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ORIGINAL RESEARCH

Multi attribute decision-making and interval-valued picture (S, T)-fuzzy graphs

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Abstract

In this manuscript, we introduce the notion of interval-valued picture (S, T)-fuzzy graphs (IVP-(S, T)-fuzzy graphs) which is the interplay between (S, T)-norms and picture fuzzy graphs. Interval-valued picture (S, T)-fuzzy graphs (IVP-(S, T)-fuzzy graphs) is also the extension of the picture fuzzy graphs. Since interval-valued picture fuzzy sets (IVPFSs) is the most extended form of the fuzzy sets to deal uncertainties, interval-valued picture (S, T)-fuzzy graphs (IVP-(S, T)-fuzzy graphs) would be more efficient to deal with the problems containing vagueness. For the sake of investigation, firstly we introduce and apply various operations like union, join, cartesian product, direct product, lexicographic product, ring sum, complement etc to interval-valued picture (S, T)-fuzzy graphs. Then, we study the structural properties of interval-valued picture (S, T)-fuzzy graphs (IVP-(S, T)-fuzzy graphs) through homomorphism, co-weak homomorphism, isomorphism etc. Afterwards, we initiate different types of interval-valued picture (S, T)-fuzzy graphs such as regular, total regular, constant etc. Moreover, we also explore some relationships among different types of interval-valued picture (S, T)-fuzzy graphs. Finally, we provide an application of interval-valued picture (S, T)-fuzzy graphs (IVP-(S, T)-fuzzy graphs) towards multiple attribute decision-making (MADM).

Keywords IVP-(S, T)-fuzzy graphs · Regular IVP-(S, T)-fuzzy graphs · Constant IVP-(S, T)-fuzzy graphs · Complement · Ring sum · MADM

Mathematics Subject Classification 05C72 · 03B52

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1 Introduction

Fuzzy sets (FSs) was first introduced by Zadeh in [1] which become an efficient tool to deal the problems containing uncertainties. Corresponding to any non-empty set, the fuzzy set gives us the degree of membership of an object. Afterwards, numerous generalized forms of fuzzy sets were introduced in the literature. Zadeh [2] himself introduced the very first generalization of fuzzy sets named interval-valued fuzzy sets (IVFSs). In IVFSs, a closed sub-interval of [0, 1] is used to express the membership degree of an object instead of a number as was considered in the case of fuzzy sets. Subsequently, Atanassov [3] proposed the term intuitionistic fuzzy sets (IFSs) in which he introduced two values that are membership and non-membership under the condition that the sum of both values must belong to a closed unit interval [0, 1]. Furthermore, he also introduced the notion of interval-valued intuitionistic fuzzy sets (IVIFSs) [4] in which both the values are the closed sub-intervals of [0, 1]. Following the work of Zadeh and Atanassov another important generalization of FSs and IFSs named picture fuzzy sets (PFSs) was introduced by Cuong [5]. He also explored few new operations and characterizations of PFSs. Actually, PFS is the direct generalization of an IFS with the addition of another value called "abstinence value". In a PFS we allocate three memberships values to an object that are positive, negative and neutral. Bo et al. [6] defined many operations and relations on PFSs. Cuong et al. [7] added different types of fuzzy logical operators in the theory of PFSs. The term interval-valued picture fuzzy sets (IVPFSs) was proposed in [8]. In IVPFSs, all of the three values consist of the closed sub-intervals of [0, 1]. Like PFS, in IVPFS the sum of the supremum of all three sub-intervals must lie within [0, 1]. Evidently, IVPFSs are more generalized form of the fuzzy sets as compared to the other generalizations of FSs like IFSs, IVFSs, IVIFSs and PFSs. Recently, W. A. Khan et al. introduced several new relations on bipolar picture fuzzy sets (BPPFSs) [9]. For more on generalizations of FSs, one may consult [10–13].

The notions of triangular norms and conorms were first introduced by Schweizer and Sklar in [14, 15]. These operators are widely used in different fields such as multi-criteria decision support, fuzzy logics, fuzzy set theory and several fields of information sciences. For further details about these operators, one may consult [16–19]. Cuong et al. [7] also studied these operators in the setting of PFSs.

The study of the graph theory based on fuzzy sets become an attractive field for the researchers due to having many applications in different field of sciences. Firstly, Rosenfeld introduced the concepts of fuzzy graphs(FGs) [20]. Bhattacharya [21] incorporated few new terms in the fuzzy graphs theory. Different new operations were explored and applied on fuzzy graphs (FGs) in [22]. Complement of fuzzy graphs (FGs) was investigated in [23]. Subsequently, the extended form of the FGs termed interval-valued fuzzy graphs (IVFGs) was introduced in [24]. Rashmanlou and Pal studied some properties of highly irregular interval-valued fuzzy graphs in [25]. Intuitionistic fuzzy graphs (IFGs) were explored in [26]. Many operations were introduced and applied to IFGs in [27]. The concepts of the complex intuitionistic fuzzy graphs along with its applications towards networking were discussed in [28]. Ring sum in product intuitionistic fuzzy graphs was discussed in [29] by Borzooei and Rashmanlou. Borzooei et al. [30] introduced the notion of t-fuzzy graphs(t-FGs). The notion of interval-valued intuitionistic (S, T)-fuzzy graphs (IVI-(S, T)-fuzzy graphs) was introduced in [31, 32]. Recently, Zuo et al. [33] introduced the term picture fuzzy graphs(PFGs). They also introduced some operations on PFGs and explored some applications of PFGs in social networking. Consequently, the notion of picture fuzzy multi-graph (PFMG) was explored in [34]. Muhiuddin et al. [35] introduced the notion of cubic graphs along with application. Poulik and Ghorai solved some real world problems using generalizations of FGs in [36, 37]. Moreover, Poulik [38] introduced Randic index for bipolar fuzzy graphs along with application. Further extended form of the PFGs termed regular picture fuzzy graphs (RPFGs) along with its applications in networking was discussed in [39]. Koczy et al. [40] further added some new terms in PFGs. They also provided evidences that the PFGs are more effective than the FGs and IFGs. Recently, the concepts of balanced picture fuzzy graphs (balanced PFGs) [41] has also been added in the literature. Amanathulla et al. [42] discussed the application of PFGs in MADM problem. Recently, Khan et al. introduced the notion of bipolar picture fuzzy graphs along with its application towards social networking [43] and the notion of Cayley picture fuzzy graphs along with its application in interconnected networks [44]. For more on fuzzy graphs theory, one may consult [21–23, 45–48].

Currently, many researchers studied different problems containing uncertainties through FGs and its generalizations. In our study, we focus to deal with such problems in the picture fuzzy environment with (S, T)-norms. Since IVPFSs is the most generalized form of FSs in which we consider the membership, neutral and non-membership degrees on the interval [0, 1]. Therefore, our newly established structure termed IVP-(S, T)-fuzzy graphs would be more useful as compared to the other generalizations of FGs to deal with uncertain real world problems.
We describe the motivations, limitations and novelty of our work as follows.

1. The generalization of FGs termed t-FGs and the generalization of IFGs named IVI-(S, T)-fuzzy graphs motivated us to introduce the notion of IVP-(S, T)-fuzzy graphs which is the generalization of PFGs.
2. Our proposed structure can be applied in such circumstances where the traditional FGs and its other generalizations fail. In such cases the notion of IVP-(S, T)-fuzzy graphs can be used by choosing the most suitable types of (S, T)-norms according to the situations. Moreover, some other norms such as Dombi (S, T)-norms, Hamacher (S, T)-norms, Einstein (S, T)-norms, Yager (S, T)-norms, Frank (S, T)-norms etc can be adjusted in the setting of IVP-(S, T)-fuzzy graphs.
3. Since interval-valued picture fuzzy sets (IVPFSs) is the most extended form of the FSs to deal with the problems containing uncertainties, interval-valued picture (S, T)-fuzzy graphs (IVP-(S, T)-fuzzy graphs) would be more efficient to deal such problems.
4. The domain of IVP-(S, T)-fuzzy graphs is very vast, we can twist it towards FGs, IVFGs, t-FGs, IFGs, IVI-(S, T)-fuzzy graphs and PFGs by assigning different memberships values. Thus the qualitative characteristics of FGs, IVFGs, t-FGs, IFGs, IVI-(S, T)-fuzzy graphs, PFGs are combined in a single IVP-(S, T)-fuzzy graphs.
5. We present the most useful method along with an efficient algorithm to solve the problems occurring in MADM through IVP-(S, T)-fuzzy graphs. In this regard,

we provide a numerical example as an evidence which reflects that IVP-(S,T)-fuzzy graphs is more efficient and is easy to apply.

This paper consists of five sections. In Sect. 2, we provide the necessary definitions and terminologies which are useful in understanding the forthcoming sections. In Sect. 3, we initiate the notion of interval-valued picture (S,T)-fuzzy graphs (IVP-(S,T)-fuzzy graphs) followed by various operations like union, join, ring sum, complement, cartesian product, direct product, lexicographic product etc to interval-valued picture (S,T)-fuzzy graphs. Homomorphism and co-weak homomorphism of IVP-(S,T)-fuzzy graphs are also discussed. Different types of IVP-(S,T)-fuzzy graphs such as regular, total regular and constant IVP-(S,T)-fuzzy graphs are also introduced. In Sect. 4, we provide the application of IVP-(S,T)-fuzzy graphs towards multi-attribute decision making theory. We also provide a numerical example and algorithm for convincing the practicality of our theoretical structure. In Sect. 5, we highlight the worth, limitations and possible extensions of the work presented in this study.

2 Preliminaries

An interval number denoted by σ is an interval expressed as $\sigma = [\sigma^-, \sigma^+]$, with $0 \le \sigma^- \le \sigma^+ \le \sigma$ and $D[0,1]$ is the collection of all interval-numbers. The interval $[\sigma, \sigma]$ is identified with the numbers $\sigma \in [0,1]$.
Now for any collection of interval numbers say $\tilde{\sigma_i} = [\sigma_i^-, \sigma_i^+] \in D[0,1]$ and $i \in I$, their supremum and infimum are

$$Inf(\tilde{\sigma_i}) = \left[\bigwedge_{i\in I} \sigma^-, \bigwedge_{i\in I} \sigma^+\right]$$

and

$$Sup(\tilde{\sigma_i}) = \left[\bigvee_{i\in I} \sigma^-, \bigvee_{i\in I} \sigma^+\right].$$

Moreover, for any two interval numbers $\tilde{\sigma_1}, \tilde{\sigma_2} \in D[0,1]$, we have

(i) $\tilde{\sigma_1} \le \tilde{\sigma_2}$ iff $\sigma_1^- \le \sigma_2^-$ and $\sigma_1^+ \le \sigma_2^+$
(ii) $\tilde{\sigma_1} = \tilde{\sigma_2}$ iff $\sigma_1^- = \sigma_2^-$ and $\sigma_1^+ = \sigma_2^+$
(iii) $\tilde{\sigma_1} < \tilde{\sigma_2}$ iff $\tilde{\sigma_1} \le \tilde{\sigma_2}$ and $\tilde{\sigma_1} \ne \tilde{\sigma_2}$
(iv) $k\tilde{\sigma} = [k\sigma^-, k\sigma^+]$ where $k \in [0,1]$.

Here, $(D[0,1], \le, \vee, \wedge)$ is a complete lattice with $0 = [0,0]$ and $1 = [1,1]$ are the smallest and largest element, respectively. Similarly, an interval number fuzzy set (i.e., F on X) is the set $F = \{(s, [\mu_{F^-}(s), \mu_{F^+}(s)]) : s \in X\}$, where μ_{F^-} and μ_{F^+} are two fuzzy subsets of s such that $\mu_{F^-} \le \mu_{F^+}$, for all $s \in X$. By stating $\mu_F = [\mu_{F^-}(s), \mu_{F^+}(s)]$ we see that $F = \{(s, \mu_F) : s \in X\}$, where $\mu_F(s) : S \to D[0,1]$.

