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ORIGINAL RESEARCH

Granulation of protein–protein interaction networks in Pythagorean fuzzy soft environment

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Abstract
Pythagorean fuzzy soft set theory is very effective to simultaneously depict the parameter-dependent uncertainty and the uncertainty of belongingness in the entities of a system. The study of information granulation in different extensions of fuzzy set theory is very close to human reasoning. We extend this concept in Pythagorean fuzzy soft set theory. Primarily, we define information system for a Pythagorean fuzzy soft set and its transformation to the information system of another Pythagorean fuzzy soft set. Additionally, we obtain Pythagorean fuzzy soft information system for a Pythagorean fuzzy soft graph which is a parameterized family of Pythagorean fuzzy information systems. The concepts of reduct, core, extended core and discernibility matrix are investigated on the basis of Pythagorean fuzzy soft indiscernibility relation. These notions are illustrated in detail through examples and results. The method for the construction of Pythagorean fuzzy soft granules in a soft graph is elaborated with the help of an example. We apply this method to protein–protein interaction networks of Parkinson's disease and consequently obtain protein complexes as Pythagorean fuzzy soft granules.

Keywords Granular computing · Pythagorean fuzzy soft graph · Indiscernibility relation · Protein–protein interaction

1 Introduction

The term granular computing was coined by Lin [18] in 1997. Granular computing is an emerging problem solving technique that makes use of information granules

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for knowledge representation and information manipulation. Information granules (simply granules) arise during the arrangement of knowledge on the basis of some similarity, physical resemblance or indistinguishability. It is obvious that whatever the data-set is under consideration, we usually try to give it conceptual description according to the problem in hand. Consider image processing as an illustration. While interpreting an image, we do not concentrate on every single pixel rather we see different collections of pixels that give a meaningful context to the picture, i.e., we notice everyday objects we deal with. Briefly, the regions occupied by an object in the picture involve a group of pixels that have proximity in semantics. This built-in quality of humans is similar to the construction, representation, interpretation and manipulation of granules which assist to withdraw definitive conclusions. Granular computing has become an autonomous area of research due to its scope in data mining [13, 16, 17, 38, 42], rough set theory [28, 29, 41], theoretical and applied sciences [30, 31], interval analysis [15] and formal concept analysis [39].

There exist connections and relationships in numerous real-world systems that can be presented effectively through networks. The granulation of networks of such models is another interesting topic. Since a graph can be considered as a generalization of crisp set therefore, for the very first time, the idea of granulation of graphs was suggested by Stell [37] in 1999. Primarily, he illustrated this concept with informal examples and then presented the complete approach for the granulation of graphs. Since a hypergraph is an extended model of graph for the visualization of multiple connections, the notion of granulation was expanded for hypergraphs as well. Liu et al. [19] proposed a clustering technique based on hypergraph partition and granular computing which itself discovers the similarities among the objects of a system. Chen et al. [9] illustrated method for the construction of multiple levels of information granulation in a hypergraph model. He also described that how to move from one level of granularity to another and how to solve different problems with the help of considered hypergraph model. Chen and Zhong [10] proposed algorithms for the formation of granular structures based on vertex degree and edge weight of a graph. Chiaselotti et al. [11] presented granular computing for simple graphs with various useful concepts. They extended the notion of information system [27, 28] of a data-set and considered the adjacency matrix as the information table of graph. Corresponding to the proposed information table, the notions of indiscernibility relation, core, extended core, reduct and discernibility matrix were also discussed for simple graphs.

To represent uncertainty in real-world systems, Zadeh [43] initiated the concept of fuzzy set in 1965 which is characterized by a mapping that takes an element of crisp universe and map it into the closed unit interval. Fuzzy sets were presented to mathematically describe the situations in which the objects do not possess the exact criteria of membership. In real scenarios, it is believed that the techniques based on fuzzy set theory play well in comparison to the corresponding crisp methods. A fuzzy set only provides the degree of membership of an element and, the degree of its non-membership is assumed to be one minus the degree of its membership value. However, this may not necessarily be true in reality. For the compensation of this difficulty, intuitionistic fuzzy set was introduced by Atanassov [8], which comprises a dependent hesitation degree as well with the limitation that the sum of degree of membership and non-membership must not exceed one for any alternative. Another non-standard fuzzy set known as Pythagorean fuzzy set was proposed afterwards by Yager [40] which provides more space for the assignment of grades as compared to intuitionistic fuzzy set. Obviously, Pythagorean fuzzy set has wide range of applicability as it permits the Pythagorean membership grades in practical problems.

In 1999, Molodtsov [23] revealed another form of uncertainty which is based on the parametric properties of elements. He defined soft set as a parameterized family of crisp sets. In order to combine the benefits of both fuzzy set theory and soft set theory, another model was suggested by Maji et al. [22]. Roy and Maji [34] highlighted many decision-making problems that found their solutions when studied in fuzzy soft theory. Pythagorean fuzzy soft sets were then proposed by Peng et al. [32].

Fuzzy graphs were proposed by Kaufmann [14] which depict uncertainty in the relationships of a network model. Akram and Nawaz [7] put forward fuzzy soft graphs and, Nawaz and Akram [25] discussed its application in oligopolistic market structure. Naz et al. [26] introduced Pythagorean fuzzy graphs and described Pythagorean fuzzy preference relation with some applications in decision-making. Akram et al. [36] presented Pythagorean fuzzy soft graph and also discussed its operations, regularity and edge-regularity. Various hybrid models of fuzzy graph theory are elaborated in [4, 6, 12, 24].

In order to solve problems which are imprecisely defined, it is remarkable to use words in place of numbers for the concept formulation in information granulation. Granules in human reasoning are fuzzy in nature and so fuzzy information granulation can be regarded as the generalization of crisp granulation to be applied in different environments. Several times, Zadeh emphasized on the study of information granulation in fuzzy set theory [44–46]. According to him, fuzzy information granulation is carried out with the technique of computing with words as the labels of fuzzy granules are the words of linguistic communication. Akram and Luqman [5] presented granular structures for the ecological networks in fuzzy soft environment. Akram et al. [2] extended the degree based models of crisp networks to fuzzy graphs and discussed fuzzy indiscernibility relation as well. The concept of information granulation has been extended in different hybrid models of fuzzy set theory [3, 20, 21].

Motivated by the aptness of Pythagorean fuzzy soft set theory, we extend the important notions of information granulation in this environment. This article adds the following notions in the existing literature.

1. It provides method for the transformation of one Pythagorean fuzzy information system to another.
2. It defines reduct, core and extended core on the basis of Pythagorean fuzzy soft indiscernibility relation for Pythagorean fuzzy soft graph. It also presents the corresponding Pythagorean fuzzy soft discernibility matrix.
3. It describes the technique for the construction of Pythagorean fuzzy soft granules in a soft graph through an example of protein–protein interaction (PPI) networks.

The research paper is structured as follows. Next section defines a Pythagorean fuzzy information system based on Pythagorean fuzzy soft set. Section 3 provides a study on the Pythagorean fuzzy soft information system of a Pythagorean fuzzy soft graph. Section 4 is devoted to produce a Pythagorean fuzzy soft discernibility matrix. Section 5 gives the application of granules formation in the PPI networks of Parkinson's disease. Last section concludes the research article.

2 Information systems for Pythagorean fuzzy soft sets

Definition 1 An information system is denoted by the quadruple $I = (U, A, f, V)$, where U is a finite set of objects, set A contains the attributes $a : U \to V_a$ for the objects of U, $f : U \times A \to V$ is the information map such that $f(u, a) \in V_a$ for all $u \in U, a \in A$, and $V = \bigcup_{a \in A} V_a$ is the value set.

An information system is generally presented in table format whose rows and columns are labeled with objects $u \in U$ and attributes $a \in A$, respectively. The (i, j)-th entry in the table is the value of attribute a_j for the object u_i given by $f(u_i, a_j)$. Next we define Pythagorean fuzzy soft set introduced by Peng et al. [32] that allows Pythagorean fuzzy values to each element of universal set corresponding to each considered parameter.

Definition 2 [32] A Pythagorean fuzzy soft set, denoted by the pair $(\mathbb{U}, Z)$, is a parameterized family of Pythagorean fuzzy subsets. It is defined by the approximate set-valued mapping $\mathbb{U} : Z \to \mathcal{P}(U)$, where $\mathcal{P}(U)$ denotes the infinite set of all Pythagorean fuzzy subsets defined over U. From a Pythagorean fuzzy soft set $(\mathbb{U}, Z)$, one can simply define a Pythagorean fuzzy information system $\mathrm{I} = (U, A, f, V)$, where $A = Z$ contains attributes of the form $a_i : U \to V_i = [0, 1]^2$ defined by $a_i(u) = (\mathbb{U}_\mu(\mathfrak{z})(u), \mathbb{U}_\nu(\mathfrak{z})(u))$, $f : U \times A \to V$ is the information map such that $f(u, a_i) = a_i(u)$ and $V = \bigcup_i V_i$ (where the index i takes the given range of parameters) denotes the Pythagorean fuzzy set of values. Conversely, if $\mathrm{I} = (U, Z, f, V)$ denotes a Pythagorean fuzzy information system then Pythagorean fuzzy soft set $(\mathbb{U}, Z)$ can be obtained by considering $Z = A$ as the set of parameters and $\mathbb{U} : Z \to \mathcal{P}(U)$ is approximate mapping defined as $\mathbb{U}(\mathfrak{z}_i) = a_i$.

Let $\mathcal{P}(U, Z)$ denotes the power set containing all possible Pythagorean fuzzy soft sets which are defined over universe U and parameter set Z.

Definition 3 Let us take into account the Pythagorean fuzzy soft power sets $\mathcal{P}(U, Z)$ and $\mathcal{P}(U', Z')$ over U and U' with parameters from Z and Z', respectively. Consider the mappings $g : U \to U'$ and $h : Z \to Z'$ and define $f = (g, h) : \mathcal{P}(U, Z) \to \mathcal{P}(U', Z')$ which maps the Pythagorean fuzzy soft set $(\mathbb{U}, Z)$ from $\mathcal{P}(U, Z)$ to Pythagorean fuzzy soft set $f(\mathbb{U}, Z)$ in $\mathcal{P}(U', Z')$ such that: for $\mathfrak{z}' \in Z'$ and $u' \in U'$

$$f_{\mathfrak{z}'}(\mathbb{U}, Z)(u') = \begin{cases} \bigvee_{u \in g^{-1}(u')} \left(\bigvee_{\mathfrak{z} \in h^{-1}(\mathfrak{z}')} \mathbb{U}(\mathfrak{z}) \right)(u), & \text{if } g^{-1}(u') \neq \varnothing, h^{-1}(\mathfrak{z}') \cap Z' \neq \varnothing, \\ (0, 0), & \text{otherwise.} \end{cases}$$

Definition 4 Let us take into account the Pythagorean fuzzy soft power sets $\mathcal{P}(U, Z)$ and $\mathcal{P}(U', Z')$ over U and U' with parameters from Z and Z', respectively. Consider the mappings $g : U \to U'$ and $h : Z \to Z'$ and define $f^{-1} = (g, h) :$

$\mathcal{P}(U', Z') \rightarrow \mathcal{P}(U, Z)$ which maps the Pythagorean fuzzy soft set $(\mathbb{U}', Z')$ from $\mathcal{P}(U', Z')$ to Pythagorean fuzzy soft set $f^{-1}(\mathbb{U}', Z')$ in $\mathcal{P}(U, Z)$ such that: for $\mathfrak{z} \in Z$ and $u \in U$

$$f_{\mathfrak{z}}^{-1}(\mathbb{U}', Z')(u) = \begin{cases} \mathbb{U}'(h(\mathfrak{z}))(g(u)), \textit{ for } h(\mathfrak{z}) \in Z', \\ \quad (0, 0), \qquad\qquad \text{otherwise.} \end{cases}$$

