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ORIGINAL RESEARCH

A note on strong edge-coloring of claw-free cubic graphs

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Abstract
A strong edge-coloring of a graph G is an edge-coloring of G such that any two edges that are either adjacent to each other or adjacent to a common edge receive distinct colors. The strong chromatic index of G, denoted by $\chi_s'(G)$, is the minimum number of colors needed to guarantee that G admits a strong edge-coloring. For any integer $n \geq 3$, let H_n denote the n-prism (i.e., the Cartesian product $C_n \square K_2$) and H_n^Δ the graph obtained from H_n by replacing each vertex with a triangle. Recently, Lin and Lin (2022) asked whether $\chi_s'(H_n^\Delta) = 6$ for any $n \geq 3$. In this short note, we answer this question in the affirmative.

Keywords Strong edge-coloring · Strong chromatic index · Claw-free · Cubic graph

Mathematics Subject Classification 05C15

1 Introduction

In this note, we only consider finite simple graphs. A *strong k-edge-coloring* of a graph G is a mapping $\phi : E(G) \to \{1, 2, \ldots, k\}$ such that if any two edges e and f are either adjacent to each other or adjacent to a common edge in G, then $\phi(e) \neq \phi(f)$. The *strong chromatic index* of G, denoted by $\chi_s'(G)$, is the minimum integer k for which G has a strong k-edge-coloring. This concept was first introduced by Fouquet and Jolivet [8, 9] and was used to solve a problem involving radio networks and their frequencies (more details on this application can be found in [18, 19]).

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A well-known conjecture of Erdős and Nešetřil [5, 6] states that for any graph G with maximum degree Δ, $\chi'_s(G) \leq \frac{5}{4}\Delta^2$ if Δ is even and $\chi'_s(G) \leq \frac{5}{4}\Delta^2 - \frac{1}{2}\Delta + \frac{1}{4}$ if Δ is odd. This conjecture is still wide open. For graphs with $\Delta \leq 3$ (such graphs are often referred to as *subcubic graphs*), the conjecture was confirmed by Andersen [1], and independently by Horák, Qing and Trotter [11]. For graphs with $\Delta = 4$, an upper bound of 21 was proved by Huang, Santana and Yu [12] (note that the conjectured bound is 20). For sufficiently large Δ, Molloy and Reed [17] showed that $\chi'_s(G) \leq 1.998\Delta^2$ by applying probabilistic techniques. This was later improved to $1.93\Delta^2$ by Bruhn and Joos [3], and then to $1.835\Delta^2$ by Bonamy, Perrett and Postle [2]. The current best known upper bound is $1.772\Delta^2$, which was recently derived by Hurley, de Joannis de Verclos and Kang [13].

The strong chromatic index of planar graphs has been intensively studied. Faudree et al. [7] proved that $\chi'_s(G) \leq 4\Delta + 4$ for any planar graph G with maximum degree Δ, and showed that there exists a planar graph G with $\chi'_s(G) = 4\Delta - 4$ for any $\Delta \geq 2$. Hocquard, Ochem and Valicov [10] proved that $\chi'_s(G) \leq 3\Delta - 3$ for any outerplanar graph G with maximum degree $\Delta \geq 3$, and constructed an example showing that the upper bound is the best possible. Confirming a conjecture of Faudree et al. [7], Kostochka et al. [14] proved that $\chi'_s(G) \leq 9$ for any subcubic planar graph G. For planar graphs with maximum degree 4, an upper bound of 19 was obtained by Wang et al. [20].

A graph is said to be *claw-free* if it does not contain an induced subgraph isomorphic to $K_{1,3}$. Dębski, Junosza-Szaniawski and Śleszyńska-Nowak [4] showed that $\chi'_s(G) \leq \frac{9}{8}\Delta^2 + \Delta$ for any claw-free graph G with maximum degree Δ. This verified the aforementioned conjecture of Erdős and Nešetřil for claw-free graphs with maximum degree at least 12. For any integer $n \geq 3$, the *n-prism* H_n is the Cartesian product $C_n \square K_2$. In 2022, Lv, Li and Zhang [16] proved that $\chi'_s(G) \leq 8$ for any connected claw-free subcubic graph G other than the 3-prism, and asked whether the upper bound 8 can be improved to 7. Very recently, Lin and Lin [15] solved this problem and constructed infinitely many connected claw-free subcubic graphs with the strong chromatic index attaining the upper bound 7.

Theorem 1.1 (Lin and Lin [15]) *If G is a connected claw-free subcubic graph not isomorphic to the* 3*-prism, then* $\chi'_s(G) \leq 7$.

