



Adomian decomposition and homotopy perturbation method for the solution of time fractional partial integro-differential equations

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Abstract

This article deals with two different methods to solve a time fractional partial integro-differential equation. The fractional derivatives are defined here in Caputo sense. The model problem is solved using the Adomian decomposition method and homotopy perturbation method. Moreover, this paper proves the convergence analysis of the solution based on the present methods. Numerical evidences are illustrated in support of the theoretical analysis.

Keywords Partial integro-differential equation · Caputo fractional derivative · Adomian decomposition method · Homotopy perturbation method · Convergence analysis

Mathematics Subject Classification 35R11 · 35R09 · 65R20 · 26A33

1 Introduction

This article considers a non-linear time fractional partial integro-differential equation (FPIDE) as follows:

$$\begin{cases} D_t^\beta w(x, t) = g(x, t) + \int_0^t \mathcal{K}(x, t-s)F(w(x, s))ds, \\ (x, t) \in ([a, b] \times (0, T]), \\ w(x, 0) = \lambda_0(x), \quad \forall x \in [a, b], \end{cases} \quad (1)$$

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where fractional order derivative D_t^β is represented in Caputo sense and $0 < \beta < 1$. Here the kernel function is symbolized as \mathcal{K} . $w(x, t)$ be the unknown continuous function defined $\forall(x, t) \in ([a, b] \times (0, T))$ and $g(x, t)$ is the given function.

The theory and generalizations of fractional calculus expanded greatly over the early twentieth century, though it was already developed in the mid eighteenth century [44]. Numerous contributions by mathematicians have given several definitions for fractional derivatives and integrals. The fractional calculus is applicable in different fields of pure and applied sciences such as aerodynamics, viscoelasticity, heat conduction, mechanics [27,45] which have attracted several researchers to investigate more in this subject. Many authors have also experimented to find the solution of integral equations, fractional differential equations (FDEs) [36] and fractional partial differential equations (FPDEs) [17]. The numerical and approximation techniques are broadly implemented on most of the FDEs since there is no exact procedure. Many methods including the finite difference method [47,49], variational iteration method (VIM) [35], homotopy perturbation method (HPM) [50], Adams Bashforth Moulton method [12], generalized differential transform method and the Adomian decomposition method (ADM) [2,5,41] have been developed to solve FDEs. Over the years many theories regarding the existence and uniqueness of solutions of FDEs, FPIDEs, fractional integro-differential equations (FIDEs) were developed. One may refer [5, 11, 18]. Recently Hussain et. al. [28] gave the existence and uniqueness of FPIDEs.

The study of FPIDEs have become an indispensable mathematical tool to describe and analyze real world problems in several areas of science and technology. One can refer [3,4,21,43,48] for the application of FPIDEs. Several numerical techniques involving collocation methods, pseudo operational matrix method have been derived to approximate the solutions of FPIDEs in [24,25]. Though several works are done with respect to numerical approximation of FPIDEs but very few contributions are there in developing a semi-analytical method to solve FPIDEs involving convergence and uniqueness results. ADM introduced by mathematician G.Adomian [1] in 1984, is a powerful tool to solve large amount of real world problems. The decomposition method is an effective technique to get semi-analytical solutions of the extensive class of dynamical systems without closure approximation, perturbation theory, assumptions of linearization and weak nonlinearity or restrictive assumptions on stochasticity. Momani and Noor [42] applied the ADM to solve the fourth-order FIDE in which the derivatives are defined in Caputo sense. Further, the solution is computed using the homotopy perturbation method (HPM) proposed by He [30] which gives the semi-analytical solution of both the linear and nonlinear problems. One may refer [21,22,33,37] for the solutions of FDE using HPM techniques. The present method requires the apriori structure of solution with higher order smoothness as series solution is used. However there are plenty of applied models where these are not available. For these type of cases, one need to use the moving mesh methods for the analysis. These methods are available in [13–16,20,46]. The He-Laplace method, which is a modification of homotopy perturbation technique is extremely effective in solving FDE as cited in [6,7,34]. The significant benefit of He's HPM includes the independent construction of perturbation equation by homotopy in topology and the independent decision of selecting the initial approximation. Further convergence of modified homo-

topy method has largely been discussed in [8,30,40]. Recently HPM was used to solve the singularly perturbed differential equations [19,33,39]. The results obtained in this paper were helpful for the optimal design of the Fangzhu device. However the reliability of a numerical method and its convergence analysis for parametric problems with singular perturbed nature are considered in several papers [10,23,38]. In this work, the time fractional partial integro-differential equation is solved using ADM and HPM techniques. The main advantage of using these methods is that, it gives analytical as well as numerical approximation of the solution which does not involve any mesh discretization. In addition, these methods do not require any large computer memory or power and are also free from rounding off errors. Further, it is noticed that the methods, ADM and HPM work smoothly when the order of the time fractional derivative $\beta \in (0, 1)$ but it may have some complication for HPM if $\beta > 1$.