Example 1 Let us take into consideration two distinct power sets of Pythagorean fuzzy soft sets $\mathcal{P}(U, Z)$ and $\mathcal{P}(U', Z')$ defined over $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$, $Z = \{\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3\}$ and $U' = \{u'_1, u'_2, u'_3, u'_4\}$, $Z' = \{\mathfrak{z}'_1, \mathfrak{z}'_2\}$, respectively. Consider the Pythagorean fuzzy soft sets $(\mathbb{U}, Z)$ and $(\mathbb{U}', Z')$ in $\mathcal{P}(U, Z)$ and $\mathcal{P}(U', Z')$ given by

$$(\mathbb{U}, Z) = (\mathbb{U}(\mathfrak{z}_1), \mathbb{U}(\mathfrak{z}_2), \mathbb{U}(\mathfrak{z}_3)),$$

where

$$\begin{aligned} \mathbb{U}(\mathfrak{z}_1) &= \{\langle u_1, (0.5, 0.8)\rangle, \langle u_2, (0.6, 0.5)\rangle, \langle u_3, (0.4, 0.7)\rangle, \langle u_4, (0.9, 0.2)\rangle, \\ &\quad \langle u_5, (0.7, 0.6)\rangle, \langle u_6, (0.5, 0.8)\rangle\}, \\ \mathbb{U}(\mathfrak{z}_2) &= \{\langle u_1, (0.4, 0.7)\rangle, \langle u_2, (0.8, 0.3)\rangle, \langle u_3, (0.6, 0.7)\rangle, \langle u_4, (0.6, 0.7)\rangle, \\ &\quad \langle u_5, (0.5, 0.8)\rangle, \langle u_6, (0.9, 0.3)\rangle\}, \\ \mathbb{U}(\mathfrak{z}_3) &= \{\langle u_1, (0.4, 0.7)\rangle, \langle u_2, (0.4, 0.9)\rangle, \langle u_3, (0.7, 0.1)\rangle, \langle u_4, (0.6, 0.7)\rangle, \\ &\quad \langle u_5, (0.5, 0.3)\rangle, \langle u_6, (0.8, 0.5)\rangle\}, \end{aligned}$$

and

$$(\mathbb{U}', Z') = (\mathbb{U}'(\mathfrak{z}'_1), \mathbb{U}'(\mathfrak{z}'_2)),$$

where

$$\begin{aligned} \mathbb{U}'(\mathfrak{z}'_1) &= \{\langle u'_1, (0.5, 0.8)\rangle, \langle u'_2, (0.7, 0.4)\rangle, \langle u'_3, (0.6, 0.3)\rangle, \langle u'_4, (0.9, 0.3)\rangle\}, \\ \mathbb{U}'(\mathfrak{z}'_2) &= \{\langle u'_1, (0.7, 0.5)\rangle, \langle u'_2, (0.3, 0.8)\rangle, \langle u'_3, (0.5, 0.6)\rangle, \langle u'_4, (0.6, 0.7)\rangle\}, \end{aligned}$$

respectively. The corresponding Pythagorean fuzzy information systems are respectively given in Tables 1 and 2.

Let us define the mappings $g : U \rightarrow U'$ and $h : Z \rightarrow Z'$ as

$$\begin{aligned} &g(u_1) = u'_4, \quad g(u_2) = u'_3, \quad g(u_3) = u'_1, \quad g(u_4) = u'_3, \quad g(u_5) = u'_2, \quad g(u_6) = u'_2, \\ &h(\mathfrak{z}_1) = \mathfrak{z}'_1, \quad h(\mathfrak{z}_2) = \mathfrak{z}'_1, \quad h(\mathfrak{z}_3) = \mathfrak{z}'_2. \end{aligned}$$

The Pythagorean fuzzy information system under $f = (g, h) : \mathcal{P}(U, Z) \rightarrow \mathcal{P}(U', Z')$ is obtained as:

Table 1 PF information system $\mathbb{I}$

$\mathbb{I}$	$\mathfrak{z}_1$	$\mathfrak{z}_2$	$\mathfrak{z}_3$
u_1	(0.5, 0.8)	(0.4, 0.7)	(0.4, 0.7)
u_2	(0.6, 0.5)	(0.8, 0.3)	(0.4, 0.9)
u_3	(0.4, 0.7)	(0.6, 0.7)	(0.7, 0.1)
u_4	(0.9, 0.2)	(0.6, 0.7)	(0.6, 0.7)
u_5	(0.7, 0.6)	(0.5, 0.8)	(0.5, 0.3)
u_6	(0.5, 0.8)	(0.9, 0.3)	(0.8, 0.5)

Table 2 PF information system $\mathbb{I}'$

$\mathbb{I}'$	$\mathfrak{z}'_1$	$\mathfrak{z}'_2$
u'_1	(0.5, 0.8)	(0.7, 0.5)
u'_2	(0.7, 0.4)	(0.3, 0.8)
u'_3	(0.6, 0.3)	(0.5, 0.6)
u'_4	(0.9, 0.3)	(0.6, 0.7)

Table 3 PF information system $f(\mathbb{I})$

$f(\mathbb{I})$	$\mathfrak{z}'_1$	$\mathfrak{z}'_2$
u'_1	(0.6, 0.7)	(0.7, 0.1)
u'_2	(0.9, 0.3)	(0.8, 0.3)
u'_3	(0.9, 0.2)	(0.6, 0.7)
u'_4	(0.5, 0.7)	(0.4, 0.7)

$$\begin{aligned}
f_{\mathfrak{z}'_1}(\mathbb{U}, Z)(u'_1) &= \bigvee_{u \in g^{-1}(u'_1)} \left(\bigvee_{\mathfrak{z} \in h^{-1}(\mathfrak{z}'_1)} \mathbb{U}(\mathfrak{z}) \right)(u) \\
&= \bigvee_{\{u_3\}} \left(\bigvee_{\{\mathfrak{z}_1, \mathfrak{z}_2\}} \mathbb{U}(\mathfrak{z}) \right)(u) \\
&= \mathbb{U}(\mathfrak{z}_1)(u_3) \vee \mathbb{U}(\mathfrak{z}_2)(u_3) \\
&= (0.4, 0.7) \vee (0.6, 0.7) = (0.6, 0.7).
\end{aligned}$$

Similarly

$$\begin{aligned}
&f_{\mathfrak{z}'_1}(\mathbb{U}, Z)(u'_2) = (0.9, 0.3), \quad f_{\mathfrak{z}'_1}(\mathbb{U}, Z)(u'_3) = (0.9, 0.2), \quad f_{\mathfrak{z}'_1}(\mathbb{U}, Z)(u'_4) = (0.5, 0.7), \\
&f_{\mathfrak{z}'_2}(\mathbb{U}, Z)(u'_1) = (0.7, 0.1), \quad f_{\mathfrak{z}'_2}(\mathbb{U}, Z)(u'_2) = (0.8, 0.3), \quad f_{\mathfrak{z}'_2}(\mathbb{U}, Z)(u'_3) = (0.6, 0.7), \\
&f_{\mathfrak{z}'_2}(\mathbb{U}, Z)(u'_4) = (0.4, 0.7).
\end{aligned}$$

Thus, the image of Pythagorean fuzzy information system $\mathbb{I}$ under f is a Pythagorean fuzzy information system $f(\mathbb{I})$ given in Table 3.

Table 4 PF information system $f^{-1}(\mathbb{I}')$

$f^{-1}(\mathbb{I}')$	$\mathfrak{z}_1$	$\mathfrak{z}_2$	$\mathfrak{z}_3$
u_1	(0.9, 0.3)	(0.9, 0.3)	(0.6, 0.7)
u_2	(0.6, 0.3)	(0.6, 0.3)	(0.5, 0.6)
u_3	(0.5, 0.8)	(0.5, 0.8)	(0.7, 0.5)
u_4	(0.6, 0.3)	(0.6, 0.3)	(0.5, 0.6)
u_5	(0.7, 0.4)	(0.7, 0.4)	(0.3, 0.8)
u_6	(0.7, 0.4)	(0.7, 0.4)	(0.3, 0.8)

Next, we calculate

$$f_{\mathfrak{z}_1}^{-1}(\mathbb{U}', Z')(u_1) = \mathbb{U}'(h(\mathfrak{z}_1))(g(u_1)) = \mathbb{U}'(\mathfrak{z}_1')(u_4') = (0.9, 0.3).$$

Similar calculations yield

$$\begin{aligned}
&f_{\mathfrak{z}_1}^{-1}(\mathbb{U}', Z')(u_2) = (0.6, 0.3), \quad f_{\mathfrak{z}_1}^{-1}(\mathbb{U}', Z')(u_3) = (0.5, 0.8),\\
&f_{\mathfrak{z}_1}^{-1}(\mathbb{U}', Z')(u_4) = (0.6, 0.3),\\
&f_{\mathfrak{z}_1}^{-1}(\mathbb{U}', Z')(u_5) = (0.7, 0.4), \quad f_{\mathfrak{z}_1}^{-1}(\mathbb{U}', Z')(u_6) = (0.7, 0.4),\\
&f_{\mathfrak{z}_2}^{-1}(\mathbb{U}', Z')(u_1) = (0.9, 0.3), \quad f_{\mathfrak{z}_2}^{-1}(\mathbb{U}', Z')(u_2) = (0.6, 0.3),\\
&f_{\mathfrak{z}_2}^{-1}(\mathbb{U}', Z')(u_3) = (0.5, 0.8),\\
&f_{\mathfrak{z}_2}^{-1}(\mathbb{U}', Z')(u_4) = (0.6, 0.3), \quad f_{\mathfrak{z}_2}^{-1}(\mathbb{U}', Z')(u_5) = (0.7, 0.4),\\
&f_{\mathfrak{z}_2}^{-1}(\mathbb{U}', Z')(u_6) = (0.7, 0.4),\\
&f_{\mathfrak{z}_3}^{-1}(\mathbb{U}', Z')(u_1) = (0.6, 0.7), \quad f_{\mathfrak{z}_3}^{-1}(\mathbb{U}', Z')(u_2) = (0.5, 0.6),\\
&f_{\mathfrak{z}_3}^{-1}(\mathbb{U}', Z')(u_3) = (0.7, 0.5),\\
&f_{\mathfrak{z}_3}^{-1}(\mathbb{U}', Z')(u_4) = (0.5, 0.6), \quad f_{\mathfrak{z}_3}^{-1}(\mathbb{U}', Z')(u_5) = (0.3, 0.8),\\
&f_{\mathfrak{z}_3}^{-1}(\mathbb{U}', Z')(u_6) = (0.3, 0.8),
\end{aligned}$$

Consequently, we get the inverse image $f^{-1}(\mathbb{I}')$ of Pythagorean fuzzy information system $\mathbb{I}'$ given in Table 4.

