M}$ is a positive real number satisfying $\max\{\sup_n \|y_n\|, \sup_n \|Fy_n\|\} \le \bar{M}$.

Under Assumption 1, we see that

$$\lim_{k\to\infty} \frac{(\|x_{n_k} - P_{C^{n_k}} x_{n_k}\|^2 + \|A_{i_n} x_{n_k} - P_{Q_{i_n}^{n_k}} A_{i_n} x_{n_k}\|^2)^2}{\|x_{n_k} + y_{n_k}^{i_n} - z_{n_k}\|^2} = 0,$$

which yields

$$\|x_{n_k} - P_{C^{n_k}} x_{n_k}\| = 0, \quad \|A_{i_n} x_{n_k} - P_{Q_{i_n}^{n_k}} A_{i_n} x_{n_k}\| = 0. \tag{75}$$

By the same process as Step 3 of Theorem 1, we obtain $\omega_w(x_n) \subset \Gamma$. Furthermore, based on the assumptions of the parameters, we obtain

$$\|x_{n_k+1} - w_{n_k}\| = \beta_{n_k}\gamma\|Fw_{n_k}\| \le \bar{M}\gamma\beta_{n_k} \to 0, \quad k \to \infty. \tag{76}$$

and

$$\begin{aligned}\|w_{n_k} - x_{n_k}\| &= t_{n_k}\|x_{n_k} + y_{n_k}^{i_n} - z_{n_k}\| \\ &= \rho_{n_k}\frac{\|x_{n_k} - P_{C^{n_k}} x_{n_k}\|^2 + \|A_{i_{n_k}} x_{n_k} - P_{Q_{i_n}^{n_k}} A_{i_{n_k}} x_{n_k}\|^2}{2\|x_{n_k} + y_{n_k}^{i_n} - z_{n_k}\|} \\ &\to 0, \quad k \to \infty.\end{aligned} \tag{77}$$

From (76) and (77), we get

$$\|x_{n_k+1} - x_{n_k}\| \to 0, \quad k \to \infty.$$

Therefore, we infer that

$$\begin{aligned}\limsup_{k\to\infty} b_{n_k} &= \limsup_{k\to\infty} -\frac{2\gamma}{\varphi}\langle Fx^*, x_{n_k+1} - x^*\rangle \\ &= -\frac{2\gamma}{\varphi}\liminf_{k\to\infty}\langle Fx^*, x_{n_k+1} - x^*\rangle \\ &= -\frac{2\gamma}{\varphi}\langle Fx^*, \hat{x} - x^*\rangle \\ &\le 0.\end{aligned} \tag{78}$$

Finally, applying Lemma 7 to (70), we conclude that $\Gamma_n \to 0$, that is $x_n \to x^*$. This completes the proof. □

Remark 1 We have the following comments for Algorithm 1–4.

- Algorithm 1–4 are based on the gradient algorithm with selection technique. In each iteration, Algorithm 1 only needs to compute two projections, one from P_{C^n} and another one from $\{P_{Q_i^n} A_i x_n\}_{i=1}^N$. Algorithm 2 only needs to compute the projection of $\{x_n - P_{C^n} x_n + A_i^* P_{Q_i^n} A_i x_n\}$ in each iteration. This technique reduces the computational effort of the projection and speeds up the rate of convergence of algorithms.
- Different selection methods of the proposed algorithms result in different representation of adaptive steps. Algorithm 1 and Algorithm 2 are two different projection and Halpern-type algorithms, which results in a different range of step sizes. Algorithm 1 and Algorithm 3 use the same step size, as do Algorithm 2 and Algorithm 4.

4 Numerical experiment

In this section, we provide some numerical examples to prove we proposed the convergence of the algorithm. All codes for the numerical computations are implemented using MATLAB R2015b. The numerical results are carried out on a personal computer with an Intel(R) Core(TM) i5-1035G1 CPU @ 1.00GHz 1.19 GHz.

Using $\|x_{n+1} - x_n\| < 10^{-6}$ as the stopping criterion, we plot the graphs of $TOL = \|x_{n+1} - x_n\|^2$ against the number of iterations for each n. The numerical results are reported in Figs. 1 and 2 and Tables 1 and 2.

