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ORIGINAL RESEARCH

A second-order difference scheme for two-dimensional two-sided space distributed-order fractional diffusion equations with variable coefficients

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Abstract
In this paper, a second-order difference scheme is developed to solve two-dimensional two-sided space distributed-order fractional diffusion equation with variable coefficients. In the spatial direction, a second-order difference scheme is proposed, the distribution-order integral is discretized by the Gauss–Legendre quadrature formula and the space fractional derivative is approximated by the weighted and shifted Grünwald–Letnikov operators. In addition, the time direction is discretized into a second-order difference scheme by the Crank–Nicolson method. Therefore, the main numerical scheme is developed. Furthermore, a small perturbation is added to the main difference scheme to construct an alternating-direction implicit scheme. Also, the stability and convergence of the numerical scheme are proved. Finally, some numerical results are provided to show the accuracy and efficiency of the proposed method.

Keywords Two-dimensional distributed-order fractional diffusion equation · Alternating direction implicit method · Variable coefficient · Stability and convergence

Mathematics Subject Classification 65M06 · 65M12

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1 Introduction

In recent years, distributed-order fractional diffusion equations (DOFDEs) have attracted considerable interest because of its ability to model the processes that become more anomalous in course of time. It is an important tool for modeling ultraslow diffusion [9, 23, 27] or accelerating superdiffusion [8, 36] where a plume of particles spreads at a logarithmic rate. DOFDEs were first proposed in [6]. During these years, the theoretical studies of DOFDEs were carried out by some literatures. For example, in [4], the authors obtained a priori estimation of the solution of the initial boundary value problems of DOFDEs by the maximal principle. In [26], the authors proved the existence of the solution of the boundary value problems of DOFDEs by constructing a formal solution using the Fourier method of variables separation. The authors of [21] discussed the well-posedness of the Cauchy problems of the abstract DOFDEs by functional calculus technique. Ansari et al. [5] used the Mittag-Leffler and Wright functions to obtain the fundamental solution of the DOFDEs with fraction Laplacian in axisymmetric cylindrical configuration. In general, the analytical solutions of many DOFDEs are not easy to gain. Therefore, different numerical methods are worth considering to solve DOFDEs. Zaky et al. [37] developed a spectral tau method based on Legendre polynomials for DOFDEs. In [3], authors studied Galerkin meshless reproducing kernel particle method for neutral delay time-space distributed-order fractional damped diffusion-wave equations. Authors of [19] presented a finite element method for the distributed order time-fractional diffusion equations. Sun et al. [32] proposed a fast and memory saving algorithm for solving distributed-order time-space fractional diffusion equations, and other methods for this type of equations can refer to [14, 16, 38, 42, 43].

As an important class of DOFDEs, the space DOFDEs have attracted extensive attentions, which can be used to simulate an accelerated superdiffusion process [22, 36]. And they mainly include two kinds, one is the Riesz space DOFDEs, the other is two-sided space DOFDEs which is more general. And under certain conditions, the two types of equations can be transformed into each other. For example, authors of [2] studied the following two-dimensional distributed-order Riesz space-fractional diffusion equation

$$\frac{\partial u(x,y,t)}{\partial t} = \int_1^2 \chi(\beta)\left[\frac{\partial^\beta u(x,y,t)}{\partial |x|^\beta} + \frac{\partial^\beta u(x,y,t)}{\partial |y|^\beta}\right]\mathrm{d}\beta + f(x,y,t), \quad (1.1)$$

where $\chi(\beta)$ is non-negative weight function of β, and $\chi(\beta)$ satisfies [1]

$$\chi(\beta) \geq 0, \quad \chi(\beta) \neq 0, \ \forall \beta \in (1,2), \quad 0 < \int_1^2 \chi(\beta)\mathrm{d}\beta < \infty.$$

$\frac{\partial^\beta u(x,y,t)}{\partial |x|^\beta}$ and $\frac{\partial^\beta u(x,y,t)}{\partial |y|^\beta}$ denote the Riesz space fractional derivative [3, 20]. (1.1) can be written as

$$\begin{aligned}\frac{\partial u(x,y,t)}{\partial t} &= \int_1^2 \chi(\beta)\frac{-1}{2\cos(\frac{\beta\pi}{2})}\left[\frac{\partial}{\partial x}\left(\frac{\partial^{\beta-1}u(x,y,t)}{\partial x^{\beta-1}} - \frac{\partial^{\beta-1}u(x,y,t)}{\partial(-x)^{\beta-1}}\right)\right.\\&\quad\left.+\frac{\partial}{\partial y}\left(\frac{\partial^{\beta-1}u(x,y,t)}{\partial y^{\beta-1}} - \frac{\partial^{\beta-1}u(x,y,t)}{\partial(-y)^{\beta-1}}\right)\right]\mathrm{d}\beta + f(x,y,t)\\&= \int_0^1 \chi(\alpha+1)\frac{-1}{2\cos\left(\frac{(\alpha+1)\pi}{2}\right)}\left[\frac{\partial}{\partial x}\left(\frac{\partial^{\alpha}u(x,y,t)}{\partial x^{\alpha}} - \frac{\partial^{\alpha}u(x,y,t)}{\partial(-x)^{\alpha}}\right)\right.\\&\quad\left.+\frac{\partial}{\partial y}\left(\frac{\partial^{\alpha}u(x,y,t)}{\partial y^{\alpha}} - \frac{\partial^{\alpha}u(x,y,t)}{\partial(-y)^{\alpha}}\right)\right]\mathrm{d}\alpha + f(x,y,t),\end{aligned} \tag{1.2}$$

let $\omega(\alpha) = \chi(\alpha+1)\frac{-1}{2\cos\left(\frac{(\alpha+1)\pi}{2}\right)}$, thus (1.1) becomes a two-dimensional two-sided space distributed-order fractional diffusion model [41]. To date there have many numerical methods to be proposed for these equations. Chen et al. [11] derived a fourth-order accurate numerical method for the distributed-order Riesz space fractional diffusion equation. Abbaszadeh et al. [2] presented a fourth-order ADI finite difference scheme to solve the two-dimensional distributed-order Riesz space-fractional diffusion equation. In [36], authors developed a novel finite volume method for the nonlinear two-sided space distributed-order diffusion equation. The general linear method and spectral Galerkin method was proposed by Zhang et al. [41] to solve the nonlinear two-sided space distributed-order diffusion equations.

However, it is worth noting that the diffusion coefficients in (1.2) are constants, and most of the previous work dealt with the problem that the diffusion coefficients are constants, but the diffusion coefficients often depend on the time or space variable in some practical problems [10, 13, 24, 29, 31, 36, 40, 44]. Based on model (1.2), we consider the following two-dimensional two-sided space distributed-order fractional diffusion model with variable diffusivity coefficients:

$$\begin{cases}\frac{\partial u(x,y,t)}{\partial t} = \int_0^1 \omega(\alpha)\left[\frac{\partial}{\partial x}\left(k_{11}(x,y,\alpha)\frac{\partial^\alpha u(x,y,t)}{\partial x^\alpha} - k_{12}(x,y,\alpha)\frac{\partial^\alpha u(x,y,t)}{\partial(-x)^\alpha}\right)\right.\\ \quad\left.+\frac{\partial}{\partial y}\left(k_{21}(x,y,\alpha)\frac{\partial^\alpha u(x,y,t)}{\partial y^\alpha} - k_{22}(x,y,\alpha)\frac{\partial^\alpha u(x,y,t)}{\partial(-y)^\alpha}\right)\right]\mathrm{d}\alpha + f(x,y,t,u),\\ \quad 0<\alpha\le 1, \quad (x,y,t)\in\Omega\times[0,T],\\ u(x,y,t)=0, \quad (x,y)\in\partial\Omega,\ t\in(0,T],\\ u(x,y,0)=\varphi(x,y), \quad (x,y)\in\Omega,\end{cases} \tag{1.3}$$

where $\Omega = (a,b)\times(c,d)$, $k_{i,j}(x,y,\alpha)$, $i, j = 1, 2$, are nonnegative diffusion coefficients, and they satisfy

$$\frac{\partial k_{11}(x,y,\alpha)}{\partial x} < 0, \quad \frac{\partial k_{12}(x,y,\alpha)}{\partial x} > 0, \quad \frac{\partial k_{21}(x,y,\alpha)}{\partial y} < 0, \quad \frac{\partial k_{22}(x,y,\alpha)}{\partial y} > 0, \tag{1.4}$$

f is a given function, and satisfies the Lipschitz condition,

$$|f(x, y, t, u) - f(x, y, t, v)| \leq L_1|u - v|, \tag{1.5}$$

where L_1 is a positive constant. Specifically, when $f(x, y, t, u) = f(x, y, t)$, Eq. (1.3) is a linear problem. And $\omega(\alpha)$ satisfies

$$\omega(\alpha) \geq 0, \quad \omega(\alpha) \neq 0, \ \forall \alpha \in (0, 1), \quad 0 < \int_0^1 \omega(\alpha)\mathrm{d}\alpha < \infty.$$

The definitions of operators $\frac{\partial^\alpha u(x,y,t)}{\partial x^\alpha}$, $\frac{\partial^\alpha u(x,y,t)}{\partial (-x)^\alpha}$, $\frac{\partial^\alpha u(x,y,t)}{\partial y^\alpha}$ and $\frac{\partial^\alpha u(x,y,t)}{\partial (-y)^\alpha}$ can be seen [25, 35].

In this paper, we first discretize the distribution-order integral by the Gauss–Legendre quadrature formula, and approximate the space fractional derivative by the weighted and shifted Grünwald–Letnikov operators [15, 34, 35]. Therefore, the second-order accuracy approximation in space can be achieved. In addition, the Crank–Nicolson method [7, 18, 28] are applied to achieve time discretization. Hence, a second-order difference scheme in all variables is developed to solve (1.3). Furthermore, to avoid solving large systems of linear equations, an alternating-direction implicit scheme is constructed by adding a small perturbation to the above second-order difference scheme. Finally, the stability and convergence of the numerical scheme are analyzed.

The outline of this paper is organized as follows. In Sect. 2, we provide the numerical method for solving the two-dimensional two-side space distributed-order fractional diffusion equation with variable diffusivity coefficients. The stability and convergence analysis are proved in Sect. 3. Section 4 presents some numerical results to show the effectiveness of our numerical method. Finally, some conclusions are made in Sect. 5.

