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ORIGINAL RESEARCH

Nonlinear two-point iterative functional boundary value problems on time scales

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Abstract

This paper is concerned with second order iterative functional boundary value problem with two-point boundary conditions on time scales. By utilizing Schauder fixed point theorem and contraction mapping principle, we establish some sufficient conditions for the existence, uniqueness and continuous dependence of bounded solutions. Finally, we provide an example to support our main results.

Keywords Time scale · Iterative BVP · Functional differential equation · Continuous dependence

Mathematics Subject Classification 34K42 · 39B12 · 39B82

1 Introduction and preliminaries

Iterative differential equation, as a special type of functional differential equations, in which the deviating argument depends on the state [19]. Many researchers have concentrated on studying first order iterative functional differential equations by different approaches such as Picard's successive approximation, fixed point theory and the technique of nonexpansive operators, see [2,8,13,21]. But the literature related to the second and higher order is very less since the presence of the iterates increases the difficulty of studying them, see [5–7,11,12,15–18,20]. This motivates us to study the following second order iterative functional boundary value problem on time scales

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$$z^{\Delta\Delta}(s) + g\big(s, z(s), z^{[2]}(s), z^{[3]}(s), \cdots, z^{[n]}(s)\big) = 0, \ s \in [a, b]_{\mathbb{T}}, \tag{1}$$

subject to the two-point boundary conditions

$$z(a) = z_a, \ z(\sigma^2(b)) = z_b, \tag{2}$$

where $\mathbb{T}$ is a time scale, $g : [a, b]_{\mathbb{T}} \times \mathbb{R}^n \to \mathbb{R}$ is a continuous function and $z^{[2]}(s) = z(z(s)), \cdots, z^{[n]}(s) = z^{[n-1]}(z(s))$. For more details about the theory of time scales, refer to [1,3]. By applying Schauder fixed point theorem and contraction mapping principle, we establish the existence and uniqueness of solutions to the BVP (1)–(2). Equation (1) in real continuous time scales describes diffusion phenomena with a source or a reaction term [12]. We refer the interested reader to [9,10,14] and the references therein for more details.

The commonly used technique in the theory of BVPs on time scales involving transforming the BVP (1)–(2) as an equivalent integral equation (see, Chapter 7 of [4])

$$\begin{aligned} z(s) &= z_a + \frac{z_b - z_a}{\sigma^2(b) - a}(s - a) \\ &+ \int_a^{\sigma(b)} G(s, t) g\big(t, z(t), z^{[2]}(t), z^{[3]}(t), \cdots, z^{[n]}(t)\big)\Delta t, \ s \in [a, \sigma^2(b)]_{\mathbb{T}}, \end{aligned} \tag{3}$$

where

$$G(s, t) = \frac{1}{\sigma^2(b) - a} \begin{cases} (s - a)(\sigma^2(b) - \sigma(t)), & if \quad s \le t, \\ (\sigma(t) - a)(\sigma^2(b) - s), & if \quad \sigma(t) \le s. \end{cases} \tag{4}$$

Moreover, we note that $G(s, t)$ is nonnegative on $[a, \sigma^2(b)]_{\mathbb{T}} \times [a, b]_{\mathbb{T}}$ and

$$G(s, t) \le G(\sigma(t), t), \ (s, t) \in [a, \sigma^2(b)]_{\mathbb{T}} \times [a, b]_{\mathbb{T}}. \tag{5}$$

We difine

$$M = \max_{s \in [a, \sigma^2(b)]_{\mathbb{T}}} \int_a^{\sigma(b)} G(s, t)\Delta t. \tag{6}$$

Lemma 1 *For any* $s_1, s_2 \in [a, \sigma^2(b)]_{\mathbb{T}}$, *the Green's function* (4) *satisfies*

$$\int_a^{\sigma^2(b)} |G(s_1, t) - G(s_2, t)|\Delta t \le N|s_1 - s_2|,$$

where $N = 4(\sigma^2(b) - a)$.