Definition 1 [49] A t-norm (s-norm) is the function $\lambda : [0,1] \times [0,1] \to [0,1]$ satisfying

1. $\lambda(s, 1) = s$ $(\lambda(s, 0) = s)$
2. $(\lambda(s, t) = (\lambda(t, s))$
3. $(\lambda(\lambda(s, t), u) = (\lambda(s, (\lambda(t, u))$
4. $\lambda(s, u) \leq \lambda(s, w)$ where $u \leq w$

for all $s, t, u, w \in [0, 1]$.

If for all $\sigma \in [0, 1]$ we have $\lambda(\sigma, \sigma) = \sigma$, then λ is an idempotent t-norm (resp., s-norm). For idempotent t-norm (resp., s-norm) λ is the map $\lambda : D[0, 1] \times D[0, 1] \rightarrow D[0, 1]$ defined by $\lambda(\widetilde{\sigma_1}, \widetilde{\sigma_2}) = [\lambda(\sigma_1^-, \sigma_2^-), \lambda(\sigma_1^+, \sigma_2^+)]$ is also idempotent and is said to be an idempotent interval t-norm (resp., s-norm).

Definition 2 [1] A FS $\bar{P}$ in T is the set $\bar{P} = \{p, \xi_{\bar{P}}(p) : p \in T\}$, where $\xi_{\bar{P}}(p) : T \rightarrow [0, 1]$ denotes the membership degree of p to FS $\bar{P}$.

Definition 3 [2] An IVFS $\bar{P}$ in T can represented as $\bar{P} = \{p, \xi_{\bar{P}}(p) : p \in T\}$, where $\xi_{\bar{P}}(p) : T \rightarrow P([0, 1])$ is a closed interval in $[0, 1]$. $\xi_{\bar{P}}(p)$ is sub-interval of $[0, 1]$ and represents the membership degree of p.

Definition 4 [3] An IFS $\bar{P}$ in T is the collection $\bar{P} = \{p, \xi_{\bar{P}}(p), \psi_{\bar{P}}(p) : p \in T\}$, where $\xi_{\bar{P}}(p) : T \rightarrow [0, 1]$ and $\psi_{\bar{P}}(p) : T \rightarrow [0, 1]$ represents the membership degree and non-membership degree of p, respectively such that $0 \leq \xi_{\bar{P}}(p) + \psi_{\bar{P}}(p) \leq 1$ for all $p \in T$.

Definition 5 [4] An IVIFS on a non-empty set T is defined by Atanassov in the form of

$$\bar{P} = \{(p, \widetilde{M_{\bar{P}}}(p), \widetilde{N_{\bar{P}}}(p) : p \in T\}$$

where $\widetilde{M_{\bar{P}}}(p)$, $\widetilde{N_{\bar{P}}}(p)$ are IVFSs on T such that

$$0 \leq Sup\widetilde{M_{\bar{P}}}(p) + Sup\widetilde{N_{\bar{P}}}(p) \leq 1$$

IVIFSs are represented by $\bar{P} = (\widetilde{M_{\bar{P}}}(p), \widetilde{N_{\bar{P}}}(p))$.

Definition 6 [5] A PFS $\bar{P}$ in T can be described as $\bar{P} = \{p, \xi_{\bar{P}}(p), \psi_{\bar{P}}(p), \zeta_{\bar{P}}(p) : p \in T\}$, where $\xi_{\bar{P}}(p) : T \rightarrow [0, 1]$, $\psi_{\bar{P}}(p) : T \rightarrow [0, 1]$ and $\zeta_{\bar{P}}(p) : T \rightarrow [0, 1]$ represents the membership, neutral membership and non- membership degree of p, respectively with $\xi_{\bar{P}}(p), \psi_{\bar{P}}(p), \zeta_{\bar{P}}(p) \leq 1$, for all $p \in T$

Definition 7 [50] An IVPFS on a non-empty set T is defined as

$$\bar{P} = \{(p, \widetilde{M_{\bar{P}}}(p), \widetilde{N_{\bar{P}}}(p), \widetilde{O_{\bar{P}}}(p) : p \in T\}$$

where $\widetilde{M_{\bar{P}}}(p)$, $\widetilde{N_{\bar{P}}}(p)$ and $\widetilde{O_{\bar{P}}}(p)$ are IVFSs ($i.e.$, $\widetilde{M_{\bar{P}}}(p) : T \rightarrow D[0, 1]$, $\widetilde{N_{\bar{P}}}(p) : T \rightarrow D[0, 1]$ and $\widetilde{O_{\bar{P}}}(p) : T \rightarrow D[0, 1]$) on T such that

$$0 \leq Sup\widetilde{M_{\bar{P}}}(p) + Sup\widetilde{N_{\bar{P}}}(p) + Sup\widetilde{O_{\bar{P}}}(p) \leq 1$$

IVPFSs is represented by $\bar{P} = (\widetilde{M_{\bar{P}}}(p), \widetilde{N_{\bar{P}}}(p), \widetilde{O_{\bar{P}}}(p))$.

Definition 8 [51] The score function for any interval-valued picture fuzzy number $\alpha = \{[p^l, p^u], [q^l, q^u], [r^l, r^u] : 0 \leq p^u + q^u + r^u \leq 1\}$ is defined as

$$scor(\alpha) = \frac{p^l - q^l - r^l + p^u - q^u - r^u}{3}$$

Definition 9 [33] A FG G for any non-empty set V of vertices is given by $G = (V, E)$, where V is fuzzy subset of V and E is a fuzzy relation (symmetric) on V, i.e., $V : V \to [0, 1]$ and $E : V \times V \to [0, 1]$ with $E(p, q) = (V(p) \cap V(q))$, for all $p, q \in E$.

Definition 10 [24] An IVFG on a graph $G^* = (\bar{V}, \bar{E})$ can be described as $G = (V, E)$, where $V = [p_V^-, p_V^+]$ is IVFS on $\bar{V}$ and $E = [p_E^-, p_E^+]$ is IVFS on $\bar{E}$.

Definition 11 [52] An IFG on any graph $G^* = (\bar{V}, \bar{E})$ is $G = (V, E)$, where $V = (\psi_V, \phi_V)$ is a IFS on $\bar{V}$ and $E = (\psi_E, \phi_E)$ is an IFS on $\bar{E} = \bar{V} \times \bar{V}$ with $jk \in \bar{E}$

$$\psi_E(j, k) \leq min(\psi_V(j), \psi_V(k))$$
$$\phi_E(j, k) \leq max(\phi_V(j), \phi_V(k))$$

Definition 12 [31] An interval-valued intuitionistic (S, T)-fuzzy graph on $G^* = (\bar{V}, \bar{E})$ is an ordered pair (V, E), where $V = (\widetilde{M_V}, \widetilde{N_V})$ is an IVIFS on $\bar{V}$ and $E = (\widetilde{P_E}, \widetilde{Q_E})$ is an IVIFS on $\bar{E}$ such that for all $j, k \in \bar{V}$, we have

$$\widetilde{P_E}\{j, k\} \leq T\{\widetilde{M_V}(j), \widetilde{M_V}(k)\}$$
$$\widetilde{Q_E}\{j, k\} \geq S\{\widetilde{N_V}(j), \widetilde{N_V}(k)\}$$

Definition 13 [33] A PFG on any graph $G^* = (\bar{V}, \bar{E})$ is given by $G = (V, E)$, where $V = (\psi_V, \phi_V, \varphi_V)$ is a PFS on $\bar{V}$ and $E = (\psi_E, \phi_E, \varphi_E)$ is a PFS on $\bar{E} = \bar{V} \times \bar{V}$ such that for each $jk \in \bar{E}$

$$\psi_E(j, k) \leq min(\psi_V(j), \psi_V(k))$$
$$\phi_E(j, k) \leq min(\phi_V(j), \phi_V(k))$$
$$\varphi_E(j, k) \geq max(\varphi_V(j), \varphi_V(k))$$

.

3 Interval-valued picture (S, T)-fuzzy graphs(IVP-(S, T)-fuzzy graphs)

In this section, we initiate the concepts of IVP-(S, T)-fuzzy graphs. Throughout, we use the norms $T(x, y) = (x_1 \wedge y_1, x_2 \vee y_2)$ and $S(x, y) = (x_1 \vee y_1, x_2 \wedge y_2)$, unless otherwise we specify.

Definition 14 An IVP-(S, T)-fuzzy graph on V is an ordered pair (A, B), where $A = (\widetilde{M_A}, \widetilde{N_A}, \widetilde{O_A})$ is an IVPFS on V and $B = (\widetilde{P_B}, \widetilde{Q_B}, \widetilde{R_B})$ is an IVPFS on E such that for all $m, n \in V$, we have

$$\widetilde{P_B}\{m,n\} \leq T\{\widetilde{M_A}(m), \widetilde{M_A}(n)\}$$
$$\widetilde{Q_B}\{m,n\} \leq T\{\widetilde{N_A}(m), \widetilde{N_A}(n)\}$$
$$\widetilde{R_B}\{m,n\} \geq S\{\widetilde{O_A}(m), \widetilde{O_A}(n)\}.$$

Example 1 Let $G^* = (V, E)$, where $V = \{u, v\}$ and $E = \{uv\}$. Then the graph $G = (A, B)$ defined on G^* shown in Fig. 1 can be described as

$$A = \left\langle V, \left(\frac{u}{[0.4, 0.4]}, \frac{v}{[0.2, 0.4]}\right), \left(\frac{u}{[0.2, 0.4]}, \frac{v}{[0.3, 0.4]}\right), \left(\frac{u}{[0.1, 0.2]}, \frac{v}{[0.3, 0.3]}\right)\right\rangle$$
$$B = \left\langle E, \left(\frac{uv}{[0.3, 0.4]}\right), \left(\frac{uv}{[0.3, 0.3]}\right), \left(\frac{uv}{[0.1, 0.3]}\right)\right\rangle$$

Clearly, A and B are interval-valued picture fuzzy subsets of V and $E = V \times V$, respectively. By using the norms $T(x, y) = (x_1 \wedge y_1, x_2 \vee y_2)$ and $S(x, y) = (x_1 \vee y_1, x_2 \wedge y_2)$, we have

$$\begin{aligned}
&\widetilde{P_B}\{m,n\} \leq T\{\widetilde{M_A}(m), \widetilde{M_A}(n)\} \\
&[0.3, 0.4] \leq T([0.4, 0.4], [0.2, 0.4]) \\
&[0.3, 0.4] \leq [0.2, 0.4] \\
&\widetilde{Q_B}\{m,n\} \leq T\{\widetilde{N_A}(m), \widetilde{N_A}(n)\} \\
&[0.3, 0.3] \leq T([0.2, 0.4], [0.3, 0.4]) \\
&[0.3, 0.4] \leq [0.2, 0.4]
\end{aligned}$$

and

$$\begin{aligned}
&\widetilde{R_B}\{m,n\} \geq S\{\widetilde{O_A}(m), \widetilde{O_A}(n)\} \\
&[0.1, 0.3] \geq S([0.1, 0.3], [0.3, 0.3]) \\
&[0.3, 0.3] \geq [0.3, 0.3].
\end{aligned}$$

Hence the graph in Fig. 1 is an IVP-(S, T)-fuzzy graph.