It is easy to observe that if G is a connected claw-free cubic graph, then $\chi'_s(G) \geq 6$. Hence, Theorem 1.1 implies that $\chi'_s(G) \in \{6, 7\}$ for any connected claw-free cubic graph G other than the 3-prism.

Let H_n^Δ denote the graph obtained from the n-prism H_n by replacing each vertex with a triangle. It is clear that H_n^Δ is a connected claw-free cubic graph. At the end of their paper, Lin and Lin [15] suggested three questions for future research, one of which is as follows.

Question 1.2 (Lin and Lin [15]) Is it true that $\chi'_s(H_n^\Delta) = 6$ for any integer $n \geq 3$?

The main objective of this short note is to give an affirmative answer to this question.

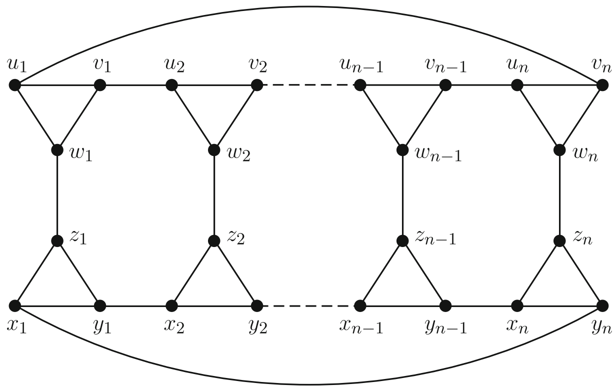


Fig. 1 The graph H_n^{Δ}

2 The proof

In this section, we prove the following theorem which answers Question 1.2 affirmatively.

Theorem 2.1 $\chi_s'(H_n^{\Delta}) = 6$ *for any integer* $n \geq 3$.

Proof For the sake of simplicity, suppose that $V(H_n^{\Delta}) = \{u_i, v_i, w_i, x_i, y_i, z_i : 1 \leq i \leq n\}$ and $E(H_n^{\Delta}) = \{u_i v_i, v_i u_{i+1}, u_i w_i, v_i w_i, x_i y_i, y_i x_{i+1}, x_i z_i, y_i z_i, w_i z_i : 1 \leq i \leq n\}$, where the subscripts are taken modulo n. An illustration is depicted in Fig. 1.

As we mentioned in the Introduction, it is easy to observe that $\chi_s'(H_n^{\Delta}) \geq 6$. (One can notice that for each $i = 1, 2, \ldots, n$, the six edges incident to u_i, v_i or w_i must be colored with distinct colors.) Hence to prove Theorem 2.1, it suffices to show that H_n^{Δ} admits a strong 6-edge-coloring which implies that $\chi_s'(H_n^{\Delta}) \leq 6$. We consider two cases according to the parity of n.

Case 1 n is even.

Then $n = 2k$ for some integer $k \geq 2$. We define an edge-coloring ϕ of H_{2k}^{Δ} as follows:

(1.1) for each $i = 1, 2, \ldots, 2k$, let $\phi(u_i w_i) = 1$, $\phi(v_i w_i) = 2$, $\phi(x_i z_i) = 3$ and $\phi(y_i z_i) = 4$;
(1.2) for each $i = 1, 3, \ldots, 2k-1$, let $\phi(w_i z_i) = 5$, $\phi(u_i v_i) = \phi(x_i y_i) = 6$, $\phi(v_i u_{i+1}) = 3$ and $\phi(y_i x_{i+1}) = 1$; and
(1.3) for each $i = 2, 4, \ldots, 2k$, let $\phi(w_i z_i) = 6$, $\phi(u_i v_i) = \phi(x_i y_i) = 5$, $\phi(v_i u_{i+1}) = 4$ and $\phi(y_i x_{i+1}) = 2$.

See Fig. 2 for an illustration. It is easy to verify that ϕ is a strong 6-edge-coloring of H_{2k}^{Δ}.

Case 2 n is odd.

