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ORIGINAL RESEARCH

Interval-valued picture fuzzy hypergraphs with application towards decision making

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Abstract

The concept of interval-valued picture fuzzy sets (IVPFSs) is the most generalized form of fuzzy sets (FSs) and is proven a useful tool to manipulate complications that arise due to incomplete information more effectively. One of the most powerful feature of IVPFSs is that it allocates the membership, non membership and neutral membership values as intervals to any element of the given data. Due to this, IVPFSs play a key role to deal uncertain data with multiple attributes. In this study, we introduce the notion of interval-valued picture fuzzy hypergraphs (IVPFHGs) which is the combination of both IVPFSs and hypergraphs and provide its application in decision making. We describe several types of IVPFHGs such as partial, simple, support, support simple, elementary IVPFHGs etc. We also initiate the concepts of dual of IVPFHGs. Moreover, $([\iota, \kappa], [\lambda, \epsilon], [\rho, \nu])$-level cuts of IVPFHGs are also addressed. We present a comparative analysis of our newly established terms with those existing in the literature and

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elaborate the superiority of IVPFHGs over the other existing fuzzy hypergraphs structures. Finally, we provide an application of IVPFHGs with algorithm and flowchart towards decision making.

Keywords IVPFHGs · Dual · Level cuts · DM

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1 Introduction

The concept of fuzzy sets (FSs) was initiated by Zadeh [1] which become useful to solve many problems related to real world with uncertainties. The classical (crisp) set has exactly two truth values True (1) and False (0) which is not suitable to deal uncertainties. Basically, FSs is the generalization of classical sets. In FSs each entity has a membership degree in [0, 1]. Since allocating a fixed number to the observation of any element in the data is again limited so assigning membership values in the form of intervals are more practical. Consequently, the term interval-valued fuzzy sets (IVFSs) was introduced in [2]. An IVFSs become more beneficial as compared to that of the FS while dealing with the problems containing uncertainties. IVFSs was broadly used in different fields like multi-valued logic [3], approximate reasoning [4, 5] etc. Another generalization of FSs termed intuitionistic fuzzy sets (IFSs) was explored in [6]. IFSs become comparatively more efficient tool to deal uncertain environments due to having one extra value termed as non membership value. An IFSs was more effectively applied for the solutions of the problems in different fields like image processing [7], decision making [8] etc. Afterwards, the generalization of IFSs called interval-valued intuitionistic fuzzy sets (IVIFSs) was introduced in [9]. In IVIFSs, the membership and non membership values allocated to entities were the suitable sub-intervals of [0, 1]. In IFSs the neutrality degree has not been taken into account. However, the neutrality degree is also important while dealing many issues in a daily life like democratic election. In fact, human thinking has many options to answers in the form of yes, no, abstain and refusal. In such situations, if we involve IFS theory then the data of casting a vote for non-candidates (refusal) will not be addressed. To deal such types of circumstances, Cuong [10] initiated the idea of picture fuzzy sets (PFSs) and it is the optimal generalized form of the FSs. PFSs comprises of membership, neutral membership and non membership values. After this, Phong et al. [11] discussed many operations on PFSs and picture fuzzy relations. Many fuzzy operators were shifted towards PFSs in [12]. Moreover, Wei [13] discussed many aggregation operators on PFSs and provided applications in MADM theory. Further generalization of PFSs named interval-valued picture fuzzy sets (IVPFSs) was also described in [10]. Khalil et al. introduced many new operations on IVPFSs along with few of its characterizations in [14]. In addition, they also proposed an algorithm based on interval-valued picture fuzzy soft sets (IVPFSs) and applied efficiently towards MADM. One can concern [15, 16] for applications of extensions of FGs.

Hypergraphs theory has got much attention of mathematicians due to having many applications in daily life. Like graph, a hypergraph is a mathematical structure which consists of vertices and edges. Different types of hypergraphs along with their applications towards various areas of sciences have been explored. An hypergraph described on a non empty set V is the pair $\bar{H} = (\bar{V}, \bar{E})$ with

1. $\bar{V} = \{\bar{x}_1, \bar{x}_2, \bar{x}_3, \ldots, \bar{x}_n\}$ is a non-empty set of vertices.
2. $\bar{E} = \{\bar{E}_1, \bar{E}_2, \bar{E}_3, \ldots, \bar{E}_r\}$ is a family of a non-empty subsets of $\bar{V} \times \bar{V}$
3. $\bigcup_k \bar{E}_k = \bar{V}_k$, where $k = 1, 2, 3, \ldots, r$.

An hypergraph is said to be simple whenever $\bar{E}_i \subseteq \bar{E}_j$ implies $i = j$. However, if the hypergraph is simple with $| \bar{E}_i \bigcap \bar{E}_j | \leq 1$, for each $\bar{E}_i, \bar{E}_j \in \bar{E}$, then we call it a linear hypergraph.
The idea of fuzzy graphs (FGs) was firstly initiated by Rosenfeld [17]. FGs gives us many useful models to explain numerous problems in better ways as compared to the crisp graphs. The concepts of fuzzy hypergraphs (FHGs) was initiated in [18]. FHGs was refined and further generalized in [19]. FHGs was also discussed in different ways in [20]. Afterwards, the extension of FHGs named interval-valued fuzzy hypergraphs (IVFHGs) was initiated in [21]. Similarly, intuitionistic fuzzy hypergraphs (IFHGs) was explored in [22]. The generalized form of IFHGs named interval-valued intuitionistic fuzzy hypergraphs (IVIFHGs) was initiated in[23]. Some new terminologies of IVIFHGs were introduced in [24]. The term picture fuzzy hypergraphs (PFHGs) was initiated in [25, 26]. PFHGs uses membership, neutral membership and non-membership degrees, respectively to deal uncertainties and to model daily life problems. Under q-rung PF environment, Luqman et al. [27] studied granulation of hyper network models and Akram et al. [26] investigated granular computing. For more on FHGs and its extensions, one can consult [26, 28–31]. Table 1 contains some notations used throughout the manuscript.

Since IVPFSs is the most extended form of FSs, it has more capability to deal uncertainties in the best possible way. Likewise, IVPFGs is the most generalized form of FGs. Many daily life problems containing uncertainties were modeled using IVPFGs and effective results were obtained. Moreover, FHGs were extended as IVFHGs, IFHGs, IVIFHGs and PFHGs. Following this, we initiate the idea of interval-valued picture fuzzy hypergrahs (IVPFHGs) in order to fill the gap in the literature. IVPFHGs is also the most extended form of FHGs. The values of vertices and edges in IVPFHGs are expressed in terms of membership, non membership and neutral membership degrees, which are the subintervals of [0, 1]. The capability of IVPFHGs to express the information through its unique structure makes it the best tool to deal uncertainties. The IVPFHGs can express the uncertain information without any loss and hence we can obtain the most accurate solution of the problem containing uncertainties.

Novelty

The novelty of our work is explained in the following steps.

1. The notion of IVPFHGs along with its different types like partial IVPFHGs, simple IVPFHGs, support IVPFHGs, support simple IVPFHGs, elementary IVPFHGs are initiated.
2. Strength of an edge of IVPHG and dual of IVPFHGs are introduced.
3. The concept of $([\iota, \kappa], [\lambda, \epsilon], [\rho, \nu])$-level cuts of IVPFHGs are described.
4. Important characterization of IVPFHGs are presented.

Table 1 Notations

Terms	Notations
Decision making	DM
Fuzzy sets	FSs
Interval valued fuzzy sets	IVFSs
Intuitionistic fuzzy	IF
Intuitionistic fuzzy sets	IFSs
Interval valued intuitionistic fuzzy sets	IVIFSs
Picture fuzzy	PF
Picture fuzzy sets	PFSs
Interval valued picture fuzzy sets	IVPFSs
Fuzzy graphs	FGs
Interval valued fuzzy graphs	IVFGs
Intuitionistic fuzzy graphs	IFGs
Interval valued intuitionistic fuzzy graphs	IVIFGs
Picture fuzzy graphs	PFGs
Interval valued picture fuzzy graphs	IVPFGs
Hypergraphs	HGs
Fuzzy hypergraphs	FHGs
Interval valued fuzzy hypergraphs	IVFHGs
Intuitionistic fuzzy hypergraphs	IFHGs
Interval valued intuitionistic fuzzy hypergraphs	IVIFHGs
Picture fuzzy hypergraphs	PFHGs
Interval valued picture fuzzy hypergraphs	IVPFHGs

5. A real life application of IVPFHGs towards DM is provided by using ($[\iota, \kappa]$, $[\lambda, \epsilon]$, $[\rho, \nu]$)-level cuts.

Motivations
The motivations of our work are as follows.

1. The existing structures in literature like IVFHGs and IVIFHGs motivated us to introduce the notion of IVPFHGs.
2. IVPFHGs can be twisted towards PFHGs, IVIFHGS, IFHGs, IVFHGs and IVFGs just by assigning different membership values. Hence the IVPFHGs combines the qualitative characteristics of all the said generalizations of FHGs.
3. IVPFHGs provides more options for representing the uncertainties.
4. The IVPFHGs are more compatible and flexible as compared to that of the crisp HGs and FHGs and also easy to apply to any system. Consequently, IVPFHGs gives more precision to the system avoiding the loss of information.

The rest of the manuscript is organized as: In Sect. 2, basic useful terminologies are provided. In Sect. 3, we begin our discussion with the definition of IVPFHGs by utilizing IVPF-relations. We present different types of IVPFHGs and describe various terms like strength of IVPFHGs, dual of IVPFHGs and ($[\iota, \kappa]$, $[\lambda, \epsilon]$, $[\rho, \nu]$)-level cuts of IVPFHGs. Throughout, we provide illustrative examples to furnish our results.

