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ORIGINAL RESEARCH

Low-density periodical burst correcting codes with decoding probability and detection capability

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Abstract
In this paper, we present low-density periodical burst correcting linear codes. Existence of such codes are studied. We also provide decoding error probability of such codes. Weight distribution and Plotkin's type bound for the set of low-density periodical burst errors are also presented. Further, we present weight distribution and Plotkin's type bound for some other periodical bursts which will be detected by such codes.

Keywords Syndrome · Low-density periodical burst · Weight distribution · Decoding error probability

Mathematics Subject Classification 94B05 · 94B65 · 94B70

1 Introduction

To protect information in memory system from noise [2], or to protect quantum information from errors due to decoherence and other quantum noise [14], or to analysis biological information about the DNA molecule [12], error control codes are used. Choice of error control code depends on what type of errors are required to be dealt with and what type of communication channels are being used.

In 1994, Lange [13] observed that the disturbances in some communication channels (e.g. power lines, car electric, compact discs, etc) are not only clustered but also periodical in nature. Also, Schmitz et al. [19] found that mixing between the two heterodyne frequencies in lithographic stages for semiconductor fabrication results in periodical errors superimposed on the desired displacement data. Such type of disturbances is called *periodical burst error* and defined as follows.

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Definition 1 [5] An s-periodical burst of length b is an n-tuple whose nonzero components are confined to distinct sets of b consecutive positions such that the sets are separated by s positions and first component of each set is nonzero.

Examples of 3-periodical bursts of length 4 in a 14-tuple are 1010 000 1100 000, 0 1000 000 1001 00, 00 1001 000 1000 0, etc.

In [13], detection of periodical bursts using **cyclic code** is studied. For correction of periodical burst, two generator polynomials are presented. In [5], the correction and Hamming weight distribution of periodical bursts using **linear code**, and error decoding probability of such linear codes is also studied. Another paper [6] presents a study on periodical bursts and multiple burst-correcting MDS codes derived by Villalba et al. [22]. In [23], Wyner introduced the concept of low-density in burst error in which disturbances within a burst of length b normally affect only a few positions. Motivated by this, we consider the number of disturbed bits within b consecutive positions of an s-periodical burst error of length b does not exceed w ($1 \leq w \leq b$). We call them as *s-periodical bursts of length b with weight up to w*. In this paper, we present a study on linear codes correcting such errors and denote such a code by $\mathbf{P_{s,b|w}BC}$-code. For similar works in this direction, one may refer to [4, 8, 10, 11, 21]. Further, to measure its goodness, we present the probability of decoding error $PD_w(E)$ of an $P_{s,b|w}BC$-code (refer Section 3.7.2, [15]). Finally, we give some periodical burst errors other than s-periodical bursts of length b of weight at most w, which will be detected by $P_{s,b|w}BC$-code. Weight distribution and upper bound on the minimum weight of the set consisting of such errors are derived. *The weight are taken in the Hamming sense only in this paper*.

The rest of the paper is organized as follows. Section 2 derives necessary and sufficient conditions for existence of a q-ary $P_{s,b|w}BC$-code. In Sect. 3, we derive the total probability of s-periodical bursts of length b in an n-tuple. Then we give the weight distribution of the error pattern and a bound on the largest attainable minimum weight by a vector in the set of the errors. We also give the decoding error probability of an $P_{s,b|w}BC$-code over a binary symmetric channel. Finally, Sect. 4 gives weight distribution and upper bound on the minimum weight of some periodical burst errors other than s-periodical bursts of length b of weight at most w that are detected by $P_{s,b|w}BC$-code.

In what follows,

$\lfloor \mathbf{x} \rfloor$: the floor function of x.

$\lceil \mathbf{x} \rceil$: the ceiling function of x.

$\mathbf{E_{(s,b|w),n,q}}$: set of all s-periodical bursts of length b with weight at most w ($w \leq b$) in an n-tuple.

: a function from $\{1, 2, \ldots, s+b-1\}$ to $\{1, 2, \ldots, b\}$ defined by

$$\gamma(r) = \begin{cases} r & if \quad 0 \leq r \leq b \\ b & if \quad b < r < b+s. \end{cases}$$

2 Existence of $P_{s,b|w}BC$-code

This section presents necessary and sufficient conditions for existence of a q-ary $P_{s,b|w}BC$-code. Examples are provided to support the results also. To prove our results, we start with the following lemma.

Lemma 1 *For given non-negative integers n, b and s* ($n \geq b+s$)*, let* $N_{(s,b|w),n,q} = |E_{(s,b|w),n,q}|$. *Then*

$$N_{(s,b|w),n,q} = \sum_{i=1}^{n} \left\{ \left[\sum_{j=0}^{w-1} \binom{b-1}{j} (q-1)^{1+j} \right]^{\left\lfloor \frac{n-i+1}{s+b} \right\rfloor} \times \sum_{j=0}^{min\{w-1, g_i-1\}} \binom{g_i-1}{j} (q-1)^{1+j} \right\},$$

where $g_i = \gamma\big((n-i+1) \mod (b+s)\big)$ *and* $\sum_{j=0}^{min\{w-1,g_i-1\}} \binom{g_i-1}{j}(q-1)^{1+j} = 1$ *if* $g_i = 0$.

Proof If periodical burst error starts from the i^{th} position ($1 \leq i \leq n$) in a vector of length n, the number of sets (excluding the last set) in which nonzero components of s-periodical burst of length b with weight at most w ($w \leq b$) are confined, is (see [5])

$$\left\lfloor \frac{n-i+1}{s+b} \right\rfloor.$$

In each set, the first component is always nonzero and remaining $b-1$ components we can choose by $\sum_{j=1}^{w-1} \binom{b-1}{j}$ ways. The number of complete sets of b consecutive components is $\left\lfloor \frac{n-i+1}{s+b} \right\rfloor$, so total number of s-periodical bursts of length b with weight at most w in these sets is

$$\left[\sum_{j=0}^{w-1} \binom{b-1}{j} (q-1)^{1+j} \right]^{\left\lfloor \frac{n-i+1}{s+b} \right\rfloor}.$$

The last set contains $g_i = \gamma\big((n-i+1) \mod (b+s)\big)$ components, out of which the first one is nonzero if $g_i > 0$. The number of ways the last set can be selected is

$$\begin{cases} 1 & if \quad g_i = 0 \\ \sum_{j=0}^{min\{w-1, g_i-1\}} \binom{g_i - 1}{j} (q-1)^{1+j} & otherwise. \end{cases}$$

Therefore, the number of s-periodical bursts of length b with weight at most w if it starts from the i^{th} position is

$$\left[\sum_{j=0}^{w-1} \binom{b-1}{j} (q-1)^{1+j}\right]^{\left\lfloor \frac{n-i+1}{s+b} \right\rfloor} \times \sum_{j=0}^{min\{w-1, g_i-1\}} \binom{g_i - 1}{j} (q-1)^{1+j},$$

where $\sum_{j=0}^{min\{w-1, g_i-1\}} \binom{g_i - 1}{j} (q-1)^{1+j} = 1$ if $g_i = 0$.