Proof Set $G_1(s,t) = \frac{(s-a)(\sigma^2(b)-\sigma(t))}{\sigma^2(b)-a}$ and $G_2(s,t) = \frac{(\sigma(t)-a)(\sigma^2(b)-s)}{\sigma^2(b)-a}$. Let $s_1, s_2 \in [a, \sigma^2(b)]_{\mathbb{T}}$ with $s_2 \le s_1$. Then

$$\begin{aligned}\int_a^{s_2} |G_1(s_1,t) - G_1(s_2,t)|\Delta t &= \frac{1}{\sigma^2(b)-a}\int_a^{s_2} |(s_1-a)(\sigma^2(b)-\sigma(t)) \\ &\quad - (s_2-a)(\sigma^2(b)-\sigma(t))|\Delta t \\ &\le \frac{1}{\sigma^2(b)-a}|s_1-s_2|\int_a^{s_2} |(\sigma^2(b)-\sigma(t))|\Delta t \\ &\le \frac{1}{\sigma^2(b)-a}|s_1-s_2|\int_a^{s_2} |(\sigma^2(b)-a)|\Delta t \\ &\le (\sigma^2(b)-a)|s_1-s_2|,\end{aligned}$$

$$\begin{aligned}\int_{s_2}^{s_1} |G_1(s_1,t) - G_2(s_2,t)|\Delta t &= \frac{1}{\sigma^2(b)-a}\int_{s_2}^{s_1} |(s_1-a)(\sigma^2(b)-\sigma(t)) \\ &\quad - (\sigma(t)-a)(\sigma^2(b)-s_2)|\Delta t \\ &\le \frac{1}{\sigma^2(b)-a}\int_{s_2}^{s_1} (s_1-a)(\sigma^2(b)-\sigma(t))\Delta t \\ &\quad + \frac{1}{\sigma^2(b)-a}\int_{s_2}^{s_1} (\sigma(t)-a)(\sigma^2(b)-s_2)\Delta t \\ &\le \int_{s_2}^{s_1} (s_1-a)\Delta t + \int_{s_2}^{s_1} (\sigma(t)-a)\Delta t \\ &\le 2(\sigma^2(b)-a)|s_1-s_2|\end{aligned}$$

and

$$\begin{aligned}\int_{s_1}^{\sigma^2(b)} |G_2(s_1,t) - G_2(s_2,t)|\Delta t &= \frac{1}{\sigma^2(b)-a}\int_a^{s_2} |(\sigma(t)-a)(\sigma^2(b)-s_1) \\ &\quad - (\sigma(t)-a)(\sigma^2(b)-s_2)|\Delta t \\ &\le \frac{1}{\sigma^2(b)-a}(\sigma(t)-a)\int_a^{s_2} |s_1-s_2|\Delta t \\ &\le (\sigma^2(b)-a)|s_1-s_2|.\end{aligned}$$

Thus,

$$\begin{aligned}&\int_a^{\sigma^2(b)} |G(s_1,t) - G(s_2,t)|\Delta t \\ &= \int_a^{s_2} |G_1(s_1,t) - G_1(s_2,t)|\Delta t + \int_{s_2}^{s_1} |G_1(s_1,t) - G_2(s_2,t)|\Delta t \\ &\quad + \int_{s_1}^{\sigma^2(b)} |G_2(s_1,t) - G_2(s_2,t)|\Delta t \le 4(\sigma^2(b)-a)|s_1-s_2|.\end{aligned}$$

□

Consider the function space

$$\mathtt{C}[a, \sigma^2(b)] = \{\mathtt{z} \mid \mathtt{z} : [a, \sigma^2(b)]_{\mathbb{T}} \to \mathbb{R} \text{ is continuous}\}$$

and for $\mathtt{L} > 0$, define

$$\mathscr{B}(\mathtt{L}) = \{\mathtt{z} \in \mathtt{C}[a, \sigma^2(b)] \mid \|\mathtt{z}\| \leq \mathtt{L},\ \forall\, \mathtt{s} \in [a, \sigma^2(b)]_{\mathbb{T}}\}.$$

Then $\mathscr{B}(\mathtt{L})$ is a Banach space with the norm

$$\|\mathtt{z}\| = \max_{\mathtt{s}\in[a,\sigma^2(b)]_{\mathbb{T}}} |\mathtt{z}(\mathtt{s})|.$$