In Sect. 4, we provide a real life application of IVPFHGs towards DM with numerical computations which reflects that our proposed structure is more reliable as compared to the other existing structures. Further to this, in Sect. 5, we conduct a comparative study and explain the superiority of the proposed structure. Finally, we conclude our study.

2 Preliminaries

Definition 1 [1] A pair $(\digamma, \bar{D})$ is a fuzzy set (FS) defined on $\bar{D}$, where $\digamma : \bar{D} \longrightarrow [0, 1]$.

Following [1, 32], $(\digamma, \bar{D})$ represents the support of a FS and is given by supp$(\digamma) = \{\bar{d} \in \bar{D} : \digamma(\bar{d}) \neq 0\}$. A function $\digamma$ is a non-trivial, if supp$(\digamma) \neq \emptyset$. $h(\digamma) = max\{\digamma(\bar{d}) \mid \bar{d} \in \bar{D}\}$ is the height of $\digamma$. A mapping $\digamma$ is called normal, if $h(\digamma)$=1. A fuzzy relation on $\bar{D}$ is the map $\nu : \bar{D} \times \bar{D} \longrightarrow [0, 1]$ with $\nu(\bar{d}, \bar{e}) \leq min\{\digamma(\bar{d}), \digamma(\bar{e})\}$, for all $\bar{d}, \bar{e} \in \bar{D}$. A family of non-trivial FSs $\{\digamma_1, \digamma_2, \ldots, \digamma_m\}$ is a fuzzy partition of $\bar{D}$, if

1. $\bigcup\{(\digamma_i) = \bar{D}, i = 1, 2, 3, \ldots, m$.
2. $\sum_{i=1}^{m} \digamma_i(\bar{d}) = 1\}$, for all $\bar{d} \in \bar{D}$.

We say that the collection $\{\digamma_1, \digamma_2, \ldots, \digamma_m\}$ is a fuzzy covering of $\bar{D}$ if it satisfies (1) and (2).

Definition 2 [33] An object $S = \{(\bar{e}, \digamma_S(\bar{e}), \omega_S(\bar{e})) \mid \bar{e} \in \bar{E}\}$ is an IFS S defined on $\bar{E}$, where $\digamma_S(\bar{e}), \omega_S(\bar{e}) \in [0, 1]$ are membership and non-membership degrees of $\bar{e}$ in S, respectively with $\digamma_S(\bar{e}) + \omega_S(\bar{e}) \leq 1$, for each $\bar{e} \in \bar{E}$.

Definition 3 [9] An object $U = \{(\bar{e}, \digamma(\bar{e}), \omega(\bar{e})) \mid \bar{e} \in \bar{E}\}$ is an IVIFS on $\bar{E}$, where $\digamma(\bar{e})$: $\bar{E} \rightarrow Int([0, 1])$ and $\omega : \bar{E} \rightarrow Int([0, 1])$ with $\digamma^+(\bar{e}) + \omega^+(\bar{e}) \leq 1$. The Supp(S)= $\{\bar{e} : \digamma^-(\bar{e}) \neq 0, \digamma^+(\bar{e}) \neq 0, \omega^-(\bar{e}) \neq 1$ and $\omega^+ \neq 1\}$.

Definition 4 [10] An object $S = \{(\bar{f}, \digamma_S(\bar{f}), \psi_S(\bar{f}), \omega_S(\bar{f})) : \bar{f} \in U\}$ is a PFS on U, where $\digamma_s(\bar{f}) \in [0, 1]$, $\psi_S(\bar{f}) \in [0, 1]$ and $\omega_S(\bar{f}) \in [0, 1]$ denotes the membership, neutral membership and non-membership degrees, respectively of $\bar{f}$ in S, such that

$$\digamma_S(\bar{f}) + \psi_S(\bar{f}) + \omega_S(\bar{f}) \leq 1).$$

Here

$$1 - (\digamma_S(\bar{f}) + \psi_S(\bar{f}) + \omega_S(\bar{f}))$$

is refusal degree of $\bar{f}$ in S.

Definition 5 [10] A picture fuzzy relation R on $\bar{G}$ and $\bar{H}$ is given by $R = \{((\bar{f}, \bar{d}), \digamma_R(\bar{f}, \bar{d}), \psi_R(\bar{f}, \bar{d}), \omega_R(\bar{f}, \bar{d})) \mid \bar{f} \in \bar{G}, \bar{d} \in \bar{H}\}$, where $\digamma_R : \bar{G} \times \bar{H} \rightarrow [0, 1]$, $\psi_R : \bar{G} \times \bar{H} \rightarrow [0, 1]$, $\omega_R : \bar{G} \times \bar{H} \rightarrow [0, 1]$, such that

$$0 \leq sup(\digamma_R(\bar{f}, \bar{d})) + sup(\psi_R(\bar{f}, \bar{d})) + sup(\omega_R(\bar{f}, \bar{d})) \leq 1,$$

for every $(\bar{f}, \bar{d}) \in \bar{G} \times \bar{H}$.

Definition 6 [14] Let $\mathcal{U}$ and $\mathcal{V}$ be two sets and $\mathcal{S}$ and $\mathcal{T}$ be two IVPFSs defined on them. Then

1. $\mathcal{S} \times_1 \mathcal{T} = \{((\bar{f}, \bar{d}), [F_{\mathcal{S}L}(\bar{f}) \cdot F_{\mathcal{T}L}(\bar{d}), F_{\mathcal{S}\mathcal{U}}(\bar{f}) \cdot F_{\mathcal{T}\mathcal{U}}(\bar{d})], [\psi_{\mathcal{S}L}(\bar{f}) \cdot \psi_{\mathcal{T}L}(\bar{d}), \psi_{\mathcal{S}\mathcal{U}}(\bar{f}) \cdot \psi_{\mathcal{T}\mathcal{U}}(\bar{d})], [\omega_{\mathcal{S}L}(\bar{f}) \cdot \omega_{\mathcal{T}L}(\bar{d}), \omega_{\mathcal{S}\mathcal{U}}(\bar{f}) \cdot \omega_{\mathcal{T}\mathcal{U}}(\bar{d})]) \mid \bar{f} \in \mathcal{U}, \bar{d} \in \mathcal{V}\}$
2. $\mathcal{S} \times_2 \mathcal{T} = \{((\bar{f}, \bar{d}), [F_{\mathcal{S}L}(\bar{f}) \wedge F_{\mathcal{T}L}(\bar{d}), F_{\mathcal{S}\mathcal{U}}(\bar{f}) \wedge F_{\mathcal{T}\mathcal{U}}(\bar{d})], [\psi_{\mathcal{S}L}(\bar{f}) \wedge \psi_{\mathcal{T}L}(\bar{d}), \psi_{\mathcal{S}\mathcal{U}}(\bar{f}) \wedge \psi_{\mathcal{T}\mathcal{U}}(\bar{d})], [\omega_{\mathcal{S}L}(\bar{f}) \vee \omega_{\mathcal{T}L}(\bar{d}), \omega_{\mathcal{S}\mathcal{U}}(\bar{f}) \vee \omega_{\mathcal{T}\mathcal{U}}(\bar{d})]) \mid \bar{f} \in \mathcal{U}, \bar{d} \in \mathcal{V}\}$.

Definition 7 [14] An IVPFS $\bar{S}$ on $\bar{V}$ is the object $\bar{S} = \{(\bar{f}, [F_{\bar{S}L}(\bar{f}), F_{\bar{S}\bar{V}}(\bar{f})], [\psi_{\bar{S}L}(\bar{f}), \psi_{\bar{S}\bar{V}}(\bar{f})], [\omega_{\bar{S}L}(\bar{f}), \omega_{\bar{S}\bar{V}}(\bar{f})]) : \bar{f} \in \bar{V}\}$, where
$F_{\bar{S}}$: $\bar{V} \rightarrow$ int([0, 1]), $F_{\bar{S}}(\bar{f}) = [F_{\bar{S}L}(\bar{f}), F_{\bar{S}\bar{V}}(\bar{f})] \in$ int([0, 1])
$\psi_{\bar{S}}$: $\bar{V} \rightarrow$ int([0, 1]), $\psi_{\bar{S}}(\bar{f}) = [\psi_{\bar{S}L}(\bar{f}), \psi_{\bar{S}\bar{V}}(\bar{f})] \in$ int([0, 1])
$\omega_{\bar{S}}$: $\bar{V} \rightarrow$ int([0, 1]), $\omega_{\bar{S}}(\bar{f}) = [\omega_{\bar{S}L}(\bar{f}), \omega_{\bar{S}\bar{V}}(\bar{f})] \in$ int([0, 1]) and for all $\bar{f} \in \bar{V}$, $F_{\bar{S}\bar{V}}(\bar{f}) + \psi_{\bar{S}\bar{V}}(\bar{f}) + \omega_{\bar{S}\bar{V}}(\bar{f}) \leq 1$.

Definition 8 [14] Let S and T be two IVPFSs. Then few basic operations on them are as follows.