Summing for i from 1 to n, we get the total number of vectors in $E_{(s,b|w),n,q}$ as

$$N_{(s,b|w),n,q} = \sum_{i=1}^{n} \left\{ \left[\sum_{j=0}^{w-1} \binom{b-1}{j} (q-1)^{1+j}\right]^{\left\lfloor \frac{n-i+1}{s+b} \right\rfloor} \times \sum_{j=0}^{min\{w-1, g_i-1\}} \binom{g_i - 1}{j} (q-1)^{1+j} \right\},$$

where $\sum_{j=0}^{min\{w-1, g_i-1\}} \binom{g_i - 1}{j} (q-1)^{1+j} = 1$ if $g_i = 0$. □

Example 1 Taking $n = 15$, $b = 2$, $s = 3$, $w = 1$ and $q = 2$ in Lemma 1, we have

$$N_{(3,2|1),15,2} = \sum_{i=1}^{15} \left\{ \left[\sum_{j=0}^{0} \binom{1}{j} 1^{1+j}\right]^{\left\lfloor \frac{16-i}{5} \right\rfloor} \times \sum_{j=0}^{0} \binom{\gamma\big((16-i) \mod 5\big) - 1}{j} 1^{1+j} \right\} = 15.$$

Then, the total number of 3-periodical bursts of length 2 with weight up to 1 in a vector of length 15 are

100001000010000, 010000100001000, 001000010000100, 000100001000010, 000010000100001, 000001000010000, 000000100001000, 000000010000100, 000 000001000010, 000000000100001, 000000000010000, 000000000001000, 000000 000000100, 000000000000010, 000000000000001.

Now, a necessary condition for existence of a q-ary $P_{s,b|w}BC$-code is given below which is equivalent to Fire bound [9] and Theorem 4.16 of [16].

Theorem 1 *For given non-negative integers n, b and s* $(n \geq b+s)$*, an* (n, k) *q-ary* $P_{s,b|w}BC$*-code* $(w \leq b)$ *must satisfy*

$$q^{n-k} \geq 1 + N_{(s,b|w),n,q}, \tag{1}$$

where $N_{(s,b|w),n,q}$ *is given by Lemma 1.*

Proof For correction of errors by a linear code, all the errors should be in different cosets of the code. So, by Lemma 1, we have

$$q^{n-k} \geq 1 + N_{(s,b|w),n,q}.$$

□

Remark 1 Equation (1) gives

$$q^k \leq \frac{q^n}{1 + N_{(s,b|w),n,q}}.$$

That is, the cardinality of an $(n,\ k)$ q-ary $P_{s,b|w}BC$-code is at most $\dfrac{q^n}{1 + N_{(s,b|w),n,q}}$.

Next, we provide a sufficient condition for existence of a q-ary $P^{w}_{s,b}BC$-code (equivalent to Varshamov-Gilbert-Sacks Bound [18] and Campopiano Bound [1] (also see Theorem 4.7 and Theorem 4.17 of [16])). The proof of the result gives a technique to construct the code in which we keep on adding the columns one after another keeping in mind that syndromes of the errors remain nonzero and distinct.

Theorem 2 *For given non-negative integers n, b and s* $(n \geq b+s)$*, we can always construct an* $(n,\ k)$ *q-ary* $P_{s,b|w}BC$*-code* $(w \leq b)$ *provided*

$$q^{n-k} > \sum_{j=0}^{w-1} \binom{b-1}{j} (q-1)^j \times \left[\sum_{j=0}^{w-1} \binom{b-1}{j} (q-1)^{1+j} \right]^{\left\lfloor \frac{n}{s+b} \right\rfloor - 1} \times \sum_{j=0}^{\min\{w-1,g-1\}} \binom{g-1}{j} (q-1)^{1+j} \times N_{(s,b|w),n-b,q},$$

where $g=\gamma\big(n \mod (s+b)\big)$, $\sum_{j=0}^{\min\{w-1,g-1\}}\binom{g-1}{j}(q-1)^{1+j}=1$ *if* $g=0$, *and* $N_{(s,b|w),n-b,q}$ *is given by Lemma 1.*

Proof Take any nonzero $(n-k)$-tuple as the first column h_1 of the $(n-k)\times n$ parity-check matrix H of the code and suppose the columns $h_2, h_3, \ldots, h_{n-1}$ are added suitably to H. Then a nonzero column h_n is added to H provided that it is not a linear combination of $w-1$ or less columns within the set of the immediately preceding $b-1$ columns together with linear combinations of columns of previous sets of b consecutive columns with at most w columns from each set, along with a linear combination of w or less columns taken from the last set of b or less consecutive b columns with the condition that the sets are also at gap of s columns. This can be written as

$$\begin{aligned} h_n \neq & \left(\sum_{i=1}^{b-1} a_{i1}h_{n-i} + \sum_{i=0}^{b-1} b_{i1}h_{n-(s+b)-i} + \sum_{i=0}^{b-1} b_{i2}h_{n-2(s+b)-i}\right. \\ & \left. + \cdots + \sum_{i=0}^{g-1} b_{i\lambda}h_{n-\lambda(s+b)-i}\right) \\ & + \left(\sum_{i=0}^{b-1} \alpha_{i1}h_{j'-i} + \sum_{i=0}^{b-1} \beta_{i1}h_{j'-(s+b)-i} + \sum_{i=0}^{b-1} \beta_{i2}h_{j'-2(s+b)-i}\right. \\ & \left. + \cdots + \sum_{i=0}^{g'-1} \beta_{i\lambda'}h_{j'-\lambda'(s+b)-i}\right), \end{aligned} \tag{2}$$

where $a_{ij}, b_{ij}, \alpha_{ij}, \beta_{ij} \in GF(q)$ such that with number of nonzero $a_{ij} \leq w-1$, and that of $b_{ij}, \alpha_{ij}, \beta_{ij} \leq w$ with $b_{0i}, \alpha_{0i}, \beta_{0i} \neq 0$; $j' \leq n-b$; $g=\gamma\big(n \mod (s+b)\big)$, $g'=\gamma\big((n-b-j'+1) \mod (s+b)\big)$, $\lambda=\left\lfloor \frac{n}{s+b}\right\rfloor$ and $\lambda'=\left\lceil \frac{n-b-j'+1}{s+b}\right\rceil$.

Note that in Expression (2), g and g' will be zero if n and $n-b-j'+1$ are multiples of $s+b$ and in that case we take $\sum_{i=0}^{g-1} b_{i\lambda}h_{n-\lambda(s+b)-i}=0$ and $\sum_{i=0}^{g'-1} b_{i\lambda'}h_{j'-\lambda'(s+b)-i}=0$.

The condition (2) ensures that syndromes of any two error patterns are distinct.

We now calculate the linear combinations on right hand side (R.H.S.) of (2) as follows:

The number of ways a_{i1}'s in the first bracket on R.H.S. of (2) can be chosen is

$$\sum_{j=0}^{w-1}\binom{b-1}{j}(q-1)^j.$$

The number of ways b_{ij}'s $(1 \leq j \leq \lambda - 1)$ can be chosen is

$$\left[\sum_{j=0}^{w-1} \binom{b-1}{j} (q-1)^{1+j}\right]^{\lambda-1}.$$

The b_{ij}'s in the last summation of the first bracket can be chosen by

$$\begin{cases} 1 & if \quad g = 0 \\ \sum_{j=0}^{\min\{w-1, g-1\}} \binom{g-1}{j} (q-1)^{1+j} & if \quad g > 0. \end{cases}$$

Therefore, total combinations of the first bracket on R.H.S. of (2) is

$$\sum_{j=0}^{w-1} \binom{b-1}{j} (q-1)^{j} \times \left[\sum_{j=0}^{w-1} \binom{b-1}{j} (q-1)^{1+j}\right]^{\lambda-1} \times \sum_{j=0}^{\min\{w-1, g-1\}} \binom{g-1}{j} (q-1)^{1+j},$$

where $\sum_{j=0}^{\min\{w-1, g-1\}} \binom{g-1}{j} (q-1)^{1+j} = 1$ if $g = 0$.