Example 1. [12] Consider $H = \mathbb{R}^{10}$, $H_1 = \mathbb{R}^{20}$, $H_2 = \mathbb{R}^{30}$ and $H_3 = \mathbb{R}^{40}$. Find a point $x^* \in \mathbb{R}^{10}$ such that

$$x^* \in \Gamma := C \cap A_1^{-1}(Q_1) \cap A_2^{-1}(Q_2) \cap A_3^{-1}(Q_3) \neq \emptyset,$$

where $C = \{x \in \mathbb{R}^{10} : \|x - c\|^2 \leq r^2\}$, $Q_i = \{x \in \mathbb{R}^{(i+1)\times 10} : \|x - c_i\|^2 \leq r_i^2\}$, and $A_1 : \mathbb{R}^{10} \to \mathbb{R}^{20}$, $A_2 : \mathbb{R}^{10} \to \mathbb{R}^{30}$, $A_3 : \mathbb{R}^{10} \to \mathbb{R}^{40}$ are bounded linear operators the elements of the representing matrices of which are randomly generated in the closed interval [-5, 5]. In this case, for any $x \in \mathbb{R}^{10}$, we have $c(x) = \|x - c\|^2 - r^2$ and $q_i(A_i x) = \|A_i x - c_i\|^2 - r_i^2$ for $i = 1, 2, 3$.

The half-spaces C^n and $Q_i^n (i = 1, 2, 3)$, are defined by

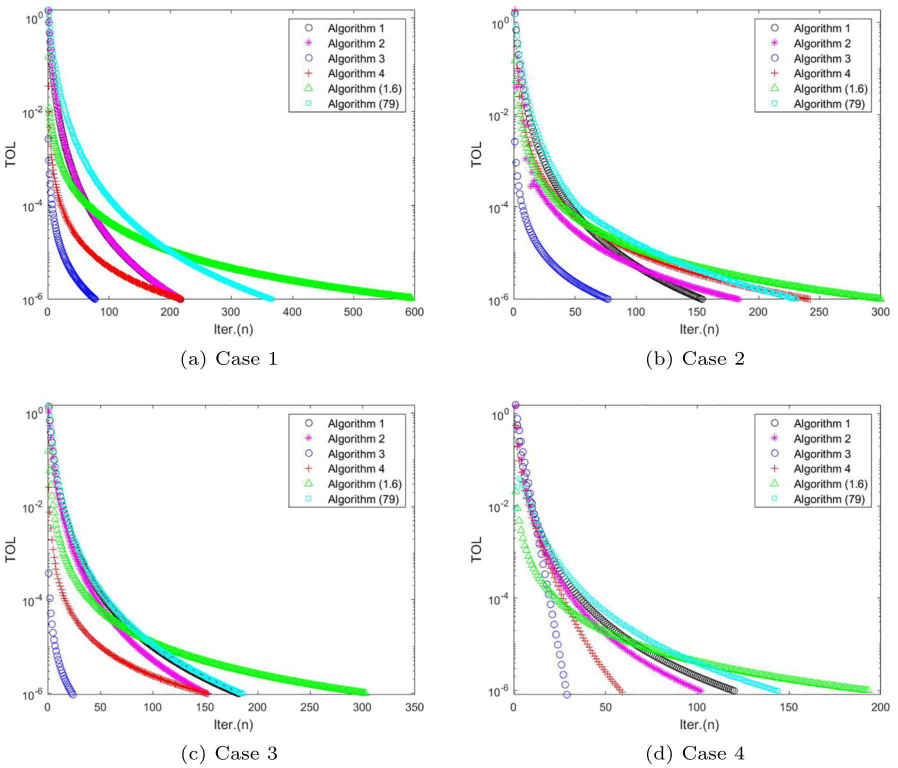
$$C^n = \{x \in \mathbb{R}^{10} : \|x_n - c\|^2 - r^2 \leq 2\langle x_n - c, x_n - x\rangle\},$$

and

$$Q_i^n = \{z \in \mathbb{R}^{20} : \|A_i x_n - c_i\|^2 - r_i^2 \leq 2\langle A_i x_n - c_i, A_i x_n - y\rangle\}.$$

The control parameters for each algorithm are chosen as follows.

- Algorithm 1 and Algorithm 2 (Our new algorithm 1 and 2): $\lambda_n = \tau_n = \frac{1}{10n+1}$ and $\alpha_n = \frac{1}{n+1}$.