3 Core, reduct and extended core of a Pythagorean fuzzy soft graph

Definition 5 [36] Let $G = (U, E)$ be a simple graph. A simple Pythagorean fuzzy soft graph $\mathbb{G}$ of graph G is denoted as $\mathbb{G} = (\mathbb{U}, \mathbb{E}, Z)$, where

(1) $(\mathbb{U}, Z)$ is a Pythagorean fuzzy soft vertex set over U,
(2) $(\mathbb{E}, Z)$ is a Pythagorean fuzzy soft edge set over E,

(3) For each $\mathfrak{z}$, $\mathbb{G}(\mathfrak{z}) = (\mathbb{U}(\mathfrak{z}), \mathbb{E}(\mathfrak{z}))$ represents Pythagorean fuzzy graph such that the membership and non-membership values of Pythagorean fuzzy edge $u_i u_j$ $(i \neq j)$ in $\mathbb{G}(\mathfrak{z})$ is given by

$$\mathbb{E}_\mu(\mathfrak{z})(u_i u_j) \leq \min\{\mathbb{U}_\mu(\mathfrak{z})(u_i), \mathbb{U}_\mu(\mathfrak{z})(u_j)\},$$
$$\mathbb{E}_\nu(\mathfrak{z})(u_i u_j) \leq \max\{\mathbb{U}_\nu(\mathfrak{z})(u_i), \mathbb{U}_\nu(\mathfrak{z})(u_j)\},$$

such that $0 \leq (\mathbb{E}_\mu(\mathfrak{z})(u_i u_j))^2 + (\mathbb{E}_\nu(\mathfrak{z})(u_i u_j))^2 \leq 1$. We denote two adjacent Pythagorean fuzzy vertices u_i and u_j in $\mathbb{G}(\mathfrak{z})$ as $u_i \sim_{\mathfrak{z}} u_j$.

Definition 6 [36] The order $\mathcal{O}(\mathbb{G})$ of a Pythagorean fuzzy soft graph $\mathbb{G} = (\mathbb{U}, \mathbb{E}, Z)$ is defined as

$$\mathcal{O}(\mathbb{G}) = \sum_{\mathfrak{z} \in Z} \left(\sum_{u \in U} \mathbb{U}_\mu(\mathfrak{z})(u), \sum_{u \in U} \mathbb{U}_\nu(\mathfrak{z})(u) \right)$$

The size $\mathcal{S}(\mathbb{G})$ of a Pythagorean fuzzy soft graph $\mathbb{G} = (\mathbb{U}, \mathbb{E}, Z)$ is defined as

$$\mathcal{S}(\mathbb{G}) = \sum_{\mathfrak{z} \in Z} \left(\sum_{u_i u_j \in E} \mathbb{E}_\mu(\mathfrak{z})(u_i u_j), \sum_{u_i u_j \in E} \mathbb{E}_\nu(\mathfrak{z})(u_i u_j) \right)$$

Definition 7 [36] The degree $d(u)$ of a vertex u in a Pythagorean fuzzy soft graph $\mathbb{G} = (\mathbb{U}, \mathbb{E}, Z)$ is defined as the sum of degrees $d_{\mathfrak{z}}(u)$ of that vertex in all Pythagorean fuzzy graphs $\mathbb{G}(\mathfrak{z})$, i.e.,

$$d(u) = \sum_{\mathfrak{z} \in Z} d_{\mathfrak{z}}(u) = \sum_{\mathfrak{z} \in Z} \left(\sum_{i \neq j} \mathbb{E}_\mu(\mathfrak{z})(u_i u_j), \sum_{i \neq j} \mathbb{E}_\nu(\mathfrak{z})(u_i u_j) \right).$$

The degree $d(u_i u_j)$ of an edge $u_i u_j$ in a Pythagorean fuzzy soft graph $\mathbb{G} = (\mathbb{U}, \mathbb{E}, Z)$ is defined as the sum of degrees $d_{\mathfrak{z}}(u_i u_j)$ of that edge in all Pythagorean fuzzy graphs $\mathbb{G}(\mathfrak{z})$, i.e.,

$$d(u_i u_j) = \sum_{\mathfrak{z} \in Z} d_{\mathfrak{z}}(u_i u_j) = \sum_{\mathfrak{z} \in Z} (d_{\mathfrak{z}}(u_i) + d_{\mathfrak{z}}(u_j) - 2(\mathbb{E}_\mu(\mathfrak{z})(u_i u_j), \mathbb{E}_\nu(\mathfrak{z})(u_i u_j))).$$

We can represent the knowledge of a Pythagorean fuzzy soft graph in the form of a Pythagorean fuzzy soft information system as described below.

Definition 8 An information system $\mathbb{I}$ of a Pythagorean fuzzy soft graph $\mathbb{G} = (\mathbb{U}, \mathbb{E}, Z)$ is a parameterized family of Pythagorean fuzzy information systems, i.e., $\mathbb{I} = \{\mathbb{I}(\mathfrak{z}) : \mathfrak{z} \in Z\}$, where $\mathbb{I}(\mathfrak{z}) = (U, A, f_{\mathfrak{z}}, V_{\mathfrak{z}})$, $\forall \mathfrak{z}$. Here, U is the universe of objects, $A = U$, $f(\mathfrak{z}) : U \times A \to V_{\mathfrak{z}}$ is the Pythagorean fuzzy information mapping defined by $f_{\mathfrak{z}}(u_i u_j) = (\mathbb{E}_\mu(\mathfrak{z})(u_i u_j), \mathbb{E}_\nu(\mathfrak{z})(u_i u_j))$ and $V_{\mathfrak{z}} = [0, 1]^2$ is the value set. Note that the parameterized collection of Pythagorean fuzzy information tables for a Pythagorean fuzzy soft graph is represented by the parameterized collection of Pythagorean fuzzy adjacency matrices given by $\mathbb{I} = \{\mathbb{I}(\mathfrak{z}) : \mathfrak{z} \in Z\}$, where

$$\mathbb{I}(\mathfrak{z}) = [(\mathbb{E}_\mu(\mathfrak{z})(u_i u_j), \mathbb{E}_\nu(\mathfrak{z})(u_i u_j))]_{n\times n},$$

denotes the information system corresponding to each Pythagorean fuzzy graph $\mathbb{G}(\mathfrak{z}) = (\mathbb{U}(\mathfrak{z}), \mathbb{E}(\mathfrak{z}))$ in $\mathbb{G}$.

Definition 9 Let $Z' \subseteq Z$ and $U' \subseteq U$. We define a parameterized family of U'-indiscernibility relations for a Pythagorean fuzzy soft graph $\mathbb{G} = (\mathbb{U}, \mathbb{E}, Z)$ as: for $u_i, u_j \in U$ and $\mathfrak{z} \in Z'$,

$$u_i R_{U'}(\mathfrak{z}) u_j \Leftrightarrow f_{\mathfrak{z}}(u_i u) = f_{\mathfrak{z}}(u_j u),$$

for all $u \in U'$. Note that $R_{U'}$ is an equivalence relation. $[u]_{U'}(\mathfrak{z})$ is an equivalence class/symmetry block in $P_{U'}(\mathfrak{z}) = \{[u]_{U'}(\mathfrak{z}) : u \in U\}$ known as the U'-indiscernibility partition relative to parameter $\mathfrak{z}$. $R_{U'}$ is also referred to as U'-indiscernibility relations of Pythagorean fuzzy soft information table $\mathbb{I}$.

Definition 10 Let $\mathbb{G} = (\mathbb{U}, \mathbb{E}, Z)$ be a Pythagorean fuzzy soft graph, then for a Pythagorean fuzzy soft vertex $u \in \mathbb{U}$, the neighborhood is defined as $\mathbb{N}(u) = \{\langle \mathfrak{z}, \mathbb{N}(\mathfrak{z})(u)\rangle : \mathfrak{z} \in Z\}$, where $\mathbb{N}(\mathfrak{z})(u) = \{\langle u', (\mathbb{E}_\mu(\mathfrak{z})(uu'), \mathbb{E}_\nu(\mathfrak{z})(uu'))\rangle : \mathbb{E}_\mu(\mathfrak{z})(uu') > 0 \textit{ or } \mathbb{E}_\nu(\mathfrak{z})(uu') > 0\}$.

Theorem 1 *If $\mathbb{G} = (\mathbb{U}, \mathbb{E}, Z)$ is a Pythagorean fuzzy soft graph over U and $\mathbb{G}(\mathfrak{z}) = (\mathbb{U}(\mathfrak{z}), \mathbb{E}(\mathfrak{z}))$ is the Pythagorean fuzzy graph corresponding to parameter $\mathfrak{z}$, then the following statements hold in each $\mathbb{G}(\mathfrak{z})$:*

(i) $u_i R_U(\mathfrak{z}) u_j$,
(ii) $u_i \sim_{\mathfrak{z}} u \Leftrightarrow u_j \sim_{\mathfrak{z}} u$ *such that* $(\mathbb{E}_\mu(\mathfrak{z})(u_i u), \mathbb{E}_\nu(\mathfrak{z})(u_i u)) = (\mathbb{E}_\mu(\mathfrak{z})(u_j u), \mathbb{E}_\nu(\mathfrak{z})(u_j u))$, *for all* $u \in U$,
(iii) $\mathbb{N}(\mathfrak{z})(u_i) = \mathbb{N}(\mathfrak{z})(u_j)$.

Proof Let $\mathbb{G} = (\mathbb{U}, \mathbb{E}, Z)$ be a Pythagorean fuzzy soft graph over U and $\mathbb{G}(\mathfrak{z}) = (\mathbb{U}(\mathfrak{z}), \mathbb{E}(\mathfrak{z}))$ be the Pythagorean fuzzy graph corresponding to parameter $\mathfrak{z}$.

(i)⇒(ii) We prove that statement (i) implies (ii). For this, suppose that for an arbitrary $u \in U$, $u_i \sim_{\mathfrak{z}} u$. We prove that $u_j \sim_{\mathfrak{z}} u$ such that $(\mathbb{E}_\mu(\mathfrak{z})(u_j u), \mathbb{E}_\nu(\mathfrak{z})(u_j u)) = (\mathbb{E}_\mu(\mathfrak{z})(u_i u), \mathbb{E}_\nu(\mathfrak{z})(u_i u))$. Note that $u_i R_U(\mathfrak{z}) u_j$ indicates that for all $u \in U$, $f_{\mathfrak{z}}(u_i u) = f_{\mathfrak{z}}(u_j u)$. Consequently, $(\mathbb{E}_\mu(\mathfrak{z})(u_i u), \mathbb{E}_\nu(\mathfrak{z})(u_i u)) = (\mathbb{E}_\mu(\mathfrak{z})(u_j u), \mathbb{E}_\nu(\mathfrak{z})(u_j u))$ which implies that $u_j \sim_{\mathfrak{z}} u$. Conversely, due to the symmetry of indiscernibility relation $R_U(\mathfrak{z})$, by assuming $u_j \sim_{\mathfrak{z}} u$, we can simply acquire $u_i \sim_{\mathfrak{z}} u$.

(ii)⇒(iii) Let $\mathbb{N}(\mathfrak{z})(u_i) = \{\langle u, (\mathbb{E}_\mu(\mathfrak{z})(u_i u), \mathbb{E}_\nu(\mathfrak{z})(u_i u))\rangle : \mathbb{E}_\mu(\mathfrak{z})(u_i u) > 0 \textit{ or } \mathbb{E}_\nu(\mathfrak{z})(u_i u) > 0\}$. We want to show that $\mathbb{N}(\mathfrak{z})(u_i) = \mathbb{N}(\mathfrak{z})(u_j)$. Note that for an arbitrary u, $u_i \sim_{\mathfrak{z}} u$ implies $u \in \mathbb{N}(\mathfrak{z})(u_i)$. But according to assumption, $u_i \sim_{\mathfrak{z}} u$ implies $u_j \sim_{\mathfrak{z}} u$ which shows that $u \in \mathbb{N}(\mathfrak{z})(u_j)$. Also $(\mathbb{E}_\mu(\mathfrak{z})(u_i u), \mathbb{E}_\nu(\mathfrak{z})(u_i u)) = (\mathbb{E}_\mu(\mathfrak{z})(u_j u), \mathbb{E}_\nu(\mathfrak{z})(u_j u))$. Therefore, $\mathbb{N}(\mathfrak{z})(u_i) = \mathbb{N}(\mathfrak{z})(u_j)$.