The second bracket on R.H.S. of (2) gives the number of s-periodical burst errors of length b with weight up to w in a vector of length $n - b$. This number, including the zero combination, is given by Lemma 1 as $N_{(s,b|w),n-b,q}$.

Thus, the total number of all possible linear combinations on R.H.S. of (2) is

$$\sum_{j=0}^{w-1} \binom{b-1}{j} (q-1)^{j} \times \left[\sum_{j=0}^{w-1} \binom{b-1}{j} (q-1)^{1+j}\right]^{\lambda-1} \\ \times \sum_{j=0}^{\min\{w-1, g-1\}} \binom{g-1}{j} (q-1)^{1+j} \times N_{(s,b|w),n-b,q},$$

where $\sum_{j=0}^{\min\{w-1, g-1\}} \binom{g-1}{j} (q-1)^{1+j} = 1$ if $g = 0$.

Since there are q^{n-k} available columns, we can add the nth column provided

$$q^{n-k} > \sum_{j=0}^{w-1} \binom{b-1}{j}(q-1)^j \times \left[\sum_{j=0}^{w-1} \binom{b-1}{j}(q-1)^{1+j}\right]^{\lambda-1} \times \sum_{j=0}^{\min\{w-1,g-1\}} \binom{g-1}{j}(q-1)^{1+j} \times N_{(s,b|w),n-b,q},$$

where $\sum_{j=0}^{\min\{w-1,g-1\}} \binom{g-1}{j}(q-1)^{1+j} = 1$ if $g = 0$. □

Now, we provide three examples of codes discussed in Theorem 2: two for binary and one for ternary case.

Example 2 Consider $n = 18$, $s = 5$, $b = 4$, $w = 2$ and $q = 2$ in Theorem 2, then $\lambda = \lfloor \frac{18}{9} \rfloor = 2$, $l = 0$. Then

$$2^{n-k} > \sum_{j=0}^{1} \binom{3}{j} \times \left[\sum_{j=0}^{1} \binom{3}{j}\right]^{1} \times \sum_{j=0}^{\min\{1,g-1\}} \binom{g-1}{j} \times N_{(5,4|2),14,2}.$$

Now

$$N_{(5,4|2),14,2} = \sum_{i=1}^{14} \left\{ \left[\sum_{j=0}^{2-1} \binom{4-1}{j}(2-1)^{1+j}\right]^{\left\lfloor \frac{n-i+1}{s+b} \right\rfloor} \times \sum_{j=0}^{min\{2-1,g_i-1\}} \binom{g_i-1}{j}(2-1)^{1+j} \right\}$$

$$= \sum_{i=1}^{14} \left\{ \left[\sum_{j=0}^{1} \binom{3}{j}(2-1)^{1+j}\right]^{\left\lfloor \frac{14-i+1}{9} \right\rfloor} \times \sum_{j=0}^{min\{1,g_i-1\}} \binom{g_i-1}{j}(2-1)^{1+j} \right\},$$

$$= 4^{\lfloor \frac{14}{9} \rfloor} \times 4 + 4^{\lfloor \frac{13}{9} \rfloor} \times 4^1 + 4^{\lfloor \frac{12}{9} \rfloor} \times 3 + 4^{\lfloor \frac{11}{9} \rfloor} \times 2 + 4^{\lfloor \frac{10}{9} \rfloor} \times 1 + 4^{\lfloor \frac{9}{9} \rfloor} \times 1$$

$$
\begin{aligned}
&+ 4^{\lfloor \frac{8}{9} \rfloor} \times 4 + 4^{\lfloor \frac{7}{9} \rfloor} \times 4 + 4^{\lfloor \frac{6}{9} \rfloor} \times 4 + 4^{\lfloor \frac{5}{9} \rfloor} \times 4 + 4^{\lfloor \frac{4}{9} \rfloor} \times 4 + 4^{\lfloor \frac{3}{9} \rfloor} \times 3 \\
&+ 4^{\lfloor \frac{2}{9} \rfloor} \times 2 + 4^{\lfloor \frac{1}{9} \rfloor} \times 1 \\
&= 4^2 + 4^2 + 4 \times 3 + 4 \times 2 + 4 \times 7 + 3 + 2 + 1 = 86,
\end{aligned}
$$

where $g_i = \gamma\big((14 - i + 1) \pmod 9\big)$ and $\sum_{j=0}^{0} \binom{g_i - 1}{j} (2-1)^{1+j} = 1$ if $g_i = 0$.

So

$$
\begin{aligned}
2^{n-k} &> 4 \times 4 \times 86 = 1376 \\
\Rightarrow n - k &> 10.
\end{aligned}
$$

We take $n - k = 11$ and this gives rise to a binary $(18, 7)$ linear code whose parity check matrix H of order 11×18 is given by

$$
H = \begin{bmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1 \\
0&1&0&0&0&0&0&0&0&0&0&1&0&0&1&0&1&0 \\
0&0&1&0&0&0&0&0&0&0&0&0&1&0&1&1&0&0 \\
0&0&0&1&0&0&0&0&0&0&0&0&0&1&1&1&0&0 \\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&1&0&0 \\
0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&1 \\
0&0&0&0&0&0&1&0&0&0&0&1&0&0&1&1&0&1 \\
0&0&0&0&0&0&0&1&0&0&0&1&0&1&0&1&1&0 \\
0&0&0&0&0&0&0&0&1&0&0&0&1&0&0&0&1&1 \\
0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&1&1&0 \\
0&0&0&0&0&0&0&0&0&0&1&0&0&1&0&0&0&1
\end{bmatrix}_{11 \times 18}.
$$

It can be verified that the syndromes of all 5-periodical burst errors of length 4 with weight up to 2 are nonzero and distinct, showing that the code can correct all 5-periodical burst errors of length 4 with weight up to 2. So, the code is a $(18, 7)$ binary $P_{5,4|2}BC$-code.

Example 3 Consider $n = 20$, $s = 5$, $b = 4$, $w = 3$ and $q = 2$ in Theorem 2, then $\lambda = \lfloor \frac{20}{9} \rfloor = 2$, $l = 2$. Then Theorem 2 gives

$$
\begin{aligned}
2^{n-k} &> \sum_{j=0}^{2} \binom{4-1}{j} (2-1)^j \times \left[\sum_{j=0}^{2} \binom{4-1}{j} (2-1)^{1+j} \right]^{\lambda - 1} \\
&\quad \times \sum_{j=0}^{\min\{3-1, 2-1\}} \binom{2-1}{j} (2-1)^{1+j} \times N_{(5,4|3),16,2} \\
\Rightarrow 2^{n-k} &> 7 \times 7 \times 2 \times 294 = 28812 \\
\Rightarrow n - k &\geq 15.
\end{aligned}
$$

Taking $n-k=18$, we get a binary $(20, 2)$ linear code whose parity check matrix H of order 18×20 is given by

$$H=\begin{bmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&1&1\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0
\end{bmatrix}_{18\times 20}.$$

Here also the syndromes of all 5-periodical burst errors of length 4 with weight up to 3 are found to be nonzero and distinct, showing that the code is a binary $(20, 2)$ $P_{5,4|3}BC$-code.