2 Numerical method

Let $\tau = \frac{T}{N}$ be the time step size, and $h_1 = \frac{b-a}{M_1}$, $h_2 = \frac{d-c}{M_2}$ be the space step size. And denote $t_n = n\tau$, $n = 0, 1, \ldots, N$, $x_i = a + ih_1$, $i = 0, 1, \ldots, M_1$, $y_j = c + jh_2$, $j = 0, 1, \ldots, M_2$. For convenience, we note

$$\begin{aligned}
A(\alpha, x, y, t) &= \frac{\partial}{\partial x}\left(k_{11}(x, y, \alpha)\frac{\partial^\alpha u(x, y, t)}{\partial x^\alpha} - k_{12}(x, y, \alpha)\frac{\partial^\alpha u(x, y, t)}{\partial (-x)^\alpha}\right) \\
&= \frac{\partial k_{11}(x, y, \alpha)}{\partial x}\frac{\partial^\alpha u(x, y, t)}{\partial x^\alpha} + k_{11}(x, y, \alpha)\frac{\partial^{\alpha+1} u(x, y, t)}{\partial x^{\alpha+1}} \\
&\quad - \frac{\partial k_{12}(x, y, \alpha)}{\partial x}\frac{\partial^\alpha u(x, y, t)}{\partial (-x)^\alpha} + k_{12}(x, y, \alpha)\frac{\partial^{\alpha+1} u(x, y, t)}{\partial (-x)^{\alpha+1}}, \\
B(\alpha, x, y, t) &= \frac{\partial}{\partial y}\left(k_{21}(x, y, \alpha)\frac{\partial^\alpha u(x, y, t)}{\partial y^\alpha} - k_{22}(x, y, \alpha)\frac{\partial^\alpha u(x, y, t)}{\partial (-y)^\alpha}\right) \\
&= \frac{\partial k_{21}(x, y, \alpha)}{\partial y}\frac{\partial^\alpha u(x, y, t)}{\partial y^\alpha} + k_{21}(x, y, \alpha)\frac{\partial^{\alpha+1} u(x, y, t)}{\partial y^{\alpha+1}}
\end{aligned}$$

$$-\frac{\partial k_{22}(x,y,\alpha)}{\partial y}\frac{\partial^{\alpha}u(x,y,t)}{\partial(-y)^{\alpha}}+k_{22}(x,y,\alpha)\frac{\partial^{\alpha+1}u(x,y,t)}{\partial(-y)^{\alpha+1}}.$$

If $\omega(\alpha)\,(A(\alpha,x,y,t)+B(\alpha,x,y,t))\in C^{\nu}[0,1]$, using Gauss–Legendre quadrature formula [30, 39] to approximate the integral of Eq. (1.3), we can write (1.3) as

$$\frac{\partial u(x,y,t)}{\partial t}=\frac{1}{2}\sum_{r=1}^{m}w_r\omega(\alpha_r)\left[A(\alpha_r,x,y,t)+B(\alpha_r,x,y,t)\right]+f(x,y,t,u)+\mathcal{O}(m^{-\upsilon}), \tag{2.1}$$

where $\alpha_r=\frac{1+p_r}{2}$, and p_r, w_r, $r=1,\ldots,m$, are quadrature points and quadrature weights, respectively. Let

$$\begin{aligned}
u_{i,j}^{n}=u(x_i,y_j,t_n),\qquad u_{i,j}^{n+\frac{1}{2}}&=\frac{u_{i,j}^{n+1}+u_{i,j}^{n}}{2},\qquad \delta_t u_{i,j}^{n+\frac{1}{2}}=\frac{u_{i,j}^{n+1}-u_{i,j}^{n}}{\tau},\\
k_{11}^{i,j,r}&=k_{11}(x_i,y_j,\alpha_r),\quad k_{12}^{i,j,r}=k_{12}(x_i,y_j,\alpha_r),\\
k_{21}^{i,j,r}&=k_{21}(x_i,y_j,\alpha_r),\quad k_{22}^{i,j,r}=k_{22}(x_i,y_j,\alpha_r),\\
k_{11,x}^{i,j,r}=\frac{\partial k_{11}(x,y,\alpha)}{\partial x}\bigg|(x_i,y_j,\alpha_r),&\quad k_{12,x}^{i,j,r}=\frac{\partial k_{12}(x,y,\alpha)}{\partial x}\bigg|(x_i,y_j,\alpha_r),\\
k_{21,y}^{i,j,r}=\frac{\partial k_{21}(x,y,\alpha)}{\partial y}\bigg|(x_i,y_j,\alpha_r),&\quad k_{22,y}^{i,j,r}=\frac{\partial k_{22}(x,y,\alpha)}{\partial y}\bigg|(x_i,y_j,\alpha_r).
\end{aligned}$$

Then, using the Crank–Nicolson method and the weighted and shifted Grünwald–Letnikov operators [25, 33, 35] to approximate Eq. (2.1), it follows that

$$\begin{aligned}
\delta_t u_{i,j}^{n+\frac{1}{2}}=\frac{1}{4}\sum_{r=1}^{m}w_r\omega(\alpha_r)\big[&A(\alpha_r,x_i,y_j,t_{n+1})+A(\alpha_r,x_i,y_j,t_n)+B(\alpha_r,x_i,y_j,t_{n+1})\\
&+B(\alpha_r,x_i,y_j,t_n)\big]+f\left(x_i,y_j,t_{n+\frac{1}{2}},u_{i,j}^{n+\frac{1}{2}}\right)+r_{ij}^{n},
\end{aligned}\tag{2.2}$$

where

$$\begin{aligned}
A(\alpha_r,x_i,y_j,t_n)=&\frac{k_{11,x}^{i,j,r}}{h_1^{\alpha_r}}\sum_{l=0}^{i+1}w_l^{\alpha_r}u_{i-l+1,j}^{n}+\frac{k_{11}^{i,j,r}}{h_1^{\alpha_r+1}}\sum_{l=0}^{i+1}w_l^{\alpha_r+1}u_{i-l+1,j}^{n}\\
&-\frac{k_{12,x}^{i,j,r}}{h_1^{\alpha_r}}\sum_{l=0}^{M_1-i+1}w_l^{\alpha_r}u_{i+l-1,j}^{n}+\frac{k_{12}^{i,j,r}}{h_1^{\alpha_r+1}}\sum_{l=0}^{M_1-i+1}w_l^{\alpha_r+1}u_{i+l-1,j}^{n},\\
B(\alpha_r,x_i,y_j,t_n)=&\frac{k_{21,y}^{i,j,r}}{h_2^{\alpha_r}}\sum_{l=0}^{j+1}w_l^{\alpha_r}u_{i,j-l+1}^{n}+\frac{k_{21}^{i,j,r}}{h_2^{\alpha_r+1}}\sum_{l=0}^{j+1}w_l^{\alpha_r+1}u_{i,j-l+1}^{n}
\end{aligned}$$

$$
-\frac{k_{22,y}^{i,j,r}}{h_2^{\alpha_r}}\sum_{l=0}^{M_2-j+1} w_l^{\alpha_r} u_{i,j+l-1}^n + \frac{k_{22}^{i,j,r}}{h_2^{\alpha_r+1}}\sum_{l=0}^{M_2-j+1} w_l^{\alpha_r+1} u_{i,j+l-1}^n,
$$

$$
\left|r_{ij}^n\right| \le C_r(m^{-v} + h_1^2 + h_2^2 + \tau^2),
$$

with C_r is a positive constant. Approximating $f\left(x_i, y_j, t_{n+\frac{1}{2}}, u_{i,j}^{n+\frac{1}{2}}\right)$ with Taylor expansion [36], (2.2) can be converted into

$$
\begin{aligned}
\delta_t u_{i,j}^{n+\frac{1}{2}} = \frac{1}{4}\sum_{r=1}^{m} w_r \omega(\alpha_r)\big[&A(\alpha_r, x_i, y_j, t_{n+1}) + A(\alpha_r, x_i, y_j, t_n) + B(\alpha_r, x_i, y_j, t_{n+1}) \\
&+ B(\alpha_r, x_i, y_j, t_n)\big] + f\left(x_i, y_j, t_{n+\frac{1}{2}}, \frac{3}{2}u_{i,j}^n - \frac{1}{2}u_{i,j}^{n-1}\right) + r_{ij}^n.
\end{aligned} \tag{2.3}
$$

And let

$$
\begin{aligned}
\delta_x u_{i,j}^n &= \frac{1}{4}\sum_{r=1}^{m}\frac{w_r\omega(\alpha_r)}{h_1^{\alpha_r}}\left(k_{11,x}^{i,j,r}\sum_{l=0}^{i+1} w_l^{\alpha_r} u_{i-l+1,j}^n - k_{12,x}^{i,j,r}\sum_{l=0}^{M_1-i+1} w_l^{\alpha_r} u_{i+l-1,j}^n\right), \\
\delta_y u_{i,j}^n &= \frac{1}{4}\sum_{r=1}^{m}\frac{w_r\omega(\alpha_r)}{h_2^{\alpha_r}}\left(k_{21,y}^{i,j,r}\sum_{l=0}^{j+1} w_l^{\alpha_r} u_{i,j-l+1}^n - k_{22,y}^{i,j,r}\sum_{l=0}^{M_2-j+1} w_l^{\alpha_r} u_{i,j+l-1}^n\right), \\
\sigma_x u_{i,j}^n &= \frac{1}{4}\sum_{r=1}^{m}\frac{w_r\omega(\alpha_r)}{h_1^{\alpha_r+1}}\left(k_{11}^{i,j,r}\sum_{l=0}^{i+1} w_l^{\alpha_r+1} u_{i-l+1,j}^n + k_{12}^{i,j,r}\sum_{l=0}^{M_1-i+1} w_l^{\alpha_r+1} u_{i+l-1,j}^n\right), \\
\sigma_y u_{i,j}^n &= \frac{1}{4}\sum_{r=1}^{m}\frac{w_r\omega(\alpha_r)}{h_2^{\alpha_r+1}}\left(k_{21}^{i,j,r}\sum_{l=0}^{j+1} w_l^{\alpha_r+1} u_{i,j-l+1}^n + k_{22}^{i,j,r}\sum_{l=0}^{M_2-j+1} w_l^{\alpha_r+1} u_{i,j+l-1}^n\right).
\end{aligned}
$$

Then, omitting r_{ij}^n, replacing u_{ij}^n with U_{ij}^n, Eq. (2.3) can be written as

$$
\begin{aligned}
&\left(1 - \tau\delta_x - \tau\sigma_x - \tau\delta_y - \tau\sigma_y\right) U_{i,j}^{n+1} = \left(1 + \tau\delta_x + \tau\sigma_x + \tau\delta_y + \tau\sigma_y\right) U_{i,j}^n \\
&\quad + \tau f\left(x_i, y_j, t_{n+\frac{1}{2}}, \frac{3}{2}U_{i,j}^n - \frac{1}{2}U_{i,j}^{n-1}\right), \qquad 1 \le i \le M_1 - 1,\ 1 \le j \le M_2 - 1.
\end{aligned} \tag{2.4}
$$

Further, Eq. (2.4) can be expressed as the following matrix

$$
(I + H)U^{n+1} = (I - H)U^n + \tau F^n, \tag{2.5}
$$

where

$$
H = (H_{i_1,j_1})_{(M_1-1)(M_2-1)\times(M_1-1)(M_2-1)} = H_1 + H_2,
$$

$H_1 = \mathrm{diag}(H_x(1), H_x(2), \ldots, H_x(M_2-1))$ with $H_x(k) = (h^x_{i,j}(k))_{(M_1-1)\times(M_1-1)}$, $1 \le k \le M_2-1$, $H_2 = (h_{i_1,j_1})_{(M_1-1)(M_2-1)\times(M_1-1)(M_2-1)} = (h_{i,j})_{(M_2-1)\times(M_2-1)}$ with $h_{i,j} = \mathrm{diag}(h^y_{i,j}(1), h^y_{i,j}(2), \ldots, h^y_{i,j}(M_1-1))$, and

$$h^x_{i,j}(k) = \begin{cases} \sum_{r=1}^{m} \frac{-\tau w_r \omega(\alpha_r)}{4h_1^{\alpha_r}} \left(\frac{1}{h_1} k_{11}^{i,k,r} w_{i-j+1}^{\alpha_r+1} + k_{11,x}^{i,k,r} w_{i-j+1}^{\alpha_r} \right), & for\ j < i-1, \\ \sum_{r=1}^{m} \frac{-\tau w_r \omega(\alpha_r)}{4h_1^{\alpha_r}} \left(\frac{1}{h_1} (k_{11}^{i,k,r} w_2^{\alpha_r+1} + k_{12}^{i,k,r} w_0^{\alpha_r+1}) + (k_{11,x}^{i,k,r} w_2^{\alpha_r} - k_{12,x}^{i,k,r} w_0^{\alpha_r}) \right), & for\ j = i-1, \\ \sum_{r=1}^{m} \frac{-\tau w_r \omega(\alpha_r)}{4h_1^{\alpha_r}} \left(\frac{1}{h_1} (k_{11}^{i,k,r} w_1^{\alpha_r+1} + k_{12}^{i,k,r} w_1^{\alpha_r+1}) + (k_{11,x}^{i,k,r} w_1^{\alpha_r} - k_{12,x}^{i,k,r} w_1^{\alpha_r}) \right), & for\ j = i, \\ \sum_{r=1}^{m} \frac{-\tau w_r \omega(\alpha_r)}{4h_1^{\alpha_r}} \left(\frac{1}{h_1} (k_{11}^{i,k,r} w_0^{\alpha_r+1} + k_{12}^{i,k,r} w_2^{\alpha_r+1}) + (k_{11,x}^{i,k,r} w_0^{\alpha_r} - k_{12,x}^{i,k,r} w_2^{\alpha_r}) \right), & for\ j = i+1, \\ \sum_{r=1}^{m} \frac{-\tau w_r \omega(\alpha_r)}{4h_1^{\alpha_r}} \left(\frac{1}{h_1} k_{12}^{i,k,r} w_{j-i+1}^{\alpha_r+1} - k_{12,x}^{i,k,r} w_{j-i+1}^{\alpha_r} \right), & for\ j > i+1, \end{cases}$$