1. **Inclusion**
   $S \subseteq T \iff \forall w \in U$, we have
$$F_{SL}(w) \leq F_{TL}(w), F_{SU}(w) \leq F_{TU}(w)$$
$$\psi_{SL}(w) \leq \psi_{TL}(w), \psi_{SU}(w) \leq \psi_{TU}(w)$$
$$\omega_{SL}(w) \geq \omega_{TL}(w), \omega_{SU}(w) \geq \omega_{TU}(w).$$
2. **Union**
   $S = T \iff S \subseteq T$ and $T \subseteq S$.
   $S \cup T = \{(w, [F_{SL}(w) \vee F_{TL}(w), F_{SU}(w) \vee F_{TU}(w)], [\psi_{SL}(w) \wedge \psi_{TL}(w), \psi_{SU}(w) \wedge \psi_{TU}(w)], [\omega_{SL}(w) \wedge \omega_{TL}(w), \omega_{SU}(w) \wedge \omega_{TU}(w)]) \mid w \in U\}$
3. **Intersection**
   $S \cap T = \{(w, [F_{SL}(w) \wedge F_{TL}(w), F_{SU}(w) \wedge F_{TU}(w)], [\psi_{SL}(w) \wedge \psi_{TL}(w), \psi_{SU}(w) \wedge \psi_{TU}(w)], [\omega_{SL}(w) \vee \omega_{TL}(w), \omega_{SU}(w) \vee \omega_{TU}(w)]) \mid w \in U)\}$.
4. **Complement**
   $Co(S) = \overline{S} = \{(w, [\omega_{SL}(w), \omega_{SU}(w)], [\psi_{SL}(w), \psi_{SU}(w)], [F_{SL}(w), F_{SU}(w)]) \mid w \in U\}$.

Definition 9 Let $\bar{F}$ be the collection of non-trivial FSs on finite set $\bar{E}$ satisfying $\bar{E} - \bigcup\{supp\mu \mid \mu \in \bar{F}\}$. Then the pair $\bar{H} = (\bar{E}, \bar{F})$ is called FHG on $\bar{E}$ and the edges of $\bar{H}$ is known as the edge set of $\bar{H}$.

Definition 10 [23] Let $\vartheta = \{\vartheta_1, \vartheta_2, \ldots, \vartheta_m\}$ be a collection of non-trivial IVIFSs defined on the vertex set $\bar{D} = \{d_1, d_2, \ldots, d_n\}$ with $\bar{D} = \bigcup_j supp\langle F_j, \psi_j \rangle$, $j = 1, 2, \ldots, m$, where F_j, ψ_j are the interval-valued membership and interval-valued non-membership values. Then we call a pair $\hat{H} = (\bar{D}, \vartheta)$ an IVIFHG defined on $\bar{D}$ and ϑ is the collection of IVIF-hyperedges of $\hat{H}$.

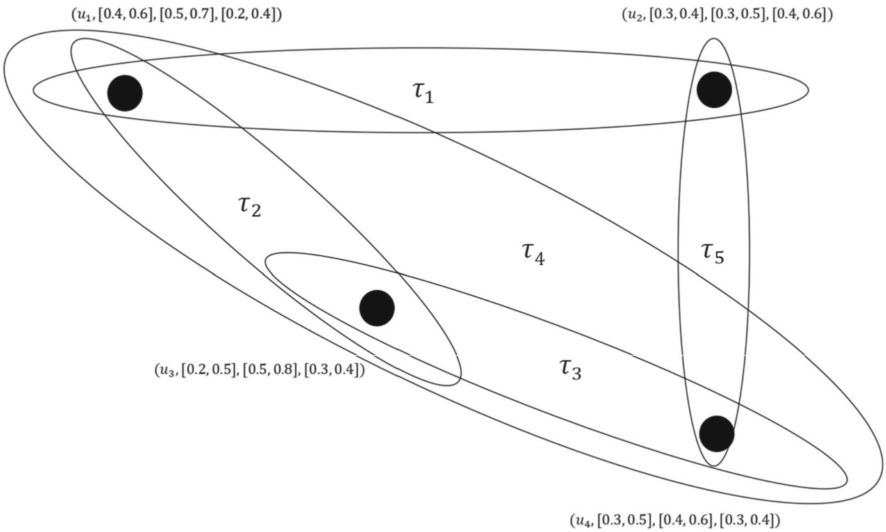

Fig. 1 Interval-valued picture fuzzy hypergraph

Definition 11 [23] The $\langle[\bar{\iota}, \bar{\kappa}], [\bar{\lambda}, \bar{\mu}]\rangle$-cut of IVIFHG $\tilde{H}$, represented by $\tilde{H}_{[\bar{\iota},\bar{\kappa}],[\bar{\lambda},\bar{\mu}]}$ and is defined as $\tilde{H}_{[\bar{\iota},\bar{\kappa}],[\bar{\lambda},\bar{\mu}]}$ $=(X_{[\bar{\iota},\bar{\kappa}],[\bar{\lambda},\bar{\mu}]}, E_{[\bar{\iota},\bar{\kappa}],[\bar{\lambda},\bar{\mu}]})$, where $X_{[\bar{\iota},\bar{\kappa}],[\bar{\lambda},\bar{\mu}]}=$ X, $E_{j,[\bar{\iota},\bar{\kappa}],[\bar{\lambda},\bar{\mu}]} = \{x_j \mid \mu_j^-(x_i) \geq \bar{\iota}, \mu_j^+(x_i) \geq \bar{\kappa}, \psi_j^-(x_i) \leq \bar{\lambda}$ and $\psi_j^-(x_i) \leq \bar{\mu}$, $j = 1, 2, \ldots, m, \}$ $E_{(m+1),[\bar{\iota},\bar{\kappa}],[\bar{\lambda},\bar{\mu}]} = \{x_j \mid \mu_j^-(x_i) < \bar{\iota}, \mu_j^+(x_i) < \bar{\kappa}, \psi_j^-(x_i) > \bar{\lambda}$ and $\psi_j^-(x_i) > \bar{\mu}, \forall j, \}$.
The hyperedge $E_{(m+1),[\bar{\iota},\bar{\kappa}],[\bar{\lambda},\bar{\mu}]}$ is added to group the elements which do not lie in any of the hyperedge $E_{j,[\bar{\iota},\bar{\kappa}],[\bar{\lambda},\bar{\mu}]}$ of $\tilde{H}_{[\bar{\iota},\bar{\kappa}],[\bar{\lambda},\bar{\mu}]}$. The hyperedges in the $\langle[\bar{\iota}, \bar{\kappa}], [\bar{\lambda}, \bar{\bar{\mu}}]\rangle$-cut hypergraph are the crisp sets.

We refer [23, 27] for further basic terms related to IVIFHGs and PFHGs, respectively.

3 Interval-valued picture fuzzy hypergraphs (IVPFHGs)

We commence this section with the definition of IVPFHG and introduce different types of it. Throughout illustrative examples are provided to furnish our results.

Definition 12 Let $\vartheta = \{\vartheta_1, \vartheta_2, \ldots, \vartheta_m\}$ be a collection of non-trivial IVPFSs defined on the vertex set $X = \{x_1, x_2, \ldots, x_n\}$ with $X = \bigcup_j \text{supp}\langle\mu_j, \psi_j, \omega_j\rangle$, j=1, 2, …, m, where μ_j, ψ_j, ω_j are the interval-valued membership, interval-valued neutral and interval-valued non-membership functions. Then we call a pair $\hat{H} = (X, \vartheta)$ an interval-valued picture fuzzy hypergraph (IVPFHG) defined on X, where ϑ is the collection of IVPF-hyperedges of $\bar{H}$.

Example 1 The graph $G = (U, \tau)$ shown in Fig. 1 with $U = \{u_1, u_2, u_3, u_4\}$ and $\tau = \{\tau_1, \tau_2, \tau_3, \tau_4, \tau_5\}$ is an interval valued picture fuzzy hypergraph.

$\tau_1 = \{(u_1, [0.4, 0.6], [0.5, 0.7], [0.2, 0.4]), (u_2, [0.3, 0.4], [0.3, 0.5], [0.4, 0.6])\}$
$\tau_2 = \{(u_1, [0.4, 0.6], [0.5, 0.7], [0.2, 0.4]), (u_3, [0.2, 0.5], [0.5, 0.8], [0.3, 0.4])\}$
$\tau_3 = \{(u_3, [0.2, 0.5], [0.5, 0.8], [0.3, 0.4]), (u_4, [0.3, 0.5], [0.4, 0.6], [0.3, 0.4])\}$
$\tau_4 = \{(u_1, [0.4, 0.6], [0.5, 0.7], [0.2, 0.4]), (u_3, [0.2, 0.5], [0.5, 0.8], [0.3, 0.4]), (u_4, [0.3, 0.5], [0.4, 0.6], [0.3, 0.4])\}$
$\tau_5 = \{(u_2, [0.3, 0.4], [0.3, 0.5], [0.4, 0.6]), (u_4, [0.3, 0.5], [0.4, 0.6], [0.3, 0.4])\}$
The corresponding incidence matrix $\mathbb{M}_{\mathbb{G}}$ is shown in 1.