Throughout the article, Ω is defined as $\{\Omega : (x, t) \in ([0, 1] \times (0, T])\}$. Caputo operator of fractional derivative is denoted by D_t^β and the fractional integral is denoted by I_t^β . For a function $w(x, t)$, defined on Ω , we define $\|w(x, t)\|_\infty = \sup_{x \in \Omega} |w(x, t)|$ is used.

The arrangement of this article is made in such a way that the basic definitions, notations and theorems are discussed in Sect. 2. Section 3 is devoted to the analysis of the proposed methods. In Sect. 4 the existence, uniqueness and convergence of solution is provided. Section 5 briefly describes the error bounds. Some test examples with rigorous numerical results are given in Sect. 6 and finally Sect. 7 is ended with some concluding remarks.

2 Preliminaries

Some fundamental definitions and associated properties are discussed here which will be helpful in the future.

Definition 1 The *Gamma function* $\Gamma(\tau)$ is specified as:

$$\Gamma(\tau) = \int_0^\infty t^{\tau-1} e^{-t} dt,$$

which is valid for all $\tau \in \mathbb{C}$, the complex field, such that $Re(\tau) > 0$. The generalization of the *factorial function* can be written as follows:

$$\Gamma(n) = (n - 1)!, \quad \forall n \in \mathbb{N}.$$

Definition 2 The *Grünwald-Letnikov fractional derivative* is a basic extension of the derivative in fractional calculus that allows one to take the derivative a non-integer number of times. It is defined as:

$${}^{GL}D^\beta w(x) = \lim_{h \rightarrow 0} \frac{1}{h^\alpha} \sum_{r=0}^{\frac{x-a}{h}} (-1)^r \frac{\Gamma(\beta + 1)}{r! \Gamma(\beta - r + 1)} w(x - rh).$$

Here β is the order of the derivative and considered as a positive real number.

Definition 3 Let $\beta \geq 0$ be a real number and suppose that $w(x, t) \in C([a, b] \times (0, T])$. The *Riemann-Liouville fractional integral* of the function denoted as $I_t^\beta w(x, t)$ is defined by:

$$I_t^\beta w(x, t) := \begin{cases} \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} w(x, s) ds, & \beta > 0, \\ w(x, t), & \beta = 0. \end{cases} \tag{2}$$

The exponent $\beta \in \mathbb{R}^+$, be the order of the integral.

Definition 4 Let $\beta \geq 0$ be any real number and $m \in \mathbb{N}$ such that $m - 1 < \beta \leq m$. If $w(x, t)$ is a function that has continuous n^{th} partial derivative with respect to t , then the *Riemann-Liouville fractional derivative* ${}^{RL}D_t^\beta$ is defined by:

$${}^{RL}D_t^\beta w(x, t) := \begin{cases} \frac{1}{\Gamma(\beta)} \frac{\partial^n}{\partial t^n} \int_0^t (t-s)^{n-\beta-1} w(x, s) ds, & \beta > 0, \\ w(x, t), & \beta = 0. \end{cases} \tag{3}$$

Definition 5 The *He's fractional derivative* of order β which is the fractal dimensions of the fractal medium is defined as:

$$\frac{\partial^\beta w}{\partial t^\beta} = \frac{1}{\Gamma(n-\beta)} \frac{d^n}{dt^n} \int_{t_0}^t (s-t)^{n-\beta-1} [w_0(s) - w(s)] ds. \tag{4}$$

In classical mechanics, the space is always assumed to be continuous, the air flow is continuous, the water flow is continuous and the continuum hypothesis works well for many practical applications. However, if we want to study, for example, molecule diffusion in water, the water becomes discontinuous, and the fractal calculus has to be adopted to describe the motion of molecules, otherwise molecule motion becomes completely unpredictable in the frame of the continuum hypothesis. Similarly, in case of Camassa–Holm (C–H) equation, when time tends to infinitively small, the classic C–H equation will not be continuous, while (4) can describe the motion well. One may refer [32] for details.

Definition 6 The *Caputo fractional derivative* of order $\beta > 0$ is defined as:

$$D_t^\beta w(x, t) := \begin{cases} \left[I_t^{m-\beta} \left(\frac{\partial^m w}{\partial t^m} \right) \right] (x, t), & m - 1 < \beta < m, \\ \frac{\partial^m w}{\partial t^m} (x, t), & \beta = m. \end{cases} \tag{5}$$