$$h^y_{i,j}(k) = \begin{cases} \sum_{r=1}^{m} \frac{-\tau w_r \omega(\alpha_r)}{4h_2^{\alpha_r}} \left(\frac{1}{h_2} k_{21}^{k,i,r} w_{i-j+1}^{\alpha_r+1} + k_{21,y}^{k,i,r} w_{i-j+1}^{\alpha_r} \right), & for\ j < i-1, \\ \sum_{r=1}^{m} \frac{-\tau w_r \omega(\alpha_r)}{4h_2^{\alpha_r}} \left(\frac{1}{h_2} (k_{21}^{k,i,r} w_2^{\alpha_r+1} + k_{22}^{k,i,r} w_0^{\alpha_r+1}) + (k_{21,y}^{k,i,r} w_2^{\alpha_r} - k_{22,y}^{k,i,r} w_0^{\alpha_r}) \right), & for\ j = i-1, \\ \sum_{r=1}^{m} \frac{-\tau w_r \omega(\alpha_r)}{4h_2^{\alpha_r}} \left(\frac{1}{h_2} (k_{21}^{k,i,r} w_1^{\alpha_r+1} + k_{22}^{k,i,r} w_1^{\alpha_r+1}) + (k_{21,y}^{k,i,r} w_1^{\alpha_r} - k_{22,y}^{k,i,r} w_1^{\alpha_r}) \right), & for\ j = i, \\ \sum_{r=1}^{m} \frac{-\tau w_r \omega(\alpha_r)}{4h_2^{\alpha_r}} \left(\frac{1}{h_2} (k_{21}^{k,i,r} w_0^{\alpha_r+1} + k_{22}^{k,i,r} w_2^{\alpha_r+1}) + (k_{21,y}^{k,i,r} w_0^{\alpha_r} - k_{22,y}^{k,i,r} w_2^{\alpha_r}) \right), & for\ j = i+1, \\ \sum_{r=1}^{m} \frac{-\tau w_r \omega(\alpha_r)}{4h_2^{\alpha_r}} \left(\frac{1}{h_2} k_{22}^{k,i,r} w_{j-i+1}^{\alpha_r+1} - k_{22,y}^{k,i,r} w_{j-i+1}^{\alpha_r} \right), & for\ j > i+1, \end{cases}$$

$$\begin{aligned} U^n &= \left(U^n_{1,1}, U^n_{2,1}, \ldots, U^n_{M_1-1,1}, \ldots, U^n_{1,M_2-1}, \ldots, U^n_{M_1-1,M_2-1}\right)^T, \\ F^n &= \left(f^n_{1,1}, f^n_{2,1}, \ldots, f^n_{M_1-1,1}, \ldots, f^n_{1,M_2-1}, \ldots, f^n_{M_1-1,M_2-1}\right)^T, \\ f^n_{i,j} &= f\left(x_i, y_j, t_{n+\frac{1}{2}}, \frac{3}{2}U^n_{i,j} - \frac{1}{2}U^{n-1}_{i,j}\right). \end{aligned}$$

To avoid solving large systems of linear equations, we add a small perturbation $\tau^3(\delta_x + \sigma_x)(\delta_y + \sigma_y)\delta_t u^{n+\frac{1}{2}}_{i,j}$ on both sides of (2.3) to construct the following ADI scheme

$$\begin{aligned} &(1 - \tau\delta_x - \tau\sigma_x)\left(1 - \tau\delta_y - \tau\sigma_y\right) u^{n+1}_{i,j} \\ &\quad = (1 + \tau\delta_x + \tau\sigma_x)\left(1 + \tau\delta_y + \tau\sigma_y\right) u^n_{i,j} + \tau f\left(x_i, y_j, t_{n+\frac{1}{2}}, \frac{3}{2}u^n_{i,j} - \frac{1}{2}u^{n-1}_{i,j}\right) + R^n_{i,j}, \end{aligned} \tag{2.6}$$

where

$$\begin{aligned} |R^n_{i,j}| &= \left| O(m^{-v} + \tau h_1^2 + \tau h_2^2 + \tau^3) + \tau^3(\delta_x + \sigma_x)(\delta_y + \sigma_y)\delta_t u^{n+\frac{1}{2}}_{i,j} \right| \\ &\le C_R(m^{-v} + h_1^2 + h_2^2 + \tau^2), \end{aligned}$$

where C_R is a constant. Omitting $R^n_{i,j}$, replacing u^n_{ij} with U^n_{ij}, (2.6) can be written as

$$
\begin{aligned}
&(1-\tau\delta_x-\tau\sigma_x)\left(1-\tau\delta_y-\tau\sigma_y\right)U^{n+1}_{i,j}\\
&\quad=(1+\tau\delta_x+\tau\sigma_x)\left(1+\tau\delta_y+\tau\sigma_y\right)U^n_{i,j}+\tau f\left(x_i,y_j,t_{n+\frac{1}{2}},\frac{3}{2}U^n_{i,j}-\frac{1}{2}U^{n-1}_{i,j}\right).
\end{aligned}
\tag{2.7}
$$

Further, (2.7) can be expressed as the following matrix

$$
(I+H_1)(I+H_2)U^{n+1}=(I-H_1)(I-H_2)U^n+\tau F^n. \tag{2.8}
$$

Then, $U^{n+1}_{i,j}$ can be solved by the following two steps:

$$
\begin{aligned}
&(1-\tau\delta_x-\tau\sigma_x)U^*_{i,j}\\
&\quad=(1+\tau\delta_x+\tau\sigma_x)\left(1+\tau\delta_y+\tau\sigma_y\right)U^n_{i,j}+\tau f\left(x_i,y_j,t_{n+\frac{1}{2}},\frac{3}{2}U^n_{i,j}-\frac{1}{2}U^{n-1}_{i,j}\right),
\end{aligned}
\tag{2.9}
$$

$$
\left(1-\tau\delta_y-\tau\sigma_y\right)U^{n+1}_{i,j}=U^*_{i,j}, \tag{2.10}
$$

where the boundary and initial conditions are

$$
\begin{aligned}
&U^n_{0,j}=u(a,y_j,t_n)=0, \qquad U^n_{i,0}=u(x_i,c,t_n)=0,\\
&U^n_{M_1,j}=u(b,y_j,t_n)=0, \qquad U^n_{i,M_2}=u(x_i,d,t_n)=0,\\
&U^0_{i,j}=\varphi(a+ih_1,c+jh_2).\\
&U^*_{0,j}=0, \qquad U^*_{M_1,j}=0, \qquad j=0,1,\ldots,M_2.
\end{aligned}
$$

3 Numerical analysis

3.1 Numerical analysis for the difference scheme (2.4)

In this subsection, we first present the following Lemma which are introduced to prove the solvability, convergence and stability of the difference scheme (2.4) will be analyzed and discussed.

Lemma 1 [25, 33, 35] *Assume that* $0<\beta<2$*, when* $0<\beta<1$*,* $\{w^\beta_l\}$ *satisfy*

$$
\begin{cases}
w^\beta_0=\frac{\beta}{2}, \quad w^\beta_1=\frac{2-\beta-\beta^2}{2}>0, \quad w^\beta_2=\frac{\beta(\beta^2+\beta-4)}{4},\\
w^\beta_2<w^\beta_3<\cdots<0, \quad \sum\limits_{l=0}^{\infty}w^\beta_l=0, \quad \sum\limits_{l=0}^{M}w^\beta_l>0, \ M\geq 1.
\end{cases}
\tag{3.1}
$$

When $1 < \beta < 2$, $\{w_l^\beta\}$ *satisfy*

$$\begin{cases} w_0^\beta = \frac{\beta}{2}, \quad w_1^\beta = \frac{2-\beta-\beta^2}{2} < 0, \quad w_2^\beta = \frac{\beta(\beta^2+\beta-4)}{4}, \\ 1 > w_0^\beta > w_3^\beta > \cdots > 0, \quad \sum_{l=0}^{\infty} w_l^\beta = 0, \quad \sum_{l=0}^{M} w_l^\beta < 0, \ M \geq 2. \end{cases} \tag{3.2}$$

Remark 1 According to Lemma 1, when $0 < \alpha_r < 1$, $\sum_{l=0}^{M_1-1} w_l^{\alpha_r} > 0$ and $\sum_{l=0,l\neq 1}^{M_1-1} w_l^{\alpha_r} < 0$, $M_1 \geq 2$. In addition, $\sum_{l=0}^{M_1-1} w_l^{\alpha_r+1} < 0$ and $\sum_{l=0,l\neq 1}^{M_1-1} w_l^{\alpha_r+1} > 0$, $M_1 \geq 3$.

Theorem 1 *Suppose that* $0 < \alpha_r < 1$, $M_1, M_2 \geq 3$, $k_{i,j}(x, y, \alpha)$, $i, j = 1, 2$, *satisfy condition* (1.4)*, then the difference scheme* (2.4) *is uniquely solvable.*

Proof In order to prove the unique solvability of the difference scheme (2.4), we need to prove $I + H$ is strictly diagonally dominant. For all $i = 1, \ldots, M_1 - 1$, $k = 1, 2, \ldots, M_2 - 1$, by means of Remark 1, we have

$$\begin{aligned}
&\sum_{j=1,j\neq i}^{M_1-1} |h_{i,j}^x(k)| \\
&= \sum_{r=1}^{m} \frac{\tau w_r \omega(\alpha_r)}{4h_1^{\alpha_r}} \sum_{j=1}^{i-2} \left(\frac{1}{h_1} k_{11}^{i,k,r} w_{i-j+1}^{\alpha_r+1} + k_{11,x}^{i,k,r} w_{i-j+1}^{\alpha_r} \right) \\
&\quad + \sum_{r=1}^{m} \frac{\tau w_r \omega(\alpha_r)}{4h_1^{\alpha_r}} \left(\frac{1}{h_1} (k_{11}^{i,k,r} w_2^{\alpha_r+1} + k_{12}^{i,k,r} w_0^{\alpha_r+1}) + (k_{11,x}^{i,k,r} w_2^{\alpha_r} - k_{12,x}^{i,k,r} w_0^{\alpha_r}) \right) \\
&\quad + \sum_{r=1}^{m} \frac{\tau w_r \omega(\alpha_r)}{4h_1^{\alpha_r}} \left(\frac{1}{h_1} (k_{11}^{i,k,r} w_0^{\alpha_r+1} + k_{12}^{i,k,r} w_2^{\alpha_r+1}) + (k_{11,x}^{i,k,r} w_0^{\alpha_r} - k_{12,x}^{i,k,r} w_2^{\alpha_r}) \right) \\
&\quad + \sum_{r=1}^{m} \frac{\tau w_r \omega(\alpha_r)}{4h_1^{\alpha_r}} \sum_{j=i+2}^{M_1-1} \left(\frac{1}{h_1} k_{12}^{i,k,r} w_{j-i+1}^{\alpha_r+1} - k_{12,x}^{i,k,r} w_{j-i+1}^{\alpha_r} \right) \\
&= \sum_{r=1}^{m} \frac{\tau w_r \omega(\alpha_r)}{4h_1^{\alpha_r+1}} \left(k_{11}^{i,k,r} \sum_{j=0,j\neq 1}^{i} w_j^{\alpha_r+1} + k_{12}^{i,k,r} \sum_{j=0,j\neq 1}^{M_1-i} w_j^{\alpha_r+1} \right) \\
&\quad + \sum_{r=1}^{m} \frac{\tau w_r \omega(\alpha_r)}{4h_1^{\alpha_r}} \left(k_{11,x}^{i,k,r} \sum_{j=0,j\neq 1}^{i} w_j^{\alpha_r} - k_{12,x}^{i,k,r} \sum_{j=0,j\neq 1}^{M_1-i} w_j^{\alpha_r} \right) \\
&< \sum_{r=1}^{m} \frac{\tau w_r \omega(\alpha_r)}{4h_1^{\alpha_r+1}} \left(-k_{11}^{i,k,r} w_1^{\alpha_r+1} - k_{12}^{i,k,r} w_1^{\alpha_r+1} \right) \\
&\quad + \sum_{r=1}^{m} \frac{\tau w_r \omega(\alpha_r)}{4h_1^{\alpha_r}} \left(-k_{11,x}^{i,k,r} w_1^{\alpha_r} + k_{12,x}^{i,k,r} w_1^{\alpha_r} \right) = h_{i,i}^x(k).
\end{aligned} \tag{3.3}$$