Definition 13 Let $\vartheta = \{\vartheta_1, \vartheta_2, \ldots, \vartheta_m\}$ be a family of finite non-trivial IVPFHG on X and R is a picture fuzzy relation on picture fuzzy subsets ϑ_i such that (i) $\mu_R(E_i) = (\mu_R\{x_1, x_2, \ldots, x_r\} \le \wedge\{\mu\vartheta_i(x_i), \mu\vartheta_i(x_2), \ldots, \mu\vartheta_i(x_r)\})$, (ii) $\psi_R(E_i) = (\psi_R\{x_1, x_2, \ldots, x_r\} \le \wedge\{\psi\vartheta_i(x_i), \psi\vartheta_i(x_2), \ldots, \psi\vartheta_i(x_r)\})$
(iii) $\omega_R(E_i) = (\omega_R\{x_1, x_2, \ldots, x_r\} \ge \vee\{\omega\vartheta_i(x_i), \omega\vartheta_i(x_2), \ldots, \omega\vartheta_i(x_r)\})$

Definition 14 Let $\vartheta = \{\vartheta_1, \vartheta_2, \ldots, \vartheta_m\}$ be a finite family of non-trivial IVPF-subsets on X and R is a picture fuzzy relation on picture fuzzy subsets ϑ_i such that (i) $\mu_R(E_i) = (\mu_R\{x_1, x_2, \ldots, x_r\} \le \wedge\{\mu\vartheta_i(x_i), \mu\vartheta_i(x_2), \ldots, \mu\vartheta_i(x_r)\})$,
(ii) $\psi_R(E_i) = (\psi_R\{x_1, x_2, \ldots, x_r\} \le \wedge\{\psi\vartheta_i(x_i), \psi\vartheta_i(x_2), \ldots, \psi\vartheta_i(x_r)\})$
(iii) $\omega_R(E_i) = (\omega_R\{x_1, x_2, \ldots, x_r\} \ge \vee\{\omega\vartheta_i(x_i), \omega\vartheta_i(x_2), \ldots, \omega\vartheta_i(x_r)\})$

Definition 15 The $\langle[\iota, \kappa], [\lambda, \epsilon], [\rho, \nu]\rangle$-cut of IVPFHG $\bar{H}$, represented by $\bar{H}_{\langle[\iota,\kappa],[\lambda,\epsilon],[\rho,\nu]\rangle}$ and defined by $\bar{H}_{\langle[\iota,\kappa],[\lambda,\epsilon],[\rho,\epsilon]\rangle} = (X_{\langle[\iota,\kappa],[\lambda,\epsilon],[\rho,\nu]\rangle}, E_{\langle[\iota,\kappa],[\lambda,\epsilon],[\rho,\nu]\rangle})$, where
$X_{\langle[\iota,\kappa],[\lambda,\epsilon],[\rho,\nu]\rangle} = X$, $E_{j\,\langle[\iota,\kappa],[\lambda,\epsilon],[\rho,\nu]\rangle} = \{x_i \mid \mu_j^-(x_i) \ge \iota, \mu_j^+(x_i) \ge \kappa, \psi_j^-(x_i) \ge \lambda\, \psi_j^+(x_i) \ge \epsilon, \omega_j^-(x_i) \le \rho, \omega_j^+(x_i) \le \nu, j = 1, 2, \ldots, m\}$, $E_{m+1\,\langle[\iota,\kappa],[\lambda,\epsilon],[\rho,\nu]\rangle} = \{x_i \mid \mu_j^-(x_i) < \iota, \mu_j^+(x_i) < \kappa, \psi_j^-(x_i) < \lambda\, \psi_j^+(x_i) < \mu, \omega_j^-(x_i) > \omega, \omega_j^+(x_i) > \nu, \forall j\}$.
The hyperedge $E_{m+1,\langle[\iota,\kappa],[\lambda,\epsilon],[\rho,\nu]\rangle}$ is added to group the elements which are not contained in any hyperedge $E_{j,\langle[\iota,\kappa],[\lambda,\epsilon],[\rho,\nu]\rangle}$ of $\bar{H}_{\langle[\iota,\kappa],[\lambda,\epsilon],[\rho,\nu]\rangle}$. The hyperedges in the $\langle[\iota, \kappa], [\lambda, \epsilon], [\rho, \nu]\rangle$-cut hypergraph are now crisp sets.

Definition 16 The dual IVPFHG of an IVPFHG $\hat{H} = (Y, \vartheta)$ is the collection $\bar{H} = (\bar{Y}, \bar{\vartheta})$, where $\bar{Y} = \{\grave{e}_1, \grave{e}_2, \ldots, \grave{e}_m\}$ is a vertices set corresponding to $\vartheta_1, \vartheta_2, \ldots, \vartheta_m$, respectively and $\{Y_1, Y_2, \ldots, Y_n\}$ is the collection of hyperedges corresponding to $y_1, y_2, \ldots, y_n$, respectively , where $Y_i(\grave{e}_j) = \vartheta_j(y_i)$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$.

Example 2 The dual IVPFHG $G^* = (U^*, \tau^*)$ of IVPFHG $G = (U, \tau)$ in Fig. 1 is shown in Fig. 2 such that $U^* = \{\tau_1^*, \tau_2^*, \tau_3^*, \tau_4^*, \tau_5^*\}$ and $\tau^* = \{u_1^*, u_2^*, u_3^*, u_4^*\}$, where
$\tau_1^* = \{(u_1^*, [0.4, 0.6], [0.5, 0.7], [0.2, 0.4]), (u_2^*, [0.3, 0.4], [0.3, 0.5], [0.4, 0.6])\}$
$\tau_2^* = \{(u_1^*, [0.4, 0.6], [0.5, 0.7], [0.2, 0.4]), (u_3^*, [0.2, 0.5], [0.5, 0.8], [0.3, 0.4])\}$
$\tau_3^* = \{(u_3^*, [0.2, 0.5], [0.5, 0.8], [0.3, 0.4]), (u_4^*, [0.3, 0.5], [0.4, 0.6], [0.3, 0.4])\}$
$\tau_4^* = \{(u_1^*, [0.4, 0.6], [0.5, 0.7], [0.2, 0.4]), (u_3^*, [0.2, 0.5], [0.5, 0.8], [0.3, 0.4]), (u_4^*, [0.3, 0.5], [0.4, 0.6], [0.3, 0.4])\}$
$\tau_5^* = \{(u_2^*, [0.3, 0.4], [0.3, 0.5], [0.4, 0.6]), (u_4^*, [0.3, 0.5], [0.4, 0.6], [0.3, 0.4])\}$
The corresponding incidence matrix is shown in Table 2

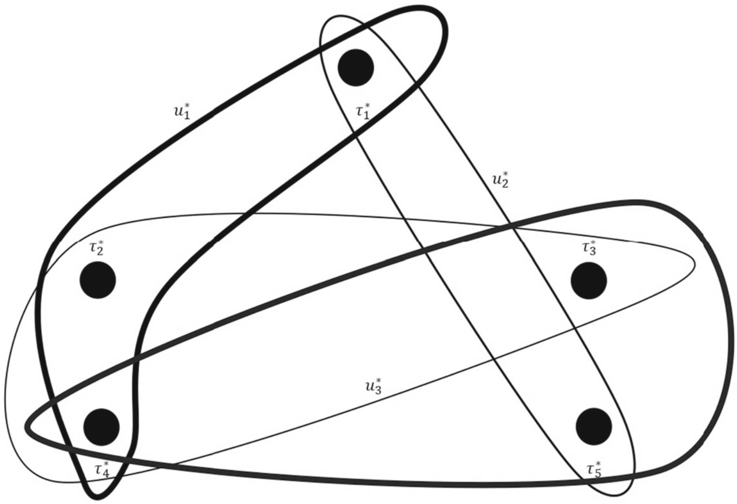

Fig. 2 Dual Interval-valued picture fuzzy hypergraph

Definition 17 The strength ζ of an hyperedge ϑ_j of an IVPFHG can be expressed as $\zeta(\vartheta_j)$= $\{\wedge(\mu_j^-(\bar{y}) \mid \mu_j^-(\bar{y}) > 0), \wedge(\mu_j^+(\bar{y}) \mid \mu_j^+(\bar{y}) > 0), \wedge(\psi_j^-(\bar{y}) \mid \psi_j^-(\bar{y}) > 0), \wedge(\psi_j^+(\bar{y}) \mid \psi_j^+(\bar{y}) > 0), \vee(\omega_j^-(\bar{y}) \mid \omega_j^-(\bar{y}) > 0), \vee(\omega_j^+(\bar{y}) \mid \omega_j^+(\bar{y}) > 0\}$.
In another words, it is the minimum of the membership values $\mu_j^-(\bar{y}), \mu_j^+(\bar{y})$, minimum of the neutral membership values $\psi_j^-(\bar{y}), \psi_j^+(\bar{y})$ and maximum of the non-membership values $\psi_j^-(\bar{y}), \psi_j^+(\bar{y})$ of the vertices exist in the hyperedge ϑ_j. It means that the group of elements lying in the hyperedge ϑ_j have a participation degree at least $\zeta(\vartheta_j)$. We call hyperedges with high strength as the strong hyperedges because of the strong cohesion in them.

Definition 18 An IVPFHG $\bar{B} = (\acute{Y}, \acute{\vartheta})$ is said to be a partial-IVPFHG of $\hat{B}$= (Y, ϑ), if $\acute{\vartheta} \subseteq \vartheta$ which can be express as $\acute{B} \subseteq \bar{B}$. If $\acute{B} \subseteq \bar{B}$ and $\acute{\vartheta} \subseteq \vartheta$, then it can be expressed as $\acute{B} \subset \bar{B}$.

Definition 19 An IVPFHG $\hat{B}$= (Y, ϑ) is said to be a simple, if ϑ has no repeated IVPF-hyperedges and whenever Y= $(\mu_y, \psi_y, \omega_y)$, Z= $(\mu_z, \psi_z, \omega_z) \in \vartheta$, and $\mu_y^-(y) \leq \mu_z^-(y)$, $\mu_y^+(y) \leq \mu_z^+(y)$, $\psi_y^-(y) \leq \psi_z^-(y)$, $\psi_y^+(y) \leq \psi_z^+(y)$ $\omega_y^-(y) \geq \omega_z^-(y)$, $\omega_y^+(y) \geq \omega_z^+(y)$, for each $y \in Y$ then $\mu_y^-(y) = \mu_z^-(y)$, $\mu_y^+(y) = \mu_z^+(y)$, $\psi_y^-(y) = \psi_z^-(y)$, $\psi_y^+(y) = \psi_z^+(y)$, $\omega_y^-(y) = \omega_z^-(y)$, $\omega_y^+(y) = \omega_z^+(y)$.