Definition 15 Let $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ be the underlying graphs of two IVP-(S, T)-fuzzy graphs $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$, respectively. Then

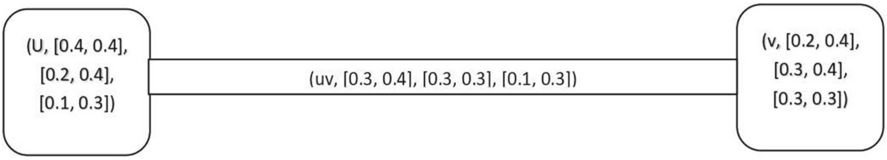

Fig. 1 IVP-(S, T)-fuzzy graph

their union $G_1 \cup G_2$ is the ordered pair (A, B), where $A = (\widetilde{M_A}, \widetilde{N_A}, \widetilde{O_A})$ and $B = (\widetilde{P_B}, \widetilde{Q_B}, \widetilde{R_B})$ are IVPFSs on $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$, respectively with

1. $\widetilde{M_A}(s) = \widetilde{M_{A_1}}(s)$, if $s \in V_1$ and $s \notin V_2$ $\widetilde{M_A}(s) = \widetilde{M_{A_2}}(s)$, if $s \in V_2$ and $s \notin V_1$ and $\widetilde{M_A}(s) = S(\widetilde{M_{A_1}}(s), \widetilde{M_{A_2}}(s))$, if $s \in V_1 \cap V_2$
2. $\widetilde{N_A}(s) = \widetilde{N_{A_1}}(s)$, if $s \in V_1$ and $s \notin V_2$ $\widetilde{N_A}(s) = \widetilde{N_{A_2}}(s)$, if $s \in V_2$ and $s \notin V_1$ and $\widetilde{N_A}(s) = S(\widetilde{N_{A_1}}(s), \widetilde{N_{A_2}}(s))$, if $s \in V_1 \cap V_2$
3. $\widetilde{O_A}(s) = \widetilde{O_{A_1}}(s)$, if $s \in V_1$ and $s \notin V_2$ $\widetilde{O_A}(s) = \widetilde{O_{A_2}}(s)$, if $s \in V_2$ and $s \notin V_1$ and $\widetilde{O_A}(s) = T(\widetilde{O_{A_1}}(s), \widetilde{O_{A_2}}(s))$, if $s \in V_1 \cap V_2$
4. $\widetilde{P_B}(st) = \widetilde{P_{B_1}}(st)$, if $st \in E_1$ and $st \notin E_2$ $\widetilde{P_B}(st) = \widetilde{P_{B_2}}(st)$, if $st \in E_2$ and $st \notin E_1$ $\widetilde{P_B}(st) = S(\widetilde{P_{B_1}}(st), \widetilde{P_{B_2}}(st))$, if $st \in E_1 \cap E_2$
5. $\widetilde{Q_B}(st) = \widetilde{Q_{B_1}}(st)$, if $st \in E_1$ and $st \notin E_2$ $\widetilde{Q_B}(st) = \widetilde{Q_{B_2}}(st)$, if $st \in E_2$ and $st \notin E_1$ $\widetilde{Q_B}(st) = S(\widetilde{Q_{B_1}}(st), \widetilde{Q_{B_2}}(st))$, if $st \in E_1 \cap E_2$
6. $\widetilde{R_B}(st) = \widetilde{R_{B_1}}(st)$, if $st \in E_1$ and $st \notin E_2$ $\widetilde{R_B}(st) = \widetilde{R_{B_2}}(st)$, if $st \in E_2$ and $st \notin E_1$ $\widetilde{R_B}(st) = T(\widetilde{R_{B_1}}(st), \widetilde{P_{B_2}}(st))$, if $st \in E_1 \cap E_2$.

Proposition 1 *The union of two IVP-(S, T)-fuzzy graphs $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ defined on $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ is an IVP-(S, T)-fuzzy graph.*

Proof Let $st \in E_1 \cap E_2$. Then we have

$$\begin{aligned}
(\widetilde{P_{B_1}} \cup \widetilde{P_{B_2}})(st) &= S(\widetilde{P_{B_1}}(st), \widetilde{P_{B_2}}(st)) \\
&\le S(T(\widetilde{M_{A_1}}(s), \widetilde{M_{A_1}}(t)), T(\widetilde{M_{A_2}}(s), \widetilde{M_{A_2}}(t)) \\
&= T(S(\widetilde{M_{A_1}}(s), \widetilde{M_{A_2}}(s)), S(\widetilde{M_{A_2}}(t), \widetilde{M_{A_2}}(t)) \\
&= T((\widetilde{M_{A_1}} \cup \widetilde{M_{A_2}})(s), (\widetilde{M_{A_1}} \cup \widetilde{M_{A_2}})(t)) \\
(\widetilde{Q_{B_1}} \cup \widetilde{Q_{B_2}})(st) &= S(\widetilde{Q_{B_1}}(st), \widetilde{Q_{B_2}}(st)) \\
&\le S(T(\widetilde{N_{A_1}}(s), \widetilde{N_{A_1}}(t)), T(\widetilde{N_{A_2}}(s), \widetilde{N_{A_2}}(t)) \\
&= T(S(\widetilde{N_{A_1}}(s), \widetilde{N_{A_2}}(s)), S(\widetilde{N_{A_2}}(t), \widetilde{N_{A_2}}(t)) \\
&= T((\widetilde{N_{A_1}} \cup \widetilde{N_{A_2}})(s), (\widetilde{N_{A_1}} \cup \widetilde{N_{A_2}})(t)) \\
(\widetilde{R_{B_1}} \cup \widetilde{R_{B_2}})(st) &= T(\widetilde{R_{B_1}}(st), \widetilde{R_{B_2}}(st)) \\
&\ge T(S(\widetilde{O_{A_1}}(s), \widetilde{O_{A_1}}(t)), S(\widetilde{O_{A_2}}(s), \widetilde{O_{A_2}}(t)) \\
&= S(T(\widetilde{O_{A_1}}(s), \widetilde{O_{A_2}}(s)), T(\widetilde{O_{A_2}}(t), \widetilde{O_{A_2}}(t)) \\
&= S((\widetilde{O_{A_1}} \cup \widetilde{O_{A_2}})(s), (\widetilde{O_{A_1}} \cup \widetilde{O_{A_2}})(t)).
\end{aligned}$$

Similarly, if $st \in E_1$ and $st \notin E_2$ or $st \in E_2$ and $st \notin E_1$, then

$$\begin{aligned}
(\widetilde{P_{B_1}} \cup \widetilde{P_{B_2}})(st) &\le T((\widetilde{M_{A_1}} \cup \widetilde{M_{A_2}})(s), (\widetilde{M_{A_1}} \cup \widetilde{M_{A_2}})(t)) \\
(\widetilde{Q_{B_1}} \cup \widetilde{Q_{B_2}})(st) &\le T((\widetilde{N_{A_1}} \cup \widetilde{N_{A_2}})(s), (\widetilde{N_{A_1}} \cup \widetilde{N_{A_2}})(t))
\end{aligned}$$

and

$$(\widetilde{R_{B_1}} \cup \widetilde{R_{B_2}})(st) \geq T((\widetilde{O_{A_1}} \cup \widetilde{O_{A_2}})(s), (\widetilde{O_{A_1}} \cup \widetilde{O_{A_2}})(t))$$

. □

Definition 16 The join of the two IVP-(S, T)-fuzzy graphs $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ over $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ is represented by $G_1 + G_2$ is the ordered pair (A, B), where $A = (\widetilde{M_A}, \widetilde{N_A}, \widetilde{O_A})$ and $B = (\widetilde{P_B}, \widetilde{Q_B}, \widetilde{R_B})$ are IVPFSs on $V = V_1 \cup V_2$ and $E = E_1 \cup E_2 \cup E'$ (where E' consists of all the edges of V_1 and V_2), respectively satisfying

1. $\widetilde{M_A}(s) = \widetilde{M_{A_1}}(s)$, if $s \in V_1$ and $s \notin V_2$
   $\widetilde{M_A}(s) = \widetilde{M_{A_2}}(s)$, if $s \in V_2$ and $s \notin V_1$
   and $\widetilde{M_A}(s) = S(\widetilde{M_{A_1}}(s), \widetilde{M_{A_2}}(s))$, if $s \in V_1 \cap V_2$
2. $\widetilde{N_A}(s) = \widetilde{N_{A_1}}(s)$, if $s \in V_1$ and $s \notin V_2$
   $\widetilde{N_A}(s) = \widetilde{N_{A_2}}(s)$, if $s \in V_2$ and $s \notin V_1$
   and $\widetilde{N_A}(s) = S(\widetilde{N_{A_1}}(s), \widetilde{N_{A_2}}(s))$, if $s \in V_1 \cap V_2$
3. $\widetilde{O_A}(s) = \widetilde{O_{A_1}}(s)$, if $s \in V_1$ and $s \notin V_2$
   $\widetilde{O_A}(s) = \widetilde{O_{A_2}}(s)$, if $s \in V_2$ and $s \notin V_1$
   and $\widetilde{O_A}(s) = T(\widetilde{O_{A_1}}(s), \widetilde{O_{A_2}}(s))$, if $s \in V_1 \cap V_2$
4. $\widetilde{P_B}(st) = \widetilde{P_{B_1}}(st)$, if $st \in E_1$ and $st \notin E_2$
   $\widetilde{P_B}(st) = \widetilde{P_{B_2}}(st)$, if $st \in E_2$ and $st \notin E_1$
   $\widetilde{P_B}(st) = S(\widetilde{P_{B_1}}(st), \widetilde{P_{B_2}}(st))$, if $st \in E_1 \cap E_2$
5. $\widetilde{Q_B}(st) = \widetilde{Q_{B_1}}(st)$, if $st \in E_1$ and $st \notin E_2$
   $\widetilde{Q_B}(st) = \widetilde{Q_{B_2}}(st)$, if $st \in E_2$ and $st \notin E_1$
   $\widetilde{Q_B}(st) = S(\widetilde{Q_{B_1}}(st), \widetilde{Q_{B_2}}(st))$, if $st \in E_1 \cap E_2$
6. $\widetilde{R_B}(st) = \widetilde{R_{B_1}}(st)$, if $st \in E_1$ and $st \notin E_2$
   $\widetilde{R_B}(st) = \widetilde{R_{B_2}}(st)$, if $st \in E_2$ and $st \notin E_1$
   $\widetilde{R_B}(st) = T(\widetilde{R_{B_1}}(st), \widetilde{P_{B_2}}(st))$, if $st \in E_1 \cap E_2$
7. $\widetilde{P_B}(st) = T(\widetilde{M_{A_1}}(st), \widetilde{M_{A_2}}(st))$
   $\widetilde{P_B}(st) = T(\widetilde{N_{A_1}}(st), \widetilde{N_{A_2}}(st))$
   $\widetilde{P_B}(st) = S(\widetilde{O_{A_1}}(st), \widetilde{O_{A_2}}(st))$, if $st \in E'$.

Remark 1 If G_1 and G_2 are two IVP-(S, T)-fuzzy graphs, then $G_1 + G_2$ is also an IVP-(S, T)-fuzzy graph.