If w is a constant function, then $D_t^\beta w = 0$. For any $\nu \in \mathbb{R}, m \in \mathbb{N}$, we have,

$$1. D_t^\beta t^\nu = \begin{cases} 0, & \text{if } 0 < \beta < 1, \nu \leq 0, \\ \frac{\Gamma(\nu+1)}{\Gamma(\nu-\beta+1)} t^{\nu-\beta}, & \text{if } 0 < \beta < 1, \nu > 0. \end{cases}$$

$$2. I_t^\beta t^\nu = \frac{\Gamma(\nu + 1)}{\Gamma(\nu + \beta + 1)} t^{\nu+\beta}, \quad \text{if } 0 < \beta < 1, \nu \geq 0.$$

For any continuous function $w(x, t)$ where $w(x, t) \in C([0, 1] \times (0, T])$, the following properties holds true:

1. $D_t^\beta I_t^\beta w(x, t) = w(x, t)$.
2. $I_t^\beta D_t^\beta w(x, t) = w(x, t) - w(x, 0), 0 < \beta < 1$.

Definition 7 Let $(\mathcal{D}, \|\cdot\|)$ be a normed space and $\mathfrak{S} : \mathcal{D} \rightarrow \mathcal{D}$ be a mapping such that $x \in \mathcal{D}$ is called a fixed point of \mathfrak{S} if $\mathfrak{S}x = x$.

Definition 8 \mathfrak{S} is called a contraction mapping on $(\mathcal{D}, \|\cdot\|)$ if there exists a real number $c \in (0, 1)$ such that,

$$\|\mathfrak{S}(x) - \mathfrak{S}(y)\| \leq c\|x - y\| \quad \forall x, y \in \mathcal{D}.$$

Theorem 1 (Banach fixed point theorem) Let $\mathfrak{S} : \mathcal{D} \rightarrow \mathcal{D}$ be a contraction mapping on a complete normed space $(\mathcal{D}, \|\cdot\|)$. Then \mathfrak{S} has a unique fixed point.

Definition 9 Let $\mathcal{F} : \mathcal{Q} \rightarrow \mathcal{D}$ be a function where $\mathcal{Q} \subseteq \mathbb{R}^n$. \mathcal{F} satisfies Lipschitz condition if \exists a constant $L \geq 0$ such that $\|\mathcal{F}y_1 - \mathcal{F}y_2\| \leq L\|y_1 - y_2\| \forall y_1, y_2 \in \mathcal{Q}$.

3 Methodology

3.1 Analysis of ADM

According to this method, the solution of (1) can be dictated as an infinite series:

$$w(x, t) = \sum_{n=0}^{\infty} w_n(x, t). \tag{6}$$

In order to solve (1) using ADM, I_t^β is operated on both sides of (1), so as to get,

$$w(x, t) = \sum_{k=0}^{m-1} \frac{\partial^k}{\partial t^k} w(x, 0) \frac{t^k}{k!} + I_t^\beta \left(g(x, t) + \int_0^t \mathcal{K}(x, t - s)F(w(x, s))ds \right). \tag{7}$$

The decomposition of nonlinear function F can be written as:

$$F = \sum_{n=0}^{\infty} \mathcal{A}_n. \tag{8}$$

Here \mathcal{A}_n , the Adomian polynomial is defined as:

$$\mathcal{A}_n = \frac{1}{n!} \left[\frac{d^n}{dv^n} F \left(\sum_{j=0}^n v^j w_j \right) \right]_{v=0}. \tag{9}$$

The Adomian polynomials [1,2,5] are one of the most important mathematical tools to approximate the nonlinear functions in the numerical algorithm which can be easily calculated by the above formula (9). The first few terms are given below:

$$\begin{aligned} \mathcal{A}_0 &= F(w_0), \\ \mathcal{A}_1 &= w_1 F'(w_0), \\ \mathcal{A}_2 &= w_2 F'(w_0) + \frac{1}{2!} w_1^2 F''(w_0), \\ \mathcal{A}_3 &= w_3 F'(w_0) + w_1 w_2 F''(w_0) + \frac{1}{3!} w_1^3 F'''(w_0). \end{aligned}$$

The components $w_0, w_1, w_2, \dots, w_n$, are determined recursively by

$$\begin{cases} w_0 = w(x, 0) + I_t^\beta g(x, t), \\ w_{n+1} = I_t^\beta \left(\int_0^t \mathcal{K}(x, t-s) \mathcal{A}_n ds \right), \quad \text{for } n = 0, 1, 2, \dots \end{cases} \tag{10}$$

Here $w(x, t) = \lim_{n \rightarrow \infty} \sum_{i=0}^n w_i(x, t)$ gives us the solution. By truncating the series, the required numerical solution is obtained by (8) up to finite (say N) number of terms. The N terms numerical approximation is defined as: $\Psi_N = \sum_{n=0}^{N-1} w_n(x, t)$. In this method w_0 is defined by the initial condition $g(x, t)$ and the other components namely w_1, w_2, \dots, w_n , are thereby derived recursively.