Thus, $H_x(k)$ is strictly diagonally dominant, furthermore, H_1 is strictly diagonally dominant. Similarly, $h^y_{i,i}(k) > \sum_{j=1,j\neq i}^{M_2-1} |h^y_{i,j}(k)|$, H_2 is strictly diagonally dominant. Therefore, $I + H$ is strictly diagonally dominant. In other words, the matrices $I + H$ is invertible, which guarantees the difference scheme (2.4) is uniquely solvable. □

Lemma 2 *Suppose that* $0 < \alpha_r < 1$*,* $M_1, M_2 \geq 3$*,* $k_{i,j}(x, y, \alpha)$*,* $i, j = 1, 2$*, satisfy condition* (1.4)*, then the eigenvalues of* $I + H$ *are all greater than* 1.

Proof Based on the Greschgorin's theorem [12, 17], the eigenvalues $\{\lambda_i\}$ of matrix $H_x(k)$ satisfy

$$\left|\lambda_i - h^x_{i,i}(k)\right| \leq \sum_{j=1,j\neq i}^{M_1-1} |h^x_{i,j}(k)|,$$

then using Theorem 1, it follows that

$$\lambda_i \geq h^x_{i,i}(k) - \sum_{j=1,j\neq i}^{M_1-1} |h^x_{i,j}(k)| > 0,$$

namely, the eigenvalues of the matrix $H_x(k)$ are all greater than 0. Furthermore, since H_1 is a block diagonal matrix, the eigenvalues of H_1 are also all greater than 0. Similarly, it can be verified that the eigenvalues of H_2 are all greater than 0. Let λ_1 and λ_2 be any eigenvalues of H_1 and H_2, respectively, then there are $\lambda_1 > 0$ and $\lambda_2 > 0$, thus the eigenvalues of $I + H$ are all greater than 1, □

Now, suppose that (1.3) has a unique sufficient smooth solution $u \in C^{5,5,3}_{x,y,t}(\Omega \times [0, T])$, let $U^n_{i,j}$ and $\widetilde{U}^n_{i,j}$ be the solution and numerical solution of the difference scheme (2.4), respectively. Define $\epsilon^n_{i,j} = U^n_{i,j} - \widetilde{U}^n_{i,j}$, substitution into (2.4) leads into

$$\begin{aligned}\left(1 - \tau\delta_x - \tau\sigma_x - \tau\delta_y - \tau\sigma_y\right)\epsilon^{n+1}_{i,j} &= \left(1 + \tau\delta_x + \tau\sigma_x + \tau\delta_y + \tau\sigma_y\right)\epsilon^n_{i,j} \\ &+ \tau\left(f\left(x_i, y_j, t_{n+\frac{1}{2}}, \frac{3}{2}U^n_{i,j} - \frac{1}{2}U^{n-1}_{i,j}\right) - f\left(x_i, y_j, t_{n+\frac{1}{2}}, \frac{3}{2}\widetilde{U}^n_{i,j} - \frac{1}{2}\widetilde{U}^{n-1}_{i,j}\right)\right).\end{aligned} \tag{3.4}$$

And (3.4) can be written as

$$(I + H)\epsilon^{n+1} = (I - H)\epsilon^n + \tau G^n, \tag{3.5}$$

where

$$\begin{aligned}\epsilon^n &= \left(\epsilon^n_{1,1}, \epsilon^n_{2,1}, \ldots, \epsilon^n_{M_1-1,1}, \ldots, \epsilon^n_{1,M_2-1}, \ldots, \epsilon^n_{M_1-1,M_2-1}\right)^T, \\ G^n &= \left(g^n_{1,1}, g^n_{2,1}, \ldots, g^n_{M_1-1,1}, \ldots, g^n_{1,M_2-1}, \ldots, g^n_{M_1-1,M_2-1}\right)^T,\end{aligned}$$

with

$$g_{i,j}^{n} = f\left(x_i, y_j, t_{n+\frac{1}{2}}, \frac{3}{2}U_{i,j}^{n} - \frac{1}{2}U_{i,j}^{n-1}\right) - f\left(x_i, y_j, t_{n+\frac{1}{2}}, \frac{3}{2}\widetilde{U}_{i,j}^{n} - \frac{1}{2}\widetilde{U}_{i,j}^{n-1}\right).$$

Let $\lambda_{i,j}$ be any eigenvalues of H, then the eigenvalues of $I+H$ and $I-H$ are $1+\lambda_{i,j}$ and $1-\lambda_{i,j}$, respectively. From Lemma 2, we know $\lambda_{i,j} > 0$, thus $1+\lambda_{i,j} > 1$. And let $\left(1+\lambda_{p,q}\right)$ be the eigenvalue corresponding to row $(M_2-1)(q-1)+p$ of matrix $I+H$, then $\left(1-\lambda_{p,q}\right)$ be the eigenvalue corresponding to row $(M_2-1)(q-1)+p$ of matrix $I-H$. Further, we have

$$\sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2-1} H_{(M_2-1)(q-1)+p,(M_2-1)(j-1)+i}\epsilon_{i,j} = \lambda_{p,q}\epsilon_{p,q}. \tag{3.6}$$

Then, we present the following Theorem based on Eq. (3.6) to illustrate the stability of the difference scheme (2.4).

Theorem 2 *Suppose that* $0<\alpha<1$, $k_{i,j}(x,y,\alpha)$, $i,\ j=1,2$, *satisfy condition* (1.4), *then the difference scheme* (2.4) *is unconditionally stable.*

Proof Denote $||\epsilon^{n+1}||_\infty = |\epsilon_{p,q}^{n+1}| = \max_{1\le i\le M_1-1, 1\le j\le M_2-1}|\epsilon_{i,j}^{n+1}|$, it follows from (3.5) and (3.6) that

$$\begin{aligned}
&\left|\left(1+\lambda_{p,q}\right)\epsilon_{p,q}^{n+1}\right| \\
&=\Bigg|\epsilon_{p,q}^{n+1} - \sum_{r=1}^{m}\frac{\tau w_r\omega(\alpha_r)}{4h_1^{\alpha_r}}\left(k_{11,x}^{p,q,r}\sum_{l=1}^{p+1}w_{p-l+1}^{\alpha_r}\epsilon_{l,q}^{n+1} - k_{12,x}^{p,q,r}\sum_{l=p-1}^{M_1-1}w_{l-p+1}^{\alpha_r}\epsilon_{l,q}^{n+1}\right) \\
&\quad - \sum_{r=1}^{m}\frac{\tau w_r\omega(\alpha_r)}{4h_2^{\alpha_r}}\left(k_{21,y}^{p,q,r}\sum_{l=0}^{q+1}w_{q-l+1}^{\alpha_r}\epsilon_{p,l}^{n+1} - k_{22,y}^{p,q,r}\sum_{l=q-1}^{M_2-1}w_{q-l+1}^{\alpha_r}\epsilon_{p,l}^{n+1}\right) \\
&\quad - \sum_{r=1}^{m}\frac{\tau w_r\omega(\alpha_r)}{4h_1^{\alpha_r+1}}\left(k_{11}^{p,q,r}\sum_{l=1}^{p+1}w_{p-l+1}^{\alpha_r+1}\epsilon_{l,q}^{n+1} + k_{12}^{p,q,r}\sum_{l=p-1}^{M_1-1}w_{l-p+1}^{\alpha_r+1}\epsilon_{l,q}^{n+1}\right) \\
&\quad - \sum_{r=1}^{m}\frac{\tau w_r\omega(\alpha_r)}{4h_2^{\alpha_r+1}}\left(k_{21}^{p,q,r}\sum_{l=0}^{q+1}w_{q-l+1}^{\alpha_r+1}\epsilon_{p,l}^{n+1} + k_{22}^{p,q,r}\sum_{l=q-1}^{M_2-1}w_{q-l+1}^{\alpha_r+1}\epsilon_{p,l}^{n+1}\right)\Bigg| \\
&=\Bigg|\epsilon_{p,q}^{n} + \sum_{r=1}^{m}\frac{\tau w_r\omega(\alpha_r)}{4h_1^{\alpha_r}}\left(k_{11,x}^{p,q,r}\sum_{l=1}^{p+1}w_{p-l+1}^{\alpha_r}\epsilon_{l,q}^{n} - k_{12,x}^{p,q,r}\sum_{l=p-1}^{M_1-1}w_{l-p+1}^{\alpha_r}\epsilon_{l,q}^{n}\right) \\
&\quad + \sum_{r=1}^{m}\frac{\tau w_r\omega(\alpha_r)}{4h_2^{\alpha_r}}\left(k_{21,y}^{p,q,r}\sum_{l=0}^{q+1}w_{q-l+1}^{\alpha_r}\epsilon_{p,l}^{n} - k_{22,y}^{p,q,r}\sum_{l=q-1}^{M_2-1}w_{q-l+1}^{\alpha_r}\epsilon_{p,l}^{n}\right) \\
&\quad + \sum_{r=1}^{m}\frac{\tau w_r\omega(\alpha_r)}{4h_1^{\alpha_r+1}}\left(k_{11}^{p,q,r}\sum_{l=1}^{p+1}w_{p-l+1}^{\alpha_r+1}\epsilon_{l,q}^{n} + k_{12}^{p,q,r}\sum_{l=p-1}^{M_1-1}w_{l-p+1}^{\alpha_r+1}\epsilon_{l,q}^{n}\right)
\end{aligned}$$

$$+\sum_{r=1}^{m}\frac{\tau w_r\omega(\alpha_r)}{4h_2^{\alpha_r+1}}\left(k_{21}^{p,q,r}\sum_{l=0}^{q+1}w_{q-l+1}^{\alpha_r+1}\epsilon_{p,l}^{n}+k_{22}^{p,q,r}\sum_{l=q-1}^{M_2-1}w_{q-l+1}^{\alpha_r+1}\epsilon_{p,l}^{n}\right)$$
$$+\tau\left(f\left(x_p,y_q,t_{n+\frac{1}{2}},\frac{3}{2}U_{p,q}^{n}-\frac{1}{2}U_{p,q}^{n-1}\right)-f\left(x_p,y_q,t_{n+\frac{1}{2}},\frac{3}{2}\widetilde{U}_{p,q}^{n}-\frac{1}{2}\widetilde{U}_{p,q}^{n-1}\right)\right)\Bigg|$$
$$=\Bigg|\left(1-\lambda_{p,q}\right)\epsilon_{p,q}^{n}$$
$$+\tau\left(f\left(x_p,y_q,t_{n+\frac{1}{2}},\frac{3}{2}U_{p,q}^{n}-\frac{1}{2}U_{p,q}^{n-1}\right)-f\left(x_p,y_q,t_{n+\frac{1}{2}},\frac{3}{2}\widetilde{U}_{p,q}^{n}-\frac{1}{2}\widetilde{U}_{p,q}^{n-1}\right)\right)\Bigg|, \tag{3.7}$$