Definition 20 An IVPFHG $\hat{H}$= (Y, ϑ) is said to be support simple, if Y= $(\mu_e, \psi_e, \omega_e)$, Z= $(\mu_z, \psi_z, \omega_z) \in \vartheta$, and $\mu_e^-(e) \leq \mu_z^-(e)$, $\mu_e^+(e) \leq \mu_z^+(e)$, $\psi_e^-(e) \leq \psi_z^-(e)$, $\psi_e^+(e) \leq \psi_z^+(e)$ $\omega_e^-(e) \geq \omega_z^-(e)$, $\omega_e^+(e) \geq \omega_z^+(e)$, for all $e \in Y$, and supp$(Y) = supp(Z)$, then $\mu_e^-(e) = \mu_z^-(e)$, $\mu_e^+(e) = \mu_z^+(e)$, $\psi_e^-(e) = \psi_z^-(e)$, $\psi_e^+(e) = \psi_z^+(e)$, $\omega_e^-(e) = \omega_z^-(e)$, $\omega_e^+(e) = \omega_z^+(e)$. An IVPFHG $\hat{H}$= (Y, ϑ) is strongly support simple if Y= $(\mu_e, \psi_e, \omega_e)$, Z= $(\mu_z, \psi_z, \omega_z) \in \vartheta$ and supp$(Y) = supp(Z)$, then $\mu_e^-(e) =$

Table 2 Incidence matrix M_G

	τ_1	τ_2	τ_3	τ_4	τ_5
u_1	[0.4,0.6][0.5,0.7][0.2,0.4]	[0.4,0.6][0.5,0.7][0.2,0.4]	[0,0] ,[0,0] ,[0,0]	[0.4,0.6][0.5,0.7][0.2,0.4]	[0,0] [0,0] [0,0]
u_2	[0.3,0.4][0.3,0.5][0.4,0.6]	[0,0] [0,0] [0,0]	[0,0] [0,0] [0,0]	[0,0] [0,0] [0,0]	[0.3,0.4][0.3,0.5][0.4,0.6]
u_3	[0,0] [0,0] [0,0]	[0.2,0.5][0.5,0.8][0.3,0.4]	[0.2,0.5][0.5,0.8][0.3,0.4]	[0.2,0.5][0.5,0.8][0.3,0.4]	[0,0] [0,0] [0,0]
u_4	[0,0] [0,0] [0,0]	[0,0] [0,0] [0,0]	[0.3,0.5][0.4,0.6][0.3,0.4]	[0.3,0.5][0.4,0.6][0.3,0.4]	[0.3,0.5][0.4,0.6][0.3,0.4]

Table 3 Incidence matrix $M_{\mathcal{H}^*}$

	u_1^*	u_2^*	u_3^*	u_4^*
τ_1^*	[0.4,0.6][0.5,0.7][0.2,0.4]	[0.3,0.4][0.3,0.5][0.4,0.6]	[0,0] [0,0] [0,0]	[0,0] [0,0] [0,0]
τ_2^*	[0.4,0.6][0.5,0.7][0.2,0.4]	[0,0] [0,0] [0,0]	[0.2,0.5][0.5,0.8][0.3,0.4]	[0,0] [0,0] [0,0]
τ_3^*	[0,0] [0,0] [0,0]	[0,0] [0,0] [0,0]	[0.2,0.5][0.5,0.8][0.3,0.4]	[0.3,0.5][0.4,0.6][0.3,0.4]
τ_4^*	[0.4,0.6][0.5,0.7][0.2,0.4]	[0,0] [0,0] [0,0]	[0.2,0.5][0.5,0.8][0.3,0.4]	[0.3,0.5][0.4,0.6][0.3,0.4]
τ_5^*	[0,0] [0,0] [0,0]	[0.3,0.4][0.3,0.5][0.4,0.6]	[0,0] [0,0] [0,0]	[0.3,0.5][0.4,0.6][0.3,0.4]

$\mu_z^-(e)$, $\mu_e^+(e) = \mu_z^+(e)$, $\psi_e^-(e) = \psi_z^-(e)$, $\psi_e^+(e) = \psi_z^+(e)$, $\omega_e^-(e) = \omega_z^-(e)$, $\omega_e^+(e) = \omega_z^+(e)$.

Definition 21 An IVPFS $\bar{Y}= \{\langle \bar{y}, \mu_{\bar{y}}(\bar{y}), \psi_{\bar{y}}(\bar{y}), \omega_{\bar{y}}(\bar{y})\rangle \mid \bar{y} \in \bar{Y}\}$ is an elementary-IVPFS, if $\bar{Y}$ is a unique valued on supp$(\bar{Y})$. An IVPFHG $\hat{H}= (\bar{Y}, \vartheta)$ whose all IVPF-hyperedges are elementary is said to be an elementary-IVPFHG.

Theorem 1 *The elementary-IVPFHG $\bar{\mathcal{H}} = (Y, \vartheta)$ is support simple if and only if it is strongly support simple.*

Proof Let $\bar{\mathcal{H}}$ is elementary, support simple. Also, let $supp(Y)= supp(Z)$. We suppose that $h(Y) \leq h(Z)$. As, $\bar{\mathcal{H}}$ is elementary, it follows that $\mu_Y^-(\bar{e}) \leq \mu_Z^-(\bar{e})$, $\mu_Y^+(\bar{e}) \leq \mu_Z^+(\bar{e})$, $\psi_Y^-(\bar{e}) \leq \psi_Z^-(\bar{e})$, $\psi_Y^+(\bar{e}) \leq \psi_Z^+(\bar{e})$ $\omega_Y^-(\bar{e}) \geq \omega_Z^-(\bar{e})$, $\omega_Y^+(\bar{e}) \geq \omega_Z^+(\bar{e})$, for each $e \in Y$ for all $x \in X$, and as $\bar{\mathcal{H}}$ is support simple so $\mu_Y^-(\bar{e}) = \mu_Z^-(\bar{e})$, $\mu_Y^+(\bar{e}) = \mu_Z^+(\bar{e})$, $\psi_Y^-(\bar{e}) = \psi_Z^-(\bar{e})$, $\psi_Y^+(\bar{e}) = \psi_Z^+(\bar{e})$, $\omega_Y^-(\bar{e}) = \omega_Z^-(\bar{e})$, $\omega_Y^+(\bar{e}) = \omega_Z^+(\bar{e})$. Hence $\bar{\mathcal{H}}$ is strongly support simple. □

Remark 1 Let $\bar{\mathcal{H}} = (Y, \vartheta)$ be an IVPFHG. Let $\iota, \kappa, \lambda, \mu, \omega, \nu, \in [0, 1]$ and

$$E_{\langle[\iota,\kappa],[\lambda,\mu],[\omega,\nu]\rangle} = \{\{X_{\langle[\iota,\kappa],[\lambda,\mu],[\omega,\nu]\rangle} \neq \emptyset \mid X \in \vartheta\}$$
$$X_{\langle[\iota,\kappa],[\lambda,\mu],[\omega,\nu]\rangle} = \bigcup_{x\in\vartheta} X_{\langle[\iota,\kappa],[\lambda,\mu],[\omega,\nu]\rangle}.$$

If $E_{\langle[\iota,\kappa],[\lambda,\mu],[\omega,\nu]\rangle} \neq \emptyset$, then the crisp hypergraph $X_{\langle[\iota,\kappa],[\lambda,\mu],[\omega,\nu]\rangle} = (X_{\langle[\iota,\kappa],[\lambda,\mu],[\omega,\nu]\rangle}, E_{\langle[\iota,\kappa],[\lambda,\mu],[\omega,\nu]\rangle})$ is the $\langle[\iota,\kappa],[\lambda,\mu],[\omega,\nu]\rangle$-level hypergraph of $\bar{\mathcal{H}}$. The collections of crisp sets (hypergraph) generated by the $\langle[\iota,\kappa],[\lambda,\mu],[\omega,\nu]\rangle$-cuts of IVPFHG share a crucial relationship among each other.

Proposition 2 *Let $\mathcal{C}$ and $\mathcal{D}$ be the two collections of sets. Then for every set $\tilde{C} \in \mathcal{C}$, there exists at least one set $\tilde{D} \in \mathcal{D}$ such that $\tilde{C} \subseteq \tilde{D}$. So, we have $\mathcal{C} \subseteq \mathcal{D}$. Since there exists a possibility that $\tilde{C} \sqsubseteq \tilde{D}$ while $\tilde{C} \bigcap \tilde{D} = \emptyset$, we have $\tilde{C} \subseteq \tilde{D}$ implies $\mathcal{C} \sqsubseteq \mathcal{D}$. However, in general the converse does not not hold true.*

Definition 22 Let $\bar{H} = (Y, \vartheta)$ be an IVPFHG and for $\langle[0,0],[0,0],[0,0]\rangle < \langle[\iota,\kappa],[\lambda,\mu],[\omega,\nu]\rangle$ $h(\bar{H})$. Suppose $H_{\langle[\iota,\kappa],[\lambda,\mu],[\omega,\nu]\rangle} = (X_{\langle[\iota,\kappa],[\lambda,\mu],[\omega,\nu]\rangle}, E_{\langle[\iota,\kappa],[\lambda,\mu],[\omega,\nu]\rangle})$ be the $\langle[\iota,\kappa],[\lambda,\mu],[\omega,\nu]\rangle$-level hypergraph of $\bar{H}$. The real numbers of the sequence $\langle[\bar{a}_i,\bar{b}_i],[\bar{c}_i,\bar{d}_i],[\bar{e}_i,\bar{f}_i]\rangle \mid 1 \leq i \leq n$, $0 < \bar{a}_n < \cdots < \bar{a}_1$, $0 < \bar{b}_n < \cdots \bar{b}_1$, $0 < \bar{c}_n < \cdots < \bar{c}_1$, $0 < \bar{d}_n < \cdots < \bar{d}_1$, and $1 > \bar{e}_n > \cdots > \bar{e}_1$, $1 > \bar{f}_n > \cdots > \bar{f}_1$, where $h(\bar{H}) = \langle[\bar{a}_1,\bar{b}_1],[\bar{c}_1,\bar{d}_1],[\bar{e}_1,\bar{f}_1]\rangle$ satisfying