3.2 Analysis of HPM

Homotopy $H : \Omega \times [0, 1] \rightarrow \mathbb{R}$ for (1) is constructed as in [30]:

$$\begin{aligned} H(w, \tilde{p}) &= (1 - \tilde{p})(D_t^\beta w(x, t) - g(x, t)) \\ &\quad + \tilde{p} \left(D_t^\beta w(x, t) - g(x, t) - \int_0^t \mathcal{K}(x, t-s) F(w(x, s)) ds \right), \end{aligned} \tag{11}$$

or

$$H(w, \tilde{p}) = D_t^\beta w(x, t) - g(x, t) - \tilde{p} \left(\int_0^t \mathcal{K}(x, t-s) F(w(x, s)) ds \right), \tag{12}$$

where embedding parameter $\tilde{p} \in [0, 1]$. Putting $\tilde{p} = 0$, now (12) turns to be a linear equation and when $\tilde{p} = 1$, then (12) boils down to (1). Further, the solution of (1) can be expressed in the form of series:

$$w(x, t) = \sum_{n=0}^{\infty} w_n(x, t) \tilde{p}^n. \tag{13}$$

Taking $\tilde{p} = 1$ in (13), approximated solution of (1) is obtained as,

$$w(x, t) = \sum_{n=0}^{\infty} w_n(x, t). \tag{14}$$

The convergence of series (14) has been proved in [31]. In order to obtain the solution, identical power of \tilde{p} are equated by substituting (14) into (12) and a set of linear equations are attained as follows:

$$\tilde{p}^0 : D_t^\beta w_0 = g(x, t), \quad w_0(0) = g(x, t). \tag{15}$$

$$\tilde{p}^1 : D_t^\beta w_1 = \int_0^t \mathcal{K}(x, t - s)F(w_0(x, s))ds, \quad w_1(0) = 0. \tag{16}$$

$$\tilde{p}^2 : D_t^\beta w_2 = \int_0^t \mathcal{K}(x, t - s)F(w_1(x, s))ds, \quad w_2(0) = 0. \tag{17}$$

⋮

Computation of approximated solution is obtained by applying I_t^β to the IVPs in (15)–(17). Finally the series solution of HPM is approximated by the following N-term truncated series:

$$\varphi_N = \sum_{n=0}^{N-1} w_n(x, t). \tag{18}$$

One can clearly observe that each non-homogeneous part of these linear differential equations gives Adomian polynomials respectively. Also because ADM assumes a series solution for (1) given by $w(x, t) = \lim_{n \rightarrow \infty} \sum_{i=0}^n w_i(x, t)$, consequently, the usage of the Taylor series expansion in HPM makes some reduction on the method and it coincides with the ADM.

4 Existence and convergence

For further analysis, the following assumptions are needed.

Assumption 1 A nonlinear function $F(w)$ is considered such that $F(w)$ satisfies Lipschitz constant L . Therefore, we have

$$\|F(w_1) - F(w_2)\| \leq L\|w_1 - w_2\|.$$

Assumption 2 The kernel \mathcal{K} defined in (1) is continuous and bounded on $(x, t) \in ([0, 1] \times (0, T])$. Then there exists $M > 0$ such that $\|\mathcal{K}\| \leq M$.

In this section, the proof of the existence and uniqueness of the solution for FPIDE (1) is established. Consider the FPIDE (1):

$$D_t^\beta w(x, t) = g(x, t) + \int_0^t \mathcal{K}(x, t - s)F(w(x, s))ds, \quad \forall(x, t) \in ([0, 1] \times (0, T]),$$

with initial condition $w(x, 0) = \lambda_0(x)$. Applying I_t^β on both the sides of (1) yields $w = \mathfrak{S}w, \forall(x, t) \in ([0, 1] \times (0, T])$, where $\mathfrak{S}w$ is defined as

$$\mathfrak{S}w(x, t) = \lambda_0(x) + I_t^\beta g(x, t) + \frac{1}{\Gamma(\beta)} \int_0^t \int_0^\zeta (t - \zeta)^{\beta-1} \mathcal{K}(x, \zeta - s)F(w(x, s))dsd\zeta. \tag{19}$$

Theorem 2 Consider that assumption (1)–(2) holds true. For the FPIDE (1) with initial condition, there exists a unique $w(x, t)$ for all $(x, t) \in ([0, 1] \times (0, T])$ if $L < \frac{\Gamma(\beta + 2)}{MT^{\beta+1}}$.