using the Lipschitz condition (1.5), we have

$$\left|\left(1+\lambda_{p,q}\right)\epsilon_{p,q}^{n+1}\right|\leq\left|1-\lambda_{p,q}\right|\left|\epsilon_{p,q}^{n}\right|+\tau L_1C_1\left(\left|\epsilon_{p,q}^{n}\right|+\left|\epsilon_{p,q}^{n-1}\right|\right), \tag{3.8}$$

where C_1 is a positive constant, further,

$$\begin{aligned}\left|\epsilon_{p,q}^{n+1}\right|&\leq\left|\frac{1-\lambda_{p,q}}{1+\lambda_{p,q}}\right|\left|\epsilon_{p,q}^{n}\right|+\frac{\tau L_1C_1}{1+\lambda_{p,q}}\left|\epsilon_{p,q}^{n}\right|+\frac{\tau L_1C_1}{1+\lambda_{p,q}}\left|\epsilon_{p,q}^{n-1}\right|\\&\leq(1+\tau L_1C_1)\left|\epsilon_{p,q}^{n}\right|+\tau L_1C_1\left|\epsilon_{p,q}^{n-1}\right|,\end{aligned} \tag{3.9}$$

namely,

$$\left\|\epsilon^{n+1}\right\|_\infty-\left\|\epsilon^{n}\right\|_\infty\leq\tau L_1C_1\left(\left\|\epsilon^{n}\right\|_\infty+\left\|\epsilon^{n-1}\right\|_\infty\right). \tag{3.10}$$

Summing up for n from 1 to k, we have

$$\begin{aligned}\left\|\epsilon^{k+1}\right\|_\infty&\leq\left\|\epsilon^{1}\right\|_\infty+\tau L_1C_1\left(\sum_{l=1}^{k}\left\|\epsilon^{l}\right\|_\infty+\sum_{l=1}^{k}\left\|\epsilon^{l-1}\right\|_\infty\right)\\&\leq\left\|\epsilon^{0}\right\|_\infty+3\tau L_1C_1\sum_{l=1}^{k}\left\|\epsilon^{l}\right\|_\infty,\end{aligned} \tag{3.11}$$

applying the Gronwall inequality [12] to above inequality yield

$$\left\|\epsilon^{n+1}\right\|_\infty\leq\exp(3L_1C_1T)\left\|\epsilon^{0}\right\|_\infty. \tag{3.12}$$

This completes the proof. □

Now, we consider the convergence of the difference scheme (2.4). Let u_{ij}^n and $U_{i,j}^n$ be the exact solution and numerical solution of (1.3), respectively. Define $e_{i,j}^n=U_{i,j}^n-u_{i,j}^n$ and $e^n=(e_{1,1}^n,e_{2,1}^n,\ldots,e_{M_1-1,1}^n,\ldots,e_{1,M_2-1}^n,\ldots,e_{M_1-1,M_2-1}^n)^T$. From (2.3) and (2.4), we get

$$\left(1-\tau\delta_x-\tau\sigma_x-\tau\delta_y-\tau\sigma_y\right)e_{i,j}^{n+1}=\left(1+\tau\delta_x+\tau\sigma_x+\tau\delta_y+\tau\sigma_y\right)e_{i,j}^{n}$$
$$+\tau\left(f\left(x_i,y_j,t_{n+\frac{1}{2}},\frac{3}{2}U_{i,j}^n-\frac{1}{2}U_{i,j}^{n-1}\right)-f\left(x_i,y_j,t_{n+\frac{1}{2}},\frac{3}{2}u_{i,j}^n-\frac{1}{2}u_{i,j}^{n-1}\right)\right)+\tau r_{i,j}^n, \tag{3.13}$$

$$e_{i,j}^0=0,\qquad 1\le i\le M_1-1,\ 1\le j\le M_2-1,$$
$$e_{0,j}^n=0,\quad e_{i,0}^n=0,\quad e_{M_1,j}^n=0,\quad e_{i,M_2}^n=0,\qquad 0\le n\le N. \tag{3.14}$$

Then, we present the following Theorem based on Eq. (3.6) to illustrate the convergence of the difference scheme (2.4).

Theorem 3 *Suppose that* $0<\alpha\le 1$ *and* τ *is small enough,* $k_{i,j}(x,y,\alpha)$, $i,j=1,2$, *satisfy condition* (1.4)*, then the error of the difference scheme* (2.4) *satisfy*

$$\left\|e^n\right\|_\infty\le C_rT\exp(3L_1C_1T)\left(m^{-v}+h_1^2+h_2^2+\tau^2\right),\qquad 0\le n\le N, \tag{3.15}$$

where C_1 *and* C_r *are positive constants.*

Proof Denote $||E^{n+1}||_\infty=|e_{p,q}^{n+1}|=\max_{1\le i\le m_1-1,1\le j\le m_2-1}|e_{i,j}^{n+1}|$, it follows (3.13) that

$$\begin{aligned}
&\left|\left(1+\lambda_{p,q}\right)e_{p,q}^{n+1}\right|\\
&=\Bigg|e_{p,q}^{n+1}-\frac{1}{4}\sum_{r=1}^{m}\frac{\tau w_r\omega(\alpha_r)}{h_1^{\alpha_r}}\left(k_{11,x}^{p,q,r}\sum_{l=1}^{p+1}w_{p-l+1}^{\alpha_r}e_{l,q}^{n+1}-k_{12,x}^{p,q,r}\sum_{l=p-1}^{M_1-1}w_{l-p+1}^{\alpha_r}e_{l,q}^{n+1}\right)\\
&\quad-\frac{1}{4}\sum_{r=1}^{m}\frac{\tau w_r\omega(\alpha_r)}{h_2^{\alpha_r}}\left(k_{21,y}^{p,q,r}\sum_{l=0}^{q+1}w_{q-l+1}^{\alpha_r}e_{p,l}^{n+1}-k_{22,y}^{p,q,r}\sum_{l=q-1}^{M_2-1}w_{q-l+1}^{\alpha_r}e_{p,l}^{n+1}\right)\\
&\quad-\frac{1}{4}\sum_{r=1}^{m}\frac{\tau w_r\omega(\alpha_r)}{h_1^{\alpha_r+1}}\left(k_{11}^{p,q,r}\sum_{l=1}^{p+1}w_{p-l+1}^{\alpha_r+1}e_{l,q}^{n+1}+k_{12}^{p,q,r}\sum_{l=p-1}^{M_1-1}w_{l-p+1}^{\alpha_r+1}e_{l,q}^{n+1}\right)\\
&\quad-\frac{1}{4}\sum_{r=1}^{m}\frac{\tau w_r\omega(\alpha_r)}{h_2^{\alpha_r+1}}\left(k_{21}^{p,q,r}\sum_{l=0}^{q+1}w_{q-l+1}^{\alpha_r+1}e_{p,l}^{n+1}+k_{22}^{p,q,r}\sum_{l=q-1}^{M_2-1}w_{q-l+1}^{\alpha_r+1}e_{p,l}^{n+1}\right)\Bigg|\\
&=\Bigg|e_{p,q}^{n}+\frac{1}{4}\sum_{r=1}^{m}\frac{\tau w_r\omega(\alpha_r)}{h_1^{\alpha_r}}\left(k_{11,x}^{p,q,r}\sum_{l=1}^{p+1}w_{p-l+1}^{\alpha_r}e_{l,q}^{n}-k_{12,x}^{p,q,r}\sum_{l=p-1}^{M_1-1}w_{l-p+1}^{\alpha_r}e_{l,q}^{n}\right)\\
&\quad+\frac{1}{4}\sum_{r=1}^{m}\frac{\tau w_r\omega(\alpha_r)}{h_2^{\alpha_r}}\left(k_{21,y}^{p,q,r}\sum_{l=0}^{q+1}w_{q-l+1}^{\alpha_r}e_{p,l}^{n}-k_{22,y}^{p,q,r}\sum_{l=q-1}^{M_2-1}w_{q-l+1}^{\alpha_r}e_{p,l}^{n}\right)\\
&\quad+\frac{1}{4}\sum_{r=1}^{m}\frac{\tau w_r\omega(\alpha_r)}{h_1^{\alpha_r+1}}\left(k_{11}^{p,q,r}\sum_{l=1}^{p+1}w_{p-l+1}^{\alpha_r+1}e_{l,q}^{n}+k_{12}^{p,q,r}\sum_{l=p-1}^{M_1-1}w_{l-p+1}^{\alpha_r+1}e_{l,q}^{n}\right)\\
&\quad+\frac{1}{4}\sum_{r=1}^{m}\frac{\tau w_r\omega(\alpha_r)}{h_2^{\alpha_r+1}}\left(k_{21}^{p,q,r}\sum_{l=0}^{q+1}w_{q-l+1}^{\alpha_r+1}e_{p,l}^{n}+k_{22}^{p,q,r}\sum_{l=q-1}^{M_2-1}w_{q-l+1}^{\alpha_r+1}e_{p,l}^{n}\right)\\
&\quad+\tau\left(f\left(x_p,y_q,t_{n+\frac{1}{2}},\frac{3}{2}U_{p,q}^n-\frac{1}{2}U_{p,q}^{n-1}\right)-f\left(x_p,y_q,t_{n+\frac{1}{2}},\frac{3}{2}u_{p,q}^n-\frac{1}{2}u_{p,q}^{n-1}\right)\right)+\tau r_{p,q}^n\Bigg|
\end{aligned}$$

$$= \Big| \left(1-\lambda_{p,q}\right) e_{p,q}^{n} + \tau \left(f\left(x_p, y_q, t_{n+\frac{1}{2}}, \frac{3}{2}U_{p,q}^{n} - \frac{1}{2}U_{p,q}^{n-1}\right) - f\left(x_p, y_q, t_{n+\frac{1}{2}}, \frac{3}{2}u_{p,q}^{n} - \frac{1}{2}u_{p,q}^{n-1}\right)\right) + \tau r_{p,q}^{n} \Big|, \tag{3.16}$$

where $r^n = (r_{1,1}^n, r_{2,1}^n, \ldots, r_{M_1-1,1}^n, \ldots, r_{1,M_2-1}^n, \ldots, r_{M_1-1,M_2-1}^n)^T$. Using the Lipschitz condition (1.5), we have

$$\left|\left(1+\lambda_{p,q}\right) e_{p,q}^{n+1}\right| \le \left|1-\lambda_{p,q}\right| \left|e_{p,q}^{n}\right| + \tau L_1 C_1 \left(\left|e_{p,q}^{n}\right| + \left|e_{p,q}^{n-1}\right|\right) + \tau \left\|r^n\right\|_{\infty}, \tag{3.17}$$

where C_1 is a positive constant, further,

$$\begin{aligned} \left|e_{p,q}^{n+1}\right| &\le \left|\frac{1-\lambda_{p,q}}{1+\lambda_{p,q}}\right| \left|e_{p,q}^{n}\right| + \frac{\tau L_1 C_1}{1+\lambda_{p,q}} \left|e_{p,q}^{n}\right| + \frac{\tau L_1 C_1}{1+\lambda_{p,q}} \left|e_{p,q}^{n-1}\right| + \frac{\tau}{1+\lambda_{p,q}} \left\|r^n\right\|_{\infty} \\ &\le \left(1+\tau L_1 C_1\right) \left|e_{p,q}^{n}\right| + \tau L_1 C_1 \left|e_{p,q}^{n-1}\right| + \tau \left\|r^n\right\|_{\infty}, \end{aligned} \tag{3.18}$$

namely,

$$\left\|e^{n+1}\right\|_{\infty} - \left\|e^{n}\right\|_{\infty} \le \tau L_1 C_1 \left(\left\|e^{n}\right\|_{\infty} + \left\|e^{n-1}\right\|_{\infty}\right) + \tau \left\|r^n\right\|_{\infty}. \tag{3.19}$$