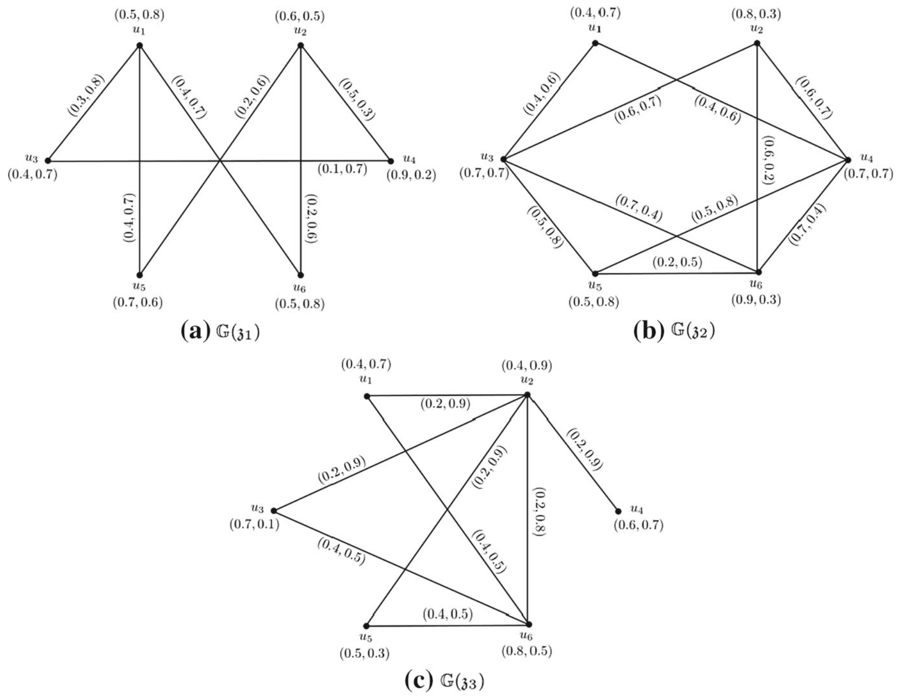

Fig. 1 Pythagorean fuzzy soft graph $\mathbb{G}$

(iii)⇒(i) Suppose that $\mathbb{N}(\mathfrak{z})(u_i) = \mathbb{N}(\mathfrak{z})(u_j)$. We want to prove that $u_i R_U(\mathfrak{z}) u_j$, i.e., $f_{\mathfrak{z}}(u_i u) = f_{\mathfrak{z}}(u_j u)$, $\forall u$. Observe that if, for any arbitrary u, $f_{\mathfrak{z}}(u_i u) = f_{\mathfrak{z}}(u_j u)$ then $u_i \sim_{\mathfrak{z}} u \Leftrightarrow u_j \sim_{\mathfrak{z}} u$ which means $u \in \mathbb{N}(\mathfrak{z})(u_i) \Leftrightarrow u \in \mathbb{N}(\mathfrak{z})(u_j)$. Since u is arbitrary, statement (i) holds for each $u \in U$.

□

From previous theorem, we can obtain following result.

Proposition 1 *Let $\mathbb{G} = (\mathbb{U}, \mathbb{E}, Z)$ be a Pythagorean fuzzy soft graph over U and $\mathbb{G}(\mathfrak{z}) = (\mathbb{U}(\mathfrak{z}), \mathbb{E}(\mathfrak{z}))$ be the Pythagorean fuzzy graph corresponding to parameter $\mathfrak{z}$. If $u_i R_U(\mathfrak{z}) u_j$, then $\mathbb{N}(\mathfrak{z})(u_i) \cap \mathbb{U}(\mathfrak{z}) = \mathbb{N}(\mathfrak{z})(u_j) \cap \mathbb{U}(\mathfrak{z})$.*

Proof The proof directly follows from Theorem 1. □

Example 2 Let $\mathbb{G} = (\mathbb{U}, \mathbb{E}, Z)$ be a Pythagorean fuzzy soft graph where $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ and $Z = \{\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3\}$ is the set of parameters. The Pythagorean fuzzy soft graph $\mathbb{G}$ is shown in Fig. 1.

The corresponding Pythagorean fuzzy soft information system $\mathbb{I}$ is a parameterized family of Pythagorean fuzzy adjacency matrices, i.e., $\mathbb{I}(\mathfrak{z})$, for all $\mathfrak{z}$ and are given in Tables 5, 6 and 7, respectively.

Let us consider $U' = U$, then

Table 5 Pythagorean fuzzy information system $\mathbb{I}(\mathfrak{z}_1)$

$\mathbb{I}(\mathfrak{z}_1)$	u_1	u_2	u_3	u_4	u_5	u_6
u_1	(0, 0)	(0, 0)	(0.3, 0.8)	(0, 0)	(0.4, 0.7)	(0.4, 0.7)
u_2	(0, 0)	(0, 0)	(0, 0)	(0.5, 0.3)	(0.2, 0.6)	(0.2, 0.6)
u_3	(0.3, 0.8)	(0, 0)	(0, 0)	(0.1, 0.7)	(0, 0)	(0, 0)
u_4	(0, 0)	(0.5, 0.3)	(0.1, 0.7)	(0, 0)	(0, 0)	(0, 0)
u_5	(0.4, 0.7)	(0.2, 0.6)	(0, 0)	(0, 0)	(0, 0)	(0, 0)
u_6	(0.4, 0.7)	(0.2, 0.6)	(0, 0)	(0, 0)	(0, 0)	(0, 0)

$$\begin{aligned}
[u_1]_U(\mathfrak{z}_1) &= \{u_1\}, [u_2]_U(\mathfrak{z}_1) = \{u_2\}, [u_3]_U(\mathfrak{z}_1) = \{u_3\}, [u_4]_U(\mathfrak{z}_1) = \{u_4\}, \\
&\qquad [u_5]_U(\mathfrak{z}_1) = \{u_5, u_6\} = [u_6]_U(\mathfrak{z}_1), \\
[u_1]_U(\mathfrak{z}_2) &= \{u_1\}, [u_2]_U(\mathfrak{z}_2) = \{u_2\}, [u_3]_U(\mathfrak{z}_2) \\
&= \{u_3, u_4\} = [u_4]_U(\mathfrak{z}_2), [u_5]_U(\mathfrak{z}_2) = \{u_5\}, [u_6]_U(\mathfrak{z}_2) = \{u_6\}, \\
[u_1]_U(\mathfrak{z}_3) &= \{u_1, u_3, u_5\} = [u_3]_U(\mathfrak{z}_3) = [u_5]_U(\mathfrak{z}_3), [u_2]_U(\mathfrak{z}_3) \\
&= \{u_2\}, [u_4]_U(\mathfrak{z}_3) = \{u_4\}, [u_6]_U(\mathfrak{z}_3) = \{u_6\},
\end{aligned}$$

and the corresponding parameterized family of $U-$indiscernibility partitions of $\mathbb{G}$ is given by

$$\begin{aligned}
P_U(\mathfrak{z}_1) &= \{\{u_1\}, \{u_2\}, \{u_3\}, \{u_4\}, \{u_5, u_6\}\}, \\
P_U(\mathfrak{z}_2) &= \{\{u_1\}, \{u_2\}, \{u_3, u_4\}, \{u_5\}, \{u_6\}\}, \\
P_U(\mathfrak{z}_3) &= \{\{u_1, u_3, u_5\}, \{u_2\}, \{u_4\}, \{u_6\}\}.
\end{aligned}$$

If $U' = \{u_1, u_2, u_5\}$, then the corresponding $U'-$indiscernibility partitions of $\mathbb{G}$ are

$$\begin{aligned}
P_{U'}(\mathfrak{z}_1) &= \{\{u_1\}, \{u_2\}, \{u_3\}, \{u_4\}, \{u_5, u_6\}\}, \\
P_{U'}(\mathfrak{z}_2) &= \{\{u_1, u_2, u_5\}, \{u_3, u_4\}, \{u_6\}\}, \\
P_{U'}(\mathfrak{z}_3) &= \{\{u_1, u_3, u_4, u_5\}, \{u_2\}, \{u_6\}\}.
\end{aligned}$$

In this way, we can find different levels of granularity with respect to each parameter by taking into account different subsets of universe U.

As already mentioned above, the Pythagorean fuzzy soft information system $\mathbb{I}$ of a Pythagorean fuzzy soft graph $\mathbb{G}$ is, in fact, a parameterized family of adjacency matrices of $\mathbb{G}$. In the following, we will discuss some concepts related to parameterized partitions of $\mathbb{G}$, which are the partitions of corresponding Pythagorean fuzzy soft information system $\mathbb{I}$ as well. So, it will not be mentioned over and over again in the context.

Table 6 Pythagorean fuzzy information system $\mathbb{I}(\mathfrak{z}_2)$

$\mathbb{I}(\mathfrak{z}_2)$	u_1	u_2	u_3	u_4	u_5	u_6
u_1	(0, 0)	(0, 0)	(0.4, 0.6)	(0.4, 0.6)	(0, 0)	(0, 0)
u_2	(0, 0)	(0, 0)	(0.6, 0.7)	(0.6, 0.7)	(0, 0)	(0.6, 0.2)
u_3	(0.4, 0.6)	(0.6, 0.7)	(0, 0)	(0, 0)	(0.5, 0.8)	(0.7, 0.4)
u_4	(0.4, 0.6)	(0.6, 0.7)	(0, 0)	(0, 0)	(0.5, 0.8)	(0.7, 0.4)
u_5	(0, 0)	(0, 0)	(0.5, 0.8)	(0.5, 0.8)	(0, 0)	(0.2, 0.5)
u_6	(0, 0)	(0.6, 0.2)	(0.7, 0.4)	(0.7, 0.4)	(0.2, 0.5)	(0, 0)

Table 7 Pythagorean fuzzy information system $\mathbb{I}(\mathfrak{z}_3)$

$\mathbb{I}(\mathfrak{z}_3)$	u_1	u_2	u_3	u_4	u_5	u_6
u_1	(0, 0)	(0.2, 0.9)	(0, 0)	(0, 0)	(0, 0)	(0.4, 0.5)
u_2	(0.2, 0.9)	(0, 0)	(0.2, 0.9)	(0.2, 0.9)	(0.2, 0.9)	(0.2, 0.8)
u_3	(0, 0)	(0.2, 0.9)	(0, 0)	(0, 0)	(0, 0)	(0.4, 0.5)
u_4	(0, 0)	(0.2, 0.9)	(0, 0)	(0, 0)	(0, 0)	(0, 0)
u_5	(0, 0)	(0.2, 0.9)	(0, 0)	(0, 0)	(0, 0)	(0.4, 0.5)
u_6	(0.4, 0.5)	(0.2, 0.8)	(0.4, 0.5)	(0, 0)	(0.4, 0.5)	(0, 0)

Definition 11 Let $\mathbb{G} = (\mathbb{U}, \mathbb{E}, Z)$ be a Pythagorean fuzzy soft graph over U and $\mathbb{G}(\mathfrak{z})$ be the corresponding Pythagorean fuzzy graph relative to parameter $\mathfrak{z}$. Let $u \in U$, then u is said to be an indispensable element for $\mathbb{G}(\mathfrak{z})$ if U-indiscernibility partition is not equals to the $U \setminus \{u\}$-indiscernibility partition with respect to that parameter, i.e., $P_U(\mathfrak{z}) \neq P_{U\setminus\{u\}}(\mathfrak{z})$. The set of all indispensable elements of U for parameter $\mathfrak{z}$ is called core of $\mathbb{G}(\mathfrak{z})$ and is denoted by $\mathcal{C}(\mathbb{G}(\mathfrak{z}))$. And, the parameterized family of core for the Pythagorean fuzzy soft graph is denoted by $\mathcal{C}(\mathbb{G})$. Additionally, the core number of $\mathbb{G}$, denoted by, $\mathcal{C}_n(\mathbb{G})$ is an m-tuple of the cardinalities $|\mathcal{C}(\mathbb{G}(\mathfrak{z}))|$ if $|Z| = m$.