1. if $\bar{a}_{i+1} < p \leq \bar{a}_i$, $\bar{b}_{i+1} < q \leq \bar{b}_i$, $\bar{c}_{i+1} < r \leq \bar{c}_i$, $\bar{d}_{i+1} < s \leq \bar{d}_i$, $\bar{e}_{i+1} > t \leq \bar{e}_i$, $\bar{f}_{i+1} > u \leq \bar{f}_i$ then $E_{\langle[p,q],[r,s],[t,u]\rangle} = E_{\langle[\bar{a}_i,\bar{b}_i],[\bar{c}_i,\bar{d}_i],[\bar{e}_i,\bar{f}_i]}$, $i = 1, 2, , \ldots, n$.
2. $E_{\langle[\bar{a}_i,\bar{b}_i],[\bar{c}_i,\bar{d}_i],[\bar{e}_i,\bar{f}_i]} \sqsubset E_{\langle[\bar{a}_{i+1},\bar{b}_{i+1}],[\bar{c}_{i+1},\bar{d}_{i+1}],[\bar{e}_{i+1},\bar{f}_{i+1}]\rangle}$, $i = 1, 2, \ldots, n-1$

is known as fundamental sequence of $\bar{H}$, represented by $F(\bar{H})$. The collection $\langle[\bar{a}_i,\bar{b}_i],[\bar{c}_i,\bar{d}_i],[\bar{e}_i,\bar{f}_i]\rangle$-level hypergraphs $\{H_{\langle[\bar{a}_i,\bar{b}_i],[\bar{c}_i,\bar{d}_i],[\bar{e}_i,\bar{f}_i]\rangle} \mid 1 \leq i \leq n\}$ is the collection of core hypergraphs of $\bar{H}$, and is represented by $C(\bar{H})$.

Definition 23 Let $\bar{\mathcal{H}}=(Y,\vartheta)$ be IVPFHG and $F(\bar{\mathcal{H}})=\{\langle[\bar{a}_i,\bar{b}_i],[\bar{c}_i,\bar{d}_i],[\bar{e}_i,\bar{f}_i]\rangle \mid 1\le i\le n\}$. Then, $\bar{\mathcal{H}}$ is sectionally elementary if for each Y, where Y is an IVPFS expressed as $\vartheta_j\in\vartheta$ and each $\langle[\bar{a}_i,\bar{b}_i],[\bar{c}_i,\bar{d}_i],[\bar{e}_i,\bar{f}_i]\rangle\in F(\bar{\mathcal{H}})$, $X_{[\iota,\kappa],[\lambda,\mu],[\omega,\nu]\rangle}=X_{\langle[\bar{a}_i,\bar{b}_i],[\bar{c}_i,\bar{d}_i],}$ $[\bar{e}_i,\bar{f}_i]\rangle$, for all $\langle[\iota,\kappa],[\lambda,\mu],[\omega,\nu]\rangle\in(\langle[\bar{a}_{i+1},\bar{b}_{i+1}],[\bar{c}_{i+1},\bar{d}_{i+1}],[\bar{e}_{i+1},\bar{f}_{i+1}]\rangle,$ $\langle[\bar{a}_i,\bar{b}_i],[\bar{c}_i,\bar{d}_i],[\bar{e}_i,\bar{f}_i]\rangle$, take $\bar{a}_{n+1}=0$, $\bar{b}_{n+1}=0$, $\bar{c}_{n+1}=0$, $\bar{d}_{n+1}=0$, $\bar{e}_{n+1}=0$, $\bar{f}_{n+1}=0)$.

Definition 24 An IVPFHG is said to be ordered, if $C(\bar{\mathcal{H}})=\{H_{\langle[\bar{a}_i,\bar{b}_i],[\bar{c}_i,\bar{d}_i],[\bar{e}_i,\bar{f}_i]\rangle}\mid 1\le i\le n\}$ and it is called simply ordered, if $C(\bar{\mathcal{H}})$ is simply ordered.

Proposition 3 *(i) An elementary-IVPFHG $\bar{\mathcal{H}}(X,\vartheta)$ is ordered.*
(ii) An ordered IVPFHG $\bar{\mathcal{H}}(X,\vartheta)$ with $C(\bar{\mathcal{H}})=\{H_{\langle[\bar{a}_i,\bar{b}_i],[\bar{c}_i,\bar{d}_i],[\bar{e}_i,\bar{f}_i]\rangle}\mid 1\le i\le n\}$ and simple $H_{\langle[a_n,b_n],[c_n,d_n],[e_n,f_n]\rangle}$, is elementary.

Proposition 4 *Let $\bar{\mathcal{H}}=(\bar{D},\vartheta)$ is a support simple IVPFHG with order n. Then $\mid\vartheta\mid$ has no upper bound.*

Proof Suppose $\bar{X}=(\bar{d},\bar{e})$, and define $\vartheta_M=\{X_i=\langle[\mu^-_{d_i},\mu^+_{d_i}][\psi^-_{d_i},\psi^+_{d_i}][\omega^-_{d_i},\omega^+_{d_i}]\rangle\mid i=1,2,\ldots,M\}$, where
$\mu^-_{d_i}(\bar{d})=1/1+i$, $\mu^+_{d_i}(\bar{d})=1/1+i$, $\psi^-_{d_i}(\bar{d})=1/1+i$, $\psi^+_{d_i}(\bar{d})=1/1+i$, $\omega^-_{d_i}(\bar{d})=1/1+i$, $\omega^+_{d_i}(\bar{d})=1/1+i$,
$\mu^-_{d_i}(\bar{e})=i/1+i$, $\mu^+_{d_i}(\bar{e})=i/1+i$, $\psi^-_{d_i}(\bar{e})=i/1+i$, $\psi^+_{d_i}(\bar{e})=i/1+i$, $\omega^-_{d_i}(\bar{e})=i/1+i$, $\omega^+_{d_i}(\bar{e})=i/1+i$.
Then $\bar{H_M}=(X,\vartheta_M)$ is simple IVPFHG with M hyperedges. □

Proposition 5 *PFGs and picture fuzzy digraphs (PFDGs) are the special cases of the PFHGs.*

Proof A PFG on a set V is a pair $\bar{G}=(\bar{V},\bar{E})$, where $\bar{E}$ is a symmetric picture fuzzy subset of $\bar{V}\times\bar{V}$ i.e., $\mu_B:\bar{V}\times\bar{V}\to[0,1]$, and for each $\bar{d}$ and $\bar{e}$ in V, we have $\mu_B(\bar{d},\bar{e})=\mu_B(\bar{e},\bar{d})$, $\psi_B(\bar{d},\bar{e})=\psi_B(\bar{e},\bar{d})$, $\omega_B(\bar{d},\bar{e})=\omega_B(\bar{e},\bar{d})$. A PFG on a picture fuzzy subset $A=(\mu_A,\psi_A,\omega_A)\in\bar{V}$ is the pair $H=(A,B)$, where the mapping(symmetric) $\mu:\bar{V}\times\bar{V}\to[0,1]$ holds $\mu_B(\bar{d},\bar{e})\le min(\mu_A(\bar{d}),\mu_(\bar{e}))$, $\psi_B(\bar{d},\bar{e})\le min(\psi_A(\bar{d}),\psi_(\bar{e}))$ and $\omega_B(\bar{d},\bar{e})\ge max(\omega_A(\bar{d}),\omega_A(\bar{e}))$, for all $\bar{d},\bar{e}\in\bar{V}$. As B is well defined, a PFG has no multiple edges. An edge is non-trivial, if $\mu_B(\bar{d},\bar{e})\ne 0$, $\psi_B(\bar{d},\bar{e})\ne 0$, $\omega_B(\bar{d},\bar{e})\ne 0$. A loop at $\bar{d}$ is $\mu_B(\bar{d},\bar{d})\ne 0$, $\psi_B(\bar{d},\bar{d})\ne 0$, $\omega_B(\bar{d},\bar{d})\ne 0$.
Alternately, a non-trivial edge denotes an elementary-PF subset of $\bar{V}$ with two(or one) elements support. Since, there are no multiple edges with each pair having distinct supports. A PFG with no loops is described by anti-reflexive relation or equivalently, by preventing fuzzy subsets with single elements support. Hence, PFG is an elementary-PFHG consisting of edges with different two vertex supports. PFDGs on a set $\bar{V}$ or a PF subset $\bar{A}$ of $\bar{V}$ are similarly defined in terms of a mapping $\bar{C}=(\mu_{\bar{C}},\psi_{\bar{C}},\omega_{\bar{C}}):\bar{V}\times\bar{V}\to[0,1]$ holds $\mu_{\bar{C}}(\bar{d},\bar{e})\le min(\mu_{\bar{A}}(\bar{d}),\mu_{\bar{A}}(\bar{e}))$, $\psi_{\bar{C}}(\bar{d},\bar{e})\le min(\psi_{\bar{A}}(\bar{d}),\psi_{\bar{A}}(\bar{e}))$ and $\omega_{\bar{C}}(\bar{d},\bar{e})\ge max(\omega_{\bar{A}}(\bar{d}),\omega_{\bar{A}}(\bar{e}))$, for all $\bar{d},\bar{e}\in\bar{V}$. Since $\bar{C}$ is well-defined, a PFDG has at most two edges having opposite orientation between any two vertices. Therefore, PFGs and PFDGs are special cases of PFHGs. □

Remark 2 IVPFGs and IVPFDGs are the specific cases of the IVPFHGs.