For $\mathtt{L} > 0$ and $\Bbbk > 0$, define the set

$$\mathscr{B}(\mathtt{L}, \Bbbk) = \{\mathtt{z} \in \mathscr{B}(\mathtt{L}) \mid \|\mathtt{z}\| \leq \mathtt{L} \text{ and } |\mathtt{z}(\mathtt{s}_1) - \mathtt{z}(\mathtt{s}_2)| \leq \Bbbk|\mathtt{s}_1 - \mathtt{s}_2|,\ \forall\, \mathtt{s}_1, \mathtt{s}_2 \in [a, \sigma^2(b)]_{\mathbb{T}}\}.$$

Then $\mathscr{B}(\mathtt{L}, \Bbbk)$ is a closed convex and bounded subset of $\mathscr{B}(\mathtt{L})$. Also, it can be seen from the definition of $\mathscr{B}(\mathtt{L}, \Bbbk)$ that for every $\psi, \phi \in \mathscr{B}(\mathtt{L}, \Bbbk)$,

$$\|\psi^{[i]} - \phi^{[i]}\| \leq \sum_{\mathtt{j}=0}^{i-1} \Bbbk^{\mathtt{j}} \|\psi - \phi\|, \quad i = 1, 2, \cdots \tag{7}$$

Now define an operator $\aleph : \mathscr{B}(\mathtt{L}, \Bbbk) \to \mathscr{B}(\mathtt{L})$ as

$$(\aleph \mathtt{z})(\mathtt{s}) = \mathtt{z}_a + \frac{\mathtt{z}_b - \mathtt{z}_a}{\sigma^2(b) - a}(\mathtt{s} - a) + \int_a^{\sigma(b)} \mathtt{G}(\mathtt{s}, \mathtt{t})\mathtt{g}\big(\mathtt{t}, \mathtt{z}(\mathtt{t}), \mathtt{z}^{[2]}(\mathtt{t}), \mathtt{z}^{[3]}(\mathtt{t}), \cdots, \mathtt{z}^{[n]}(\mathtt{t})\big)\Delta \mathtt{t},\ \ \mathtt{s} \in [a, \sigma^2(b)]_{\mathbb{T}} \tag{8}$$

Then $\mathtt{z}$ is a solution of (1)–(2) if and only if $\mathtt{z}$ is a fixed point of $\aleph$.

2 Existence and uniqueness of solutions

This section deals with existence and uniqueness of solutions for (1)–(2). In order to reach our goal, we assume the following condition hold:

($\mathcal{H}_1$) Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be positive constants such that

$$|\mathtt{g}(\mathtt{t}, \mathtt{z}_1, \mathtt{z}_2, \cdots, \mathtt{z}_n) - \mathtt{g}(\mathtt{t}, \widehat{\mathtt{z}}_1, \widehat{\mathtt{z}}_2, \cdots, \widehat{\mathtt{z}}_n)| \leq \sum_{\mathtt{j}=1}^{n} \alpha_{\mathtt{j}} \|\mathtt{z}_{\mathtt{j}} - \widehat{\mathtt{z}}_{\mathtt{j}}\|$$

Lemma 2 *Suppose* $(\mathcal{H}_1)$ *holds. Then operator* $\aleph$ *is continuous and compact on* $\mathscr{B}(\mathtt{L}, \Bbbk)$.

Proof Let $\mathtt{z}, \widehat{\mathtt{z}} \in \mathscr{B}(\mathtt{L}, \Bbbk)$ and $\mathtt{s} \in [a, \sigma^2(b)]_{\mathbb{T}}$. Then by $(\mathcal{H}_1)$, (6) and (7),

$$\begin{aligned}
\|(\aleph\mathtt{z})(\mathtt{s}) - (\aleph\widehat{\mathtt{z}})(\mathtt{s})\| \leq & \int_a^{\sigma(b)} |\mathtt{G}(\mathtt{s},\mathtt{t})| \Big| \mathtt{g}\big(\mathtt{t}, \mathtt{z}(\mathtt{t}), \mathtt{z}^{[2]}(\mathtt{t}), \mathtt{z}^{[3]}(\mathtt{t}), \cdots, \mathtt{z}^{[n]}(\mathtt{t})\big) \\
& - \mathtt{g}\big(\mathtt{t}, \widehat{\mathtt{z}}(\mathtt{t}), \widehat{\mathtt{z}}^{[2]}(\mathtt{t}), \widehat{\mathtt{z}}^{[3]}(\mathtt{t}), \cdots, \widehat{\mathtt{z}}^{[n]}(\mathtt{t})\big) \Big| \Delta\mathtt{t} \\
\leq & \int_a^{\sigma(b)} |\mathtt{G}(\mathtt{s},\mathtt{t})| \sum_{\mathtt{j}=1}^{n} \alpha_{\mathtt{j}} \|\mathtt{z}^{[\mathtt{j}]} - \widehat{\mathtt{z}}^{[\mathtt{j}]}\| \Delta\mathtt{t} \\
\leq & \mathtt{M} \sum_{\mathtt{j}=1}^{n} \alpha_{\mathtt{j}} \sum_{i=0}^{\mathtt{j}-1} \Bbbk^i \|\mathtt{z} - \widehat{\mathtt{z}}\|.
\end{aligned}$$