Definition 12 Let $\mathbb{G} = (\mathbb{U}, \mathbb{E}, Z)$ be a Pythagorean fuzzy soft graph over U and $\mathbb{G}(\mathfrak{z})$ be the corresponding Pythagorean fuzzy graph with respect to parameter $\mathfrak{z}$. The reduct of $\mathbb{G}(\mathfrak{z})$ is a set $U' \subseteq U$ satisfying the following conditions:

(i) $P_U(\mathfrak{z}) = P_{U'}(\mathfrak{z})$,
(ii) $P_U(\mathfrak{z}) \neq P_{U'\setminus\{u\}}(\mathfrak{z})$, for all $u \in U'$.

We represent the family of all reducts of $\mathbb{G}(\mathfrak{z})$ by $\mathcal{R}(\mathbb{G}(\mathfrak{z}))$ and, the parameterized collection of reduct families for all parameters $\mathfrak{z}$ is the reduct of $\mathbb{G}$ denoted by $\mathcal{R}(\mathbb{G})$.

Example 3 Consider the Pythagorean fuzzy soft graph $\mathbb{G} = (\mathbb{U}, \mathbb{E}, Z)$ given in Fig. 1 together with the corresponding Pythagorean fuzzy soft information system $\mathbb{I}$. Observe that u_6 is an indispensable element in $\mathbb{G}(\mathfrak{z}_3)$ since $P_U(\mathfrak{z}_3) \neq P_{U\setminus\{u_6\}}(\mathfrak{z}_3)$. The core of each Pythagorean fuzzy graph $\mathbb{G}(\mathfrak{z})$ is given by

$$\mathcal{C}(\mathbb{G}(\mathfrak{z}_1)) = \varnothing, \quad \mathcal{C}(\mathbb{G}(\mathfrak{z}_2)) = \varnothing, \quad \mathcal{C}(\mathbb{G}(\mathfrak{z}_3)) = \{u_6\},$$

and consequently, the core of $\mathbb{G}$ is a soft set

$$\mathcal{C}(\mathbb{G}) = \left\{ \frac{\varnothing}{\mathfrak{z}_1}, \frac{\varnothing}{\mathfrak{z}_2}, \frac{\{u_6\}}{\mathfrak{z}_3} \right\}.$$

The core number of the considered Pythagorean fuzzy soft graph $\mathbb{G}$ is a 3-tuple $\mathcal{C}_n(\mathbb{G}) = (0, 0, 1)$. Further, using Definition 12, the reduct of $\mathbb{G}$ is obtained as

$$\mathcal{R}(\mathbb{G}) = \left\{ \frac{\{\{u_1, u_3\}, \{u_1, u_5\}, \{u_1, u_6\}, \{u_2, u_4\}, \{u_2, u_5\}, \{u_2, u_6\}, \{u_3, u_4\}, \{u_3, u_5, u_6\}, \{u_4, u_5, u_6\}\}}{\mathfrak{z}_1}, \right.$$
$$\left. \frac{\{\{u_3\}, \{u_4\}, \{u_2, u_6\}, \{u_5, u_6\}\}}{\mathfrak{z}_2}, \frac{\{\{u_1, u_6\}, \{u_2, u_6\}, \{u_3, u_6\}, \{u_5, u_6\}\}}{\mathfrak{z}_3} \right\}.$$

We now introduce the concept of extended core for a Pythagorean fuzzy soft graph $\mathbb{G}$ since we have seen in previous example that the ordinary core of $\mathbb{G}(\mathfrak{z})$ is an empty set for parameters $\mathfrak{z}_1, \mathfrak{z}_2$.

Definition 13 Let $\mathbb{G} = (\mathbb{U}, \mathbb{E}, Z)$ be a Pythagorean fuzzy soft graph over U and $\mathbb{G}(\mathfrak{z})$ be the corresponding Pythagorean fuzzy graph with respect to parameter $\mathfrak{z}$. A subset $U' \subseteq U$ is said to be an essential of $\mathbb{G}(\mathfrak{z})$ when:

(i) $P_{U \setminus U'}(\mathfrak{z}) \neq P_U(\mathfrak{z})$,
(ii) For all other proper subsets $V \subset U'$, $P_{U \setminus V}(\mathfrak{z}) = P_U(\mathfrak{z})$.

The family of all essential subsets of $\mathbb{G}(\mathfrak{z})$ is its extended core denoted by $\mathrm{ESS}(\mathbb{G}(\mathfrak{z}))$. Further, $\mathrm{ESS}_i(\mathbb{G}(\mathfrak{z}))$, $1 \leq i \leq n$, where $n = |U|$ is the collection

$$\mathrm{ESS}_i(\mathbb{G}(\mathfrak{z})) = \{U' \in \mathrm{ESS}(\mathbb{G}(\mathfrak{z})) : |U'| = i\}.$$

One can also write the essential numerical sequence of $\mathbb{G}(\mathfrak{z})$ as an n-tuple

$$\mathrm{ENS}(\mathbb{G}(\mathfrak{z})) = (|\mathrm{ESS}_1(\mathbb{G}(\mathfrak{z}))|, |\mathrm{ESS}_2(\mathbb{G}(\mathfrak{z}))|, ..., |\mathrm{ESS}_n(\mathbb{G}(\mathfrak{z}))|).$$

Moreover, $\mathrm{Edim}(\mathbb{G}(\mathfrak{z})) = \min\{i : |\mathrm{ESS}_i(\mathbb{G}(\mathfrak{z}))| \neq 0\}$ is the essential dimension of $\mathbb{G}(\mathfrak{z})$.

We present all these notions in the following example.

Example 4 Let us take into consideration the Pythagorean fuzzy soft graph $\mathbb{G} = (\mathbb{U}, \mathbb{E}, Z)$ of Example 2. For all $\mathfrak{z}$, $\mathrm{ESS}_1(\mathbb{G}(\mathfrak{z}))$ is given by

$$\mathrm{ESS}_1(\mathbb{G}(\mathfrak{z}_1)) = \varnothing, \quad \mathrm{ESS}_1(\mathbb{G}(\mathfrak{z}_2)) = \varnothing, \quad \mathrm{ESS}_1(\mathbb{G}(\mathfrak{z}_3)) = \{\{u_6\}\}$$

or

$$\mathrm{ESS}_1(\mathbb{G}) = \left\{ \frac{\varnothing}{\mathfrak{z}_1}, \frac{\varnothing}{\mathfrak{z}_2}, \frac{\{\{u_6\}\}}{\mathfrak{z}_3} \right\}.$$

Similarly, for all i, the essential subsets obtained are given in Table 8.

Table 8 Essential of Pythagorean fuzzy soft graph in Fig. 1

$\mathrm{ESS}_i(\mathbb{G}(\mathfrak{z}))$	$\mathfrak{z}_1$	$\mathfrak{z}_2$	$\mathfrak{z}_3$
$i=1$	$\varnothing$	$\varnothing$	$\{u_6\}$
$i=2$	$\varnothing$	$\varnothing$	$\varnothing$
$i=3$	$\varnothing$	$\{u_3, u_4, u_6\}$	$\varnothing$
$i=4$	$\{u_1, u_2, u_3, u_4\}, \{u_1, u_2, u_3, u_5\}, \{u_1, u_2, u_3, u_6\}, \{u_1, u_2, u_4, u_5\}, \{u_1, u_2, u_4, u_6\}, \{u_1, u_4, u_5, u_6\}, \{u_2, u_3, u_5, u_6\}, \{u_3, u_4, u_5, u_6\}$	$\{u_1, u_3, u_5, u_6\}, \{u_2, u_3, u_4, u_5\}$	$\{u_1, u_2, u_3, u_5\}$
$i=5$	$\varnothing$	$\varnothing$	$\varnothing$
$i=6$	$\varnothing$	$\varnothing$	$\varnothing$

The corresponding essential numerical sequences are

$$\mathrm{ENS}(\mathbb{G}(\mathfrak{z}_1)) = (0, 0, 0, 8, 0, 0), \quad \mathrm{ENS}(\mathbb{G}(\mathfrak{z}_2)) = (0, 0, 1, 2, 0, 0),$$
$$\mathrm{ENS}(\mathbb{G}(\mathfrak{z}_3)) = (1, 0, 0, 1, 0, 0).$$

Finally, the essential dimensions of Pythagorean fuzzy graphs relative to each parameter are

$$\mathrm{Edim}(\mathbb{G}(\mathfrak{z}_1)) = 4, \quad \mathrm{Edim}(\mathbb{G}(\mathfrak{z}_2)) = 3, \quad \mathrm{Edim}(\mathbb{G}(\mathfrak{z}_3)) = 1.$$

As a consequence, $\mathrm{Edim}(\mathbb{G}) = (4, 3, 1)$ is the essential dimension of $\mathbb{G}$.

Proposition 2 *Let $\mathbb{G} = (\mathbb{U}, \mathbb{E}, Z)$ be a Pythagorean fuzzy soft graph over U. Then*

$$\mathcal{C}(\mathbb{G}) = \left\{ \bigcup U' \in \mathrm{ESS}_1(\mathbb{G}(\mathfrak{z})) : \mathfrak{z} \in Z \right\}.$$

Proof It can be deduced directly from the definitions of $\mathcal{C}(\mathbb{G})$ and $\mathrm{ESS}_1(\mathbb{G})$. □

4 Pythagorean fuzzy soft discernibility matrix

We now define the concept of discernibility matrix for a Pythagorean fuzzy soft information system of a Pythagorean fuzzy soft graph. For a Pythagorean fuzzy soft graph $\mathbb{G} = (\mathbb{U}, \mathbb{E}, Z)$, we can obtain a family of discernibility matrices termed as Pythagorean fuzzy soft discernibility matrix Δ. Corresponding to each parameter $\mathfrak{z}$, the entries $\Delta^{\mathfrak{z}}_{ij}$ of a discernibility matrix $\Delta^{\mathfrak{z}}$ should be Pythagorean fuzzy sets such that every element belonging to this set possesses a Pythagorean membership grade which depicts the extent to which the attributes i and j are discernible (through membership degree) and indiscernible (through non-membership degree) with one another. To obtain the entries of $\Delta^{\mathfrak{z}}$, we will use the Pythagorean fuzzy discernibility measure given by $\Delta^{\mathfrak{z}}_{ij} = \mathcal{N}(\mu_S(\mathfrak{z})(u_i u_j), \nu_S(\mathfrak{z})(u_i u_j))$, where $\mathcal{N}$ represents the negation of

Pythagorean fuzzy set $(\mu_S(\mathfrak{z})(u_i u_j), \nu_S(\mathfrak{z})(u_i u_j))$, where

$$\begin{aligned}
&(\mu_S(\mathfrak{z})(u_i u_j), \nu_S(\mathfrak{z})(u_i u_j)) \\
&= (\sqrt{1 - \max\{(\mathbb{E}_\mu(\mathfrak{z})(u_i u) - \mathbb{E}_\mu(\mathfrak{z})(u_j u))^2, (\mathbb{E}_\nu(\mathfrak{z})(u_i u) - \mathbb{E}_\nu(\mathfrak{z})(u_j u))^2\}}, \\
&\sqrt{\min\{(\mathbb{E}_\mu(\mathfrak{z})(u_i u) - \mathbb{E}_\mu(\mathfrak{z})(u_j u))^2, (\mathbb{E}_\nu(\mathfrak{z})(u_i u) - \mathbb{E}_\nu(\mathfrak{z})(u_j u))^2\}} : u \in U), \qquad (1)
\end{aligned}$$

obtained by using Pythagorean fuzzy similarity measure [1].