(a) Case 1

(b) Case 2

(c) Case 3

(d) Case 4

Fig. 1 Example 1: Comparison of all algorithms in different cases

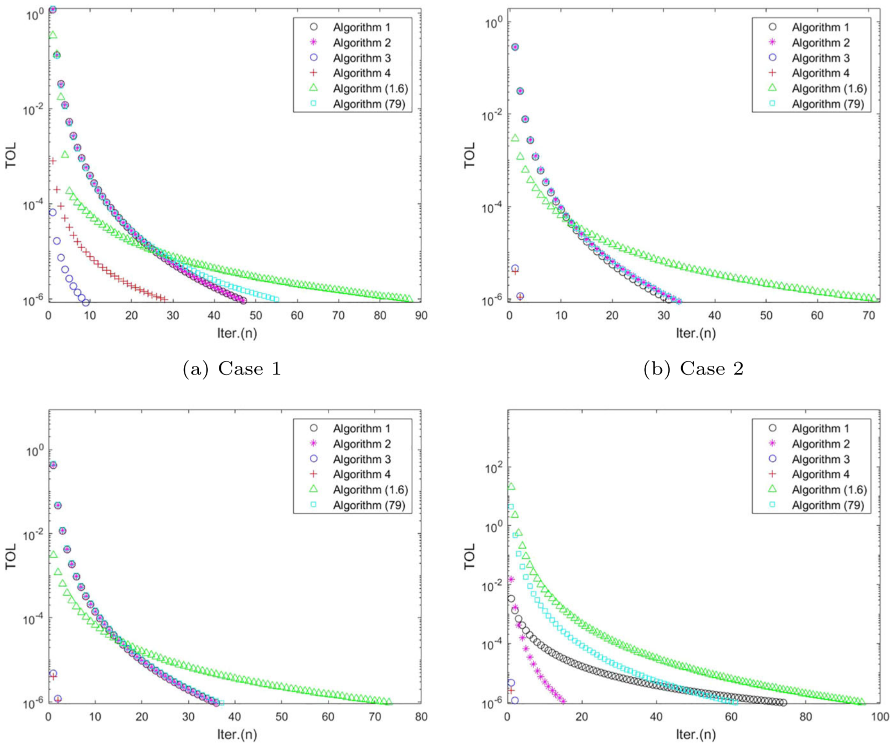
- Algorithm 3 and Algorithm 4 (Our new algorithm 3 and 4): $\gamma = 0.15$, $\beta_n = \frac{1}{100n+1}$, and $Fx = 2x$ for all $x \in \mathbb{R}^{10}$.
- Algorithm (1.6) (in (4), [12]): $\lambda_1 = \frac{1}{6}$, $\lambda_2 = \frac{1}{3}$, $\lambda_3 = \frac{1}{2}$, and $f(x) = 0.975x$ for all $x \in \mathbb{R}^{10}$.
- Algorithm (79) (in [16]) is as follows:

$$x_{n+1} = \alpha_n u + (1-\alpha_n)(x_n - \rho_1^n (I - P_{C^n})x_n - \tau_n \sum_{i=1}^{N} \vartheta_i A_i^*(I - P_{Q_i^n})A_i x_n)),$$

where $\tau_n := \frac{\rho_2^n \sum_{i=1}^{N} \vartheta_i \|(I-P_{Q_i^n})A_i x_n\|^2}{\bar{\tau_n}^2}$, $\bar{\tau_n} := \max\{\| \sum_{i=1}^{N} \vartheta_i A_i^*(I - P_{Q_i^n})A_i x_n\|, \beta\}$.

We take $\rho_1^n = \rho_2^n = \frac{1}{10n+1}$, $\vartheta_1 = \frac{1}{6}$, $\vartheta_2 = \frac{1}{3}$ $\vartheta_3 = \frac{1}{2}$, and $\beta = 0.05$.
Now we bring the results of the iterations for four cases, where $e_1 = (1, 1, ..., 1) \in \mathbb{R}^{10}$.

Case 1: Take $u = 10e_1$ and $x_1 = 5e_1$;
Case 2: Take $u = 3e_1$ and $x_1 = 0.5e_1$;
Case 3: Take $u = 3e_1$ and $x_1 = -0.5e_1$;
Case 4: Take $u = e_1$ and $x_1 = -0.2e_1$.