Definition 17 Let $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ be the underlying graphs of two IVP-(S, T)-fuzzy graphs $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$, respectively. Then their ring sum $G_1 \oplus G_2$ is the ordered pair (A, B), where $A = (\widetilde{M_A}, \widetilde{N_A}, \widetilde{O_A})$ and $B = (\widetilde{P_B}, \widetilde{Q_B}, \widetilde{R_B})$ are IVPFSs on $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$, respectively with

1. $\widetilde{M_A}(s) = \widetilde{M_{A_1}}(s)$, if $s \in V_1$ and $s \notin V_2$
   $\widetilde{M_A}(s) = \widetilde{M_{A_2}}(s)$, if $s \in V_2$ and $s \notin V_1$
   and $\widetilde{M_A}(s) = S(\widetilde{M_{A_1}}(s), \widetilde{M_{A_2}}(s))$, if $s \in V_1 \cap V_2$

2. $\widetilde{N_A}(s) = \widetilde{N_{A_1}}(s)$, if $s \in V_1$ and $s \notin V_2$
   $\widetilde{N_A}(s) = \widetilde{N_{A_2}}(s)$, if $s \in V_2$ and $s \notin V_1$
   and $\widetilde{N_A}(s) = S(\widetilde{N_{A_1}}(s), \widetilde{N_{A_2}}(s))$, if $s \in V_1 \cap V_2$
3. $\widetilde{O_A}(s) = \widetilde{O_{A_1}}(s)$, if $s \in V_1$ and $s \notin V_2$
   $\widetilde{O_A}(s) = \widetilde{O_{A_2}}(s)$, if $s \in V_2$ and $s \notin V_1$
   and $\widetilde{O_A}(s) = T(\widetilde{O_{A_1}}(s), \widetilde{O_{A_2}}(s))$, if $s \in V_1 \cap V_2$
4. $\widetilde{P_B}(st) = \widetilde{P_{B_1}}(st)$, if $st \in E_1$ and $st \notin E_2$
   $\widetilde{P_B}(st) = \widetilde{P_{B_2}}(st)$, if $st \in E_2$ and $st \notin E_1$
   $\widetilde{P_B}(st) = 0$, if $st \in E_1 \cap E_2$
5. $\widetilde{Q_B}(st) = \widetilde{Q_{B_1}}(st)$, if $st \in E_1$ and $st \notin E_2$
   $\widetilde{Q_B}(st) = \widetilde{Q_{B_2}}(st)$, if $st \in E_2$ and $st \notin E_1$
   $\widetilde{Q_B}(st) = 0$, if $st \in E_1 \cap E_2$
6. $\widetilde{R_B}(st) = \widetilde{R_{B_1}}(st)$, if $st \in E_1$ and $st \notin E_2$
   $\widetilde{R_B}(st) = \widetilde{R_{B_2}}(st)$, if $st \in E_2$ and $st \notin E_1$
   $\widetilde{R_B}(st) = 0$, if $st \in E_1 \cap E_2$.

Remark 2 If G_1 and G_2 are two IVP-(S, T)-fuzzy graphs, then $G_1 \oplus G_2$ is also an IVP-(S, T)-fuzzy graph.

Definition 18 Let $G = (A, B)$ be an IVP-(S, T)-fuzzy graph, where $A = (\widetilde{M_A}, \widetilde{N_A}, \widetilde{O_A}) = ([M_A^-, M_A^+], [N_A^-, N_A^+], [O_A^-, O_A^+])$ is an IVPFS on V and $B = (\widetilde{P_B}, \widetilde{Q_B}, \widetilde{R_B}) = ([P_B^-, P_B^+], [Q_B^-, Q_B^+], [Q_B^-, Q_B^+])$ is an IVPFS on E. Then the complement of G is $\bar{G} = (\bar{A}, \bar{B})$, where $\bar{A} = A$ and for $\bar{B}$ we have

$$\bar{P}_B^-(m, n) = 0, \text{ if } P_B^- \geq 0 \text{ and } P_B^-(m, n) = \min\,(M_A^-(m), M_A^-(n)), \text{ otherwise}$$
$$\bar{P}_B^+(m, n) = 0, \text{ if } P_B^+ \geq 0 \text{ and } P_B^+(m, n) = \min\,(M_A^+(m), M_A^+(n)), \text{ otherwise}$$
$$\bar{Q}_B^-(m, n) = 0, \text{ if } Q_B^- \geq 0 \text{ and } Q_B^-(m, n) = \min\,(N_A^-(m), N_A^-(n)), \text{ otherwise}$$
$$\bar{Q}_B^+(m, n) = 0, \text{ if } Q_B^+ \geq 0 \text{ and } Q_B^+(m, n) = \min\,(N_A^+(m), N_A^+(n)), \text{ otherwise}$$
$$\bar{R}_B^-(m, n) = 0, \text{ if } R_B^- \geq 0 \text{ and } R_B^-(m, n) = \min\,(O_A^-(m), O_A^-(n)), \text{ otherwise}$$
$$\bar{R}_B^+(m, n) = 0, \text{ if } R_B^+ \geq 0 \text{ and } R_B^+(m, n) = \min\,(O_A^+(m), O_A^+(n)), \text{ otherwise}$$

for all $m, n \in V$.

Example 2 Let $G^* = (V, E)$, where $V = \{u, v, w, x\}$ and $E = \{uv, vw, uw, wx\}$. An IVP-(S, T)-fuzzy graph $G = (A, B)$ defined on G^* shown in Fig. 2 is described as

$$A = \left\langle V, \left(\frac{u}{[0.3, 0.5]}, \frac{v}{[0.1, 0.4]}, \frac{w}{[0.2, 0.3]}, \frac{x}{[0.2, 0.4]}\right), \left(\frac{u}{[0.4, 0.6]}, \frac{v}{[0.4, 0.7]}, \frac{w}{[0.2, 0.4]}, \frac{x}{[0.3, 0.5]}\right), \left(\frac{u}{[0.1, 0.3]}, \frac{v}{[0.3, 0.2]}, \frac{w}{[0.3, 0.5]}, \frac{x}{[0.2, 0.3]}\right)\right\rangle$$

$$B = \left\langle E, \left(\frac{uv}{[0.3, 0.4]}, \frac{vw}{[0.2, 0.4]}, \frac{uw}{[0.3, 0.4]}, \frac{wx}{[0.2, 0.3]}\right),\right.$$

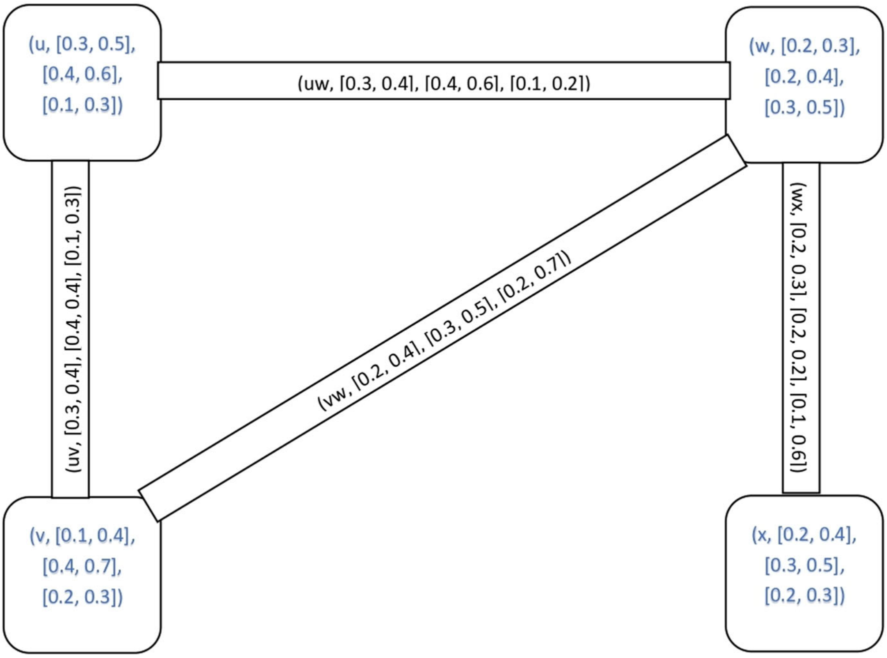

Fig. 2 IVP-(S, T)-fuzzy graph

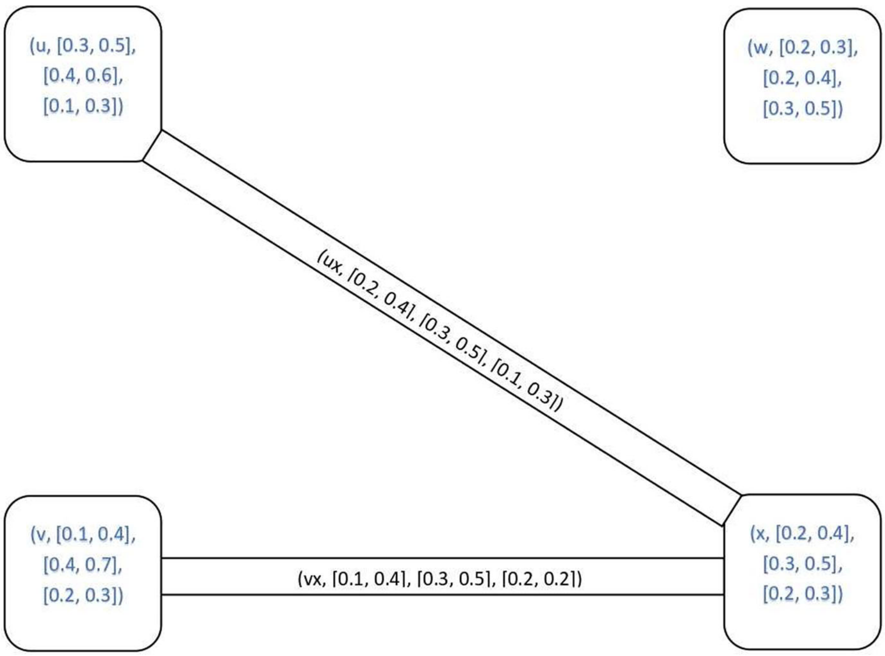

Fig. 3 Complement of IVP-(S, T)-fuzzy graph

$$\left(\frac{uv}{[0.4, 0.4]}, \frac{vw}{[0.3, 0.5]}, \frac{uw}{[0.4, 0.6]}, \frac{wx}{[0.2, 0.2]}\right),$$
$$\left.\left(\frac{uv}{[0.1, 0.3]}, \frac{vw}{[0.2, 0.7]}, \frac{uw}{[0.1, 0.2]}, \frac{wx}{[0.1, 0.6]}\right)\right\rangle$$

The complement of G denoted by $\bar{G} = (\bar{A}, \bar{B})$ is shown in Fig. 3 and is verified as

$$\begin{aligned}\bar{A} =& \left\langle V, \left(\frac{u}{[0.3, 0.5]}, \frac{v}{[0.1, 0.4]}, \frac{w}{[0.2, 0.3]}, \frac{x}{[0.2, 0.4]}\right),\right.\\ &\left(\frac{u}{[0.4, 0.6]}, \frac{v}{[0.4, 0.7]}, \frac{w}{[0.2, 0.4]}, \frac{x}{[0.3, 0.5]}\right),\\ &\left.\left(\frac{u}{[0.1, 0.3]}, \frac{v}{[0.3, 0.2]}, \frac{w}{[0.3, 0.5]}, \frac{x}{[0.2, 0.3]}\right)\right\rangle\\ \bar{B} =& \left\langle E, \left(\frac{ux}{[0.2, 0.4]}, \frac{vx}{[0.1, 0.4]}\right),\right.\\ &\left.\left(\frac{ux}{[0.3, 0.5]}, \frac{vx}{[0.3, 0.5]}\right), \left(\frac{ux}{[0.1, 0.3]}, \frac{vx}{[0.2, 0.2]}\right)\right\rangle\end{aligned}$$

Clearly, $\bar{G}$ is an IVP-(S, T)-fuzzy graph.