Summing up for n from 1 to k, and notice $\left\|e^0\right\|_{\infty} = 0$, we have

$$\begin{aligned} \left\|e^{k+1}\right\|_{\infty} &\le \left\|e^{1}\right\|_{\infty} + \tau L_1 C_1 \left(\sum_{l=1}^{k} \left\|e^{l}\right\|_{\infty} + \sum_{l=1}^{k} \left\|e^{l-1}\right\|_{\infty}\right) + \sum_{l=1}^{k} \tau \left\|r^{l}\right\|_{\infty} \\ &\le \left\|e^{0}\right\|_{\infty} + 3\tau L_1 C_1 \sum_{l=1}^{k} \left\|e^{l}\right\|_{\infty} + \sum_{l=1}^{k} \tau \left\|r^{l}\right\|_{\infty} \\ &= 3\tau L_1 C_1 \sum_{l=1}^{k} \left\|e^{l}\right\|_{\infty} + \sum_{l=1}^{k} \tau \left\|r^{l}\right\|_{\infty}, \end{aligned} \tag{3.20}$$

applying the Gronwall inequality [12] to above inequality yield

$$\left\|e^{n+1}\right\|_{\infty} \le C_r T \exp(3L_1 C_1 T) \left(m^{-v} + h_1^2 + h_2^2 + \tau^2\right). \tag{3.21}$$

This completes the proof. □

3.2 Numerical analysis for the difference scheme (2.7)

In this subsection, the solvability, convergence and stability of the difference scheme (2.7) will be analyzed and discussed.

Theorem 4 *Suppose that* $0 < \alpha_r < 1$*,* $k_{i,j}(x, y, \alpha)$*,* $i, j = 1, 2$*, satisfy condition* (1.4)*, then the difference scheme* (2.7) *is uniquely solvable.*

Proof In order to prove the unique solvability of the difference scheme (2.7), we need to prove $I + H_1$ and $I + H_2$ are strictly diagonally dominant. By means of Theorem 1, we know H_1 and H_2 are strictly diagonally dominant. Therefore, $I + H_1$ and $I + H_2$ are strictly diagonally dominant. In other words, the matrices $I + H_1$ and $I + H_2$ are invertible, which guarantees the difference scheme (2.7) is uniquely solvable. □

Lemma 3 *Suppose that* $0 < \alpha_r < 1$*,* $M_1, M_2 \geq 3$*,* $k_{i,j}(x, y, \alpha)$*,* $i, j = 1, 2$*, satisfy condition* (1.4)*, then the eigenvalues of* $I + H_1$ *and* $I + H_2$ *are all greater than* 1.

Proof Using Lemma 2, we know the eigenvalues of H_1 and H_2 are all greater than 0. Thus, the eigenvalues of $I + H_1$ and $I + H_2$ are all greater than 1, □

Suppose that (1.3) has a unique sufficient smooth solution $u \in C^{5,5,3}_{x,y,t}(\Omega \times [0, T])$, let $U^n_{i,j}$ and $\widetilde{U}^n_{i,j}$ be the solution and numerical solution of the difference scheme (2.7), respectively. Define $\epsilon^n_{i,j} = U^n_{i,j} - \widetilde{U}^n_{i,j}$, substitution into (2.7) leads into

$$\begin{aligned}
&(1 - \tau\delta_x - \tau\sigma_x)\left(1 - \tau\delta_y - \tau\sigma_y\right)\epsilon^{n+1}_{i,j} \\
&\quad = (1 + \tau\delta_x + \tau\sigma_x)\left(1 + \tau\delta_y + \tau\sigma_y\right)\epsilon^n_{i,j} \\
&\qquad + \tau\left(f\left(x_i, y_j, t_{n+\frac{1}{2}}, \frac{3}{2}U^n_{i,j} - \frac{1}{2}U^{n-1}_{i,j}\right) - f\left(x_i, y_j, t_{n+\frac{1}{2}}, \frac{3}{2}\widetilde{U}^n_{i,j} - \frac{1}{2}\widetilde{U}^{n-1}_{i,j}\right)\right).
\end{aligned} \tag{3.22}$$

Further, (3.22) can be expressed as the following matrix

$$W\epsilon^{n+1} = K\epsilon^n + \tau G^n, \tag{3.23}$$

where

$$W = (W_{i_1,j_1})_{(M_1-1)(M_2-1)\times(M_1-1)(M_2-1)} = (I + H_1)(I + H_2),$$

and

$$K = (K_{i_1,j_1})_{(M_1-1)(M_2-1)\times(M_1-1)(M_2-1)} = (I - H_1)(I - H_2).$$

Let $\lambda_{i,j}$ and $\mu_{i,j}$ be any eigenvalues of $I + H_1$ and $I + H_2$, respectively, then the eigenvalues of W and K are $(1+\lambda_{i,j})(1+\mu_{i,j})$ and $(1-\lambda_{i,j})(1-\mu_{i,j})$, respectively. From Lemma 3, we know $1+\lambda_{i,j} > 1$ and $1+\mu_{i,j} > 1$. And let $\left(1 + \lambda_{p,q}\right)\left(1 + \mu_{p,q}\right)$ be the eigenvalue corresponding to row $(M_2 - 1)(q - 1) + p$ of matrix W, then $\left(1 - \lambda_{p,q}\right)\left(1 - \mu_{p,q}\right)$ be the eigenvalue corresponding to row $(M_2 - 1)(q - 1) + p$ of matrix K. Further, we have

$$\sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2-1} W_{(M_2-1)(q-1)+p,(M_2-1)(j-1)+i}\epsilon_{i,j} = \left(1 + \lambda_{p,q}\right)\left(1 + \mu_{p,q}\right)\epsilon_{p,q}, \tag{3.24}$$

and

$$\sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2-1} K_{(M_2-1)(q-1)+p,(M_2-1)(j-1)+i}\epsilon_{i,j} = \left(1-\lambda_{p,q}\right)\left(1-\mu_{p,q}\right)\epsilon_{p,q}. \tag{3.25}$$

Then, we present the following Theorem based on Eqs. (3.24), (3.25) to illustrate the stability of the difference scheme (2.7).

Theorem 5 *Suppose that* $0<\alpha<1$, $k_{i,j}(x,y,\alpha)$, $i,j=1,2$, *satisfy condition* (1.4), *then the difference scheme* (2.7) *is unconditionally stable.*

Proof Denote $||\epsilon^{n+1}||_\infty = |\epsilon_{p,q}^{n+1}| = \max\limits_{1\le i\le M_1-1, 1\le j\le M_2-1} |\epsilon_{i,j}^{n+1}|$, it follows from (3.23), (3.24) and (3.25) that

$$\begin{aligned}&\left|\left(1+\lambda_{p,q}\right)\left(1+\mu_{p,q}\right)\epsilon_{p,q}^{n+1}\right|\\ &\quad=\Bigg|\left(1-\lambda_{p,q}\right)\left(1-\mu_{p,q}\right)\epsilon_{p,q}^{n}\\ &\qquad+\tau\left(f\left(x_p,y_q,t_{n+\frac{1}{2}},\frac{3}{2}U_{p,q}^{n}-\frac{1}{2}U_{p,q}^{n-1}\right)-f\left(x_p,y_q,t_{n+\frac{1}{2}},\frac{3}{2}\widetilde{U}_{p,q}^{n}-\frac{1}{2}\widetilde{U}_{p,q}^{n-1}\right)\right)\Bigg|.\end{aligned} \tag{3.26}$$

Using the Lipschitz condition (1.5) yiled

$$\begin{aligned}&\left|\left(1+\lambda_{p,q}\right)\left(1+\mu_{p,q}\right)\epsilon_{p,q}^{n+1}\right|\\ &\quad\le\left|\left(1-\lambda_{p,q}\right)\left(1-\mu_{p,q}\right)\epsilon_{p,q}^{n}\right|+\tau L_1C_1\left|\epsilon_{p,q}^{n}\right|+\tau L_1C_1\left|\epsilon_{p,q}^{n-1}\right|,\end{aligned} \tag{3.27}$$

where C_1 is a positive constant. Using Lemma 3, it follows that

$$\begin{aligned}\left|\epsilon_{p,q}^{n+1}\right|&\le\left|\frac{\left(1-\lambda_{p,q}\right)\left(1-\mu_{p,q}\right)}{\left(1+\lambda_{p,q}\right)\left(1+\mu_{p,q}\right)}\right|\left|\epsilon_{p,q}^{n}\right|+\frac{\tau L_1C_1}{\left(1+\lambda_{p,q}\right)\left(1+\mu_{p,q}\right)}\left|\epsilon_{p,q}^{n}\right|\\ &\quad+\frac{\tau L_1C_1}{\left(1+\lambda_{p,q}\right)\left(1+\mu_{p,q}\right)}\left|\epsilon_{p,q}^{n-1}\right|\\ &\le\left|\epsilon_{p,q}^{n}\right|+\tau L_1C_1\left|\epsilon_{p,q}^{n}\right|+\tau L_1C_1\left|\epsilon_{p,q}^{n-1}\right|,\end{aligned} \tag{3.28}$$

namely,

$$\left\|\epsilon^{n+1}\right\|_\infty-\left\|\epsilon^{n}\right\|_\infty\le\tau L_1C_1\left(\left\|\epsilon^{n}\right\|_\infty+\left\|\epsilon^{n-1}\right\|_\infty\right). \tag{3.29}$$

Summing up for n from 1 to k, we have

$$\begin{aligned}\left\|\epsilon^{k+1}\right\|_\infty &\le \left\|\epsilon^{1}\right\|_\infty + \tau L_1 C_1 \left(\sum_{l=1}^{k}\left\|\epsilon^{l}\right\|_\infty + \sum_{l=1}^{k}\left\|\epsilon^{l-1}\right\|_\infty\right)\\ &\le \left\|\epsilon^{0}\right\|_\infty + 3\tau L_1 C_1 \sum_{l=1}^{k}\left\|\epsilon^{l}\right\|_\infty, \end{aligned} \tag{3.30}$$

applying the Gronwall inequality [12] to above inequality yield

$$\left\|\epsilon^{n+1}\right\|_\infty \le \exp(3L_1C_1T)\left\|\epsilon^{0}\right\|_\infty. \tag{3.31}$$

This completes the proof. □

Now, we consider the convergence of the difference scheme (2.7). Let u_{ij}^n and $U_{i,j}^n$ be the exact solution and numerical solution of (1.3), respectively. Define $e_{i,j}^n = U_{i,j}^n - u_{i,j}^n$ and $e^n = (e_{1,1}^n, e_{2,1}^n, \dots, e_{M_1-1,1}^n, \dots, e_{1,M_2-1}^n, \dots, e_{M_1-1,M_2-1}^n)^T$. From (2.3) and (2.7), we get

$$\begin{aligned}&(1-\tau\delta_x-\tau\sigma_x)\left(1-\tau\delta_y-\tau\sigma_y\right)e_{i,j}^{n+1} = (1+\tau\delta_x+\tau\sigma_x)\left(1+\tau\delta_y+\tau\sigma_y\right)e_{i,j}^{n}\\ &\quad+\tau\left(f\left(x_i, y_j, t_{n+\frac{1}{2}}, \frac{3}{2}U_{i,j}^n - \frac{1}{2}U_{i,j}^{n-1}\right) - f\left(x_i, y_j, t_{n+\frac{1}{2}}, \frac{3}{2}u_{i,j}^n - \frac{1}{2}u_{i,j}^{n-1}\right)\right) + \tau R_{i,j}^n,\end{aligned} \tag{3.32}$$

$$\begin{aligned}&e_{i,j}^0 = 0, \quad 1\le i\le M_1-1,\ 1\le j\le M_2-1,\\ &e_{0,j}^n = 0,\quad e_{i,0}^n = 0,\quad e_{M_1,j}^n = 0,\quad e_{i,M_2}^n = 0,\quad 0\le n\le N.\end{aligned} \tag{3.33}$$

Then, we present the following Theorem based on Eqs. (3.24), (3.25) to illustrate the stability of the difference scheme (2.7).