Example 5 Let us take into consideration the Pythagorean fuzzy soft graph $\mathbb{G} = (\mathbb{U}, \mathbb{E}, Z)$ of Example 2. We can obtain the Pythagorean fuzzy soft discernibility matrix by taking the negation of the Pythagorean fuzzy set computed by using Equation 1. The entries $\Delta^{\mathfrak{z}}_{ij}$ of this matrix $\Delta^{\mathfrak{z}}$ for all parameters $\mathfrak{z}$ are given in Table 9. Note that Δ is the collection of symmetric Pythagorean fuzzy discernibility matrices.

5 Application: protein–protein interaction networks

Proteins direct almost all activities within the cell of an organism. It includes distinct biological activities like the acceleration of metabolic reactions, transportation of molecules, replication of DNA and the maintenance of a cell's structure. Proteins work independently as well as interact with each other to mutually carry out various biological functions. The study of PPIs is essential to apprehend protein functioning inside the living cell. This is because PPIs are involved in approximately all biological processes which result in the construction of protein complexes.

A protein complex is a group of multiple proteins that are linked with each other through non-covalent PPIs. Protein complexes are in the form of quaternary structures and can be regraded as multimolecular machines that play a significant role in signal transduction, mRNA transcription, DNA translation and in other biological processes. There are numerous methods for the detection and identification of protein complexes. Some of these methods determine them in the PPI networks of the corresponding problem. Here a PPI network is, in fact, a graph each of whose vertex represents a protein and its edges join the interacting protein pairs. The interaction prediction methods can efficiently discover the interacting protein pairs in a PPI network. A PPI network might be a disconnected graph consisting of isolated components out of which the largest component is referred to as main component.

PPI networks depict an enormous number of physical interactions among proteins and are worthwhile because proteins possessing functional similarity are responsible for complex formation. Additionally, the study of these networks assists in the understanding of those processes that either initiate or continue the disease development. They can also be used for the examination of diseases. Since the disease genes try to interact with one another therefore, PPI networks are significant to identify the neighbors of known facilitating genes.

Table 9 Entries of Pythagorean fuzzy discernibility matrices $\Delta^{\mathfrak{z}}$

	$\mathfrak{z}_1$	$\mathfrak{z}_2$	$\mathfrak{z}_3$
$\Delta^{\mathfrak{z}}_{12}$	$\{\langle u_3, (0.3, 0.6)\rangle, \langle u_4, (0.3, 0.87)\rangle, \langle u_5, (0.1, 0.98)\rangle, \langle u_6, (0.1, 0.98)\rangle\}$	$\{\langle u_3, (0.1, 0.98)\rangle, \langle u_4, (0.1, 0.98)\rangle, \langle u_6, (0.2, 0.8)\rangle\}$	$\{\langle u_1, (0.2, 0.43)\rangle, \langle u_2, (0.2, 0.43)\rangle, \langle u_3, (0.2, 0.43)\rangle, \langle u_4, (0.2, 0.43)\rangle, \langle u_5, (0.2, 0.43)\rangle, \langle u_6, (0.2, 0.95)\rangle\}$
$\Delta^{\mathfrak{z}}_{13}$	$\{\langle u_1, (0.3, 0.6)\rangle, \langle u_3, (0.3, 0.6)\rangle, \langle u_4, (0.1, 0.71)\rangle, \langle u_5, (0.4, 0.71)\rangle, \langle u_6, (0.4, 0.71)\rangle\}$	$\{\langle u_1, (0.4, 0.8)\rangle, \langle u_2, (0.6, 0.71)\rangle, \langle u_3, (0.4, 0.8)\rangle, \langle u_4, (0.4, 0.8)\rangle, \langle u_5, (0.5, 0.6)\rangle, \langle u_6, (0.4, 0.71)\rangle\}$	$\{\}$
$\Delta^{\mathfrak{z}}_{14}$	$\{\langle u_2, (0.3, 0.87)\rangle, \langle u_3, (0.1, 0.98)\rangle, \langle u_5, (0.4, 0.71)\rangle, \langle u_6, (0.4, 0.71)\rangle\}$	$\{\langle u_1, (0.4, 0.8)\rangle, \langle u_2, (0.6, 0.71)\rangle, \langle u_3, (0.4, 0.8)\rangle, \langle u_4, (0.4, 0.8)\rangle, \langle u_5, (0.5, 0.6)\rangle, \langle u_6, (0.4, 0.71)\rangle\}$	$\{\langle u_6, (0.4, 0.87)\rangle\}$
$\Delta^{\mathfrak{z}}_{15}$	$\{\langle u_1, (0.4, 0.71)\rangle, \langle u_2, (0.2, 0.8)\rangle, \langle u_3, (0.3, 0.6)\rangle, \langle u_5, (0.4, 0.71)\rangle, \langle u_6, (0.4, 0.71)\rangle\}$	$\{\langle u_3, (0.1, 0.98)\rangle, \langle u_4, (0.1, 0.98)\rangle, \langle u_6, (0.2, 0.87)\rangle\}$	$\{\}$
$\Delta^{\mathfrak{z}}_{16}$	$\{\langle u_1, (0.4, 0.71)\rangle, \langle u_2, (0.2, 0.8)\rangle, \langle u_3, (0.3, 0.6)\rangle, \langle u_5, (0.4, 0.71)\rangle, \langle u_6, (0.4, 0.71)\rangle\}$	$\{\langle u_2, (0.2, 0.8)\rangle, \langle u_3, (0.2, 0.95)\rangle, \langle u_4, (0.2, 0.95)\rangle, \langle u_5, (0.2, 0.87)\rangle\}$	$\{\langle u_1, (0.4, 0.87)\rangle, \langle u_2, (0.0, 0.99)\rangle, \langle u_3, (0.4, 0.87)\rangle, \langle u_5, (0.4, 0.87)\rangle, \langle u_6, (0.4, 0.87)\rangle\}$
$\Delta^{\mathfrak{z}}_{23}$	$\{\langle u_1, (0.3, 0.6)\rangle, \langle u_4, (0.4, 0.92)\rangle, \langle u_5, (0.2, 0.8)\rangle, \langle u_6, (0.2, 0.8)\rangle\}$	$\{\langle u_1, (0.4, 0.8)\rangle, \langle u_2, (0.6, 0.71)\rangle, \langle u_3, (0.6, 0.71)\rangle, \langle u_4, (0.6, 0.71)\rangle, \langle u_5, (0.5, 0.6)\rangle, \langle u_6, (0.1, 0.98)\rangle\}$	$\{\langle u_1, (0.2, 0.43)\rangle, \langle u_2, (0.2, 0.43)\rangle, \langle u_3, (0.2, 0.43)\rangle, \langle u_4, (0.2, 0.43)\rangle, \langle u_5, (0.2, 0.43)\rangle, \langle u_6, (0.2, 0.95)\rangle\}$
$\Delta^{\mathfrak{z}}_{24}$	$\{\langle u_2, (0.3, 0.87)\rangle, \langle u_3, (0.1, 0.71)\rangle, \langle u_4, (0.3, 0.87)\rangle, \langle u_5, (0.2, 0.8)\rangle, \langle u_6, (0.2, 0.8)\rangle\}$	$\{\langle u_1, (0.4, 0.8)\rangle, \langle u_2, (0.6, 0.71)\rangle, \langle u_3, (0.6, 0.71)\rangle, \langle u_4, (0.6, 0.71)\rangle, \langle u_5, (0.5, 0.6)\rangle, \langle u_6, (0.1, 0.98)\rangle\}$	$\{\langle u_1, (0.2, 0.43)\rangle, \langle u_2, (0.2, 0.43)\rangle, \langle u_3, (0.2, 0.43)\rangle, \langle u_4, (0.2, 0.43)\rangle, \langle u_5, (0.2, 0.43)\rangle, \langle u_6, (0.2, 0.6)\rangle\}$

Table 9 continued

	$\mathfrak{z}1$	$\mathfrak{z}2$	$\mathfrak{z}3$
$\Delta^{\mathfrak{z}}_{25}$	$\{\langle u_1, (0.4, 0.71)\rangle, \langle u_2, (0.2, 0.8)\rangle, \langle u_4, (0.3, 0.87)\rangle, \langle u_5, (0.2, 0.8)\rangle, \langle u_6, (0.2, 0.8)\rangle\}$	$\{\langle u_3, (0.1, 0.99)\rangle, \langle u_4, (0.1, 0.99)\rangle, \langle u_6, (0.3, 0.92)\rangle\}$	$\{\langle u_1, (0.2, 0.43)\rangle, \langle u_2, (0.2, 0.43)\rangle, \langle u_3, (0.2, 0.43)\rangle, \langle u_4, (0.2, 0.43)\rangle, \langle u_5, (0.2, 0.43)\rangle, \langle u_6, (0.2, 0.95)\rangle\}$
$\Delta^{\mathfrak{z}}_{26}$	$\{\langle u_1, (0.4, 0.71)\rangle, \langle u_2, (0.2, 0.8)\rangle, \langle u_4, (0.3, 0.87)\rangle, \langle u_5, (0.2, 0.8)\rangle, \langle u_6, (0.2, 0.8)\rangle\}$	$\{\langle u_2, (0.2, 0.8)\rangle, \langle u_3, (0.1, 0.95)\rangle, \langle u_4, (0.1, 0.95)\rangle, \langle u_5, (0.2, 0.87)\rangle, \langle u_6, (0.2, 0.8)\rangle\}$	$\{\langle u_1, (0.2, 0.92)\rangle, \langle u_2, (0.2, 0.6)\rangle, \langle u_3, (0.2, 0.92)\rangle, \langle u_4, (0.2, 0.43)\rangle, \langle u_5, (0.2, 0.92)\rangle, \langle u_6, (0.2, 0.6)\rangle\}$
$\Delta^{\mathfrak{z}}_{34}$	$\{\langle u_1, (0.3, 0.6)\rangle, \langle u_2, (0.3, 0.87)\rangle, \langle u_3, (0.1, 0.71)\rangle, \langle u_4, (0.1, 0.71)\rangle\}$	$\{\}$	$\{\langle u_6, (0.4, 0.87)\rangle\}$
$\Delta^{\mathfrak{z}}_{35}$	$\{\langle u_1, (0.1, 0.99)\rangle, \langle u_2, (0.2, 0.8)\rangle, \langle u_4, (0.1, 0.71)\rangle\}$	$\{\langle u_1, (0.4, 0.8)\rangle, \langle u_2, (0.6, 0.71)\rangle, \langle u_3, (0.5, 0.6)\rangle, \langle u_4, (0.5, 0.6)\rangle \langle u_5, (0.5, 0.6)\rangle, \langle u_6, (0.1, 0.87)\rangle\}$	$\{\}$
$\Delta^{\mathfrak{z}}_{36}$	$\{\langle u_1, (0.1, 0.99)\rangle, \langle u_2, (0.2, 0.8)\rangle, \langle u_4, (0.1, 0.71)\rangle\}$	$\{\langle u_1, (0.4, 0.8)\rangle, \langle u_2, (0.0, 0.87)\rangle, \langle u_3, (0.4, 0.71)\rangle, \langle u_4, (0.4, 0.71)\rangle, \langle u_5, (0.3, 0.95)\rangle, \langle u_6, (0.4, 0.71)\rangle\}$	$\{\langle u_1, (0.4, 0.87)\rangle, \langle u_2, (0.0, 0.99)\rangle, \langle u_3, (0.4, 0.87)\rangle, \langle u_5, (0.4, 0.87)\rangle, \langle u_6, (0.4, 0.87)\rangle\}$
$\Delta^{\mathfrak{z}}_{45}$	$\{\langle u_1, (0.4, 0.71)\rangle, \langle u_2, (0.3, 0.95)\rangle, \langle u_3, (0.1, 0.71)\rangle\}$	$\{\langle u_1, (0.4, 0.8)\rangle, \langle u_2, (0.6, 0.71)\rangle, \langle u_3, (0.5, 0.6)\rangle, \langle u_4, (0.5, 0.6)\rangle \langle u_5, (0.5, 0.6)\rangle, \langle u_6, (0.1, 0.87)\rangle\}$	$\{\langle u_6, (0.4, 0.87)\rangle\}$
$\Delta^{\mathfrak{z}}_{46}$	$\{\langle u_1, (0.4, 0.71)\rangle, \langle u_2, (0.3, 0.95)\rangle, \langle u_3, (0.1, 0.71)\rangle\}$	$\{\langle u_1, (0.4, 0.8)\rangle, \langle u_2, (0.0, 0.87)\rangle, \langle u_3, (0.4, 0.71)\rangle, \langle u_4, (0.4, 0.71)\rangle, \langle u_5, (0.3, 0.95)\rangle, \langle u_6, (0.4, 0.71)\rangle\}$	$\{\langle u_1, (0.4, 0.87)\rangle, \langle u_2, (0.0, 0.99)\rangle, \langle u_3, (0.4, 0.87)\rangle, \langle u_5, (0.4, 0.87)\rangle\}$
$\Delta^{\mathfrak{z}}_{56}$	$\{\}$	$\{\langle u_2, (0.2, 0.8)\rangle, \langle u_3, (0.2, 0.92)\rangle, \langle u_4, (0.2, 0.92)\rangle, \langle u_5, (0.2, 0.87)\rangle, \langle u_6, (0.2, 0.87)\rangle\}$	$\{\langle u_1, (0.4, 0.87)\rangle, \langle u_2, (0.0, 0.99)\rangle, \langle u_3, (0.4, 0.87)\rangle, \langle u_5, (0.4, 0.87)\rangle, \langle u_6, (0.4, 0.87)\rangle\}$