Proof The set of all continuously differentiable functions defined over the region Ω form a complete normed space with supremum norm. Also, as it is seen that (19) is given in operator form $\mathfrak{S}w = w$ and therefore, it remains to show that \mathfrak{S} is a contraction mapping and for this purpose we take $w_1, w_2 \in C(\Omega)$. So,

$$\begin{aligned} & \| \mathfrak{S}w_1(x, t) - \mathfrak{S}w_2(x, t) \| \\ &= \left\| \lambda_0(x) + I_t^\beta g(x, t) + \frac{1}{\Gamma(\beta)} \int_0^t \int_0^\zeta (t - \zeta)^{\beta-1} \mathcal{K}(x, \zeta - s)F(w_1(x, s))dsd\zeta \right. \\ &\quad \left. - \lambda_0(x) + I_t^\beta g(x, t) + \frac{1}{\Gamma(\beta)} \int_0^t \int_0^\zeta (t - \zeta)^{\beta-1} \mathcal{K}(x, \zeta - s)F(w_2(x, s))dsd\zeta \right\| \\ &\leq \left\| \frac{1}{\Gamma(\beta)} \int_0^t \int_0^\zeta (t - \zeta)^{\beta-1} [\mathcal{K}(x, \zeta - s)F(w_1(x, s)) \right. \\ &\quad \left. - \mathcal{K}(x, \zeta - s)F(w_2(x, s))]dsd\zeta \right\| \\ &\leq \frac{ML}{\Gamma(\beta)} \int_0^t \int_0^\zeta (t - \zeta)^{\beta-1} \|w_1(x, s) - w_2(x, s)\|dsd\zeta \\ &\leq \frac{ML}{\Gamma(\beta)} \|w_1 - w_2\| \int_0^t \int_0^\zeta (t - \zeta)^{\beta-1} dsd\zeta \\ &\leq \frac{ML}{\Gamma(\beta + 2)} T^{\beta+1} \|w_1 - w_2\|, \tag{20} \end{aligned}$$

and since $L < \frac{\Gamma(\beta + 2)}{MT^{\beta+1}}$, which implies that $\frac{MLT^{\beta+1}}{\Gamma(\beta + 2)} < 1$. Therefore, by general Banach contraction mapping principle, \mathfrak{S} has a unique fixed point, which means that (1) has a unique solution. Hence, this proves the uniqueness and existence of solution for the model problem. One may refer [28] for more details. \square

4.1 Convergence analysis

Theorem 3 *Suppose that assumption (1) and (2) hold. If the series solution*

$$w(x, t) = \sum_{i=0}^{\infty} w_i(x, t), \tag{21}$$

and $\|w_1\|_{\infty} < \infty$ obtained by q th-order deformation is convergent, then it converges to the exact solution of (1).

Proof Let $C(\Omega, \|\cdot\|)$ be the Banach space of all continuous functions with $|w_1(x, t)| \leq \infty \forall (x, t) \in ([0, 1] \times (0, T])$.

First we define the sequence of partial sums as s_p . Let s_p and s_q be arbitrary partial sums with $p \geq q$. We claim that

$$s_p = \sum_{i=0}^n w_i(x, t),$$

is a Cauchy sequence in this Banach space. To do so,

$$\begin{aligned} \|s_p - s_q\|_{\infty} &= \max_{\forall x \in \Omega} |s_p - s_q| \\ &= \max_{\forall x \in \Omega} \left| \sum_{i=0}^n w_i(x, t) - \sum_{i=0}^m w_i(x, t) \right| \\ &= \max_{\forall x \in \Omega} \left| \sum_{i=m+1}^n w_i(x, t) \right| \\ &= \max_{\forall x \in \Omega} \left| \sum_{i=m+1}^n \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \left[\int_0^t \mathcal{K}(x, t-s) \sum_{i=m}^{n-1} \mathcal{A}_i(x, s) ds \right] d\zeta \right|. \end{aligned} \tag{22}$$

From (8) and (9), we have

$$\begin{aligned} \sum_{i=m}^{n-1} \mathcal{A}_i &= F(s_{p-1}) - F(s_{q-1}), \\ \sum_{i=m}^{n-1} w_i &= w(s_{p-1}) - w(s_{q-1}). \end{aligned}$$

So,

$$\begin{aligned} \|s_p - s_q\|_\infty &= \max_{\forall x \in \Omega} \left(\left| \int_0^t \mathcal{K}(x, t-s)(F(s_{p-1}) - F(s_{q-1}))ds \right| d\zeta \right) \\ &\leq \max_{\forall x \in \Omega} \frac{1}{\Gamma(\beta)} \left[\int_0^t |\mathcal{K}(x, t-s)| |(F(s_{p-1}) - f(s_{q-1}))| ds \right] d\zeta \\ &\leq \frac{MLT^{\beta+1}}{\Gamma(\beta + 2)} \|s_{p-1} - s_{q-1}\|_\infty \\ &= \vartheta \|s_{p-1} - s_{q-1}\|_\infty, \end{aligned}$$

where,

$$\vartheta = \left(\frac{MLT^{\beta+1}}{\Gamma(\beta + 2)} \right). \tag{23}$$

Let $p = q + 1$, then

$$\begin{aligned} \|s_p - s_q\|_\infty &\leq \vartheta \|s_q - s_{q-1}\|_\infty \\ &\leq \vartheta^2 \|s_{q-1} - s_{q-2}\| \\ &\vdots \\ &\leq \vartheta^q \|s_1 - s_0\|. \end{aligned} \tag{24}$$