If $n = 3$, then a strong 6-edge-coloring of H_3^{Δ} is given in Fig. 3. So we may assume that $n \geq 5$, and hence $n = 2k + 1$ for some integer $k \geq 2$. We define an edge-coloring ψ of H_{2k+1}^{Δ} as follows:

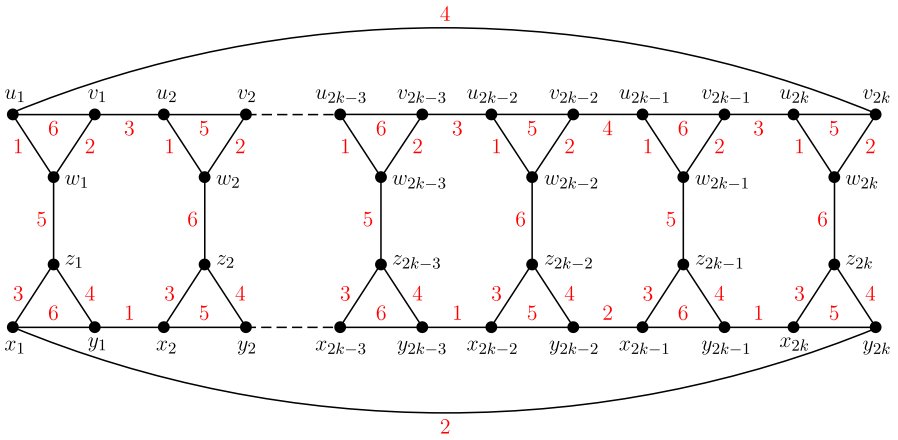
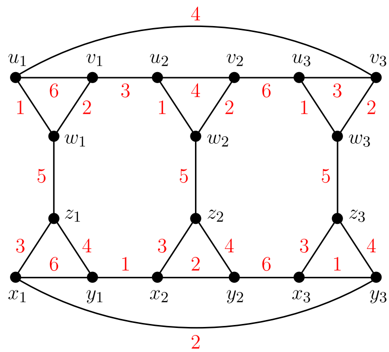


Fig. 2 A strong 6-edge-coloring of H^{Δ}_{2k}

Fig. 3 A strong 6-edge-coloring of H^{Δ}_{3}



(2.1) for each $i = 1, 2, \ldots, 2k+1$, let $\psi(u_i w_i) = 1$, $\psi(v_i w_i) = 2$, $\psi(x_i z_i) = 3$ and $\psi(y_i z_i) = 4$;

(2.2) for each $i = 1, 3, \ldots, 2k-1$, let $\psi(w_i z_i) = 5$, $\psi(u_i v_i) = \psi(x_i y_i) = 6$, $\psi(v_i u_{i+1}) = 3$ and $\psi(y_i x_{i+1}) = 1$;

(2.3) for each $i = 2, 4, \ldots, 2k-2$, let $\psi(w_i z_i) = 6$, $\psi(u_i v_i) = \psi(x_i y_i) = 5$, $\psi(v_i u_{i+1}) = 4$ and $\psi(y_i x_{i+1}) = 2$; and

(2.4) for the remaining uncolored edges, let $\psi(x_{2k+1} y_{2k+1}) = 1$, $\psi(x_{2k} y_{2k}) = \psi(y_{2k+1} x_1) = 2$, $\psi(u_{2k+1} v_{2k+1}) = 3$, $\psi(u_{2k} v_{2k}) = \psi(v_{2k+1} u_1) = 4$, $\psi(w_{2k} z_{2k}) = \psi(w_{2k+1} z_{2k+1}) = 5$, and $\psi(v_{2k} u_{2k+1}) = \psi(y_{2k} x_{2k+1}) = 6$.

Please refer to Fig. 4 for a detailed illustration. One can easily check that ψ is a strong 6-edge-coloring of H^{Δ}_{2k+1}. This completes the proof of the theorem. $\square$

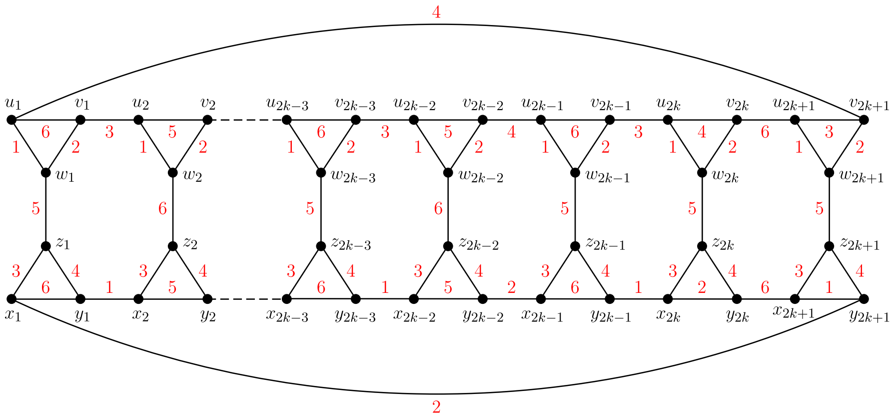


Fig. 4 A strong 6-edge-coloring of H^{Δ}_{2k+1}

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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