Thus, $\aleph$ is continuous. It can be seen by Arzela-Ascoli theorem that $\aleph$ is compact. □

Lemma 3 *Suppose* $(\mathcal{H}_1)$ *and the following hold.*

$(\mathcal{H}_2)$ $2|\mathtt{z}_a| + |\mathtt{z}_b| + \mathtt{M}\left[\mathtt{g}^\star + \mathtt{L}\sum_{\mathtt{j}=1}^{n} \alpha_{\mathtt{j}} \sum_{i=0}^{\mathtt{j}-1} \Bbbk^i\right] \leq \mathtt{L}$, *where* $\mathtt{g}^\star = \max_{\mathtt{t}\in[a,\sigma^2(b)]_{\mathbb{T}}} |\mathtt{g}(\mathtt{t}, 0, 0, \cdots, 0)|$.

Then $|(\aleph\mathtt{z})(\mathtt{s})| \leq \mathtt{L}$ *for all* $\mathtt{s} \in [a, \sigma^2(b)]_{\mathbb{T}}$ *and* $\mathtt{z} \in \mathscr{B}(\mathtt{L}, \Bbbk)$.

Proof Let $\mathtt{z} \in \mathscr{B}(\mathtt{L}, \Bbbk)$ and $\mathtt{s} \in [a, \sigma^2(b)]_{\mathbb{T}}$. Then

$$\begin{aligned}
|(\aleph\mathtt{z})(\mathtt{s})| = & \Bigg| \mathtt{z}_a + \frac{\mathtt{z}_b - \mathtt{z}_a}{\sigma^2(b) - a}(\mathtt{s} - a) \\
& + \int_a^{\sigma(b)} \mathtt{G}(\mathtt{s},\mathtt{t}) \mathtt{g}\big(\mathtt{t}, \mathtt{z}(\mathtt{t}), \mathtt{z}^{[2]}(\mathtt{t}), \mathtt{z}^{[3]}(\mathtt{t}), \cdots, \mathtt{z}^{[n]}(\mathtt{t})\big) \Delta\mathtt{t} \Bigg| \\
\leq & \ 2\mathtt{z}_a + \mathtt{z}_b + \int_a^{\sigma(b)} |\mathtt{G}(\mathtt{s},\mathtt{t})| |\mathtt{g}\big(\mathtt{t}, \mathtt{z}(\mathtt{t}), \mathtt{z}^{[2]}(\mathtt{t}), \mathtt{z}^{[3]}(\mathtt{t}), \cdots, \mathtt{z}^{[n]}(\mathtt{t})\big) \\
& - \mathtt{g}(\mathtt{t}, 0, 0, \cdots, 0)| \Delta\mathtt{t} \\
& + \int_a^{\sigma(b)} |\mathtt{G}(\mathtt{s},\mathtt{t})| |\mathtt{g}(\mathtt{t}, 0, 0, \cdots, 0)| \Delta\mathtt{t} \\
\leq & \ 2\mathtt{z}_a + \mathtt{z}_b + \mathtt{M} \sum_{\mathtt{j}=1}^{n} \alpha_{\mathtt{j}} \sum_{i=0}^{\mathtt{j}-1} \Bbbk^i \|\mathtt{z}\| + \mathtt{M}\mathtt{g}^\star \\
\leq & \ 2\mathtt{z}_a + \mathtt{z}_b + \mathtt{M}\left[\mathtt{L} \sum_{\mathtt{j}=1}^{n} \alpha_{\mathtt{j}} \sum_{i=0}^{\mathtt{j}-1} \Bbbk^i + \mathtt{g}^\star\right] \\
\leq & \ \mathtt{L}.
\end{aligned}$$