4 Application of IVPFHGs in decision making

Over the last few years decision making got much attention of the researchers specially mathematicians. They provided the solution to many real world problems using different theories like fuzzy theory, set pair analysis, soft set theory etc. Fuzzy graph theory and fuzzy hypergraph theory play a vital role in solving complex uncertain real world problems. Here we have provided an algorithm followed by an application of IVPFHG towards multi-attribute decision-making. We also provide a flowchart related to this.

DECISION MAKING ALGORITHM.

1. Input the membership, neutral membership and non-membership interval values of all IVPFHG edges.
2. Compute the membership, neutral membership and non-membership interval values of IVPFH edges such that
   $\digamma_R\{m, n\} \leq \bigwedge\{\digamma_S(m), \digamma_S(n)\}$
   $\psi_R\{m, n\} \leq \bigwedge\{\psi_S(m), \psi_S(n)\}$
   $\omega_R\{m, n\} \geq \bigvee\{\omega_S(m), \omega_S(n)\}$.
3. Calculate the $[\iota, \kappa]$, $[\lambda, \epsilon]$, $[\rho, \nu]$-cuts $\tau_j^{[\iota,\kappa],[\lambda,\epsilon],[\rho,\nu]}$ of IVPFH edges such that $\mu_j^-(x_i) \geq \iota$, $\mu_j^+(x_i) \geq \kappa$, $\psi_j^-(x_i) \geq \lambda$ $\psi_j^+(x_i) \geq \epsilon$ and $\omega_j^-(x_i) \leq \rho$, $\omega_j^+(x_i) \leq \nu$ for all j=1, 2, …, m.
4. Find out the crisp sets describing the most suitable appliance according to the customers satisfaction levels.

Now we will solve a DM problem using the above algorithm. We consider a problem in which a person Mr. X wants to purchase an electric appliance (say refrigerator) which is available of many companies like Haier, Dawlance, Singer etc in the market. He needs the appliance which is most appropriate according to his desires and criteria. Let say that Mr. X consider the following four companies namely $U = \{u_1, u_2, u_3, u_4\}$ from which the appliance can be chosen to be purchased. We will discuss how the $\langle[\iota, \kappa], [\lambda, \epsilon], [\rho, \nu]\rangle$-level cuts can be applied to IVPFHG to make a perfect decision to buy a perfect appliancei.e., refrigerator.

As IVPFS is the most generalized form of the FSs to express uncertainty. We consider the companies as the vertices of an IVPFHG and hyper edges as the common features of the appliances, which are (as vertices) contained in that hyperedge.

The membership, neutral membership and non-membership degrees of vertices (which denotes the companies) depict that how much the company fulfills the costumers requirements, how much the product do not affect the customers choice and how much product is not suitable for customer. This is shown in Table 4.

The attributes are the hyperedges $L = \{L_1, L_2, L_3, L_4\}$ of IVPFHGs used to express the features of various companies such as durability L_1, quality L_2, functionality L_3, and marketability L_4. As L_2 is considered as quality, so the membership degrees [0.5, 0.7], [0.6, 0.8], [0.3, 0.5] of τ_1 represents that the appliances manufactured by company u_1 are 50–70% of good quality and are as per customers wish, 60–80% appliances are of normal quality which do not effect the customers wish and 30–50% of the appliances manufactured by the company lack the quality level as per

Table 4 Incidence matrix

$\mathcal{I}$	τ_1	τ_2	τ_3	τ_4
u_1	$\langle[0.5, 0.7][0.6, 0.8][0.3, 0.5]\rangle$	$\langle[0.5, 0.7][0.6, 0.8][0.3, 0.5]\rangle$	$\langle[0, 0]\ [0, 0]\ [0, 0]\rangle$	$\langle[0.5, 0.7][0.6, 0.8][0.3, 0.5]\rangle$
u_2	$\langle[0.4, 0.5][0.4, 0.6][0.5, 0.7]\rangle$	$\langle[0, 0]\ [0, 0]\ [0, 0]\rangle$	$\langle[0.4, 0.5][0.4, 0.6][0.5, 0.7]\rangle$	$\langle[0.4, 0.5][0.4, 0.6][0.5, 0.7]\rangle$
u_3	$\langle[0, 0]\ [0, 0]\ [0, 0]\rangle$	$\langle[0.3, 0.6][0.6, 0.9][0.4, 0.5]\rangle$	$\langle[0, 0]\ [0, 0]\ [0, 0]\rangle$	$\langle[0.3, 0.6][0.6, 0.9][0.4, 0.5]\rangle$
u_4	$\langle[0, 0]\ [0, 0]\ [0, 0]\rangle$	$\langle[0, 0]\ [0, 0]\ [0, 0]\rangle$	$\langle[0.4, 0.6][0.5, 0.7][0.4, 0.5]\rangle$	$\langle[0, 0]\ [0, 0]\ [0, 0]\rangle$

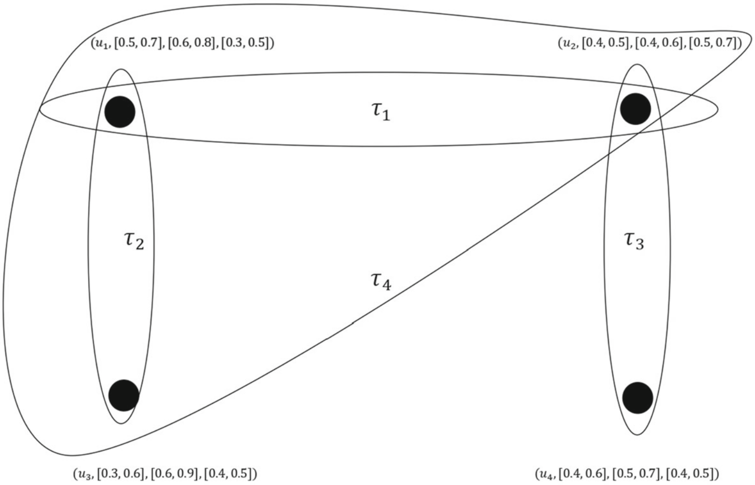

Fig. 3 Interval-valued picture fuzzy hypergraph

Table 5 Incidence matrix

$\tau^{[0.4,0.5][0.5,0.6][0.2,0.3]}$-level sets	Appliances
$\tau_1^{[0.4,0.5][0.5,0.6][0.2,0.3]}$	u_1
$\tau_2^{[0.4,0.5][0.5,0.6][0.2,0.3]}$	u_1, u_3
$\tau_3^{[0.4,0.5][0.5,0.6][0.2,0.3]}$	u_1
$\tau_4^{[0.4,0.5][0.5,0.6][0.2,0.3]}$	u_1, u_3

customers desire. Similarly, all the vertices consist of different values which describe the characteristics of all appliances manufactured by different companies.
To select the most suitable appliance, we will find the $\langle[\iota, \kappa], [\lambda, \epsilon], [\rho, \nu]\rangle$-level cuts of all hyperedges. We choose the values of $[\iota, \kappa]$, $[\lambda, \epsilon]$ and $[\rho, \nu]$ in such a way that they will be chosen according to the customers demand. Let $[\iota, \kappa], [\lambda, \epsilon], [\rho, \nu] = [0.4, 0.5][0.5, 0.6][0.2, 0.3]$, this means that the customer will consider the product, which will satisfy 40–50% or more of the characteristics mentioned, 50–60% of the product features do not effect the customers desire and 20–30% features are deficient in the product. The $\langle[\iota, \kappa], [\lambda, \epsilon], [\rho, \nu]$-level cuts of all hyperedges are given as:

Here $\tau_1^{[0.4,0.5][0.5,0.6][0.2,0.3]}$ level set represents that u_1 appliance is the most durable among other, $\tau_2^{[0.4,0.5][0.5,0.6][0.2,0.3]}$ level set represents that u_1 and u_3 appliances are best quality wise the most durable among other, $\tau_3^{[0.4,0.5][0.5,0.6][0.2,0.3]}$ level set represents that u_1 appliance is the best functionality wise and $\tau_4^{[0.4,0.5][0.5,0.6][0.2,0.3]}$ level set represents that u_1 appliance has the best marketability as compared to the other appliances.

Hence by considering different $\langle[\iota, \kappa], [\lambda, \epsilon], [\rho, \nu]\rangle$-levels corresponding to the demand of customer, one can choose the most suitable appliance fulfilling the demands

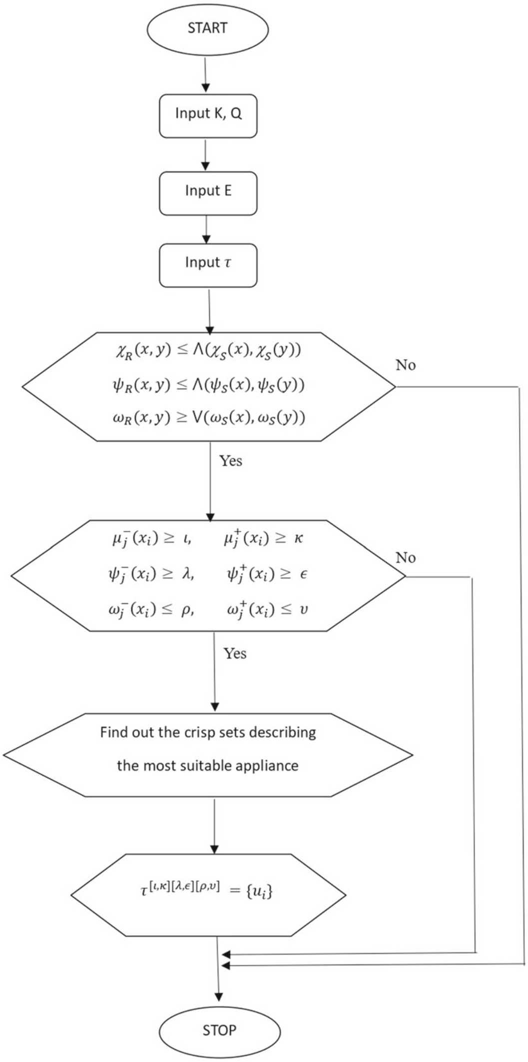

Fig. 4 Flowchart

of a customer. A flow chart in Fig. 4 can help to understand our proposed algorithm in a better way.