5.1 Parkinson's disease

Parkinson's disease is a progressive neuro-degenerative disorder of human's central nervous system which affects nearly 1% of adults of age 60 and above. It initiates due to the neural degeneration in midbrain and its cause is unknown in most cases. Its symptoms are tremors at rest, cogwheel or plastic rigidity in muscles and bradykinesia in which the diseased person has difficulty in the initiation of voluntary movement that reduces walking speed. Postural instability can also be witnessed in severe cases. Genetic mutations are also believed to be one of the major causes of Parkinson's disease as almost 15% of victims already have a close relative with same disease. Genes are the segments of DNA that are involved in the production of polypeptide chains of proteins therefore, there exist a close relationship between genes associated with disease and proteins [35]. A very little research has been done on protein interactions which are associated to disease state.

Network science has made contributions in the study of many biological and biochemical processes as it gives visual representation to the interacting entities. As an example of Pythagorean fuzzy soft granular structures, we consider PPI networks established on the basis of gene expression profiles of Parkinson's disease [33]. Gene expression profile is an approach to discover all those genes which are used in the synthesis of proteins. With this technique, it is possible to investigate how body reacts to a disease or to its treatment. Two statistical approaches namely, 2-tailed t-test (2ttt) and significance analysis of microarrays (SAM) were adapted to acquire each possible differentially expressed gene. Afterwards, the PPI networks were developed in such a way that each protein appears only one time in either network.

The PPI networks whose data-sets were obtained from 2ttt and SAM are here symbolically denoted by $G(\mathfrak{z}_1)$ and $G(\mathfrak{z}_2)$, respectively. We call it collectively a soft graph $G = \{G(\mathfrak{z}_1), G(\mathfrak{z}_2)\}$ whose visual representation is given in Figs. 2 and 3, respectively. There are 406 vertices, 690 edges and 47 is largest degree of vertex in $G(\mathfrak{z}_1)$. Likewise, $G(\mathfrak{z}_2)$ possesses 121 vertices, 172 edges and 21 is the highest degree. The cumulative degree distribution for $G(\mathfrak{z}_1)$ and $G(\mathfrak{z}_2)$ are given in Figs. 4 and 5, respectively. In order to construct granules in a network, there must be some form of similarity in its entities. Since the PPI networks here are very complex therefore, we assume protein complexes in these networks as granules. Table 10 reports protein complexes in the considered PPI networks.

Each granule in a granular structure has a center c which is the vertex of highest degree among all vertices of granule. In a granule, the distance of each vertex with its center is the length of shortest path between them. We will use this distance so as to determine the Pythagorean membership grades of each vertex of a granule. These Pythagorean membership grades can be determined with the help of following Pythagorean membership function:

$$\mathcal{G}_{k\mu}(\mathfrak{z})(u) = \frac{1}{1+dist(c_k,u)}, \quad \mathcal{G}_{k\nu}(\mathfrak{z})(u) = 1 - \left(\frac{1}{1+dist(c_k,u)}\right)^2, \tag{2}$$

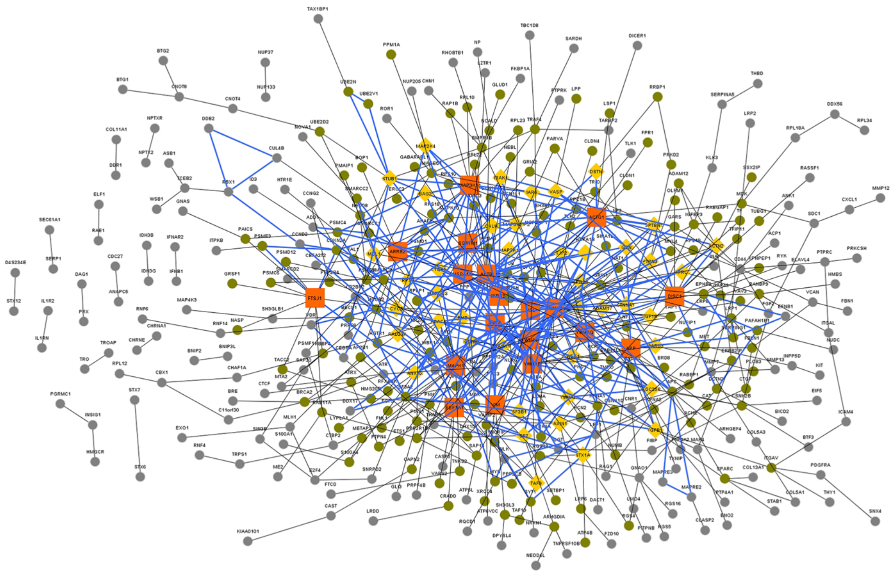

Fig. 2 PPI network $G(\mathfrak{z}_1)$ [33]

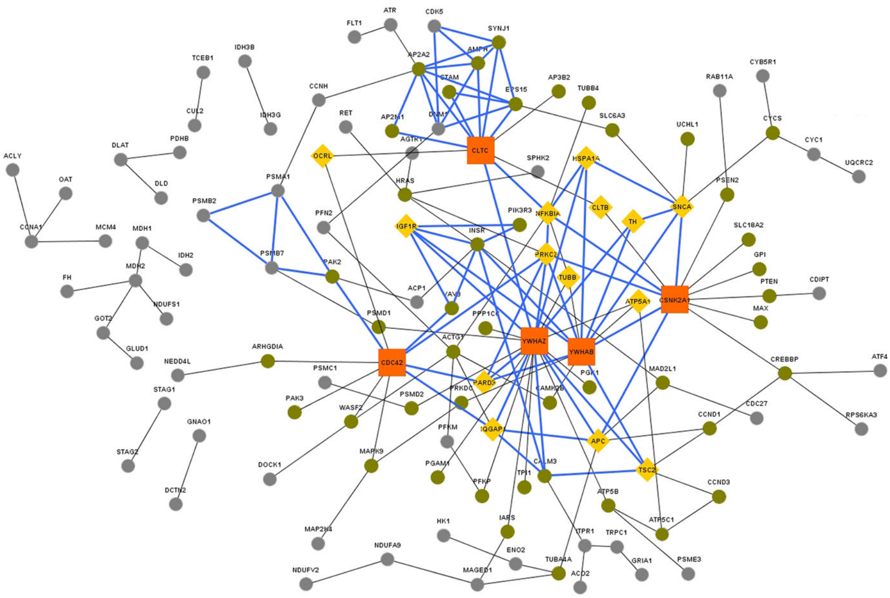

Fig. 3 PPI network $G(\mathfrak{z}_2)$ [33]

Table 10 List of complexes in $G(\mathfrak{z}_1)$ and $G(\mathfrak{z}_2)$ [33]

$G(\mathfrak{z}_1)$		$G(\mathfrak{z}_2)$	
Complex No.	Genes	Complex No.	Genes
1	CD2BP2, DDX17, NF2, PRPF8, SF3A2, SF3B1, SRRM2, WBP11	1	AMPH, AP2A2, AP2M1, EPS15, TH
2	ACTB, ACTG1, NF2, SMARCA4, SMARCC1, SMARCC2	2	AMPH, CLTC, DNM1, EPS15, SYNJ1
3	MAP2K1, MAPK1, YWHAB, YWHAE, YWHAH, YWHAZ	3	AMPH, CLTC, DNM1, SYNJ1 TH
4	ACTB, NF2, SMARCA4, SMARCC1, SMARCC2	4	APC, PSMA1, PSMB2, PSMB7, TH
5	ACTB, NF2, SMARCC1, SMARCC2, VDR	5	APC, CDC42, IQGAP1, TH
6	APC, APP, PSMC4, PSMD1, PSMD4	6	CDC42, PARD3, PRKCZ, TH
7	EEF1A1, MYB, RPLP0, RPLP1, RPS3	7	CSNK2A1, TH, YWHAB, YWHAZ
8	MAP2K1, YWHAB, YWHAE, YWHAH, YWHAZ		
9	NF2, SMARCA4, SMARCC1, SMARCC2, VDR		
10	PNN, PRPF8, SF3A2, SF3B1, SRRM2		

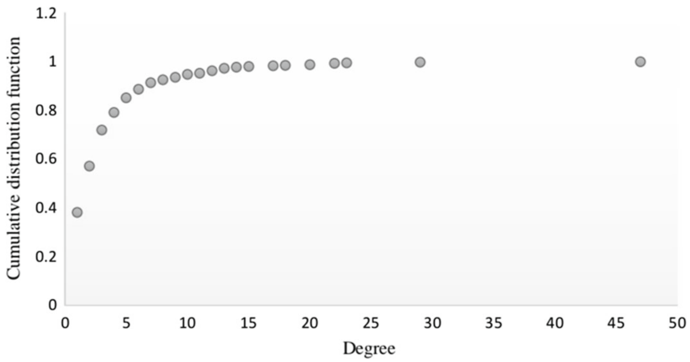

Fig. 4 Cumulative degree distribution of $G(\mathfrak{z}_1)$

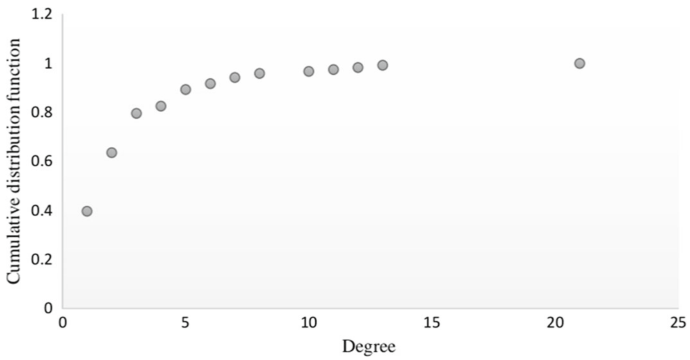

Fig. 5 Cumulative degree distribution of $G(\mathfrak{z}_2)$

where k denotes the k^{th} complex and $dist(c_k, u)$ is the distance between c_k and u in $G(\mathfrak{z})$. The granules (protein complexes) of $G(\mathfrak{z}_1)$ and $G(\mathfrak{z}_2)$ with their Pythagorean fuzzy grades are given in Tables 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26 and 27, respectively. The bold genes in each table represent the center of granule and, degree of each vertex, distance of each vertex from center and the Pythagorean fuzzy membership grades (computed using Eq. 2) are given in each Table. Notice that each Pythagorean fuzzy granule $\mathcal{G}(\mathfrak{z})$ of each graph $G(\mathfrak{z})$ is a Pythagorean fuzzy set.