Example 4 Consider $n=23$, $s=5$, $b=4$, $w=1$ and $q=3$ in Theorem 2, then $\lambda=\lfloor\frac{23}{9}\rfloor=2$, $l=5$. Then Theorem 2 gives

$$\begin{aligned}
&3^{n-k}>1\times 2\times 2\times 62\\
\Rightarrow\ & n-k>5.
\end{aligned}$$

Considering $n-k=6$, we get a ternary $(23, 17)$ linear code whose parity check matrix H of order 6×23 is given by

$$H=\begin{bmatrix}
1&0&1&0&0&0&0&0&2&0&0&0&0&0&0&1&1&0&0&1&0&0&1\\
0&1&0&0&0&0&1&0&0&1&0&0&0&0&1&0&1&1&0&0&0&0&0\\
0&0&2&0&0&0&2&0&0&0&1&0&0&2&0&0&1&0&1&0&0&0&0\\
0&0&0&1&1&0&1&2&0&0&0&2&0&0&0&0&0&0&2&0&0&0&0\\
1&2&0&0&1&0&1&0&0&0&0&0&2&0&1&0&0&0&0&1&1&0&0\\
0&0&0&0&1&2&0&1&0&1&0&0&0&0&0&1&0&0&0&1&0&1&0
\end{bmatrix}_{6\times 23}.$$

Here also the syndromes of all 5-periodical burst errors of length 4 with weight up to 1 are nonzero and distinct, showing that the code is a $(23, 17)$ ternary $P_{5,4|1}BC$-code.

3 Weight distribution and error decoding probability

This section presents the weight distribution of the errors in $E_{(s,b|w),n,q}$. One may refer to [3, 7, 20] and their references for weight distribution of other type of errors. We also provide an upper bound on the minimum weight of a vector in the set $E_{(s,b|w),n,q}$ which is equivalent to Plotkin bound [17] (also Lemma 4.1 of Peterson and Weldon [16]). We then derive the total probability of the error pattern and decoding error probability of an $P_{s,b|w}BC$-code in a binary symmetric channel.

Lemma 2 *For* $0 \leq j \leq n$, *let* $E_{(s,b|w),n,q}(j) = \{e \in E_{(s,b|w),n,q} \text{such that weight of } e \text{ is } j\}$ *and* $N_{(s,b|w),n,q}(j) = |E_{(s,b|w),n,q}(j)|$. *Then*

$$N_{(s,b|w),n,q}(j) = \sum_{i=1}^{n} \sum_{j_1, j_2, \ldots, j_{l_i}, j_{l'}} \binom{b-1}{j_1}\binom{b-1}{j_2} \cdots \binom{b-1}{j_{l_i}}\binom{g_i-1}{j_{l'}}(q-1)^{\lambda_i + j_1 + j_2 + \cdots + j_{l_i} + j_{l'}},$$

where $g_i = \gamma\big((n-i+1) \mod (b+s)\big)$, $\lambda_i = \left\lceil \frac{n-i+1}{s+b} \right\rceil$, $l_i = \left\lfloor \frac{n-i+1}{s+b} \right\rfloor$ *and* $j_1, j_2, \ldots, j_{l_i}, j_{l'}$ *are nonnegative integers such that* $\lambda_i + j_1 + j_2 + \cdots + j_{l_i} + j_{l'} = j$, $0 \leq j_1, j_2, \ldots, j_{l_i} \leq w-1$ *and* $0 \leq j_{l'} \leq min\{g_i - 1, w-1\}$.

Proof The nonzero components of the error pattern that starts from i^{th} position ($1 \leq i \leq n$) are confined to $l_i = \left\lfloor \frac{n-i+1}{s+b} \right\rfloor$ sets of b consecutive components followed by the last set consisting of $g_i = \gamma\big((n-i+1) \mod (b+s)\big)$ consecutive components, first position of each set is nonzero. Then we can select any j_i positions ($i = 1, 2, \ldots, l_i$) from $b-1$ positions for nonzero components by $\binom{b-1}{j_i}$ ways and $j_{l'}$ positions from the last set by $\binom{g_i-1}{j_{i'}}$ ways, where $j_i \leq w-1$ and $j_{l'} \leq min\{g_i - 1, w-1\}$. Therefore the total number of s-periodical burst errors of length b with weight at most w that has weight j is

$$\begin{aligned}
&\binom{b-1}{j_1}(q-1)^{1+j_1} \times \binom{b-1}{j_2}(q-1)^{1+j_2} \times \ldots \\
&\quad \times \binom{b-1}{j_{l_i}}(q-1)^{1+j_{l_i}} \times \binom{g_i-1}{j_{l'}}(q-1)^{\delta+j_{l'}} \\
&= \binom{b-1}{j_1}\binom{b-1}{j_2}\cdots\binom{b-1}{j_{l_i}}\binom{g_i-1}{j_{l'}}(q-1)^{l_i+\delta+j_1+j_2+\cdots+j_{l_i}+j_{l'}},
\end{aligned}$$

where $l_i + \delta + j_1 + j_2 + \cdots + j_{l_i} + j_{l'} = j$, $\delta = \begin{cases} 0 \text{ if } g_i = 0 \\ 1 \text{ otherwise} \end{cases}$, $0 \leq j_1, j_2, \ldots, j_{l_i} \leq w-1$ and $0 \leq j_{l'} \leq min\{g_i - 1, w-1\}$.

So, total number of vectors in $E_{(s,b|w),n,q}(j)$ is

$$N_{(s,b|w),n,q}(j) = \sum_{i=1}^{n} + \sum_{j_1,j_2,\ldots,j_{l_i},j_{l'}} \binom{b-1}{j_1}\binom{b-1}{j_2}\cdots\binom{b-1}{j_{l_i}}\binom{g_i-1}{j_{l'}}(q-1)^{\lambda_i+j_1+j_2+\cdots+j_{l_i}+j_{l'}},$$

where $\lambda_i = l_i + \delta = \left\lceil \frac{n-i+1}{s+b} \right\rceil$, and $j_1, j_2, \ldots, j_{l_i}, j_{l'}$ are nonnegative integers such that $\lambda_i + j_1 + j_2 + \cdots + j_{l_i} + j_{l'} = j$, $0 \le j_1, j_2, \ldots, j_{l_i} \le w-1$ and $0 \le j_{l'} \le min\{g_i - 1, w-1\}$. □

Remark 2 Observe that for given non-negative integers n, b and s ($n \ge s+b$), the maximum number of nonzero components in a vector of $E_{(s,b|w),n,q}$ can be found when the periodical burst starts from the first position. This number W_{max} is given by

$$W_{max} = \left\lfloor \frac{n}{s+b} \right\rfloor w + \gamma'(g), \text{ where } \gamma'(g) = \begin{cases} g & if \quad g \le w \\ w & otherwise. \end{cases}$$

So, $E_{(s,b|w),n,q}(j) = 0$ for $W_{max} < j \le n$.