Theorem 1 *The complement of an IVP-(S, T)-fuzzy graph is an IVP-(S, T)-fuzzy graph.*

Proof Straightforward. □

Definition 19 Let $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ be the underlying graphs of two IVP-(S, T)-fuzzy graphs $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$, respectively. Then their cartesian product $G_1 \times G_2$ is the ordered pair (A, B), where $A = (\widetilde{M_A}, \widetilde{N_A}, \widetilde{O_A})$ and $B = (\widetilde{P_B}, \widetilde{Q_B}, \widetilde{R_B})$ are IVPFSs in $V = V_1 \times V_2$ and $E = \{(s, s_2)(s, t_2) : s \in V_1, s_2t_2 \in E_2\} \cup \{(s_1, s_2)(t_1, t_2) : s \in V_1, s_1t_1 \in E_1, s_2t_2 \in E_2\}$, respectively satisfying

1.

$$\begin{aligned}\widetilde{M_A}(s_1, s_2) &= T(\widetilde{M_A}(s_1), \widetilde{M_A}(s_2)), \widetilde{N_A}(s_1, s_2)\\ &= T(\widetilde{N_A}(s_1), \widetilde{N_A}(s_2), \widetilde{OA}(s_1, s_2)\\ &= S(\widetilde{N_A}(s_1), \widetilde{N_A}(s_2)\end{aligned}$$

for all $(s_1, s_2) \in V_1 \times V_2$

2.

$$\begin{aligned}\widetilde{P_B}((s, s_2), (s, t_2)) &= T(\widetilde{M_{A_1}}(s), \widetilde{P_{B_2}}(s_2t_2))\\ \widetilde{Q_B}((s, s_2), (s, t_2)) &= T(\widetilde{M_{A_1}}(s), \widetilde{Q_{B_2}}(s_2t_2))\end{aligned}$$

and

$$\widetilde{R_B}((s, s_2), (s, t_2)) = S(\widetilde{M_{A_1}}(s), \widetilde{R_{B_2}}(s_2t_2))$$

where $s \in V_1, s_2t_2 \in E_2$

3.

$$\widetilde{P_B}((s_1, u), (t_1, u)) = T(\widetilde{P_{B_1}}(s_1t_1), \widetilde{P_{A_2}}(u))$$
$$\widetilde{Q_B}((s_1, u), (t_1, u)) = T(\widetilde{Q_{B_1}}(s_1t_1), \widetilde{Q_{A_2}}(u))$$

and

$$\widetilde{R_B}((s_1, u), (t_1, u)) = S(\widetilde{R_{B_1}}(s_1t_1), \widetilde{R_{A_2}}(u))$$

where $u \in V_2, s_1t_1 \in E_1$.

Remark 3 If G_1 and G_2 are two IVP-(S, T)-fuzzy graphs, then $G_1 \times G_2$ is also an IVP-(S, T)-fuzzy graph.

Definition 20 Let $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ be the underlying graphs of two IVP-(S, T)-fuzzy graphs $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$, respectively. Then their direct product $G_1 * G_2$ is the ordered pair (A, B), where $A = (\widetilde{M_A}, \widetilde{N_A}, \widetilde{O_A})$ and $B = (\widetilde{P_B}, \widetilde{Q_B}, \widetilde{R_B})$ are IVPFSs on $V = V_1 \cup V_2$ and $E = \{(s_1, s_2)(t_1, t_2) : s_1, t_1 \in E_1, s_2t_2 \in E_2\}$, respectively satisfying

1. $\widetilde{M_A}(s_1, s_2) = T(\widetilde{M_A}(s_1), \widetilde{M_A}(s_2))$
   $\widetilde{N_A}(s_1, s_2) = T(\widetilde{N_A}(s_1), \widetilde{N_A}(s_2))$
   $\widetilde{O_A}(s_1, s_2) = S(\widetilde{O_A}(s_1), \widetilde{O_A}(s_2))$
   where $(s_1, s_2) \in V_1 \times V_2$
2. $\widetilde{P_B}((s_1, s_2), (t_1, t_2)) = T(\widetilde{P_{B_1}}(s_1t_1), \widetilde{P_{B_2}}(s_2t_2))$
   $\widetilde{Q_B}((s_1, s_2), (t_1, t_2)) = T(\widetilde{Q_{B_1}}(s_1t_1), \widetilde{Q_{B_2}}(s_2t_2))$
   $\widetilde{R_B}((s_1, s_2), (t_1, t_2)) = S(\widetilde{R_{B_1}}(s_1t_1), \widetilde{R_{B_2}}(s_2t_2))$
   where $s_1t_1 \in E_1$ and $s_2t_2 \in E_2$.

Proposition 2 *Let $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ be the underlying graphs of two IVP-(S, T)-fuzzy graphs $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$, respectively., then $G_1 * G_2$ is also an IVP-(S, T)-fuzzy graph.*

Proof Let $s_1t_1 \in E_1$ and $s_2t_2 \in E_2$. Then, we have

$$\begin{aligned}
(\widetilde{P_{B_1}} * \widetilde{P_{B_2}})((s_1, s_2), (t_1, t_2)) &= T(\widetilde{P_{B_1}}(s_1t_1), \widetilde{P_{B_2}}(s_2t_2)) \\
&\leq T(T(\widetilde{M_{A_1}}(s_1), \widetilde{M_{A_1}}(t_1)), T(\widetilde{M_{A_2}}(s_2), \widetilde{M_{A_2}}(t_2)) \\
&= T(T(\widetilde{M_{A_1}}(s_1), \widetilde{M_{A_2}}(s_2)), T(\widetilde{M_{A_1}}(t_1), \widetilde{M_{A_2}}(t_2)) \\
&= T((\widetilde{M_{A_1}} \times \widetilde{M_{A_2}})(s_1, s_2), (\widetilde{M_{A_1}} \times \widetilde{M_{A_2}})(t_1, t_2)) \\
(\widetilde{Q_{B_1}} * \widetilde{Q_{B_2}})((s_1, s_2), (t_1, t_2)) &= T(\widetilde{Q_{B_1}}(s_1t_1), \widetilde{Q_{B_2}}(s_2t_2))
\end{aligned}$$

$$
\begin{aligned}
&\le T(T(\widetilde{N_{A_1}}(s_1), \widetilde{N_{A_1}}(t_1)), T(\widetilde{N_{A_2}}(s_2), \widetilde{N_{A_2}}(t_2))\\
&= T(T(\widetilde{N_{A_1}}(s_1), \widetilde{N_{A_2}}(s_2)), T(\widetilde{N_{A_1}}(t_1), \widetilde{N_{A_2}}(t_2))\\
&= T((\widetilde{N_{A_1}} \times \widetilde{N_{A_2}})(s_1, s_2), (\widetilde{N_{A_1}} \times \widetilde{N_{A_2}})(t_1, t_2))\\
(\widetilde{R_{B_1}} * \widetilde{R_{B_2}})((s_1, s_2), (t_1, t_2)) &= S(\widetilde{R_{B_1}}(s_1t_1), \widetilde{R_{B_2}}(s_2t_2))\\
&\ge S(S(\widetilde{O_{A_1}}(s_1), \widetilde{O_{A_1}}(t_1)), S(\widetilde{O_{A_2}}(s_2), \widetilde{O_{A_2}}(t_2))\\
&= S(S(\widetilde{O_{A_1}}(s_1), \widetilde{O_{A_2}}(s_2)), S(\widetilde{O_{A_1}}(t_1), \widetilde{O_{A_2}}(t_2))\\
&= S((\widetilde{O_{A_1}} \times \widetilde{O_{A_2}})(s_1, s_2), (\widetilde{O_{A_1}} \times \widetilde{O_{A_2}})(t_1, t_2)).
\end{aligned}
$$

This completes the proof. □

Definition 21 Let $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ be the underlying graphs of two IVP-(S, T)-fuzzy graphs $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$, respectively. Then their lexicographic product $G_1 \cdot G_2$ is the ordered pair (A, B), where $A = (\widetilde{M_A}, \widetilde{N_A}, \widetilde{O_A})$ and $B = (\widetilde{P_B}, \widetilde{Q_B}, \widetilde{R_B})$ are IVPFs on $V = V_1 \times V_2$ and $E = \{(s_1, s_2)(t_1, t_2) : s \in V_1, s_2t_2 \in E_2\} \cup \{(s_1, s_2)(t_1, t_2) : s_1, t_1 \in E_1, s_2t_2 \in E_2\}$, respectively satisfying

1. $\widetilde{M_A}(s_1, s_2) = T(\widetilde{M_{A_1}}(s_1), \widetilde{M_{A_2}}(s_2))$
   $\widetilde{N_A}(s_1, s_2) = T(\widetilde{N_{A_1}}(s_1), \widetilde{N_{A_2}}(s_2))$
   $\widetilde{O_A}(s_1, s_2) = S(\widetilde{O_{A_1}}(s_1), \widetilde{O_{A_2}}(s_2))$
   where $(s_1, s_2) \in V_1 \times V_2$
2. $\widetilde{P_B}((s, s_2), (s, t_2)) = T(\widetilde{M_{A_1}}(s_1), \widetilde{P_{B_2}}(s_2t_2))$
   $\widetilde{Q_B}((s, s_2), (s, t_2)) = T(\widetilde{N_{A_1}}(s_1), \widetilde{Q_{B_2}}(s_2t_2))$
   $\widetilde{R_B}((s, s_2), (s, t_2)) = S(\widetilde{R_{A_1}}(s_1), \widetilde{R_{B_2}}(s_2t_2))$
   where $s \in V_1, s_2t_2 \in E_2$
3. $\widetilde{P_B}((s_1, s_2), (t_1, t_2)) = T(\widetilde{P_{B_1}}(s_1t_1), \widetilde{P_{B_2}}(s_2t_2))$
   $\widetilde{Q_B}((s_1, s_2), (t_1, t_2)) = T(\widetilde{Q_{B_1}}(s_1t_1), \widetilde{Q_{B_2}}(s_2t_2))$
   $\widetilde{R_B}((s_1, s_2), (t_1, t_2)) = S(\widetilde{R_{B_1}}(s_1t_1), \widetilde{R_{B_2}}(s_2t_2))$
   where $s_1t_1 \in E_1, s_2t_2 \in E_2$.

Proposition 3 *Let $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ be the underlying graphs of two IVP-(S, T)-fuzzy graphs $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$, respectively. Then $G_1 \cdot G_2$ is also an IVP-(S, T)-fuzzy graph.*

Proof Let $s \in V_1, s_2t_2 \in E_2$. Then, we have

$$
\begin{aligned}
(\widetilde{P_{B_1}} \cdot \widetilde{P_{B_2}})((s, s_2), (s, t_2)) &= T(\widetilde{M_{A_1}}(s), \widetilde{P_{B_2}}(s_2t_2))\\
&\le T(\widetilde{M_{A_1}}(s), T(T(\widetilde{M_{A_2}}(s_2), T(\widetilde{M_{A_2}}(t_2))\\
&= T(T(\widetilde{M_{A_1}}(s), \widetilde{M_{A_2}}(s_2)), T(\widetilde{M_{A_1}}(s), \widetilde{M_{A_2}}(t_2)))\\
&= T((\widetilde{M_{A_1}}, \widetilde{M_{A_2}})(s, s_2), (\widetilde{M_{A_1}}, \widetilde{M_{A_2}})(s, t_2))\\
(\widetilde{Q_{B_1}} \cdot \widetilde{Q_{B_2}})((s, s_2), (s, t_2)) &= T(\widetilde{N_{A_1}}(s), \widetilde{Q_{B_2}}(s_2t_2))\\
&\le T(\widetilde{N_{A_1}}(s), T(T(\widetilde{N_{A_2}}(s_2), T(\widetilde{N_{A_2}}(t_2))
\end{aligned}
$$

$$
\begin{aligned}
&= T(T(\widetilde{N_{A_1}}(s), \widetilde{N_{A_2}}(s_2)), T(\widetilde{N_{A_1}}(s), \widetilde{N_{A_2}}(t_2)))\\
&= T((\widetilde{N_{A_1}}, \widetilde{N_{A_2}})(s, s_2), (\widetilde{N_{A_1}}, \widetilde{N_{A_2}})(s, t_2))\\
(\widetilde{R_{B_1}} \cdot \widetilde{R_{B_2}})((s, s_2), (s, t_2)) &= S(\widetilde{O_{A_1}}(s), \widetilde{R_{B_2}}(s_2t_2))\\
&\geq S(\widetilde{O_{A_1}}(s), S(S(\widetilde{O_{A_2}}(s_2), S(\widetilde{O_{A_2}}(t_2))\\
&= S(S(\widetilde{O_{A_1}}(s), \widetilde{O_{A_2}}(s_2)), S(\widetilde{O_{A_1}}(s), \widetilde{O_{A_2}}(t_2)))\\
&= S((\widetilde{O_{A_1}}, \widetilde{O_{A_2}})(s, s_2), (\widetilde{O_{A_1}}, \widetilde{O_{A_2}})(s, t_2)).
\end{aligned}
$$