We can also define the granular degree as the Pythagorean fuzzy cardinality of Pythagorean fuzzy granule. The granular degrees $d_{\mathcal{G}_k}(\mathfrak{z})$ of $\mathcal{G}_k(\mathfrak{z})$, $\forall \mathfrak{z}$ are given by

$$
\begin{aligned}
&d_{\mathcal{G}_1}(\mathfrak{z}_1) = (4.20, 5.46), \quad d_{\mathcal{G}_2}(\mathfrak{z}_1) = (2.91, 3.28), \quad d_{\mathcal{G}_3}(\mathfrak{z}_1) = (2.74, 4.36),\\
&\qquad d_{\mathcal{G}_4}(\mathfrak{z}_1) = (2.41, 3.47),\\
&d_{\mathcal{G}_5}(\mathfrak{z}_1) = (2.24, 3.61), \quad d_{\mathcal{G}_6}(\mathfrak{z}_1) = (2.16, 3.66), \quad d_{\mathcal{G}_7}(\mathfrak{z}_1) = (2.41, 3.47),\\
&\qquad d_{\mathcal{G}_8}(\mathfrak{z}_1) = (2.41, 3.47),\\
&d_{\mathcal{G}_9}(\mathfrak{z}_1) = (2.75, 3.19), \quad d_{\mathcal{G}_{10}}(\mathfrak{z}_1) = (3.00, 3.00),\\
&d_{\mathcal{G}_1}(\mathfrak{z}_2) = (2.75, 3.19), \quad d_{\mathcal{G}_2}(\mathfrak{z}_2) = (2.83, 3.14), \quad d_{\mathcal{G}_3}(\mathfrak{z}_2) = (2.83, 3.14),\\
&\qquad d_{\mathcal{G}_4}(\mathfrak{z}_2) = (1.98, 3.75),
\end{aligned}
$$

Table 11 $\mathcal{G}_1(\mathfrak{z}_1)$

Genes	Degree	Distance from center	Pythagorean fuzzy grades
CD2BP2	5	1	(0.5, 0.75)
DDX17	2	1	(0.5, 0.75)
NF2	5	4	(0.2, 0.96)
PRPF8	2	1	(0.5, 0.75)
SF3A2	**11**	**0**	**(1,0)**
SF3B1	4	1	(0.5, 0.75)
SRRM2	4	1	(0.5, 0.75)
WBP11	5	1	(0.5, 0.75)

Table 12 $\mathcal{G}_2(\mathfrak{z}_1)$

Genes	Degree	Distance from center	Pythagorean fuzzy grades
ACTB	**29**	**0**	**(1,0)**
ACTG1	23	1	(0.5, 0.75)
NF2	5	3	(0.25, 0.94)
SMARCA4	10	1	(0.5, 0.75)
SMARCC1	4	2	(0.33, 0.89)
SMARCC2	4	2	(0.33, 0.89)

Table 13 $\mathcal{G}_3(\mathfrak{z}_1)$

Genes	Degree	Distance from center	Pythagorean fuzzy grades
MAP2K1	4	3	(0.25, 0.94)
MAPK1	20	2	(0.33, 0.89)
YWHAB	22	2	(0.33, 0.89)
YWHAE	22	1	(0.5, 0.75)
YWHAH	6	2	(0.33, 0.89)
YWHAZ	**47**	**0**	**(1,0)**

Table 14 $\mathcal{G}_4(\mathfrak{z}_1)$

Genes	Degree	Distance from center	Pythagorean fuzzy grades
ACTB	**29**	**0**	**(1,0)**
NF2	5	3	(0.25, 0.94)
SMARCA4	10	1	(0.5, 0.75)
SMARCC1	4	2	(0.33, 0.89)
SMARCC2	4	2	(0.33, 0.89)

Table 15 $\mathcal{G}_5(\mathfrak{z}_1)$

Genes	Degree	Distance from center	Pythagorean fuzzy grades
ACTB	**29**	**0**	**(1,0)**
NF2	5	3	(0.25, 0.94)
SMARCC1	4	2	(0.33, 0.89)
SMARCC2	4	2	(0.33, 0.89)
VDR	4	2	(0.33, 0.89)

Table 16 $\mathcal{G}_6(\mathfrak{z}_1)$

Genes	Degree	Distance from center	Pythagorean fuzzy grades
APC	11	2	(0.33, 0.89)
APP	**15**	**0**	**(1,0)**
PSMC4	5	3	(0.25, 0.94)
PSMD1	4	2	(0.33, 0.89)
PSMD4	9	3	(0.25, 0.94)

Table 17 $\mathcal{G}_7(\mathfrak{z}_1)$

Genes	Degree	Distance from center	Pythagorean fuzzy grades
EEF1A1	**12**	**0**	**(1,0)**
MYB	4	3	(0.25, 0.94)
RPLP0	3	2	(0.33, 0.89)
RPLP1	2	1	(0.5, 0.75)
RPS3	5	2	(0.33, 0.89)

Table 18 $\mathcal{G}_8(\mathfrak{z}_1)$

Genes	Degree	Distance from center	Pythagorean fuzzy grades
MAP2K1	4	3	(0.25, 0.94)
YWHAB	22	2	(0.33, 0.89)
YWHAE	22	1	(0.5, 0.75)
YWHAH	6	2	(0.33, 0.89)
YWHAZ	47	**0**	**(1,0)**

$$d_{\mathcal{G}_5}(\mathfrak{z}_2) = (2.08, 2.58), \quad d_{\mathcal{G}_6}(\mathfrak{z}_2) = (2.25, 2.44), \quad d_{\mathcal{G}_7}(\mathfrak{z}_2) = (2.16, 2.53).$$

Table 19 $\mathcal{G}_9(\mathfrak{z}_1)$

Genes	Degree	Distance from center	Pythagorean fuzzy grades
NF2	5	3	(0.25, 0.94)
SMARCA4	**10**	**0**	**(1,0)**
SMARCC1	4	1	(0.5, 0.75)
SMARCC2	4	1	(0.5, 0.75)
VDR	4	1	(0.5, 0.75)

Table 20 $\mathcal{G}_{10}(\mathfrak{z}_1)$

Genes	Degree	Distance from center	Pythagorean fuzzy grades
PNN	6	1	(0.5, 0.75)
PRPF8	2	1	(0.5, 0.75)
SF3A2	**11**	**0**	**(1,0)**
SF3B1	4	1	(0.5, 0.75)
SRRM2	4	1	(0.5, 0.75)

Table 21 $\mathcal{G}_1(\mathfrak{z}_2)$

Genes	Degree	Distance from center	Pythagorean fuzzy grades
AMPH	5	1	(0.5, 0.75)
AP2A2	**8**	**0**	**(1,0)**
AP2M1	2	1	(0.5, 0.75)
EPS15	6	1	(0.5, 0.75)
TH	3	3	(0.25, 0.94)

Table 22 $\mathcal{G}_2(\mathfrak{z}_2)$

Genes	Degree	Distance from center	Pythagorean fuzzy grades
AMPH	5	1	(0.5, 0.75)
CLTC	**11**	**0**	**(1,0)**
DNM1	5	2	(0.33, 0.89)
EPS15	6	1	(0.5, 0.75)
SYNJ1	5	1	(0.5, 0.75)

6 Conclusions

Information granulation is a technique whereby the objects of a system, which seem to be different at the considered level of explanation, are put together in certain groups on the basis of some coherency to form a granule at a lower level. Granules in real-world systems are fuzzy in nature. For if we consider human head as a universe of discourse then the granules of organs of head do not possess sharply defined boundaries. This

Table 23 $\mathcal{G}_3(\mathfrak{z}_2)$

Genes	Degree	Distance from center	Pythagorean fuzzy grades
AMPH	5	1	(0.5, 0.75)
CLTC	**11**	**0**	**(1,0)**
DNM1	5	1	(0.5, 0.75)
SYNJ1	5	1	(0.5, 0.75)
TH	3	2	(0.33, 0.89)

Table 24 $\mathcal{G}_4(\mathfrak{z}_2)$

Genes	Degree	Distance from center	Pythagorean fuzzy grades
APC	**6**	**0**	**(1,0)**
PSMA1	4	4	(0.2, 0.96)
PSMB2	2	4	(0.2, 0.96)
PSMB7	4	3	(0.25, 0.94)
TH	3	2	(0.33, 0.89)

Table 25 $\mathcal{G}_5(\mathfrak{z}_2)$

Genes	Degree	Distance from center	Pythagorean fuzzy grades
APC	6	2	(0.33, 0.89)
CDC42	**10**	**0**	**(1,0)**
IQGAP1	5	1	(0.5, 0.75)
TH	3	3	(0.25, 0.94)

Table 26 $\mathcal{G}_6(\mathfrak{z}_2)$

Genes	Degree	Distance from center	Pythagorean fuzzy grades
CDC42	10	**0**	**(1,0)**
PARD3	4	1	(0.5, 0.75)
PRKCZ	7	1	(0.5, 0.75)
TH	3	3	(0.25, 0.94)

Table 27 $\mathcal{G}_7(\mathfrak{z}_2)$

Genes	Degree	Distance from center	Pythagorean fuzzy grades
CSNK2A1	12	2	(0.33, 0.89)
TH	3	1	(0.5, 0.75)
YWHAB	13	2	(0.33, 0.89)
YWHAZ	21	**0**	**(1,0)**

scrutiny urged the researchers to study the concept of granularity in different extensions of fuzzy set theory. Motivated by the advantages of Pythagorean fuzzy set theory and soft set theory, we have considered the Pythagorean fuzzy soft environment to study information granulation. Pythagorean fuzzy soft set theory efficiently represents both parameter-wise and membership dependent uncertainties. In this manuscript, we have described the transformation of Pythagorean fuzzy information systems obtained from Pythagorean fuzzy soft sets into one another. Additionally, for a Pythagorean fuzzy soft graph, Pythagorean fuzzy soft information system has been defined which is a parameterized family of Pythagorean fuzzy information systems. The Pythagorean fuzzy soft indiscernibility relation and the corresponding notions of reduct, core and extended core are well explained with examples. The Pythagorean fuzzy soft discernibility matrix obtained in this article is also very appealing since the discernibility among attributes obtained is in the form of Pythagorean fuzzy sets. At the end, we have presented protein complexes emerged in the PPI networks of Parkinson's disease as Pythagorean fuzzy soft granules. As the considered networks are constructed on the basis of gene expression profiles, any diagnosis and treatment of disease can easily be viewed in the victim. In future, we plan to extend our research in the following topics: (i) Rough Pythagorean fuzzy granular structures, (ii) Rough bipolar neutrosophic hypergraphs and (iii) Granulation of Pythagorean fuzzy soft hypergraphs.

Declarations

Conflict of interest The authors declare no conflicts of interest.

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