So,

$$\begin{aligned} \|s_p - s_q\|_\infty &\leq \|s_{q+1} - s_q\|_\infty + \|s_{q+2} - s_{q+1}\|_\infty + \dots + \|s_p - s_{p-1}\|_\infty \\ &\leq [\vartheta^q + \vartheta^{q+1} + \dots + \vartheta^{p-1}] \|s_1 - s_0\|_\infty \\ &\leq \vartheta^m [1 + \vartheta + \vartheta^2 + \dots + \vartheta^{p-q-1}] \|s_1 - s_0\| \\ &\leq \vartheta^m \left(\frac{1 - \vartheta^{p-q}}{1 - \vartheta} \|w_1\|_\infty \right). \end{aligned}$$

Since $0 < \vartheta < 1$, we have $(1 - \vartheta^{p-q}) < 1$, and then

$$\|s_p - s_q\|_\infty \leq \frac{\vartheta^q}{1 - \vartheta} \|w_1\|_\infty.$$

But $\|w_1(x, t)\|_\infty < \infty$, so as $m \rightarrow \infty$, then

$$\|s_p - s_q\|_\infty \rightarrow 0.$$

We conclude that s_p is a Cauchy sequence in $C([0, 1] \times (0, T])$. Therefore

$$w = \lim_{n \rightarrow \infty} w_n.$$

Then, the series is convergent. Readers may go through [9] for the proof of ADM for nonlinear problems. □

5 Error bounds

The exact solution for (1) is given by $w(x, t) = \lim_{N \rightarrow \infty} \Psi_N$ and the numerical solution can be obtained by truncating the series (8) up to finite number of terms. If Ψ_N gives the N terms approximate solution then the absolute point-wise error bound depends on the partial sum $\sum_{n=0}^{N-1} w_n(x, t)$ and which is bounded by $M \frac{\vartheta^q}{1 - \vartheta}$, where ϑ is defined in (23) satisfying $0 < \vartheta < 1$. Refer [29].

6 Illustrative examples

Here are some test problems to show the effectiveness of proposed methods.

Example 1 Consider the following test problem:

$$\begin{cases} D_t^\beta w(x, t) + \int_0^t \mathcal{K}(x, t - s)w(x, s)ds = f(x, t), \\ \forall(x, t) \in ([0, 1] \times (0, T]), \\ w(x, 0) = 0, \quad \forall x \in [0, 1], \end{cases}$$

where $0 < \beta < 1$. $f(x, t)$ and \mathcal{K} are given by $f(x, t) = \frac{(\sin \pi x)t^{0.25}}{\Gamma(1.25)} - \frac{1}{4}xt^4 \sin^2 \pi x$ and $\mathcal{K} = xt$ respectively. The exact solution for the given test problem is $w(x, t) = t \sin \pi x$.

Example 2 Consider the following nonlinear FPIDE:

$$\begin{cases} D_t^\beta w(x, t) + \int_0^t \mathcal{K}(x, t - s)w(x, s)ds = f(x, t), \\ \forall(x, t) \in ([0, 1] \times (0, T]), \\ w(x, 0) = 0, \quad \forall x \in [0, 1], \end{cases}$$

where the kernel function is given by $\mathcal{K} = te^x$ and $\beta \in (0, 1)$. The exact solution for the given test problem is $w(x, t) = e^x t^2$.

Example 3 Consider the following test problem:

$$\begin{cases} D_t^\beta w(x, t) + \int_0^t \mathcal{K}(x, t - s)w(x, s)ds = f(x, t), \\ \forall(x, t) \in ([0, 1] \times (0, T]), \\ w(x, 0) = 0, \quad \forall x \in [0, 1]. \end{cases}$$

Table 1 \mathbb{E}_N^∞ with $\beta = 0.2$ for Example 1 using ADM

x	\mathbb{E}_1^∞	\mathbb{E}_2^∞
0.1	5.6874e-2	4.4619e-3
0.2	1.0818e-1	8.4870e-3
0.3	1.4890e-1	1.1681e-2
0.4	1.7504e-1	1.3732e-2
0.5	1.8405e-1	1.4439e-2
0.6	1.7504e-1	1.3732e-2
0.7	1.4890e-1	1.1681e-2
0.8	1.0818e-1	8.4870e-3
0.9	5.6874e-2	4.4619e-3
1.0	2.2539e-17	1.7683e-18

Table 2 \mathbb{E}_N^∞ with $\beta = 0.75$ for Example 1 using HPM

x	\mathbb{E}_1^∞	\mathbb{E}_2^∞
0.1	7.4089e-3	9.5827e-5
0.2	1.4093e-2	1.8227e-4
0.3	1.9397e-2	2.5088e-4
0.4	2.2802e-2	2.9493e-4
0.5	2.3976e-2	3.1010e-4
0.6	2.2802e-2	2.9493e-4
0.7	1.9397e-2	2.5088e-4
0.8	1.4093e-2	1.8227e-4
0.9	7.4089e-3	9.5827e-5
1.0	2.9362e-18	3.7977e-20

The exact solution for the given test problem is $w(x, t) = xt^\beta$. $0 < \beta < 1$ and the kernel function is given by $\mathcal{K} = e^x$.