If $s_1t_1 \in E_1, s_2t_2 \in E_2$, then we have

$$
\begin{aligned}
(\widetilde{P_{B_1}} \cdot \widetilde{P_{B_2}})((s_1, s_2), (t_1, t_2)) &= T(\widetilde{P_{B_1}}(s_1t_1), \widetilde{P_{B_2}}(s_2t_2))\\
&\leq T(T(\widetilde{M_{A_1}}(s_1), \widetilde{M_{A_1}}(t_1)), T(\widetilde{M_{A_2}}(s_2), \widetilde{M_{A_2}}(t_2)))\\
&= T(T(\widetilde{M_{A_1}}(s_1), \widetilde{M_{A_1}}(s_2)), T(\widetilde{M_{A_2}}(s_2), \widetilde{M_{A_2}}(t_2)))\\
&= T((\widetilde{M_{A_1}}, \widetilde{M_{A_2}})(s_1, s_2), (\widetilde{M_{A_2}}, \widetilde{M_{A_2}})(t_1, t_2))\\
(\widetilde{Q_{B_1}} \cdot \widetilde{Q_{B_2}})((s_1, s_2), (t_1, t_2)) &= T(\widetilde{Q_{B_1}}(s_1t_1), \widetilde{Q_{B_2}}(s_2t_2))\\
&\leq T(T(\widetilde{N_{A_1}}(s_1), \widetilde{N_{A_1}}(t_1)), T(\widetilde{N_{A_2}}(s_2), \widetilde{N_{A_2}}(t_2)))\\
&= T(T(\widetilde{N_{A_1}}(s_1), \widetilde{N_{A_1}}(s_2)), T(\widetilde{N_{A_2}}(s_2), \widetilde{N_{A_2}}(t_2)))\\
&= T((\widetilde{N_{A_1}}, \widetilde{N_{A_2}})(s_1, s_2), (\widetilde{N_{A_2}}, \widetilde{N_{A_2}})(t_1, t_2))\\
(\widetilde{R_{B_1}} \cdot \widetilde{R_{B_2}})((s_1, s_2), (t_1, t_2)) &= S(\widetilde{R_{B_1}}(s_1t_1), \widetilde{R_{B_2}}(s_2t_2))\\
&\geq S(S(\widetilde{O_{A_1}}(s_1), \widetilde{O_{A_1}}(t_1)), S(\widetilde{O_{A_2}}(s_2), \widetilde{O_{A_2}}(t_2)))\\
&= S(S(\widetilde{O_{A_1}}(s_1), \widetilde{O_{A_1}}(s_2)), S(\widetilde{O_{A_2}}(s_2), \widetilde{O_{A_2}}(t_2)))\\
&= S((\widetilde{O_{A_1}}, \widetilde{O_{A_2}})(s_1, s_2), (\widetilde{O_{A_2}}, \widetilde{O_{A_2}})(t_1, t_2)).
\end{aligned}
$$

□

Definition 22 Let $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ be the underlying graphs of two IVP-(S, T)-fuzzy graphs $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$, respectively. Then

1. a homomorphism f is a mapping $f : V_1 \to V_2$ with
   (a) $\widetilde{M_{A_1}}(s_1) \leq \widetilde{M_{A_2}}(f(s_1)), \widetilde{N_{A_1}}(s_1) \leq \widetilde{N_{A_2}}(f(s_1)), \widetilde{O_{A_1}}(s_1) \geq \widetilde{O_{A_2}}(f(s_1))$, for all $s_1 \in V_1$
   (b) $\widetilde{P_{B_1}}(s_1s_2) \leq \widetilde{P_{B_2}}(f(s_1))(f(s_2)), \widetilde{Q_{B_1}}(s_1s_2) \leq \widetilde{Q_{B_2}}(f(s_1))(f(s_2))$,
   $\widetilde{R_{B_1}}(s_1s_2) \geq \widetilde{R_{B_2}}(f(s_1))(f(s_2))$, for all $s_1s_2 \in E_1$
2. a bijective homomorphism $f : G_1 \to G_2$ is called a weak isomorphism, if
   $\widetilde{M_{A_1}}(s_1) = \widetilde{M_{A_2}}(f(s_1)), \widetilde{N_{A_1}}(s_1) = \widetilde{N_{A_2}}(f(s_1)), \widetilde{O_{A_1}}(s_1) = \widetilde{O_{A_2}}(f(s_1))$, for all $s_1 \in V_1$
3. co-weak isomorphism is a bijective homomorphism $f : G_1 \to G_2$, if
   $\widetilde{P_{B_1}}(s_1s_2) = \widetilde{P_{B_2}}(f(s_1))(f(s_2)), \widetilde{Q_{B_1}}(s_1s_2) = \widetilde{Q_{B_2}}(f(s_1))(f(s_2))$,
   $\widetilde{R_{B_1}}(s_1s_2) = \widetilde{R_{B_2}}(f(s_1))(f(s_2))$, for all $s_1s_2 \in E_1$.

A bijective homomorphism $g : G_1 \to G_2$ is called an isomorphism, if it satisfies conditions (2&3).

3.1 Regular IVP-(S, T)-fuzzy graphs

Definition 23 The degree of vertex s of any IVP-(S,T)-fuzzy graph is described as $deg(s) = (deg_M(s), deg_N(s), deg_O(s))$, where $deg_M(s) = \sum_{s\neq t} \widetilde{P_B(st)}$, $deg_N(s) = \sum_{s\neq t} \widetilde{Q_B(st)}$ and $deg_O(s) = \sum_{s\neq t} \widetilde{R_B(st)}$. If $deg_M(v_i) = k_i$, $deg_N(v_j) = k_j$ and $deg_O(v_l) = k_l$, for all v_i, v_j, v_l. Then such a graph is called regular IVP-(S,T)-fuzzy graph of degree (k_i, k_j, k_l).

Definition 24 The closed neighborhood degree of vertex s of any IVP-(S,T)-fuzzy graph $G = (A, B)$ is defined by $deg[s] = (d_M[s], d_N[s], d_O[s])$, where

$$d_M[s] = deg_M(s) + \widetilde{M_A(s)}$$

$$d_N[s] = deg_N(s) + \widetilde{N_A(s)}$$

and

$$d_O[s] = deg_O(s) + \widetilde{O_A(s)}.$$

If in G, every vertex has same closed neighborhood degree i.e., $m = (m_1^*, m_2^*, m_3^*)$, then such graph is a totally regular IVP-(S,T)-fuzzy graph with degree m.

Proposition 4 *Let G_1 and G_2 be two IVP-(S,T)-fuzzy graphs. If G_1 is isomorphic to G_2 and G_1 is regular(totally regular) IVP-(S,T)-fuzzy graph, then G_2 is also regular(totally regular).*

Proof For both the cases, let G_1 is isomorphic G_2. For the first case, let G_1 is an $n = (n_1, n_2, n_3)$-regular IVP-(S,T)-fuzzy graph. Since

$$\begin{aligned} deg[s] &= (deg_M(s), deg_N(s), deg_O(s)) \\ &= \left(\sum_{s\neq y} \widetilde{P_B(sy)}, \sum_{s\neq y} \widetilde{Q_B(sy)}, \sum_{s\neq y} \widetilde{R_B(sy)} \right) \\ &= (n_1, n_2, n_3) \end{aligned}$$

we have

$$\begin{aligned} n_1 = deg_M(s) &= \widetilde{P_B(sy)} = P_B(\widetilde{g(s)g}(y)) = deg_M g(s) \\ n_2 = deg_N(s) &= \widetilde{Q_B(sy)} = Q_B(\widetilde{g(s)g}(y)) = deg_N g(s) \\ n_3 = deg_O(s) &= \widetilde{R_B(sy)} = R_B(\widetilde{g(s)g}(y)) = deg_O g(s). \end{aligned}$$

Hence, G_2 is an n-regular IVP-(S, T)-fuzzy graph.
Now for second case let G_1 is an $m = (m_1, m_2, m_3)$ totally regular IVP-(S, T)-fuzzy graph. We defined earlier that $deg[s] = (d_M[s], d_N[s], d_O[s])$ where $d_M[s] = deg_M(s) + \widetilde{M_A(s)}$, $d_N[s] = deg_N(s) + \widetilde{N_A(s)}$ and $d_O[s] = deg_O(s) + \widetilde{O_A(s)}$.
Therefore,

$$m_1 = deg_M(s) + \widetilde{M_A(s)} = deg_M g(s) + \widetilde{M_A(g(s))} = d_M[g(s)]$$
$$m_2 = deg_N(s) + \widetilde{N_A(s)} = deg_N g(s) + \widetilde{N_A(g(s))} = d_N[g(s)]$$
$$m_3 = deg_O(s) + \widetilde{O_A(s)} = deg_O g(s) + \widetilde{O_A(g(s))} = d_O[g(s)].$$

It implies that G_2 is an m-totally regular IVP-(S, T)-fuzzy graph. □

Definition 25 For any IVP-(S, T)-fuzzy graph $G = (A, B)$

1. Order of G is $O(G) = (O_M(G), O_N(G), O_O(G))$, where

$$O_M(G) = \sum_{v \in V} M_A(v)$$
$$O_N(G) = \sum_{v \in V} N_A(v)$$

and

$$O_O(G) = \sum_{v \in V} O_A(v).$$

2. Size of G is $S(G) = (S_M(G), S_N(G), S_O(G))$, where

$$S_M(G) = \sum_{u \neq v} P_B(uv)$$
$$S_N(G) = \sum_{u \neq v} Q_B(uv)$$

and

$$S_O(G) = \sum_{u \neq v} R_B(uv).$$

Definition 26 An IVP-(S, T)-fuzzy graph $G = (A, B)$ on $G^* = (V, E)$ is called constant IVP-(S, T)-fuzzy graph of degree (k_i, k_j, k_k), if for all $v_i, v_j, v_k \in V$, we have $d_M(v_i) = k_i$, $d_N(v_j) = k_j$ and $d_O(v_k) = k_k$.

Definition 27 The total degree of a vertex $v \in V$ for any IVP-(S, T)- fuzzy graph $G = (A, B)$ on $G^* = (V, E)$ is

$$td(v) = [d_M(v) + M_A(v), d_N(v) + N_A(v), d_O(v) + O_A(v)]$$

.

Remark 4 We say that G is an IVP-(S, T)-fuzzy graph of total degree (r_1, r_2, r_3), if total degree of all vertices is same i.e., (r_1, r_2, r_3).