□

Lemma 4 *Suppose* $(\mathcal{H}_1)$ *and the following hold.*

$(\mathcal{H}_3)$ $\frac{|z_b - z_a|}{\sigma^2(b)-a} + N\left[g^\star + L\sum_{j=1}^{n} \alpha_j \sum_{i=0}^{j-1} \Bbbk^i\right] \le \Bbbk.$

Then $|(\aleph z)(s_1) - (\aleph z)(s_2)| \le \Bbbk|s_1 - s_2|$ *for all* $s_1, s_2 \in [a, \sigma^2(b)]_{\mathbb{T}}$ *and* $z \in \mathscr{B}(L, \Bbbk)$.

Proof Let $s_1, s_2 \in [a, \sigma^2(b)]_{\mathbb{T}}$ and $z \in \mathscr{B}(L, \Bbbk)$ with $s_1 \le s_2$. Then

$$\begin{aligned}|(\aleph z)(s_1) - (\aleph z)(s_2)| &\le \frac{|z_b - z_a|}{\sigma^2(b) - a}|s_1 - s_2| + \int_a^{\sigma(b)} |G(s_1, t) \\ &\quad - G(s_2, t)||g\big(t, z(t), z^{[2]}(t), z^{[3]}(t), \cdots, z^{[n]}(t)\big)|\Delta t \\ &\le \frac{|z_b - z_a|}{\sigma^2(b) - a}|s_1 - s_2| + \int_a^{\sigma(b)} |G(s_1, t) \\ &\quad - G(s_2, t)|\Big|g\big(t, z(t), z^{[2]}(t), z^{[3]}(t), \cdots, z^{[n]}(t)\big)|\Delta t \\ &\quad - g(t, 0, 0, \cdots, 0)\Big|\Delta t + \int_a^{\sigma(b)} |G(s_1, t) \\ &\quad - G(s_2, t)||g(t, 0, 0, \cdots, 0)\Big|\Delta t \\ &\le \frac{|z_b - z_a|}{\sigma^2(b) - a}|s_1 - s_2| \\ &\quad + N\sum_{j=1}^{n} \alpha_j \sum_{i=0}^{j-1} \Bbbk^i \|z\||s_1 - s_2| + Ng^\star|s_1 - s_2| \\ &\le \frac{|z_b - z_a|}{\sigma^2(b) - a}|s_1 - s_2| \\ &\quad + N\left[L\sum_{j=1}^{n} \alpha_j \sum_{i=0}^{j-1} \Bbbk^i + g^\star\right]|s_1 - s_2| \\ &\le \Bbbk|s_1 - s_2|.\end{aligned}$$

□

Lemma 5 *Suppose* $(\mathcal{H}_1)$*–*$(\mathcal{H}_3)$ *hold. Then* $\aleph(\mathscr{B}(L, \Bbbk)) \subset \mathscr{B}(L, \Bbbk)$.

Proof It is clear from Lemmas 3 and 4 that $\aleph$ maps $\mathscr{B}(L, \Bbbk)$ into itself. □

Theorem 1 *Suppose* $(\mathcal{H}_1)$*–*$(\mathcal{H}_3)$ *hold. Then BPV* (1)*–*(2) *has a solution in* $\mathscr{B}(L, \Bbbk)$.

Proof From Lemmas 2 to 5, we see that all the conditions of Schauder's fixed point theorem are satisfied on $\mathscr{B}(L, \Bbbk)$. thus there exists a fixed point $z^\star$ in $\mathscr{B}(L, \Bbbk)$ such that $\aleph z^\star = z^\star$. Therefore, $z^\star$ is a solution of (1)–(2). This completes the proof. □

Theorem 2 *Suppose* $(\mathcal{H}_1)$*-*$(\mathcal{H}_3)$ *and the following hold.*

$(\mathcal{H}_4)$ $M\sum_{j=1}^{n} \alpha_j \sum_{i=0}^{j-1} \Bbbk^i < 1.$

Then BPV (1)–(2) *has a unique solution in* $\mathscr{B}(\mathtt{L}, \Bbbk)$.