Theorem 3 *The minimum weight of a vector in the set* $E_{(s,b|w),n,q}$ *is at most*

$$\frac{\sum_{j=1}^{w_{max}} jN_{(s,b|w),n,q}(j)}{N_{(s,b|w),n,q}},$$

where $N_{(s,b|w),n,q}$ *is given by Lemma 1 and* $N_{(s,b|w),n,q}(j)$ *by Lemma 2.*

Proof By Lemma 1 and Lemma 2, the average weight of a vector in $E_{(s,b|w),n,q}$ is

$$\frac{\sum_{j=1}^{w_{max}} jN_{(s,b|w),n,q}(j)}{N_{(s,b|w),n,q}}.$$

As the minimum weight of a vector in a set can be at most the average weight, this follows the theorem. □

Remark 3 If w is odd, we consider the two s-periodical bursts of length b:

$$\Big(\underbrace{x_1'0x_3'0x_5'0\ldots0x_w'00\ldots0}_{b}\ \underbrace{00\ldots0}_{s}\ \underbrace{x_1''0x_3''0x_5''0\ldots0x_w''00\ldots0}_{b}\ \underbrace{00\ldots0}_{s}\ldots\ldots\Big) \text{ and}$$

$$\Big(\underbrace{0x_2'0x_4'0x_6'0\ldots x_{w-1}'00\ldots0}_{b}\ \underbrace{00\ldots0}_{s}\ \underbrace{0x_2''0x_4''0x_6''0\ldots x_{w-1}''00\ldots0}_{b}\ \underbrace{00\ldots0}_{s}\ldots\ldots\Big),$$

where $x_i', x_i'' \in GF(q) \setminus \{0\}$.

If w is even, then consider the two s-periodical bursts of length b:

$$\Big(\underbrace{x_1'0x_3'0x_5'0\ldots x_{w-1}'00\ldots0}_{b}\ \underbrace{00\ldots0}_{s}\ \underbrace{x_1''0x_3''0x_5''0\ldots x_{w-1}''00\ldots0}_{b}\ \underbrace{00\ldots0}_{s}\ldots\ldots\Big) \text{ and}$$

$$\Big(\underbrace{0x_2'0x_4'0x_6'0\ldots 0x_w'00\ldots0}_{b}\ \underbrace{00\ldots0}_{s}\ \underbrace{0x_2''0x_4''0x_6''0\ldots 0x_w''00\ldots0}_{b}\ \underbrace{00\ldots0}_{s}\ldots\ldots\Big),$$

where $x_i', x_i'' \in GF(q)\setminus\{0\}$.

In both cases, difference of the two vectors gives an s-periodical burst of length b with weight W_{max}. So, the minimum distance of the set $E_{(s,b|w),n,q} \leq W_{max}$ and the maximum distance of the set $E_{(s,b|w),n,q} \geq W_{max}$.

Now, total probability of vectors of $E_{(s,b|w),n,q}$ in a binary symmetric channel is given in the following theorem.

Theorem 4 *The total probability $P_w(E)$ of errors from the set $E_{(s,b|w),n,2}$ over a memoryless binary symmetric channel with transition probability ϵ is given by*

$$P_w(E) = \sum_{i=1}^{n}\Bigg[\sum_{j_1,j_2,\ldots,j_{l_i},j_{l'}}\binom{b-1}{j_1}\binom{b-1}{j_2}\cdots\binom{b-1}{j_{l_i}}\binom{g_i-1}{j_{l'}} \times \epsilon^{\lambda_i+(j_1+j_2+\cdots+j_{l_i}+j_{l'})}(1-\epsilon)^{n-\lambda_i-j_1-j_2-\cdots-j_{l_i}-j_{l'}}\Bigg],$$

where $g_i = \gamma\big((n-i+1) \mod (b+s)\big)$, $\lambda_i = \left\lceil\frac{n-i+1}{s+b}\right\rceil$, $l_i = \left\lfloor\frac{n-i+1}{s+b}\right\rfloor$, $0 \leq j_1, j_2, \ldots, j_{l_i} \leq w-1$ and $0 \leq j_{l'} \leq min\{g_i-1, w-1\}$.

Proof As the first position of each set of the error pattern has nonzero component, the number of always nonzero components in a periodical burst that starts from i^{th} position ($1 \leq i \leq n$) is $\lambda_i = \left\lceil\frac{n-i+1}{s+b}\right\rceil$. The other nonzero components come from the remaining positions such that each set contains not more than w nonzero components. As the nonzero components are confined to $l_i = \left\lfloor\frac{n-i+1}{s+b}\right\rfloor$ sets of b consecutive components followed by a set of $g_i = \gamma\big((n-i+1) \mod (b+s)\big)$ consecutive components, the total probability of s-periodical burst errors of length b of weight at most w that starts from i^{th} position is given by

$$\sum_{j_1,j_2,\ldots,j_{l_i},j_{l'}}\binom{b-1}{j_1}\binom{b-1}{j_2}\cdots\binom{b-1}{j_{l_i}}\binom{g_i-1}{j_{l'}}\times \epsilon^{\lambda_i+(j_1+j_2+\cdots+j_{l_i}+j_{l'})}(1-\epsilon)^{n-\lambda_i-j_1-j_2-\cdots-j_{l_i}-j_{l'}},$$

where $0 \leq j_1, j_2, \ldots, j_{l_i} \leq w-1$ and $0 \leq j_{l'} \leq min\{g_i-1, w-1\}$.

Taking $i = 1, 2, \ldots, n$ gives the result. □

Next, we give the probability of decoding error for a binary $P_{s,b|w}BC$-code as follows.

Theorem 5 *Let $PD_w(E)$ be the probability of decoding error of an (n,k) binary $P_{s,b|w}BC$-code on a memoryless binary symmetric channel with transition probability ϵ, then*

$$PD_w(E) = 1 - \sum_{j=1}^{W_{max}} N_{(s,b|w),n,q}(j).\epsilon^j(1-\epsilon)^{n-j},$$

where $N_{(s,b|w),n,q}(j)$ is given by Lemma 2.

Proof As the probability of correcting an error is the probability that the error is a coset leader in the standard array for the code, the probability of a vector of $E_{(s,b|w),n,q}(j)$ being one of the coset leaders is

$$N_{(s,b|w),n,q}(j).\epsilon^j(1-\epsilon)^{n-j}.$$

Therefore, the probability $PD_w(E)$ of decoding error of the code is given by

$$PD_w(E) = 1 - \sum_{j=1}^{W_{max}} N_{(s,b|w),n,q}(j).\epsilon^j(1-\epsilon)^{n-j}.$$

□

4 Detection and weight distribution of some periodical bursts

In this section, we give detection of some periodical burst errors by $P_{s,b|w}BC$-code (other than correctable errors). Weight distribution of those errors and upper bound on the minimum weight of the set of such errors are given. For this, we first define two sets.