Theorem 2 *Let $G^* = (V, E)$ be the underlying graph of an IVP-(S, T)-fuzzy graph $G = (A, B)$ and $A = (\widetilde{M_A}, \widetilde{N_A}, \widetilde{O_A})$ be any constant function. Then the below statements are equivalent.*

1. *G is a regular IVP-(S, T)-fuzzy graph*

2. *G is totally regular IVP-(S, T)-fuzzy graph.*

Proof Let $A = (\widetilde{M_A}, \widetilde{N_A}, \widetilde{O_A})$ be any constant function, where $\widetilde{M_A} = c_1$, $\widetilde{N_A} = c_2$ and $\widetilde{O_A} = c_3$, for all $s \in V$.
$(1) \implies (2)$: Assume that G is an n-regular IVP-(S, T)-fuzzy graph, then $(deg_M(s) = n_m$, $(deg_N(s) = n_n$ and $(deg_O(s) = n_o$, for all $s \in V$.
So,

$$d_M[s] = deg_M(s) + \widetilde{M_A(s)} = n_m + c_1,$$
$$d_N[s] = deg_N(s) + \widetilde{N_A(s)} = n_n + c_2$$

and

$$d_O[s] = deg_O(s) + \widetilde{O_A(s)} = n_o + c_3$$

for all $s \in V$. Thus G is totally regular IVP-(S, T)-fuzzy graph.
$(2) \implies (1)$: Let G is a totally regular IVP-(S, T)-fuzzy graph, then $d_M[s] = k_1$, $d_N[s] = k_2$, $d_O[s] = k_3$ or $dec_M(s) + \widetilde{M_A(s)} = k_1$, $deg_N(s) + \widetilde{N_A(s)} = k_2$, $deg_O(s) + \widetilde{O_A(s)} = k_3$ or $deg_M(s) + c_1 = k_1, deg_N(s) + c_2 = k_2, deg_O(s) + c_3 = k_3$, for all $s \in V$, or $deg_M(s) = k_1 - c_1$, $deg_N(s) = k_2 - c_2$, $deg_O(s) = k_3 - c_3$, for all $s \in V$. Thus, G is a regular IVP-(S, T)-fuzzy graph. □

Proposition 5 *If an IVP-(S, T)-fuzzy graph G is both the regular and totally regular, then $A = (\widetilde{M_A}, \widetilde{N_A}, \widetilde{O_A})$ is a constant function.*

Proof Let G be a regular and totally regular IVP-(S, T)-fuzzy graph. Then

$$deg_M(s) = n_1, deg_N(s) = n_2 \text{ and } deg_O(s) = n_3, \text{ for all } s \in V_1,$$
$$d_M[s] = k_1, d_N[s] = k_2 \text{ and } d_O[s] = k_3 \text{ for all } s \in V_1.$$

It follows that

$$\begin{aligned} d_M[s] = k_1 &\Leftrightarrow deg_M(s) + \widetilde{M_A(s)} = k_1 \\ &\Leftrightarrow n_1 + \widetilde{M_A(s)} = k_1 \\ &\Leftrightarrow \widetilde{M_A(s)} = k_1 - n_1, \end{aligned}$$

for all $s \in V$. Similarly, $\widetilde{N_A(s)} = k_2 - n_2$ and $\widetilde{O_A(s)} = k_3 - n_3$, for all $s \in V$. Hence, $A = (\widetilde{M_A}, \widetilde{N_A}, \widetilde{O_A})$ is a constant function. □

Remark 5 1. if n is odd, G is regular iff B is a constant function.
2. if n is even, G is regular iff $\widetilde{P_B}(v_{i-1}v_i) = \widetilde{P_B}(v_{i+1}v_{i+2})$, $\widetilde{Q_B}(v_{i-1}v_i) = \widetilde{Q_B}(v_{i+1}v_{i+2})$ and $\widetilde{R_B}(v_{i-1}v_i) = \widetilde{R_B}(v_{i+1}v_{i+2})$, $1 \le i \le n$, where $i+1$ and $i+2$ are in module n.

4 MADM using IVP-(S, T)-fuzzy graphs

Among the other generalizations of fuzzy sets, IVPFSs is the best and become an efficient tool to deal with uncertainty. In this section, we deal MADM problem through interval-valued picture (S, T)-fuzzy graph. For illustration we introduce an algorithm to solve MADM problem in the interval-valued picture fuzzy environment.
Let $A = \{A_1, A_2, ..., A_m\}$ be the set of alternatives and $C = \{C_1, C_2, ..., C_n\}$ be the set of attributes. $w = \{w_1, w_2, ..., w_n\}$ be a weighted vector of attribute C_i, $i = 1, 2, ..., n$, where $w_i \ge 0$ for $i = 1, 2, ..., n$ and $\sum_{i=1}^{n} w_i = 1$.

Let $M = (b_{kj})_{m\times n} = (\rho_{p_{kj}, q_{kj}, r_{kj}})_{m\times n}$ be an interval-valued picture fuzzy decision matrix, where p_{kj}, q_{kj} and r_{kj} denote positive membership, negative membership and neutral membership values of alternatives A_j to attributes C_j given by the decision maker. Here positive membership value represents the extent to which any alternative A_j satisfy attribute C_j given by decision maker, negative membership value represents the extent to which any alternative A_j does not fulfill attribute C_j given by decision maker and neutral membership value represents the extent to which any alternative A_j does not satisfy attribute C_j, given by decision maker, where $p_{kj}, q_{kj}, r_{kj} \in [0, 1]$ and $0 \le p_{kj} + q_{kj} + r_{kj} \le 1$.

The interval-valued picture (S, T)-fuzzy relation between two attributes $C_i = (\widetilde{M_{A_i}}, \widetilde{N_{A_i}}, \widetilde{O_{A_i}}) = ([p_i^l, p_i^u], [q_i^l, q_i^u], [r_i^l, r_i^u])$ and $C_j = (\widetilde{M_{A_j}}, \widetilde{N_{A_j}}, \widetilde{O_{A_j}}) = ([p_j^l, p_j^u], [q_j^l, q_j^u], [r_j^l, r_j^u])$ is defined as $\psi_{ij} = (\widetilde{P}_{B_{ij}}, \widetilde{Q}_{B_{ij}}, \widetilde{R}_{B_{ij}})$, where

$$\begin{aligned}
\widetilde{P}_{B_{ij}} &\le T\{\widetilde{M_{A_i}}, \widetilde{M_{A_j}}\}\\
\widetilde{Q}_{B_{ij}} &\le T\{\widetilde{N_{A_i}}, \widetilde{N_{A_j}}\}\\
\widetilde{R}_{B_{ij}} &\ge S\{\widetilde{O_{A_i}}, \widetilde{O_{A_j}}\}.
\end{aligned}$$

For all $i, j = 1, 2, ..., m$. Otherwise $\psi_{ij} = ([0, 0], [0, 0], [1, 1])$.
Now we provide an algorithm to solve a MADM problem using IVP-(S, T)-fuzzy graphs. It consist of following steps.

1. Determine the impact coefficient between attributes C_i and C_j using

$$\lambda_{ij} = \frac{(p_{ij}^l + p_{ij}^u) + 2 - (q_{ij}^l + q_{ij}^u) + 2 - (r_{ij}^l + r_{ij}^u)}{6}$$

where $\psi_{ij} = ([p_{ij}^l, p_{ij}^u], [q_{ij}^l, q_{ij}^u], [r_{ij}^l, r_{ij}^u])$ is an interval-valued picture fuzzy edge between vertices C_i and C_j, for $i, j = 1, 2, ..., m$. We have $\psi_{ii} = 1$ and $\psi_{ij} = \psi_{ji}$ for $i = j$.

2. Find the attribute of the alternative A_k by

$$A_k = 1/3(\tilde{p_k}, \tilde{q_k}, \tilde{r_k}) = 1/3(\Sigma_{j=1}^n w_j(\Sigma_{i=1}^n \lambda_{ij} b_{ki}))$$

where $b_{ki} = ([p_{ki}^l, p_{ki}^u], [q_{ki}^l, q_{ki}^u], [r_{ki}^l, r_{ki}^u])$ is an interval-valued picture fuzzy number.

3. Find the score function of the alternative $A_k = [p_k^l, p_k^u], [q_k^l, q_k^u], [r_k^l, r_k^u]$ by

$$scor(A_k) = \frac{p_k^l - q_k^l - r_k^l + p_k^u - q_k^u - r_k^u}{3}$$

4. Rank all the alternatives A_k depending upon the values of their score function and select the best alternative.
5. End

4.1 Numerical example

A famous university has three campuses (i.e., E_1, E_2, E_3) in its major cities. At the end of every year the university has to select a "student of year" based on the overall performance of the student throughout the year. For this, each campus first shortlist the name of their best student. Then, from these three students the group of judges (decision makers) have to decide the best one. All the selected students are highly competent. Firstly, the overall performance of the student is expressed by the judges in term of an interval-valued picture fuzzy number $(\widetilde{M_A}, \widetilde{N_A}, \widetilde{O_A}) = ([p^l, p^u], [q^l, q^u], [r^l, r^u])$, where $\widetilde{M_A} = [p^l, p^u]$ represents the positive attitude of student, $\widetilde{N_A} = [q^l, q^u]$ shows negative attitude of the student and $\widetilde{O_A} = [r^l, r^u]$ is for the neutral attitude of student in campus.

As the judges (decision makers) have to select the best student for the "student of year" award, so three measurable alternatives are given below.

1. Campus one student A_1
2. Campus two student A_2
3. Campus three student A_3

The panel of judges(decision makers) makes decision on the basis of the following three attributes:

1. Academic record of the student C_1
2. Behavior of the student in campus C_2
3. Performance in extra-curricular activities C_3

The weight vector of the criteria is given by $w = (w_1, w_2, w_3)^T = (0.20, 0.25, 0.55)^T$. On the basis of three attributes the overall performance of alternatives is measured by judges and they give their results in the form of interval-valued picture fuzzy information. The interval-valued picture fuzzy decision matrix M is shown in Fig. 4 below

$$\begin{bmatrix} \langle [0.2,0.4],[0.2,0.5],[0.2,0.7] \rangle & \langle [0.1,0.4],[0.3,0.7],[0.3,0.8] \rangle & \langle [0.2,0.3],[0.1,0.7],[0.2,0.6] \rangle \\ \langle [0.1,0.4],[0.3,0.6],[0.3,0.7] \rangle & \langle [0.1,0.5],[0.2,0.5],[0.2,0.4] \rangle & \langle [0.3,0.5],[0.2,0.7],[0.1,0.5] \rangle \\ \langle [0.3,0.4],[0.1,0.3],[0.2,0.5] \rangle & \langle [0.2,0.5],[0.1,0.6],[0.3,0.4] \rangle & \langle [0.1,0.4],[0.5,0.7],[0.2,0.4] \rangle \end{bmatrix}$$

Fig. 4 Interval-valued picture fuzzy decision matrix

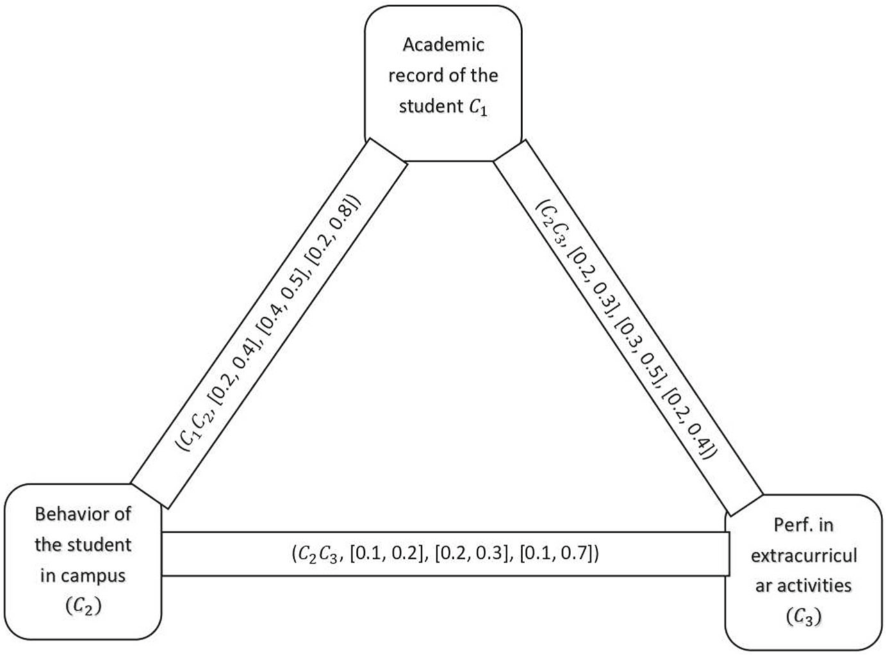

Fig. 5 Graph relationship between criteria

The relationship among the said attributes can be described by using a complete graph $G = (V, E)$, where V is the set of edges representing the attributes and E is the set of edges representing the relationships among the attributes.
$E = (\psi_{12}, \psi_{23}, \psi_{13})$, where $\psi_{12} = ([0.2, 0.4], [0.4, 0.5], [0.2, 0.8])$, $\psi_{23} = ([0.1, 0.2], [0.2, 0.3], [0.1, 0.7])$ and $\psi_{13} = ([0.2, 0.3], [0.3, 0.5], [0.2, 0.4])$.