Proof Let $\mathtt{z}, \widehat{\mathtt{z}} \in \mathscr{B}(\mathtt{L}, \Bbbk)$ and $\mathtt{s} \in [a, \sigma^2(b)]_{\mathbb{T}}$. Then by Lemma 2,

$$\|(\aleph \mathtt{z})(\mathtt{s}) - (\aleph \widehat{\mathtt{z}})(\mathtt{s})\| \leq \mathtt{M} \sum_{\mathtt{j}=1}^{n} \alpha_{\mathtt{j}} \sum_{i=0}^{\mathtt{j}-1} \Bbbk^{i} \|\mathtt{z} - \widehat{\mathtt{z}}\|$$

Therefore, by the contraction mapping principle $\aleph$ has a unique fixed point in $\mathscr{B}(\mathtt{L}, \Bbbk)$. This completes the proof.

3 Continuous dependence

In this section, we establish continuous dependence of the unique solution on $\mathtt{g}$.

Theorem 3 *Suppose* $(\mathcal{H}_1)$–$(\mathcal{H}_4)$ *hold. Then unique solution of* (1)–(2) *obtained in Theorem* 2 *depends continuously on* $\mathtt{g}$.

Proof Let $\mathtt{g}$ and $\widehat{\mathtt{g}}$ be two given functions and consider the corresponding operators $\aleph$ and $\widehat{\aleph}$ defined by (8). Next by Theorem 2, there exist two unique functions $\mathtt{z}(\mathtt{s})$ and $\widehat{\mathtt{z}}(\mathtt{s})$ in $\mathscr{B}(\mathtt{L}, \Bbbk)$ such that $\mathtt{z} = \aleph \mathtt{z}$ and $\widehat{\mathtt{z}} = \widehat{\aleph} \widehat{\mathtt{z}}$. Then,

$$\begin{aligned}
\|\aleph \widehat{\mathtt{z}} - \widehat{\aleph} \widehat{\mathtt{z}}\| \leq & \int_a^{\sigma(b)} |\mathtt{G}(\mathtt{s}, \mathtt{t})| \Big| \mathtt{g}\big(\mathtt{t}, \widehat{\mathtt{z}}(\mathtt{t}), \widehat{\mathtt{z}}^{[2]}(\mathtt{t}), \widehat{\mathtt{z}}^{[3]}(\mathtt{t}), \cdots, \widehat{\mathtt{z}}^{[n]}(\mathtt{t})\big) \\
& - \widehat{\mathtt{g}}\big(\mathtt{t}, \widehat{\mathtt{z}}(\mathtt{t}), \widehat{\mathtt{z}}^{[2]}(\mathtt{t}), \widehat{\mathtt{z}}^{[3]}(\mathtt{t}), \cdots, \widehat{\mathtt{z}}^{[n]}(\mathtt{t})\big) \Big| \Delta \mathtt{t} \\
\leq & \int_a^{\sigma(b)} |\mathtt{G}(\mathtt{s}, \mathtt{t})| \Big[\Big| \mathtt{g}\big(\mathtt{t}, \widehat{\mathtt{z}}(\mathtt{t}), \widehat{\mathtt{z}}^{[2]}(\mathtt{t}), \widehat{\mathtt{z}}^{[3]}(\mathtt{t}), \cdots, \widehat{\mathtt{z}}^{[n]}(\mathtt{t})\big) \\
& - \widehat{\mathtt{g}}\big(\mathtt{t}, \mathtt{z}(\mathtt{t}), \mathtt{z}^{[2]}(\mathtt{t}), \mathtt{z}^{[3]}(\mathtt{t}), \cdots, \mathtt{z}^{[n]}(\mathtt{t})\big) \Big| \\
& + \Big| \widehat{\mathtt{g}}\big(\mathtt{t}, \mathtt{z}(\mathtt{t}), \mathtt{z}^{[2]}(\mathtt{t}), \mathtt{z}^{[3]}(\mathtt{t}), \cdots, \mathtt{z}^{[n]}(\mathtt{t})\big) \\
& - \widehat{\mathtt{g}}\big(\mathtt{t}, \widehat{\mathtt{z}}(\mathtt{t}), \widehat{\mathtt{z}}^{[2]}(\mathtt{t}), \widehat{\mathtt{z}}^{[3]}(\mathtt{t}), \cdots, \widehat{\mathtt{z}}^{[n]}(\mathtt{t})\big) \Big| \Big] \Delta \mathtt{t} \\
\leq & \int_a^{\sigma(b)} |\mathtt{G}(\mathtt{s}, \mathtt{t})| \Big[\|\mathtt{g} \\
& - \widehat{\mathtt{g}}\| + \sum_{\mathtt{j}=1}^{n} \alpha_{\mathtt{j}} \sum_{i=0}^{\mathtt{j}-1} \Bbbk^{i} \|\mathtt{z} - \widehat{\mathtt{z}}\| \Big] \Delta \mathtt{t} \\
= & \mathtt{M} \Big[\|\mathtt{g} - \widehat{\mathtt{g}}\| + \sum_{\mathtt{j}=1}^{n} \alpha_{\mathtt{j}} \sum_{i=0}^{\mathtt{j}-1} \Bbbk^{i} \|\mathtt{z} - \widehat{\mathtt{z}}\| \Big].
\end{aligned}$$