For $s > b$, let $\mathbb{A}$ be the collection of all $(s-b)$-periodical bursts of length $2b$ of the form:

$$\Big(\underbrace{x_1 \bullet\bullet\cdots\bullet}_{b}\underbrace{x_2 \bullet\bullet\cdots\bullet}_{b}\underbrace{00\ldots0}_{s-b}\underbrace{x_3 \bullet\bullet\cdots\bullet}_{b}\underbrace{x_4 \bullet\bullet\cdots\bullet}_{b}\underbrace{00\ldots0}_{s-b}\underbrace{x_5 \bullet\bullet\cdots\bullet}_{b}\underbrace{x_6 \bullet\bullet\cdots\bullet}_{b}\ldots\ldots\Big),$$

and for $s \leq b$, $\mathbb{A}'$ be the collection of all 1-periodical burst errors of length $b+s-1$ of the form:

$$\Big(\underbrace{x_1 \bullet\bullet\cdots\bullet}_{b}\underbrace{x_2 \bullet\bullet\cdots\bullet}_{s-1}0\underbrace{x_3 \bullet\bullet\cdots\bullet}_{b}\underbrace{x_4 \bullet\bullet\cdots\bullet}_{s-1}0\underbrace{x_5 \bullet\bullet\cdots\bullet}_{b}\underbrace{x_6 \bullet\bullet\cdots\bullet}_{s-1}0\ldots\ldots\Big),$$

where $x_i \in GF(q)\setminus\{0\}$ and $\bullet \in GF(q)$ such that consecutive $b-1$ bullets have at most $w-1$ nonzero components.

Theorem 6 *An q-ary $P_{s,b|w}BC$-code detects all periodical burst errors from the set $\mathbb{A}$ and $\mathbb{A}'$.*

Proof As every member of $\mathbb{A}$ or $\mathbb{A}'$ can be expressed as the sum (difference) of two s-periodical bursts of length b of weight at most w. So, no element of $\mathbb{A}$ or $\mathbb{A}'$ can be a codeword of $P_{s,b|w}BC$-code. This follows the theorem. □

Now we give weight distribution of vectors of $\mathbb{A}$ and $\mathbb{A}'$.

Lemma 3 *If A_j be the collections of the vectors of $\mathbb{A}$ having weight j and $g^{(1)} = \gamma_1\big(n \mod (b+s)\big)$ where $\gamma_1(r) = \begin{cases} r & if\ n \pmod{(b+s)} \le 2b \\ 2b & otherwise. \end{cases}$ Then*

1. *if $g^{(1)} = \gamma_1\big(n \mod (b+s)\big) = 0$,*

$$|A_j| = \sum_{j_1, j_2, \ldots, j_{2l}} \left[\prod_{\rho=1}^{2l} \binom{b-1}{j_\rho}\right] (q-1)^{2l+j_1+j_2+\cdots+j_{2l}},$$

where $2l + j_1 + j_2 + \cdots + j_{2l} = j$ such that $0 \le j_1, j_2, \ldots, j_{2l} \le w-1$ and $l = \left\lfloor \dfrac{n}{s+b} \right\rfloor$.

2. *if $1 \le g^{(1)} = \gamma_1\big(n \mod (b+s)\big) \le b$,*

$$|A_j| = \sum_{j_1, j_2, \ldots, j_{2l}, j_{l'}} \left[\prod_{\rho=1}^{2l} \binom{b-1}{j_\rho}\right] \binom{g^{(1)}-1}{j_{l'}} (q-1)^{2l+j_1+j_2+\cdots+j_{2l}+j_{l'}+1},$$

where $2l + j_1 + j_2 + \cdots + j_{2l} + j_{l'} + 1 = j$ such that $0 \le j_1, j_2, \ldots, j_l \le w-1$, $0 \le j_{l'} \le min\{g^{(1)}-1, w-1\}$ and $l = \left\lfloor \dfrac{n}{s+b} \right\rfloor$.

3. *if $b+1 \le g^{(1)} = \gamma_1\big(n \mod (b+s)\big) \le 2b$,*

$$|A_j| = \sum_{j_1, j_2, \ldots, j_{2l}, j_{l'}, j_{l''}} \left[\prod_{\rho=1}^{2l} \binom{b-1}{j_\rho}\right] \binom{b-1}{j_{l'}} \times \binom{g^{(1)}-b-1}{j_{l'}} (q-1)^{2l+j_1+j_2+\cdots+j_{2l}+j_{l'}+j_{l''}+2},$$

where $2l + j_1 + j_2 + \cdots + j_{2l} + j_{l'} + j_{l''} + 2 = j$ such that $0 \le j_1, j_2, \ldots, j_l, j_{l'} \le w-1$, $0 \le j_{l''} \le min\{g^{(1)}-b-1, w-1\}$ and $l = \left\lfloor \dfrac{n}{s+b} \right\rfloor$.

Further, maximum weight of elements of $\mathbb{A}$ is

$$W^1_{max} = 2w\left\lfloor \frac{n}{s+b} \right\rfloor + \gamma_1''(g^{(1)}),$$

$$where\ \gamma_1''(g^{(1)}) = \begin{cases} g^{(1)} & if\ 0 \le g^{(1)} \le w \\ w & if\ w+1 \le g^{(1)} \le b \\ w + g^{(1)} - b & if\ b+1 \le g^{(1)} \le b+w \\ 2w & if\ b+w+1 \le g^{(1)} \le 2b. \end{cases}$$

Proof Observe that if $n \pmod{(b+s)} = 0$, all nonzero components in any vector of $\mathbb{A}$ are confined to $\left\lfloor \frac{n}{s+b} \right\rfloor$ sets that are separated by $s-b$ consecutive zeros. If $n \pmod{(b+s)} \neq 0$, nonzero components in a vector of $\mathbb{A}$ are confined to $\left\lfloor \frac{n}{s+b} \right\rfloor + 1$ sets that are separated by $s-b$ consecutive zeros, where each of $\left\lfloor \frac{n}{s+b} \right\rfloor$ sets has $2b$ consecutive components and the last set has $g^{(1)} = \gamma_1\big(n \mod (b+s)\big)$ components, where $\gamma_1(r) = \begin{cases} r & if\ n \pmod{(b+s)} \le 2b \\ 2b & otherwise \end{cases}$. Then the cardinality of the set A_j of the vectors of $\mathbb{A}$ having weight j is calculated as follows.

Sub-case (i). If $g^{(1)} = 0$, the number $|A_j|$ of the vectors of $\mathbb{A}$ having weight j is given by

$$\begin{aligned} &\sum_{j_1, j_2, \ldots, j_{2l}} \binom{b-1}{j_1}(q-1)^{1+j_1} \times \binom{b-1}{j_2}(q-1)^{1+j_2} \times \cdots \times \binom{b-1}{j_{2l}}(q-1)^{1+j_{2l}} \\ &\quad = \sum_{j_1, j_2, \ldots, j_{2l}} \left[\prod_{\rho=1}^{2l} \binom{b-1}{j_\rho}\right] (q-1)^{2l+j_1+j_2+\cdots+j_{2l}}, \end{aligned}$$

where $2l + j_1 + j_2 + \cdots + j_{2l} = j$ such that $0 \le j_1, j_2, \ldots, j_{2l} \le w-1$ and $l = \left\lfloor \frac{n}{s+b} \right\rfloor$.