(a) Case 1 (b) Case 2

(c) Case 3 (d) Case 4

Fig. 2 Example 2: Comparison of all algorithms in different cases

Table 1 Numerical results of all algorithms with different x_0 and u

Algorithm	Case 1		Case 2		Case 3		Case 4	
	CPU(s)	Iter.(n)	CPU(s)	Iter.(n)	CPU(s)	Iter.(n)	CPU(s)	Iter.(n)
Alg. 1	0.1024	220	0.1417	157	0.1275	182	0.0668	115
Alg. 2	0.1165	225	0.1439	193	0.1123	155	0.0568	104
Alg. 3	0.0434	77	0.0973	124	0.0398	24	0.0756	29
Alg. 4	0.1594	215	0.1327	224	0.1143	153	0.0506	60
Alg. (1.6)	0.2311	596	0.0913	300	0.1374	303	0.1211	193
Alg. (79)	0.1399	356	0.1006	235	0.1061	190	0.0765	146

Example 2. Let $H = H_1 = L_2([0, 1])$. For all $y \in L_2([0, 1])$, $z \in L_2([0, 1])$, $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ are defined by $\langle y, z \rangle := \int_0^1 y(t)z(t)dt$ and $\|y\| := (\int_0^1 |y(t)|^2 dt)^{\frac{1}{2}}$, respectively. Furthermore, we consider the half-spaces

$$C := \{x \in L_2([0, 1]) | \int_0^1 x(t)dt \leq 1\}$$

Table 2 Numerical results of all algorithms with different x_0 and u

Algorithm	Case 1		Case 2		Case 3		Case 4	
	CPU(s)	Iter.(n)	CPU(s)	Iter.(n)	CPU(s)	Iter.(n)	CPU(s)	Iter.(n)
Alg. 1	15.3756	47	23.5493	31	43.9618	36	57.0047	74
Alg. 2	50.3079	47	41.8029	31	67.0135	36	2.7160	3
Alg. 3	3.0016	9	0.8664	3	2.3713	3	2.7160	3
Alg.4	23.8211	28	2.6605	3	2.8622	3	0.1429	2
Alg. (1.6)	142.2924	87	128.1796	71	106.4480	73	239.2246	95
Alg.(79)	80.0467	55	132.4689	33	103.2844	37	214.5914	61

and

$$Q := \{x \in L_2([0,1]) | \int_0^1 |y(t) - sint|^2 dt \leq 16\}.$$

And $A : L_2([0,1]) \to L_2([0,1])$, where $(Ax)(t) = \int_0^1 x(t)dt, \forall t \in [0,1], x \in L_2([0,1])$. Then, A is a bounded linear operator, $\|A\| = \frac{2}{\pi}$ and $(A^*x)(t) = \int_1^t x(s)ds$.

The metric projections P_C and P_Q are defined by

$$P_C(x(t)) = \begin{cases} x(t) + 1 - a, & a > 1, \\ x(t), & a \leq 1, \end{cases} \tag{79}$$

and

$$P_Q(y(t)) = \begin{cases} sin(t) + \frac{4(y(t) - sin(t))}{\sqrt{b}}, & b > 16, \\ y(t), & b \leq 16, \end{cases} \tag{80}$$

where $a = \int_0^1 x(t)dt$ and $b = \int_0^1 |y(t) - sin(t)|^2 dt$.

Now we bring the results of the iterations for four cases,

Case 1: Take $u = t$ and $x_1 = e^{3t}$;

Case 2: Take $u = t^3 + 2t$ and $x_1 = sin(3t)$;

Case 3: Take $u = -t$ and $x_1 = \frac{2^t}{2}$;

Case 4: Take $u = \frac{e^{5t}}{5}$ and $x_1 = sin(3t)$.

Remark 2 The preliminary results presented in Tables 1 and 2 and Figs. 1 and 2 demonstrate the advantages and computational efficiency of the proposed methods over some known schemes.

- The suggested Algorithms 1–4 require fewer iterations and CPU time than Alg.(1.6) and Alg.(79) in reaching the same stopping criterion.
- The step size of Alg. (1.6) depends on the criterion of the transfer operator, and thus its performance is significantly weaker than that of our algorithms 1–4 with adaptive step sizes.

- The advantage of our proposed algorithm compared to Alg. (79) lies in the application of the selection technique, which is shown to be advantageous when n is sufficiently large. We have only discussed the case where $n = 3$.