5 Comparative study and superiority of our presented model

Zadeh introduced FSs in 1965 in which each element has only membership value that lie in interval [0, 1]. Using the FSs and Fuzzy relations Rosenfeld introduced the concept of FGs. To express the uncertain data in a more suitable way FHGs were introduced. FSs were firstly generalized to IVFs. Using IVFs, the concept of IVFHGs was introduced in which vertices and edges were expressed by using only membership

Table 6 Fuzzy hypergraphs and their generalizations

Authors	Reference	Articles
Lee-Kwang and Lee	[19]	FHGs and fuzzy partition
Chen	[21]	IVFHGs and fuzzy partition
Akram and Dudek	[24]	IVHGs with applications
Akram and Alshehri	[34]	Tempered IVFHGs
Gong and Hua	[35]	Bipolar IVFS in graphs and HGs settings
Akram and Luqman	[36]	Intuitionistic single-valued neutrosophic hypergraphs
Akram et al.	[37]	Single-valued neutrosophic HGs
Naz et al.	[23]	HGs and Transversals of HGs in IVIF Setting
Akram et al.	[38]	A novel DM approach based on HGs in IF environment
Samantha and Pal	[39]	Bipolar FHGs
Wang and Gong	[40]	An application of FHGs and HGs in granular computing
Samantha and Pal	[27]	Granulation of hypernetwork models under the q-rung PF environment
Akram et al.	[31]	Granular computing based on q-rung PFHGs

values which were intervals. Attanasov introduced IFSs which was the generalization of FSs. Based on the idea of IFSs, the notion of IFHGs was introduced in which the values of vertices and edges were expressed in terms of membership and non membership degrees. IFHGs expresses the uncertain information better than FHGs and IVFHGs. Afterwards, using IVIFSs, the notion of IVIFHGs was introduced as the generalization of IFHGs, IVFHGs and FHGs. Further to this, PFSs were introduced by Coung. PFSs were introduced as the generalization of FSs and IFSs. Using PFSs, PFHGs were introduced. In PFHGs, vertices and edges were expressed using PF numbers which include the membership, non membership and neutral membership values. PFSs were generalized to interval-valued picture fuzzy sets (IVPFSs) which was the most generalized form of fuzzy sets (FSs). It allocates the membership, non membership and neutral membership values to its each element. The values are the intervals which were subintervals of [0, 1]. In this manuscript, using IVPFSs we introduce the concept of IVPFHGs. The introduced notion IVPFHGs is the most generalized form of FHGs as it is proven to be a useful tool to manipulate complications that arise due to incomplete information. IVPFHGs play a key role to deal the data with uncertainties. We also conduct a comparative analysis between IVPFHGs and the other structures highlighted in Table 6. Like IVPFSs and IVPFGs, it is established that IVPFHGs is the most extended form of FHGs.

Superiority of our presented model

Using IVPFSs we introduce the notion of IVPFHGs which fills up the gap existing in the literature and IVPFHGs is the most generalized form of FGs. The introduced notion of IVPFHGs is better than the IVFHGs and IVIFHGs existing in literature like if we consider FHGs, then it has only $\langle[\iota, \kappa]\rangle$-level cuts and they represent only the

Table 7 Generalizations of fuzzy sets

Authors	Reference	Notion introduced
Zadeh	[1]	Fuzzy sets
Zadeh	[2]	Interval valued fuzzy sets
Attanasov	[28]	Intuitionistic fuzzy sets
Attanasov	[9]	Interval valued intuitionistic fuzzy sets
Cuong	[10]	Picture fuzzy sets
Cuong	[10]	Interval valued picture fuzzy sets

membership values. Like in the above application, if we represent the information using IVFHGs, than for selecting the suitable appliance, we find $\langle[\iota, \kappa]\rangle$-level cuts of hyperedges. The chosen values for $[\iota, \kappa]$ are according to the customer demand. These values only represent those features of the appliance which are according to the desires of the customer. But IVFHGs take only membership values, means the result may be good but it is also possible that the selected appliance may have features which customer do not want in the selected appliance. Hence to overcome this one may use IVIFHGs, in which by using $\langle[\iota, \kappa], [\lambda, \epsilon]\rangle$-level cuts one can express the desired features for the appliance as well as those features which he does not want in the selected appliance. In IVIFHGs, $[\iota, \kappa]$ represent membership values i.e., features as per customer desire and $[\lambda, \epsilon]$ represents non-membership values i.e., features which customer don't want in the appliance. This overall makes IVIFHGs better than IVFHGs. Beside the membership values which are representing the desirous features necessary in the appliance for customer and non-membership values which are representing the features which customer don't want in the appliance, there may be some extra features in the appliance such that their presence may increase options for the customer but don't effect the customer's choice. These are additional features in the appliance i.e., their presence in the appliance increase the features of it. These features may be according to the desire of customer or not but there presence is not necessary as per the criteria of the customer. He can use the features according to the need but if he do not need these features he may simply not use them. Their is a need of bigger structure to represent these all features. The introduced structure i.e., IVPFHGs is best to represent all the features in a best possible way. These features are adjusted in the neutral function. As we have shown in the above application, the $\langle[\iota, \kappa], [\lambda, \epsilon], [\rho, \nu]\rangle$-level cuts of hyperedges of IVPFHGs express all the features in the best possible way. $[\iota, \kappa]$ represents membership values i.e., features as per customer desires and $[\lambda, \epsilon]$ represents non-membership values i.e., features which customer do not want in the appliance and the additional features are named as neutral values and these are expressed using $[\rho, \nu]$. Tables 7, 8 and 9 show the details of the generalizations of FSs, FGs and FHGs, respectively. Figure 5 shows all the possible generalizations of FGs and FHGs.

Table 8 Generalizations of fuzzy graphs

Authors	Reference	Notion introduced
Rosenfeld	[41]	Fuzzy graphs
Akram and Dudek	[42]	Interval valued fuzzy graphs
Parvathi and Karunambigai	[43]	Intuitionistic fuzzy graphs
Atanassov	[44]	Interval valued intuitionistic fuzzy graphs
Zuo et al.	[45]	Picture fuzzy graphs
Jayalakshmi and Kamali	[46]	Interval valued picture fuzzy graphs

Table 9 Generalizations of fuzzy hypergraphs

Authors	Reference	Notion introduced
Kaufmann	[18]	Fuzzy hypergraphs
Chen	[21]	Interval valued fuzzy hypergraphs
Parvathi et al.	[22]	Intuitionistic fuzzy hypergraphs
Naz et al.	[23]	Interval valued intuitionistic fuzzy hypergraphs
Luqman et al.	[25]	Picture fuzzy hypergraphs

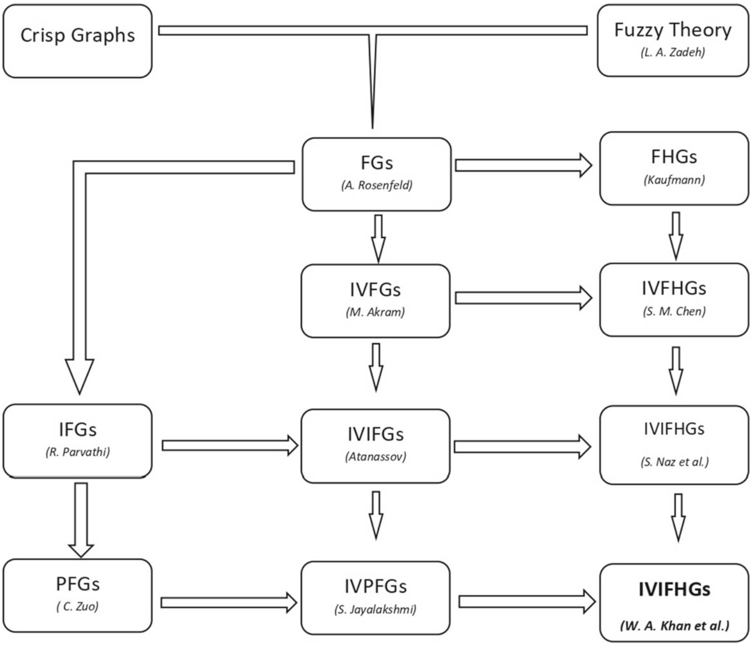

Fig. 5 Generalizations of fuzzy graphs

6 Conclusion

In this study, we have introduced IVPFHGs which is the generalization of FHGs, IVFHGs, IFHGs, IVIFHGs and PFHGs. Actually, IVIFHG also combines all the features of existing fuzzy graphs structures within a single framework. By assigning different membership values, IVPFHGs can be converted into any of the existing fuzzy graphs structure depending on the circumstances. IVPFHGs can handle uncertainties more effectively than any other type of FHGs. Recently, FGs and DM become an important area of research. In this manuscript, we have introduced and discussed the notion of IVPFHGs. We have discussed various types of IVPFHGs and have explored some interrelationships among them. We have also introduced the dual of IVPFHGs. Throughout, we have furnished our results with suitable examples. Finally, we have provided an application of IVPFHGs in DM using the $[\iota, \kappa]$, $[\lambda, \epsilon]$, $[\rho, \nu]$-level cuts. We have also provided a comparative analysis and found that like IVPFSs and IVPFGs, IVPFHGs is the most extended form of FHGs. The concepts presented in this article can be shifted towards other FHGs structures like bipolar PFHGs, interval-valued bipolar PFHGs etc.

Declarations

Conflict of interest The authors declare that there is no any conflict of interest.

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