The absolute point-wise error is defined by:

$$\mathbb{E}_N^\infty = \left| w(x, t) - \Phi_N \right| = \left| w(x, t) - \sum_{n=0}^{N-1} w_n(x, t) \right|. \tag{25}$$

6.1 Results and discussion

The absolute point-wise errors for Example 1 with $\beta = 0.2$ is presented in Table 1 using ADM and Table 2 with $\beta = 0.75$ using HPM. Similarly, the absolute point wise errors for Example 2 at $t = 0.5$ and $\beta = 0.75$ are viewed in tabular form in Table 3 using ADM. Also, Table 4 represent the errors for Example 2 using HPM for $t = 0.8$ at $\beta = 0.25$. Incase of Example 3 for $t \in (0, 2]$, absolute point-wise errors are represented in Table 5 for $\beta = 0.85$. Further Table 6 show the errors for Example 3 at

Table 3 \mathbb{E}_N^∞ with $\beta = 0.75$ at $t = 0.5$ for Example 2 using ADM

x	\mathbb{E}_1^∞	\mathbb{E}_2^∞	\mathbb{E}_3^∞
0.1	4.7642e-3	2.9543e-5	1.2390e-5
0.2	6.4309e-3	4.4035e-5	1.5808e-5
0.3	8.6809e-3	6.5617e-5	1.9183e-5
0.4	1.1718e-2	9.7734e-5	2.1391e-5
0.5	1.5818e-2	1.4549e-4	2.0052e-5
0.6	2.1351e-2	2.1642e-4	1.0469e-5
0.7	2.8821e-2	3.2161e-4	1.6224e-5
0.8	3.8905e-2	4.7725e-4	1.6268e-5
0.9	5.2516e-2	7.0688e-4	1.4875e-4
1.0	7.0889e-2	1.0443e-3	4.3486e-4

Table 4 \mathbb{E}_N^∞ with $\beta = 0.25$ at $t = 0.8$ for Example 2 using HPM

x	\mathbb{E}_1^∞	\mathbb{E}_2^∞	\mathbb{E}_3^∞
0.1	3.4758e-2	1.7386e-3	9.1131e-4
0.2	4.6919e-2	2.8562e-3	1.5121e-3
0.3	6.3333e-2	4.6883e-3	2.5118e-3
0.4	8.5491e-2	7.6879e-3	4.1778e-3
0.5	1.1540e-1	1.2591e-2	6.9590e-3
0.6	1.5578e-1	2.0589e-2	1.1609e-2
0.7	2.1027e-1	3.3604e-2	1.9397e-2
0.8	2.8384e-1	5.4716e-2	3.2456e-2
0.9	3.8315e-1	8.8827e-2	5.4371e-2
1.0	5.1719e-1	1.4366e-1	9.1139e-2

Table 5 \mathbb{E}_N^∞ with $\beta = 0.85$ and $x = 0.5$ for Example 3 using ADM

t	\mathbb{E}_1^∞	\mathbb{E}_2^∞
0.2	0.0007	0.0000
0.4	0.0047	0.0001
0.6	0.0139	0.0005
0.8	0.0303	0.0018
1.0	0.0554	0.0050
1.2	0.0906	0.0114
1.4	0.1374	0.0230
1.6	0.1970	0.0422
1.8	0.2708	0.0721
2.0	0.3599	0.1165

Table 6 \mathbb{E}_N^∞ with $\beta = 0.85$ and $x = 0.3$ for Example 3 using HPM

t	\mathbb{E}_1^∞	\mathbb{E}_2^∞
0.3	0.0003	0.00001
0.6	0.0018	0.0001
0.9	0.0053	0.0003
1.2	0.0115	0.0012
1.5	0.0209	0.0034
1.8	0.0343	0.0077
2.1	0.0520	0.0155
2.4	0.0745	0.0285
2.7	0.1024	0.0487
3.0	0.1361	0.0787