Sub-case (ii). If $1 \le g^{(1)} \le b$, the number $|A_j|$ of the vectors of $\mathbb{A}$ having weight j is given by

$$\begin{aligned} &\sum_{j_1, j_2, \ldots, j_{2l}, j_{l'}} \binom{b-1}{j_1}(q-1)^{1+j_1} \times \binom{b-1}{j_2}(q-1)^{1+j_2} \\ &\quad \times \cdots \times \binom{b-1}{j_{2l}}(q-1)^{1+j_{2l}} \times \binom{g^{(1)}-1}{j_{l'}}(q-1)^{1+j_{l'}} \\ &\quad = \sum_{j_1, j_2, \ldots, j_{2l}, j_{l'}} \left[\prod_{\rho=1}^{2l} \binom{b-1}{j_\rho}\right] \binom{g^{(1)}-1}{j_{l'}} (q-1)^{2l+j_1+j_2+\cdots+j_{2l}+j_{l'}+1}, \end{aligned}$$

where $2l + j_1 + j_2 + \cdots + j_{2l} + j_{l'} + 1 = j$ such that $0 \le j_1, j_2, \ldots, j_l \le w-1$, $0 \le j_{l'} \le min\{g^{(1)} - 1, w-1\}$ and $l = \left\lfloor \frac{n}{s+b} \right\rfloor$.

Sub-case (iii). If $b+1 \leq g^{(1)} \leq 2b$, the number $|A_j|$ of the vectors of $\mathbb{A}$ having weight j is given by

$$\sum_{j_1, j_2, \ldots, j_{2l}, j_{l'}, j_{l''}} \binom{b-1}{j_1}(q-1)^{1+j_1} \times \binom{b-1}{j_2}(q-1)^{1+j_2} \times \ldots$$
$$\times \binom{b-1}{j_{2l}}(q-1)^{1+j_{2l}} \times \binom{b-1}{j_{l'}}(q-1)^{1+j_{l'}} \times \binom{g^{(1)}-b-1}{j_{l''}}(q-1)^{1+j_{l''}}$$
$$= \sum_{j_1, j_2, \ldots, j_{2l}, j_{l'}, j_{l''}} \left[\prod_{\rho=1}^{2l} \binom{b-1}{j_\rho}\right] \binom{b-1}{j_{l'}}$$
$$\times \binom{g^{(1)}-b-1}{j_{l''}}(q-1)^{2l+j_1+j_2+\cdots+j_{2l}+j_{l'}+j_{l''}+2},$$

where $2l+j_1+j_2+\cdots+j_{2l}+j_{l'}+j_{l''}+2 = j$ such that $0 \leq j_1, j_2, \ldots, j_l, j_{l'} \leq w-1$, $0 \leq j_{l''} \leq min\{g^{(1)}-b-1, w-1\}$ and $l = \left\lfloor \frac{n}{s+b} \right\rfloor$.

Maximum weight can be calculated by taking $2w$ weight in each $\left\lfloor \frac{n}{s+b} \right\rfloor$ sets of complete $2b$ components and the last set having maximum weight

$$\gamma_1''(g^{(1)}) = \begin{cases} g^{(1)} & if\ 0 \leq g^{(1)} \leq w \\ w & if\ w+1 \leq g^{(1)} \leq b \\ w+g^{(1)}-b & if\ b+1 \leq g^{(1)} \leq b+w \\ 2w & if\ b+w+1 \leq g^{(1)} \leq 2b. \end{cases}$$

This shows that

$$W_{max}^1 = 2w\left\lfloor \frac{n}{s+b} \right\rfloor + \gamma_1''(g^{(1)}).$$

□

Lemma 4 *Let A'_j be the set of all vectors of $\mathbb{A}'$ whose weight is j and $g^{(2)} = \gamma_2\big(n \mod (b+s)\big)$ where $\gamma_2(r) = \begin{cases} r & if\ n \pmod{(b+s)} \leq b+s-1 \\ b+s-1 & otherwise. \end{cases}$*
Then

1. if $g^{(2)} = \gamma_2\big(n \mod (b+s)\big) = 0$,

$$|A'_j| = \sum_{j_1, j_2, \ldots, j_{2l}} \left[\prod_{\rho=1}^{l} \binom{b-1}{j_{2\rho-1}} \prod_{\rho=1}^{l} \binom{s-2}{j_{2\rho}}\right] (q-1)^{2l+j_1+j_2+\cdots+j_{2l}},$$

where $2l+j_1+j_2+\cdots+j_{2l} = j$ such that $0 \leq j_1, j_3, \ldots, j_{2l-1} \leq w-1$, $0 \leq j_2, j_4, \ldots, j_{2l} \leq min\{w-1, s-2\}$ and $l = \lfloor \frac{n}{s+b} \rfloor$.

2. *if* $1 \le g^{(2)} = \gamma_2\big(n \mod (b+s)\big) \le b$,

$$|A'_j| = \sum_{j_1, j_2, \ldots, j_{2l}, j_{l'}} \left[\prod_{\rho=1}^{l} \binom{b-1}{j_{2\rho-1}} \prod_{\rho=1}^{l} \binom{s-2}{j_{2\rho}} \right] \times \binom{g^{(2)}-1}{j_{l'}} (q-1)^{2l+j_1+j_2+\cdots+j_{2l}+j_{l'}+1},$$

where $2l+j_1+j_2+\cdots+j_{2l}+j_{l'}+1 = j$ *such that* $0 \le j_1, j_3, \ldots, j_{2l-1} \le w-1$, $0 \le j_2, j_4, \ldots, j_{2l} \le min\{w-1, s-2\}$, $0 \le j_{l'} \le min\{w-1, g^{(2)}-1\}$ *and* $l = \left\lfloor \frac{n}{s+b} \right\rfloor$.

3. *if* $b+1 \le g^{(2)} = \gamma_2\big(n \mod (b+s)\big) \le b+s-1$,

$$|A'_j| = \sum_{j_1, j_2, \ldots, j_{2l}, j_{l'}, j_{l''}} \left[\prod_{\rho=1}^{l} \binom{b-1}{j_{2\rho-1}} \prod_{\rho=1}^{l} \binom{s-2}{j_{2\rho}} \right] \times \binom{b-1}{j_{l'}} \binom{g^{(2)}-b-1}{j_{l''}} (q-1)^{2l+j_1+j_2+\cdots+j_{2l}+j_{l'}+j_{l''}+2},$$

where $2l+j_1+j_2+\cdots+j_{2l}+j_{l'}+j_{l''}+2 = j$ *such that* $0 \le j_1, j_3, \ldots, j_{2l-1}, j_{l'} \le w-1$, $0 \le j_2, j_4, \ldots, j_{2l} \le min\{w-1, s-2\}$, $0 \le j_{l''} \le min\{w-1, g^{(2)}-b-1\}$ *and* $l = \left\lfloor \frac{n}{s+b} \right\rfloor$.