Therefore,

$$\begin{aligned}\|\mathtt{z}-\widehat{\mathtt{z}}\| &\leq \|\aleph\mathtt{z}-\aleph\widehat{\mathtt{z}}\|+\|\aleph\widehat{\mathtt{z}}-\widehat{\aleph}\widehat{\mathtt{z}}\| \\ &\leq \mathtt{M}\sum_{\mathtt{j}=1}^{n}\alpha_{\mathtt{j}}\sum_{i=0}^{\mathtt{j}-1}\mathbb{k}^{i}\|\mathtt{z}-\widehat{\mathtt{z}}\|+\mathtt{M}\Big[\|\mathtt{g}-\widehat{\mathtt{g}}\|+\sum_{\mathtt{j}=1}^{n}\alpha_{\mathtt{j}}\sum_{i=0}^{\mathtt{j}-1}\mathbb{k}^{i}\|\mathtt{z}-\widehat{\mathtt{z}}\|\Big].\end{aligned}$$

That is

$$\|\mathtt{z}-\widehat{\mathtt{z}}\| \leq \mathtt{M}\left[1-2\mathtt{M}\sum_{\mathtt{j}=1}^{n}\alpha_{\mathtt{j}}\sum_{i=0}^{\mathtt{j}-1}\mathbb{k}^{i}\right]^{-1}\|\mathtt{g}-\widehat{\mathtt{g}}\|.$$

This completes the proof. □

Example 1 Consider the time scale $\mathbb{T}=\{10^m \mid m\in\mathbb{Z}\}\cup\{0\}$ and $\mathtt{z}_a=1$, $\mathtt{z}_b=2$, $a=0$, $b=1$ in the (1)–(2). It follows from (6) that, for $\mathtt{s}=\frac{1}{10^m}$, $m=-1,0,1,\cdots$,

$$\begin{aligned}\max_{\mathtt{s}\in[0,100]_{\mathbb{T}}}\int_0^{10}\mathtt{G}(\mathtt{s},\mathtt{t})\Delta\mathtt{t} &= \max_{m}\Bigg[\lim_{k\to+\infty}\int_{\frac{1}{10^k}}^{\frac{1}{10^m}}10\mathtt{t}(10^2-\frac{1}{10^m})\Delta\mathtt{t} \\ &\quad+\int_{\frac{1}{10^m}}^{10}\frac{1}{10^m}(10^2-10\mathtt{t})\Delta\mathtt{t}\Bigg] \\ &=\frac{10}{11}\max_{m}\Big[101\times10^{-2m}-10^{-3m}-10^{-2}\Big] \\ &=8181.818182.\end{aligned}$$

So, $\mathtt{M}=8181.818182$ and $\mathtt{N}=400$. Next, consider the function

$$\begin{aligned}\mathtt{g}(\mathtt{t},\mathtt{z}(\mathtt{t}),\mathtt{z}^{[2]}(\mathtt{t}),\mathtt{z}^{[3]}(\mathtt{t})) &= \cos(\mathtt{t})+\frac{\pi}{154\times10^3}\mathtt{z}(\mathtt{t}) \\ &\quad+\frac{\pi}{256\times10^3}\sin(\mathtt{z}^{[2]}(\mathtt{t}))+\frac{\pi}{323\times10^3}\cos(\mathtt{z}^{[3]}(\mathtt{t})).\end{aligned}$$