Table 7 Comparison of results with $\beta = 0.5$ for Example 3

t	\mathbb{E}_2^∞ using ADM	\mathbb{E}_2^∞ using VIM
0.2	1.5266e-5	2.0033e-3
0.4	2.9048e-4	1.1333e-2
0.6	1.6274e-3	3.1229e-2
0.8	5.5270e-3	6.4107e-2
1.0	1.4268e-2	1.1199e-1
1.2	3.0965e-2	1.7666e-1
1.4	5.9621e-2	2.5972e-1
1.6	1.0516e-1	3.6264e-1
1.8	1.7349e-1	4.8681e-1
2.0	2.7148e-1	6.3351e-1

Table 8 Comparison of results with $\beta = 0.75$ for Example 3

t	\mathbb{E}_2^∞ using HPM	\mathbb{E}_2^∞ using VIM
0.2	2.7809e-6	1.7903e-4
0.4	3.1463e-5	7.1611e-4
0.6	1.3005e-4	1.6112e-3
0.8	3.5596e-4	2.8644e-3
1.0	7.7730e-4	4.4757e-3
1.2	1.4714e-3	6.4450e-3
1.4	2.5237e-3	8.7723e-3
1.6	4.0272e-3	1.1458e-2
1.8	6.0819e-3	1.4501e-2
2.0	8.7941e-3	1.7903e-2

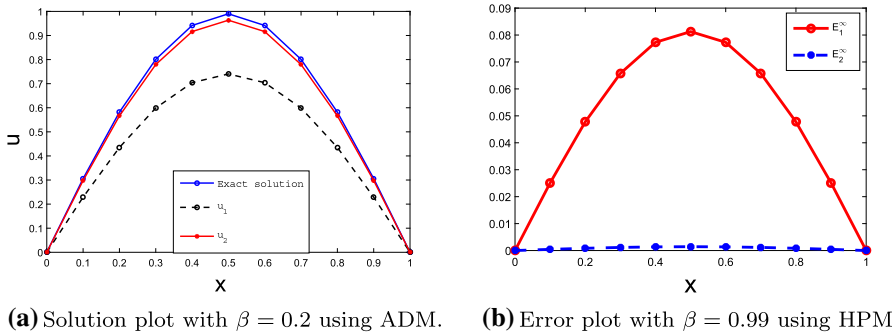


Fig. 1 Solution and error plots with different values of β for Example 1

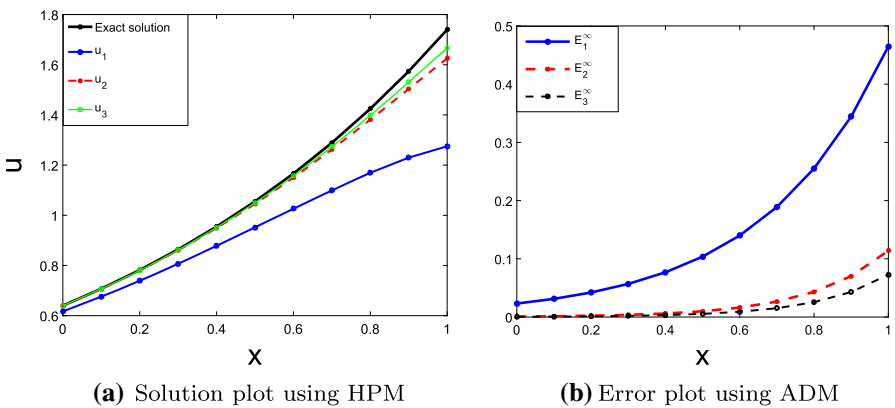


Fig. 2 Solution and error plots at $t = 0.8$ for Example 2

$\beta = 0.85$ for $t \in (0, 3]$. It can be easily seen that for different values of β , error E_N^∞ decreases as the number of terms in the series increases. Comparison of results for both the methods are made. Table 7 compares the error using two terms between ADM and VIM. Similarly comparison between HPM and VIM is shown in Table 8 which proves the robustness of our proposed methods. One may refer [26] for the details on VIM.

The graphical representation of the absolute point-wise errors is shown in Figs. 1 and 2. The graphs depict the convergence of the proposed methods for various values of β . Fig. 1(a) represent the exact solution along with the approximated solutions for two terms taking $\beta = 0.2$ using ADM and Fig. 1(b) represent the error plot using HPM by taking two terms in the series. Similarly in case of nonlinear FPIDE, Fig. 2(a) shows the exact solution and the approximated solutions for Example 2 using three terms in the series with $\beta = 0.75$ using HPM. It can be observed that approximated solutions gradually converges with the exact solution. The error plot for Example 2 is shown in Fig. 2(b) which represents that the error curves moves towards zero as the number of terms in the series increases.

7 Conclusion

This article successfully operated the Adomian decomposition and the homotopy perturbation method to solve the time fractional partial integro-differential equation. The methods are free from discretization. In addition, the reliability of the methods and the reduction in the computational work scale gives them greater applicability. The uniqueness and convergence analysis of the technique is briefly described. Finally convergence to the exact solution is shown with the help of graphs and tables.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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