The corresponding IVP-(S, T)-fuzzy graph is shown in Fig. 5.

We use the algorithm defined above to find out the best alternative that is the winner of the year award.
The impact coefficient between the attributes is given below.

$$\lambda_{12} = \frac{(0.2 + 0.4) + 2 - (0.4 + 0.5) + 2 - (0.2 + 0.8)}{6} = 0.45$$
$$\lambda_{23} = \frac{(0.1 + 0.2) + 2 - (0.2 + 0.3) + 2 - (0.1 + 0.7)}{6} = 0.5$$
$$\lambda_{13} = \frac{(0.2 + 0.3) + 2 - (0.3 + 0.5) + 2 - (0.2 + 0.4)}{6} = 0.516$$

Now we find overall criterion of alternatives given by $A_k = 1/3(\Sigma_{j=1}^{3} w_j(\Sigma_{i=1}^{3}\lambda_{ij}b_{ki}))$. So, for $k = 1$, we have

$$
\begin{aligned}
A_1 &= 1/3(\Sigma_{j=1}^{n} w_j(\Sigma_{i=1}^{n}\lambda_{ij}b_{1i}))\\
&= 1/3(w_1 \times (\lambda_{11}b_{11} + \lambda_{21}b_{12} + \lambda_{31}b_{13}) + w_2 \times (\lambda_{12}b_{11} + \lambda_{22}b_{12} + \lambda_{32}b_{13})\\
&\quad + w_3 \times (\lambda_{13}b_{11} + \lambda_{23}b_{12} + \lambda_{33}b_{13}))\\
&= 1/3(0.20 \times ([0.2, 0.4], [0.2, 0.5], [0.2, 0.7] + (0.45 \times [0.1, 0.4], [0.3, 0.7],\\
&[0.3, 0.8]) + (0.516 \times [0.2, 0.3], [0.1, 0.7], [0.2, 0.6])) + 0.25 \times ((0.45\\
&\quad \times [0.2, 0.4], [0.2, 0.5], [0.2, 0.7] + [0.1, 0.4], [0.3, 0.7], [0.3, 0.8] + (0.5\\
&\quad \times [0.2, 0.3], [0.1, 0.7], [0.2, 0.6]))) + 0.55 \times ((0.516 \times [0.2, 0.4], [0.2, 0.5],\\
&[0.2, 0.7]) + (0.5 \times [0.1, 0.4], [0.3, 0.7], [0.3, 0.8]) + [0.2, 0.3], [0.1, 0.7],\\
&[0.2, 0.6]))\\
&= 1/3(0.20 \times ([0.3482, 0.7348], [0.3866, 1.1762], [0.4382, 1.3696]) + 0.25\\
&\quad \times ([0.29, 0.73], [0.44, 1.275], [0.49, 1.415]) + 0.55 \times ([0.3532, 0.7064],\\
&[0.352, 1.308], [0.4532, 1.3612]))\\
&= 1/3([0.06964, 0.14696], [0.07732, 0.23524], [0.08764, 0.27392]\\
&\quad + [0.0725, 0.1825], [0.11, 0.31875], [0.1225, 0.35375] + [0.19426, 0.38852],\\
&[0.1936, 0.7194], [0.24926, 0.74866])\\
&= 1/3([0.3364, 0.71798], [0.38092, 1.27339], [0.4594, 1.37633])\\
A_1 &= [0.1121, 0.2393], [0.1269, 0.4244], [0.1531, 0.4587].
\end{aligned}
$$

For $k = 2$

$$
\begin{aligned}
A_2 &= 1/3(\Sigma_{j=1}^{n} w_j(\Sigma_{i=1}^{n}\lambda_{ij}b_{2i}))\\
&= 1/3(w_1 \times (\lambda_{11}b_{21} + \lambda_{21}b_{22} + \lambda_{31}b_{23}) + w_2 \times (\lambda_{12}b_{21} + \lambda_{22}b_{22} + \lambda_{32}b_{23})\\
&\quad + w_3 \times (\lambda_{13}b_{21} + \lambda_{23}b_{22} + \lambda_{33}b_{23}))\\
&= 1/3(0.20 \times ([0.1, 0.4], [0.3, 0.6], [0.3, 0.7] + (0.45 \times [0.1, 0.5], [0.2, 0.5],\\
&[0.2, 0.4]) + (0.516 \times [0.3, 0.5], [0.2, 0.7], [0.1, 0.5])) + 0.25 \times ((0.45\\
&\quad \times [0.1, 0.4], [0.3, 0.6], [0.3, 0.7]) + [0.1, 0.5], [0.2, 0.5], [0.2, 0.4] + (0.5\\
&\quad \times [0.3, 0.5], [0.2, 0.7], [0.1, 0.5])) + 0.55 \times ((0.516 \times [0.1, 0.4], [0.3, 0.6],\\
&[0.3, 0.7]) + (0.5 \times [0.1, 0.5], [0.2, 0.5], [0.2, 0.4]) + [0.3, 0.5],\\
&[0.2, 0.7], [0.1, 0.5]))\\
&= 1/3(0.20 \times ([0.2998, 1.399], [0.4932, 1.1862], [0.4416, 1.318]) + 0.25\\
&\quad \times ([0.295, 0.93], [0.435, 1.12], [0.385, 0.965]) + 0.55 \times ([0.4016, 0.9564],\\
&[0.4548, 1.2596], [0.3548, 1.2596]))\\
&= 1/3([0.0598, 0.2798], [0.09864, 0.23724], [0.08832, 0.22636]\\
&\quad + [0.07375, 0.2325], [0.10875, 0.28], [0.09625, 0.24125] + [0.22088, 0.52602],\\
&[0.25014, 0.69278], [0.19514, 0.58366])
\end{aligned}
$$

$$
\begin{aligned}
&= 1/3([0.35443, 1.03832], [0.45753, 1.21002], [0.35271, 1.05127]) \\
A_2 &= [0.1181, 0.3461], [0.1525, 0.4033], [0.1175, 0.3504].
\end{aligned}
$$

For $k = 3$

$$
\begin{aligned}
A_3 &= 1/3(\Sigma_{j=1}^{n} w_j (\Sigma_{i=1}^{n} \lambda_{ij} b_{3i})) \\
&= 1/3(w_1 \times (\lambda_{11} b_{31} + \lambda_{21} b_{32} + \lambda_{31} b_{33}) + w_2 \times (\lambda_{12} b_{31} + \lambda_{22} b_{32} + \lambda_{32} b_{33}) \\
&\quad + w_3 \times (\lambda_{13} b_{31} + \lambda_{23} b_{32} + \lambda_{33} b_{33})) \\
&= 1/3(0.20 \times ([0.3, 0.4], [0.1, 0.3], [0.2, 0.5] + (0.45 \times [0.2, 0.5], [0.1, 0.6], \\
&[0.3, 0.4]) + (0.516 \times 0.1, 0.4], [0.5, 0.7], [0.2, 0.4])) + 0.25 \times ((0.45 \\
&\quad \times [0.3, 0.4], [0.1, 0.3], [0.2, 0.5]) + [0.2, 0.5], [0.1, 0.6], [0.3, 0.4] + (0.5 \\
&\quad \times [0.1, 0.4], [0.5, 0.7], [0.2, 0.4]))) + 0.55 \times ((0.516 \times [0.3, 0.4], [0.1, 0.3], \\
&[0.2, 0.5]) + (0.5 \times [0.2, 0.5], [0.1, 0.6], [0.3, 0.4]) + [0.1, 0.4], [0.5, 0.7], \\
&[0.2, 0.4])) \\
&= 1/3(0.20 \times ([0.4416, 0.8314], [0.4030, 0.9312], [0.4385, 0.8864]) + 0.25 \\
&\quad \times ([0.3850, 0.8800], [0.3950, 1.0850], [0.4900, 0.8250]) + 0.55 \\
&\quad \times ([0.3548, 0.8564], [0.6016, 1.1548], [0.4532, 0.8580])) \\
&= 1/3([0.08832, 0.16628], [0.0806, 0.18624], [0.0877, 0.17728]+ \\
&[0.09625, 0.2200], [0.09875, 0.27125], [0.1225, 0.20625] + [0.19514, 0.47102], \\
&[0.33088, 0.63514], [0.24926, 0.4719]) \\
&= 1/3([0.37459, 0.8573], [0.51023, 1.09263], [0.45946, 0.85543]) \\
A_3 &= [0.1248, 0.2857], [0.17007, 0.3642], [0.1531, 0.28521].
\end{aligned}
$$

The score value of each alternative by using the score function is given by

$$
\begin{aligned}
sco(A_1) &= -0.2705 \\
sco(A_2) &= -0.1865 \\
sco(A_3) &= -0.1873
\end{aligned}
$$

Clearly, we have

$$
sco(A_2) \geq sco(A_3) \geq sco(A_1).
$$

Hence, the student of the year award goes to the candidate from campus E_2.

5 Conclusion

In different fields of sciences fuzzy graph is proven an effective tool to model any uncertain real world problem occur in decision making theory, computer science, pattern recognition etc. Many generalizations of fuzzy graphs have been introduced to deal with the uncertain real life problems. In our study, we have provided the generalization of PFGs termed IVP-(S, T)-fuzzy graphs. Initially, we have defined and applied different operations which include complement and ring sum to IVP-(S, T)-fuzzy graphs. Homomorphism and co-weak homomorphism of IVP-(S, T)-fuzzy graphs have also defined. We introduced different types of IVP-(S, T)-fuzzy graphs like regular, total regular and constant IVP-(S, T)-fuzzy graphs. We have explored through application that to model the uncertain real world problems IVP-(S, T)-fuzzy graphs is the best among the other existing generalized FGs. As an evidences, we have provided the solution of the problem related to MADM. In addition, we have also provided an algorithm to handle MADM problems using IVP-(S, T)-fuzzy graphs. In this regard, we have provided the analytic and numerical solutions of the problem occur in MADM by using IVP-(S, T)-fuzzy graphs. During this, we have observed that our analysis is more perfect and efficient than any other solution of such type of problem in MADM. Moreover, the domain of IVP-(S, T)-fuzzy graphs is very vast, one can twist it towards FGs, IVFGs, t-FGs, IFGs, IVI-(S, T)-fuzzy graphs and PFGs by giving different memberships values. Thus, the qualitative characteristics of FGs, IVFGs, t-FGs, IFGs, IVI-(S, T)-fuzzy graphs and PFGs can be viewed in a single IVP-(S, T)-fuzzy graphs. One could extend this study towards balanced picture fuzzy graphs, incidence picture fuzzy graphs, bipolar picture fuzzy graphs etc.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest regarding the publication of this article.

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