Further, maximum weight of elements of $\mathbb{A}'$ *is*

$$W^2_{max} = 2w \left\lfloor \frac{n}{s+b} \right\rfloor + \gamma''_2(g^{(2)}),$$

$$\text{where } \gamma''_2(g^{(2)}) = \begin{cases} g^{(2)} & if \quad 0 \le g^{(2)} \le w \\ w & if \quad w+1 \le g^{(2)} \le b \\ w + g^{(2)} - b & if \quad b+1 \le g^{(2)} \le b+w \\ 2w & if \quad b+w+1 \le g^{(2)} \le b+s-1. \end{cases}$$

Proof In this case also, if $n \pmod{(b+s)} = 0$, all nonzero components in any vector of $\mathbb{A}$ are confined to $\left\lfloor \frac{n}{s+b} \right\rfloor$ sets that are separated by one zero. If $n \pmod{(b+s)} \neq 0$, nonzero components in a vector of $\mathbb{A}$ are confined to $\left\lfloor \frac{n}{s+b} \right\rfloor + 1$ sets that are separated by one zero, where each of $\left\lfloor \frac{n}{s+b} \right\rfloor$ sets has $b+s-1$ consecutive components and the last set has $g^{(2)} = \gamma_2\big(n \mod (b+s)\big)$ components, where $\gamma_2(r) = \begin{cases} r & if\ n \pmod{(b+s)} \le b+s-1 \\ b+s-1 & otherwise \end{cases}$. Then the cardinality of the set A'_j of the vectors of $\mathbb{A}'$ having weight j is calculated as follows.

Sub-case (i). If $g^{(2)} = 0$, the number $|A'_j|$ of the vectors of $\mathbb{A}'$ having weight j is given by

$$\sum_{j_1, j_2, \ldots, j_{2l}} \binom{b-1}{j_1}(q-1)^{1+j_1} \times \binom{s-2}{j_2}(q-1)^{1+j_2} \times \binom{b-1}{j_3}(q-1)^{1+j_3} \\ \times \binom{s-2}{j_4}(q-1)^{1+j_4} \times \cdots \times \binom{b-1}{j_{2l-1}}(q-1)^{1+j_{2l-1}} \\ \times \binom{s-2}{j_{2l}}(q-1)^{1+j_{2l}} \\ = \sum_{j_1, j_2, \ldots, j_{2l}} \left[\prod_{\rho=1}^{l}\binom{b-1}{j_{2\rho-1}}\prod_{\rho=1}^{l}\binom{s-2}{j_{2\rho}}\right](q-1)^{2l+j_1+j_2+\cdots+j_{2l}},$$

where $2l + j_1 + j_2 + \cdots + j_{2l} = j$ such that $0 \leq j_1, j_3, \ldots, j_{2l-1} \leq w-1$, $0 \leq j_2, j_4, \ldots, j_{2l} \leq min\{w-1, s-2\}$ and $l = \lfloor \frac{n}{s+b} \rfloor$.

Sub-case (ii). If $1 \leq g^{(2)} \leq b$, the number $|A'_j|$ of the vectors of $\mathbb{A}'$ having weight j is given by

$$\sum_{j_1, j_2, \ldots, j_{2l}, j_{l'}} \binom{b-1}{j_1}(q-1)^{1+j_1} \times \binom{s-2}{j_2}(q-1)^{1+j_2} \\ \times \binom{b-1}{j_3}(q-1)^{1+j_3} \times \binom{s-2}{j_4}(q-1)^{1+j_4} \\ \times \cdots \times \binom{b-1}{j_{2l-1}}(q-1)^{1+j_{2l-1}} \\ \times \binom{s-2}{j_{2l}}(q-1)^{1+j_{2l}} \times \binom{g^{(2)}-1}{j_{l'}}(q-1)^{1+j_{l'}} \\ = \sum_{j_1, j_2, \ldots, j_{2l}, j_{l'}} \left[\prod_{\rho=1}^{l}\binom{b-1}{j_{2\rho-1}}\prod_{\rho=1}^{l}\binom{s-2}{j_{2\rho}}\right] \\ \times \binom{g^{(2)}-1}{j_{l'}}(q-1)^{2l+j_1+j_2+\cdots+j_{2l}+j_{l'}+1},$$

where $2l + j_1 + j_2 + \cdots + j_{2l} + j_{l'} + 1 = j$ such that $0 \leq j_1, j_3, \ldots, j_{2l-1}, \leq w-1$, $0 \leq j_2, j_4, \ldots, j_{2l} \leq min\{w-1, s-2\}$, $0 \leq j_{l'} \leq min\{w-1, g^{(2)}-1\}$ and $l = \left\lfloor \dfrac{n}{s+b} \right\rfloor$.

Sub-case (iii). If $b + 1 \leq g^{(2)} \leq b + s - 1$, the number $|A'_j|$ of the vectors of $\mathbb{A}'$ having weight j is given by

$$\sum_{j_1,j_2,\dots,j_{2l},j_{l'},j_{l''}} \binom{b-1}{j_1}(q-1)^{1+j_1}$$
$$\times \binom{s-2}{j_2}(q-1)^{1+j_2} \times \binom{b-1}{j_3}(q-1)^{1+j_3} \times \binom{s-2}{j_4}(q-1)^{1+j_4}$$
$$\times \dots\dots \times \binom{b-1}{j_{2l-1}}(q-1)^{1+j_{2l-1}} \times \binom{s-2}{j_{2l}}(q-1)^{1+j_{2l}}$$
$$\times \binom{b-1}{j_{l'}}(q-1)^{1+j_{l'}} \times \binom{g^{(2)}-b-1}{j_{l''}}(q-1)^{1+j_{l''}}$$
$$= \sum_{j_1,j_2,\dots,j_{2l},j_{l'},j_{l''}} \left[\prod_{\rho=1}^{l}\binom{b-1}{j_{2\rho-1}}\prod_{\rho=1}^{l}\binom{s-2}{j_{2\rho}}\right]$$
$$\times \binom{b-1}{j_{l'}}\binom{g^{(2)}-b-1}{j_{l''}}(q-1)^{2l+j_1+j_2+\dots+j_{2l}+j_{l'}+j_{l''}+2},$$

where $2l+j_1+j_2+\dots+j_{2l}+j_{l'}+j_{l''}+2=j$ such that $0 \le j_1, j_3, \dots, j_{2l-1}, j_{l'} \le w-1$, $0 \le j_2, j_4, \dots, j_{2l} \le min\{w-1, s-2\}$, $0 \le j_{l''} \le min\{w-1, g^{(2)}-b-1\}$ and $l = \left\lfloor \frac{n}{s+b} \right\rfloor$.

Maximum weight of a vector of $\mathbb{A}'$ can be calculated in the same way as Lemma 3. □

Finally, we put Plotkin's type of bound for the set $\mathbb{A}$ and $\mathbb{A}'$ whose proof is similar to Theorem 3.

Theorem 7 *The minimum weight of a vector is bounded by* $\frac{\sum_{j=1}^{W^1_{max}} jA(j)}{\sum_{j=1}^{W^1_{max}} A(j)}$ *for the set* $\mathbb{A}$ *and* $\frac{\sum_{j=1}^{W^2_{max}} jA'(j)}{\sum_{j=1}^{W^2_{max}} A'(j)}$ *for the set* $\mathbb{A}'$*, where* $A(j)$*,* $A'(j)$*,* W^1_{max} *and* W^2_{max} *are given by Lemma 3-4.*

5 Conclusion

This paper presents the conditions for existence of low-density periodical burst correcting linear codes along with its decoding error probability. Weight distribution and Plotkin's type bound for the error set are also presented. The same is also studied for some other periodical bursts which will be detected by such codes. There may be a more systematic way of constructing such an error correcting code which can be investigated. Optimum codes which correct only such errors and no others can be interesting to look for.

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