Theorem 6 *Suppose that $0 < \alpha \le 1$ and τ is small enough, $k_{i,j}(x, y, \alpha)$, $i, j = 1, 2$, satisfy condition (1.4), then the error of the difference scheme (2.7) satisfy*

$$\left\|e^n\right\|_\infty \le C_R T \exp(3L_1C_1T)\left(m^{-\upsilon} + h_1^2 + h_2^2 + \tau^2\right), \qquad 0 \le n \le N, \tag{3.34}$$

where C_1 and C_R are positive constants.

Proof Denote $||E^{n+1}||_\infty = |e_{p,q}^{n+1}| = \max_{1\le i\le m_1-1, 1\le j\le m_2-1} |e_{i,j}^{n+1}|$, it follows from (3.32), (3.24) and (3.25) that

$$\begin{aligned}&\left|(1+\lambda_{p,q})\left(1+\mu_{p,q}\right)e_{p,q}^{n+1}\right|\\ &\quad= \left|\ (1-\lambda_{p,q})\left(1-\mu_{p,q}\right)e_{p,q}^{n}\right.\end{aligned}$$

$$
\begin{aligned}
&+\tau\left(f\left(x_p, y_q, t_{n+\frac{1}{2}}, \frac{3}{2}U_{p,q}^n-\frac{1}{2}U_{p,q}^{n-1}\right)-f\left(x_p, y_q, t_{n+\frac{1}{2}}, \frac{3}{2}u_{p,q}^n-\frac{1}{2}u_{p,q}^{n-1}\right)\right)+\tau R_{p,q}^n\Big| \\
&\leq \left|\left(1-\lambda_{p,q}\right)\left(1-\mu_{p,q}\right)\right|\left|e_{p,q}^n\right|+\tau L_1 C_1\left|e_{p,q}^n\right|+\tau L_1 C_1\left|e_{p,q}^{n-1}\right|+\tau\left\|R^n\right\|_{\infty}, \qquad (3.35)
\end{aligned}
$$

where $R^n=(R_{1,1}^n, R_{2,1}^n, \ldots, R_{M_1-1,1}^n, \ldots, R_{1,M_2-1}^n, \ldots, R_{M_1-1,M_2-1}^n)^T$. Using Lemma 3, it follows that

$$
\begin{aligned}
\left|e_{p,q}^{n+1}\right| \leq& \left|\frac{\left(1-\lambda_{p,q}\right)\left(1-\mu_{p,q}\right)}{\left(1+\lambda_{p,q}\right)\left(1+\mu_{p,q}\right)}\right|\left|e_{p,q}^n\right|+\frac{\tau L_1 C_1}{\left(1+\lambda_{p,q}\right)\left(1+\mu_{p,q}\right)}\left|e_{p,q}^n\right| \\
&+\frac{\tau L_1 C_1}{\left(1+\lambda_{p,q}\right)\left(1+\mu_{p,q}\right)}\left|e_{p,q}^n\right|+\frac{\tau}{\left(1+\lambda_{p,q}\right)\left(1+\mu_{p,q}\right)}\left\|R^n\right\|_{\infty} \\
\leq& \left|e_{p,q}^n\right|++\tau L_1 C_1\left|e_{p,q}^n\right|+\tau L_1 C_1\left|e_{p,q}^{n-1}\right|+\tau\left\|R^n\right\|_{\infty}, \qquad (3.36)
\end{aligned}
$$

namely,

$$
\left\|e^{n+1}\right\|_{\infty}-\left\|e^n\right\|_{\infty} \leq \tau L_1 C_1\left(\left\|e^n\right\|_{\infty}+\left\|e^{n-1}\right\|_{\infty}\right)+\tau\left\|R^n\right\|_{\infty}. \qquad (3.37)
$$

Summing up for n from 1 to k, and notice $\left\|e^0\right\|_{\infty}=0$, we have

$$
\begin{aligned}
\left\|e^{k+1}\right\|_{\infty} &\leq \left\|e^1\right\|_{\infty}+\tau L_1 C_1\left(\sum_{l=1}^k\left\|e^l\right\|_{\infty}+\sum_{l=1}^k\left\|e^{l-1}\right\|_{\infty}\right)+\sum_{l=1}^k \tau\left\|R^l\right\|_{\infty} \\
&\leq \left\|e^0\right\|_{\infty}+3\tau L_1 C_1 \sum_{l=1}^k\left\|e^l\right\|_{\infty}+\sum_{l=1}^k \tau\left\|R^l\right\|_{\infty} \\
&= 3\tau L_1 C_1 \sum_{l=1}^k\left\|e^l\right\|_{\infty}+\sum_{l=1}^k \tau\left\|R^l\right\|_{\infty} \qquad (3.38)
\end{aligned}
$$

applying the Gronwall inequality [12] to above inequality yield

$$
\left\|e^{n+1}\right\|_{\infty} \leq C_R T \exp(3L_1 C_1 T)\left(m^{-v}+h_1^2+h_2^2+\tau^2\right). \qquad (3.39)
$$

This completes the proof. □

4 Numerical examples

In this section, we give three examples to demonstrate the accuracy and efficiency of the numerical schemes (2.4) and (2.7). All numerical results are carried out by Matlab R2018a software, and the computation time (CPU) is measured in seconds. The errors are given by

$$E_\infty(h_1, h_2, \tau) = \max_{\substack{1\le i\le M_1, 1\le j\le M_2\\ 1\le n\le N}} |u_{i,j}^n - u(x_i, y_j, t_n)|.$$

And in Example 1 and Example 2, the spatial direction and the temporal direction convergence orders are given by

$$\text{order}_h = \log_2\left(\frac{E_\infty(h_1, h_2, \tau)}{E_\infty(h_1/2, h_2/2, \tau)}\right) \quad \text{or} \quad \text{order}_\tau = \log_2\left(\frac{E_\infty(h_1, h_2, \tau)}{E_\infty(h_1, h_2, \tau/2)}\right).$$

In Examples 1–3, when $\tau = h_1 = h_2$, the convergence orders are given by

$$\text{order}_\infty = \log_2\left(\frac{E_\infty(h_1, h_2, \tau)}{E_\infty(h_1/2, h_2/2, \tau/2)}\right).$$

Example 1. Consider the model (1.3) with $T = 2$, $\Omega = (0, 1) \times (0, 1)$, $\omega(\alpha) = 2\cos(\frac{\pi\alpha}{2})$, $k_{11}(x, y, \alpha) = k_{21}(x, y, \alpha) = \frac{3-e^{xy\alpha}}{2}$, $k_{12}(x, y, \alpha) = k_{22}(x, y, \alpha) = \frac{3+e^{xy\alpha}}{2}$, $f(x, y, t, u) = f(x, y, t)$, $f(x, y, t)$ and $\varphi(x, y)$ are determined by the exact solution $u(x, y, t) = e^{-t}x^2(1-x)^2y^2(1-y)^2$.

This is a linear numerical example. The errors and the convergence orders of the numerical schemes (2.4) and (2.7) in the spatial and temporal directions are shown in Tables 1 and 2, respectively. As can be seen from Tables 1 and 2 that the convergence orders are closed to theoretical results. And as can be seen from the numerical results, numerical scheme (2.7) has a slightly higher error than numerical scheme (2.4), but the CPU time of numerical scheme (2.7) is lower than that of numerical scheme (2.4), especially when the number of nodes is larger, which indicates that numerical scheme (2.7) is more suitable for dealing with large and sparse systems of linear equations. Figure 1 give the maximum errors of the numerical schemes (2.4) and (2.7) for solving Example 1 in distributed order. From it, we can observe that the error hardly changes anymore when the number of quadrature nodes increases to a certain number, this suggests that errors in the numerical scheme at this point are dominated by errors in space and time. Therefore, in the calculation process, we compute distribution-order integral using Gauss–Legendre quadrature rule with $m = 4$. Table 3 gives the maximum errors and convergence orders of numerical scheme (2.7) based on different α when the number of spatial nodes is the same as the number of temporal nodes, in which case the model (1.3) can be reduced to two-sided space fractional diffusion equations with variable coefficients [13]. Figures 2 and 3 present the approximation solution at $\tau = \frac{1}{40}$, $m = 4$ as can be seen from the Figs. 2 and 3, the numerical solution is in well accordance with the exact solution, it's verified the validity of the two numerical schemes.

Example 2. Consider the model (1.3) with $T = 1$, $\Omega = (0, 1) \times (0, 1)$, $\omega(\alpha) = -2\Gamma(4-\alpha)\cos\left(\frac{\pi(\alpha+1)}{2}\right)$, $k_{11}(x, y, \alpha) = k_{21}(x, y, \alpha) = k_{12}(x, y, \alpha) = k_{22}(x, y, \alpha) = \frac{-1}{2\cos\left(\frac{\pi(\alpha+1)}{2}\right)}$, $f(x, y, t, u) = f(x, y, t) + g(u)$, $f(x, y, t, u)$ and

Table 1 The numerical results of numerical scheme (2.4) and numerical scheme (2.7) in the spatial direction with $N = 1000$, $m = 4$

$h_1 = h_2$	The numerical scheme (2.4)			The numerical scheme (2.7)		
	E	order$_h$	CPU	E	order$_h$	CPU
$\frac{1}{10}$	2.3206e−05	–	3.1295	2.3212e−05	–	3.1023
$\frac{1}{20}$	5.7156e−06	2.0215	13.2993	5.7268e−06	2.0191	10.9897
$\frac{1}{40}$	1.4096e−06	2.0196	179.5699	1.4153e−06	2.0166	69.2099
$\frac{1}{80}$	3.4148e−07	2.0454	>1 h	3.4722e−07	2.0272	874.2657
$\frac{1}{160}$	–	–	–	8.5267e−08	2.0258	>2 h

Table 2 The numerical results of numerical scheme (2.4) and numerical scheme (2.7) in the temporal direction with $M_1 = M_2 = 100$, $m = 4$

τ	The numerical scheme (2.4)			The numerical scheme (2.7)		
	E	order$_\tau$	CPU	E	order$_\tau$	CPU
$\frac{1}{10}$	2.2500e−05	–	58.2319	3.1152e−05	–	17.8608
$\frac{1}{20}$	5.5509e−06	2.0191	116.7743	7.6905e−06	2.0182	35.0599
$\frac{1}{40}$	1.3750e−06	2.0133	233.1773	1.8940e−06	2.0216	70.9473
$\frac{1}{80}$	3.4435e−07	1.9975	532.4373	4.6960e−07	2.0119	143.0522
$\frac{1}{160}$	–	–	–	1.3189e−07	1.8321	286.9074