5 Conclusions

In this paper, based on the relaxed CQ method and Halpern-type method, we proposed two new adaptive iterative algorithms to discover solutions of the split feasibility problem with multiple output sets in infinite dimensional Hilbert spaces. More importantly, according to different selection conditions, we give two different adaptive step sizes without the prior knowledge of the operator norm of the involved operator. Moreover, as a generalization, we construct two new algorithms to solve the variational inequalities over the solution set of split feasibility problem with multiple output sets. Under some suitable conditions, we established the strong convergence theorems of the suggested algorithms. Finally, the advantages of the proposed algorithms are confirmed by two numerical examples. It is interesting to extend the results obtained in this paper to Banach spaces or more complex spaces.

Declarations

Conflict of interest No potential conflict of interest was reported by the author(s).

References

1. Alakoya, T.O., Mewomo, O.T.: A relaxed inertial tsengs extragradient method for solving split variational inequalities with multiple output sets. Mathematics **11**(2), 386 (2023)
2. Bauschke, H.H., Combettes, P.L.: Convex Analysis and Monotone Operator Theory in Hilbert Spaces, 2nd edn. Springer, Berlin (2017)
3. Browder, F.E.: Nonlinear mappings of nonexpansive and accretive-type in Banach spaces. B. Am. Math. Soc. **73**(6), 875–882 (1967)
4. Byrne, C.: Iterative oblique projection onto convex sets and the split feasibility problem. Inverse Probl. **18**(2), 441 (2002)
5. Censor, Y., Bortfeld, T., Martin, B., Trofimov, A.: A unified approach for inversion problems in intensity-modulated radiation therapy. Phys. Med. Biol. **51**(10), 2353 (2006)
6. Censor, Y., Elfving, T.: A multiprojection algorithm using Bregman projections in a product space. Numer. Algorithms **8**, 221–239 (1994)
7. Censor, Y., Elfving, T., Kopf, N., Bortfeld, T.: The multiple-sets split feasibility problem and its applications for inverse problems. Inverse Probl. **21**(6), 2071 (2005)
8. Censor, Y., Segal, A.: Iterative projection methods in biomedical inverse problems. Math. Methods Biomed. Imaging Intensity-Modul. Radiat. Therapy (IMRT) **10**, 65–96 (2008)
9. Fukushima, M.: A relaxed projection method for variational inequalities. Math. Program. **35**(1), 5870 (1986)
10. Goebel, K., Simeon, R.: Uniform convexity, hyperbolic geometry, and nonexpansive mappings. Dekker (1984)
11. López, G., Martín-Márquez, V., Wang, F., Xu, H.K.: Solving the split feasibility problem without prior knowledge of matrix norms. Inverse Probl. **28**(8), 085004 (2012)
12. Reich, S., Truong, M.T., Mai, T.N.H.: The split feasibility problem with multiple output sets in Hilbert spaces. Optim. Lett. **14**(8), 2335–2353 (2020)

13. Reich, S., Tuyen, T.M.: A generalized cyclic iterative method for solving variational inequalities over the solution set of a split common fixed point problem. Numer. Algorithms **91**(1), 1–17 (2022)
14. Reich, S., Tuyen, T.M., Thuy, N.T.T., Ha, M.T.N.: A new self-adaptive algorithm for solving the split common fixed point problem with multiple output sets in Hilbert spaces. Numer. Algorithms **89**(3), 1031–1047 (2022)
15. Saejung, S., Yotkaew, P.: Approximation of zeros of inverse strongly monotone operators in Banach spaces. Nonlinear Anal. **75**(2), 742–750 (2012)
16. Taddele, G.H., Kumam, P., Sunthrayuth, P., Gebrie, A.G.: Self-adaptive algorithms for solving split feasibility problem with multiple output sets. Numer. Algorithms **92**(2), 1335–1366 (2023)
17. Wang, J., Hu, Y., Li, C., Yao, J.C.: Linear convergence of CQ algorithms and applications in gene regulatory network inference. Inverse Probl. **33**(5), 055017 (2017)
18. Yamada, I.: The hybrid steepest descent method for the variational inequality problem over the intersection of fixed point sets of nonexpansive mappings. Inherently parallel algorithm. Feasibility Optim. Appl. **8**, 473–504 (2001)
19. Yang, Q.: The relaxed CQ algorithm solving the split feasibility problem. Inverse Probl. **20**(4), 1261–1266 (2004)
20. Yao, Y., Postolache, M., Zhu, Z.: Gradient methods with selection technique for the multiple-sets split feasibility problem. Optimization **69**(2), 269–281 (2019)