Then

$$\begin{aligned}&|\mathtt{g}(\mathtt{t},\mathtt{z}_1,\mathtt{z}_2,\mathtt{z}_3)-\mathtt{g}(\mathtt{t},\widehat{\mathtt{z}}_1,\widehat{\mathtt{z}}_2,\widehat{\mathtt{z}}_3)| \leq \alpha_1\|\mathtt{z}_1-\widehat{\mathtt{z}}_1\| \\ &\quad+\alpha_2\|\mathtt{z}_2-\widehat{\mathtt{z}}_2\|+\alpha_3\|\mathtt{z}_3-\widehat{\mathtt{z}}_3\|,\end{aligned}$$

where $\alpha_1=\frac{\pi}{154\times10^3}$, $\alpha_2=\frac{\pi}{256\times10^3}$, $\alpha_3=\frac{\pi}{323\times10^3}$ and

$$\mathtt{g}(\mathtt{t},0,0,0)=\cos(\mathtt{t})+\frac{\pi}{323\times10^3}\leq 1.000009726 := \mathtt{g}^{\star}.$$

Further, let $\mathtt{L} = \frac{\pi}{2.42\times 10^{-4}}$ and $\Bbbk = \frac{\pi}{4.95\times 10^{-3}}$. Then

$$2|\mathtt{z}_a| + |\mathtt{z}_b| + \mathtt{M}\left[\mathtt{g}^\star + \mathtt{L}\sum_{\mathtt{j}=1}^{n}\alpha_\mathtt{j}\sum_{i=0}^{\mathtt{j}-1}\Bbbk^i\right] = 12692.87935 \le \mathtt{L},$$

$$\frac{|\mathtt{z}_b - \mathtt{z}_a|}{\sigma^2(b) - a} + \mathtt{N}\left[\mathtt{g}^\star + \mathtt{L}\sum_{\mathtt{j}=1}^{n}\alpha_\mathtt{j}\sum_{i=0}^{\mathtt{j}-1}\Bbbk^i\right] = 620.3552127 \le \Bbbk$$

and

$$\mathtt{M}\sum_{\mathtt{j}=1}^{n}\alpha_\mathtt{j}\sum_{i=0}^{\mathtt{j}-1}\Bbbk^i = 0.3471772648 < 1.$$

Thus, all assumptions $(\mathcal{H}_1)$–$(\mathcal{H}_4)$ hold. Therefore, the BVP

$$\begin{cases} \mathtt{z}^{\Delta\Delta}(\mathtt{s}) + \cos(\mathtt{s}) + \dfrac{\pi}{154\times 10^3}\mathtt{z}(\mathtt{s}) \\ \quad + \dfrac{\pi}{256\times 10^3}\sin(\mathtt{z}^{[2]}(\mathtt{s})) + \dfrac{\pi}{323\times 10^3}\cos(\mathtt{z}^{[3]}(\mathtt{s})) = 0, \ \mathtt{s}\in[0,1]_{\mathbb{T}}, \\ \mathtt{z}(0) = 1,\ \mathtt{z}(\sigma^2(1)) = 2, \end{cases}$$

has a unique solution in $\mathscr{B}\left(\frac{\pi}{2.42\times 10^{-4}}, \frac{\pi}{4.95\times 10^{-3}}\right)$ and depends continuously on the function $\mathtt{g}$.

4 Conclusion

Iterative differential equation, as a special type of functional differential equations, in which the deviating arguments depend on the state. Many researchers have concentrated on studying first order iterative functional differential equations by different approaches such as Picard's successive approximation, fixed point theory and the technique of nonexpansive operators. But the literature related to the second and higher order is very less since the presence of the iterates increases the difficulty of studying them. This work gives a criteria for the existence, uniqueness and continuous dependence of solutions for nonlinear second order iterative functional boundary values with two-point boundary conditions on time scales. In the future, we study higher order iterative functional boundary value problems on time scales and fractional order iterative boundary value problems on time scales.

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Declarations

Conflict of interest It is declared that authors has no competing interests.

Human and animal right This article does not contain any studies with human participants or animals performed by any of the authors.

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