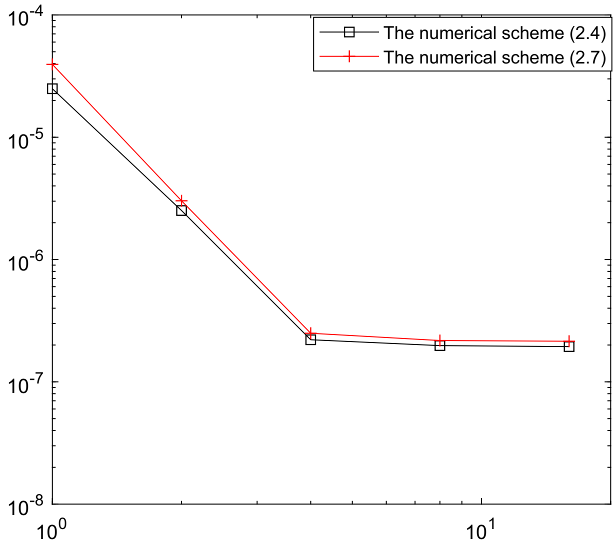
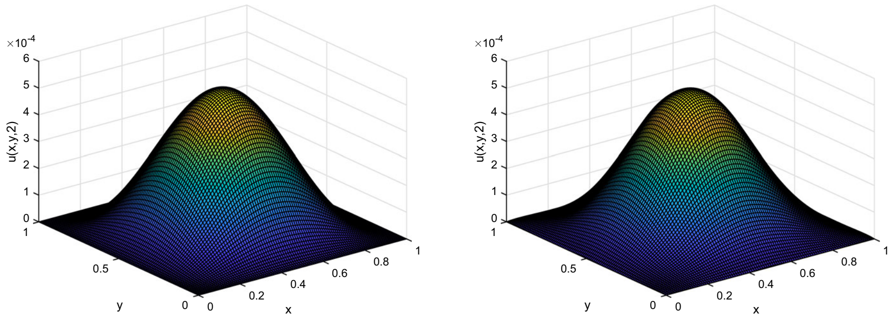


Fig. 1 Distributed-order errors of the numerical schemes (2.4) and (2.7) for Example 1

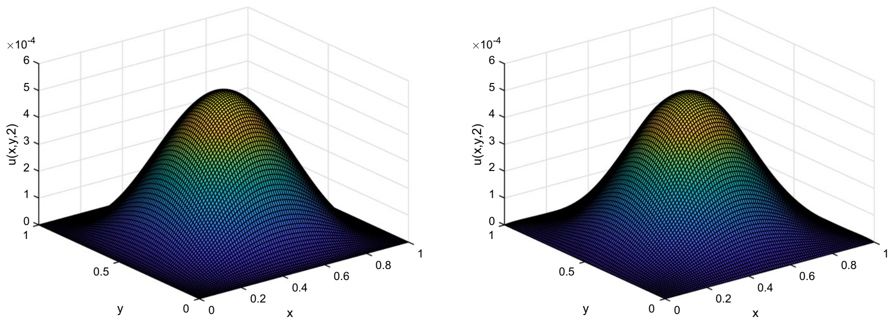
Table 3 The numerical results of numerical scheme (2.7) for Example 1 with $m = 4$

$\tau = h_1 = h_2$	$\alpha = 0.3$		$\alpha = 0.5$		$\alpha = 0.8$	
	E	order$_\infty$	E	order$_\infty$	E	order$_\infty$
$\frac{1}{10}$	3.2084e−05	–	3.5139e−05	–	3.2407e−05	–
$\frac{1}{20}$	7.8786e−06	2.0258	8.6153e−06	2.0281	7.9141e−06	2.0338
$\frac{1}{40}$	1.8963e−06	2.0548	2.1145e−06	2.0266	1.9582e−06	2.0149
$\frac{1}{80}$	4.9157e−07	1.9477	5.3167e−07	1.9917	4.8990e−07	1.9990
$\frac{1}{160}$	1.5189e−07	1.6944	1.7036e−07	1.6419	1.3189e−07	1.6494



(a) Approximation solution (b) Exact solution

Fig. 2 Exact solution and approximation solution of numerical scheme (2.7) for Example 1



(a) Approximation solution (b) Exact solution

Fig. 3 Exact solution and approximation solution of numerical scheme (2.7) for Example 1

$\varphi(x, y)$ are determined by the exact solution $u(x, y, t) = t^3x^4(1-x)^4y^4(1-y)^4$.

This example can be reduced to model (1.1) with $\chi(\beta) = -2\Gamma(5-\beta)\cos(\frac{\pi\beta}{2})$. When $g(u) = 0$, this is a linear numerical example. The errors and the convergence orders of the method in [2] and the numerical scheme (2.7) in the spatial and temporal directions are shown in Tables 4 and 5, respectively. From Table 4, it can be seen that when the number of spatial nodes is small, the numerical scheme (2.7) has less error

Table 4 The errors and the convergence orders of Example 2 in the spatial direction with $N = 1000, m = 4$

$h_1 = h_2$	The method in [2]			The numerical scheme (2.7)		
	E	order_h	CPU	E	order_h	CPU
$\frac{1}{10}$	2.0588e−05	–	1	8.3009e−07	–	2.6963
$\frac{1}{20}$	1.2891e−06	3.9973	4	2.0937e−07	1.9872	8.7535
$\frac{1}{40}$	8.0608e−08	3.9993	15	5.2633e−08	1.9920	47.1824
$\frac{1}{80}$	5.0386e−09	3.9998	31	1.3059e−08	2.0109	434.2656
$\frac{1}{160}$	3.1460e−10	4.0014	55	3.1036e−09	2.0731	>1 h

Table 5 The errors and the convergence orders of Example 2 in the temporal direction with $M_1 = M_2 = 100, m = 4$

τ	The method in [2]			The numerical scheme (2.7)		
	E	order_τ	CPU	E	order_τ	CPU
$\frac{1}{10}$	1.8571e−04	–	3	1.9356e−06	–	10.8942
$\frac{1}{20}$	5.2127e−05	1.8329	8	5.3777e−07	1.8477	21.3948
$\frac{1}{40}$	1.3143e−05	1.9877	17	1.3243e−07	2.0218	43.6896
$\frac{1}{80}$	3.2928e−06	1.9969	29	2.7006e−08	2.2939	68.4498
$\frac{1}{160}$	8.2467e−07	1.9974	53	5.2637e−09	2.3591	136.3546

than the method in [2], but when the number of nodes is large, the method in [2] is more advantageous than the numerical scheme (2.7) and the convergence order is higher than that of the numerical scheme (2.7). As can be seen from Table 5, the numerical scheme (2.7) has a lower error than the method in [2], but the CPU time is more than the method in [2]. When $g(u) = -u(1+u)$, this is a nonlinear numerical example. And assume $\tau = h_1 = h_2$, $m = 4$ in this example. The errors and the convergence orders of the numerical scheme (2.4) and the numerical scheme (2.7) are shown in Table 6. The numerical results show that the convergence order is consistent with the theoretical value, verifying the accuracy of the numerical scheme. And Fig. 4a gives the max absolute errors of the numerical schemes (2.4) and (2.7) for different values of τ.

Example 3. Consider the model (1.3) with $T = 1.5$, $\Omega = (0,1) \times (0,1)$, $\omega(\alpha) = 2e^{\alpha}$, $k_{11}(x, y, \alpha) = \frac{1-x^2 y\alpha}{2}$, $k_{21}(x, y, \alpha) = \frac{1-xy^2\alpha}{2}$, $k_{12}(x, y, \alpha) = \frac{1+x^2 y\alpha}{2}$, $k_{22}(x, y, \alpha) = \frac{1+xy^2\alpha}{2}$, $f(x, y, t, u) = g(u)$, $g(u) = u(1+u)$, $\varphi(x, y) = \sin(1)x^3(1-x)^3 y^3(1-y)^3$.

This is a nonlinear numerical example without exact solution, thus, we take the approximate solutions with $m = 4$, $N = 1000$ and $M_1 = M_2 = 100$ as reference solutions in this example. And assume $\tau = h_1 = h_2$, $m = 4$ in this example. Table 7 give some numerical results when different values of τ are taken. From Table 7, it can be concluded that the convergence orders of both two numerical schemes is close to 2, and the error of numerical scheme (2.4) is lower than that of numerical scheme

Table 6 The numerical results of numerical scheme (2.4) and numerical scheme (2.7) for Example 2

$\tau = h_1 = h_2$	The numerical scheme (2.4)			The numerical scheme (2.7)		
	E	order$_\infty$	CPU	E	order$_\infty$	CPU
$\frac{1}{10}$	7.1119e−07	–	0.0359	1.1755e−06	–	0.0249
$\frac{1}{20}$	1.8097e−07	1.9745	0.2909	3.3620e−07	1.8059	0.2380
$\frac{1}{40}$	4.5778e−08	1.9830	3.7275	8.7050e−08	1.9494	1.8084
$\frac{1}{80}$	1.1521e−08	1.9904	107.6790	2.1919e−08	1.9897	30.9019
$\frac{1}{160}$	–	–	–	5.4843e−09	1.9988	832.9684

Table 7 The numerical results of numerical scheme (2.4) and numerical scheme (2.7) for Example 3

$\tau = h_1 = h_2$	The numerical scheme (2.4)			The numerical scheme (2.7)		
	E	order$_\infty$	CPU	E	order$_\infty$	CPU
$\frac{1}{10}$	5.9528e−06	–	0.0520	1.1781e−05	–	0.0326
$\frac{1}{20}$	1.4523e−06	2.0352	0.3551	2.9456e−06	1.9998	0.2810
$\frac{1}{40}$	3.6289e−07	2.0004	5.8638	7.4071e−07	1.9916	2.5310
$\frac{1}{80}$	9.0869e−08	1.9980	155.0424	1.8882e−07	1.9719	49.7262
$\frac{1}{160}$	–	–	–	5.5687e−08	1.7616	1391.6883

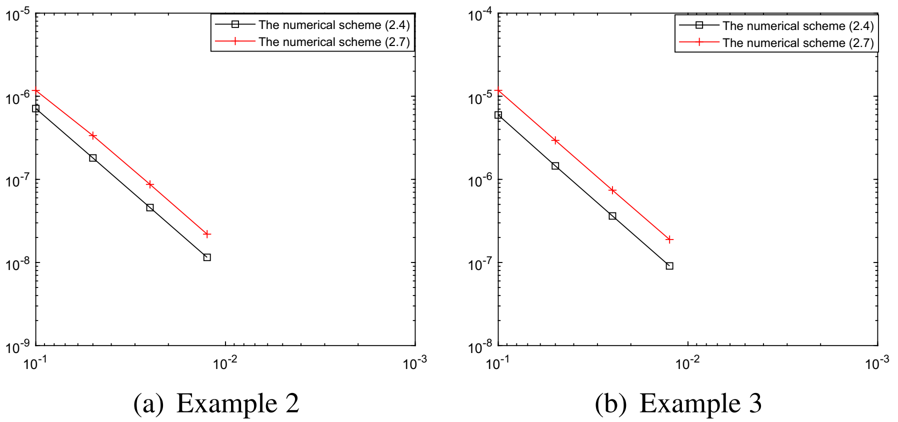


Fig. 4 The max absolute errors of the numerical schemes (2.4) and (2.7)

(2.7), which is consistent with the theoretical analysis. Moreover, the CPU time of the numerical scheme (2.7) is lower than that of the numerical scheme (2.4). Especially when the number of nodes is large. This indicates that the numerical scheme (2.7) is better suited to handle large systems of sparse linear equations. Figure 4b gives the max absolute errors of the numerical schemes (2.4) and (2.7) for different values of τ.

5 Conclusion

In this paper, we developed a second order in both space and time numerical scheme for two-dimensional two-sided space distributed-order fractional diffusion equation with variable coefficients. In addition, a small perturbation is added to this numerical scheme to construct an alternating-direction implicit scheme. Subsequently, we proved that the difference scheme is unconditionally stable and convergent with the accuracy of $O(m^{-v} + h_1^2 + h_2^2 + \tau^2)$. Finally, some numerical results are given to show the stability and convergence of our numerical scheme. And the numerical results indicates that the numerical scheme (2.7) is better suited to handle large systems of sparse linear equations. However, the model in this paper is too restrictive on the diffusion coefficient, and the computational cost of the numerical scheme is too high. In future work, we will work on developing higher-order numerical methods or developing fast iterative algorithms to solve models with less restriction on diffusion coefficients.

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Declarations

Conflict of interest The authors declare no competing interests.

Ethical approval Not applicable.

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