Journal of Applied Mathematics and Computing (2022) 68:2003–2014
https://doi.org/10.1007/s12190-021-01610-0

ORIGINAL RESEARCH

On r-circulant matrices with generalized bi-periodic Fibonacci numbers

Mehmet Dağlı[1] · Elif Tan[2] · Oktay Ölmez[2]

Received: 14 February 2021 / Revised: 20 July 2021 / Accepted: 21 July 2021 /
Published online: 25 July 2021

Abstract
In this paper, we calculate the Frobenius norm, and give upper and lower bounds for the spectral norm of r-circulant matrices whose entries are defined in terms of generalized bi-periodic Fibonacci numbers. We also provide explicit formulas for the computation of eigenvalues and determinants of these matrices.

Keywords r-circulant matrix · Spectral norm · Fibonacci numbers · Bi-periodic Fibonacci numbers

Mathematics Subject Classification 15A60 · 15B05 · 11B39

1 Introduction

The *generalized bi-periodic Fibonacci sequence* $\{w_n\} = \{w_n(w_0, w_1; a, b)\}$, with arbitrary initial values w_0 and w_1, is defined [7] by the recurrence relation

$$w_n = a^{\xi(n+1)} b^{\xi(n)} w_{n-1} + w_{n-2}, \quad n \geq 2,$$

This research was supported by The Scientific Research Coordination Unit of Amasya University. Project Number: FMB-BAP 20-0474.

✉ Mehmet Dağlı
mehmet.dagli@amasya.edu.tr

Elif Tan
etan@ankara.edu.tr

Oktay Ölmez
oolmez@ankara.edu.tr

1 Department of Mathematics, Amasya University, 05000 Amasya, Turkey

2 Department of Mathematics, Ankara University, 06100 Ankara, Turkey

where $\xi(n) = (1-(-1)^n)/2$, and a and b are nonzero real numbers. Note that $\xi(n)$ returns to 0 when n is even, and to 1 when n is odd. Several well-known integer sequences are its special cases. For example, this sequence is reduced to the bi-periodic Fibonacci sequence $\{q_n\}$ for $w_0 = 0$, $w_1 = 1$, and to the bi-periodic Lucas sequence $\{p_n\}$ for $w_0 = 2$, $w_1 = b$. We refer to [4,7,13,18–21] for basic properties of these sequences and their generalizations.

Several recent works have been dedicated to the study of r-circulant matrices with special entries such as Fibonacci-like numbers. Solak [15,16] obtained some bounds for the spectral norm of circulant matrices whose entries are Fibonacci and Lucas numbers. Shen and Cen [14] generalized the results of Solak to r-circulant matrices. Nalli and Sen [12] investigated the norms of circulant matrices with generalized Fibonacci numbers. Alptekin et al. [1] obtained the spectral norm and eigenvalues of circulant matrices whose entries are Horadam numbers. Yazlik and Taskara [23] found upper and lower bounds on the norms of r-circulant matrices with generalized k-Horadam numbers. They also provided formulas for the computation of the determinant and eigenvalues of such matrices. We refer to [2,3,5,10,11] for related studies.

Recently, Köme and Yazlik [9] obtained upper and lower bounds for the spectral norm of r-circulant matrices whose entries are bi-periodic Fibonacci and Lucas numbers. In the same spirit, we shall calculate the Frobenius norm, find upper and lower bounds on the spectral norm, and calculate the eigenvalues and determinants of r-circulant matrices whose entries are generalized bi-periodic Fibonacci numbers. To this purpose, we review the background material concerning the basic definitions and facts of r-circulant matrices and matrix norms in the rest of this section.

For $n > 0$, the Binet formula of the sequence $\{w_n\}$ can be written as

$$w_n = \frac{a^{\xi(n+1)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(X\alpha^n - Y\beta^n \right),$$

where

$$X = \frac{w_1 - (\beta/a)w_0}{\alpha - \beta} \quad \text{and} \quad Y = \frac{w_1 - (\alpha/a)w_0}{\alpha - \beta}$$

[19,20]. The numbers

$$\alpha = \frac{ab + \sqrt{a^2b^2 + 4ab}}{2} \quad \text{and} \quad \beta = \frac{ab - \sqrt{a^2b^2 + 4ab}}{2} \tag{1}$$

are the roots of the polynomial $x^2 - abx - ab$, and they satisfy

$$\alpha + \beta = ab, \quad \alpha - \beta = \sqrt{a^2b^2 + 4ab}, \quad \alpha\beta = -ab.$$

By the Binet formula, the sequences $\{q_n\}$ and $\{p_n\}$ are given by

$$q_n = \frac{a^{\xi(n+1)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\alpha^n - \beta^n}{\alpha - \beta} \right) \quad \text{and} \quad p_n = \frac{a^{-\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\alpha^n + \beta^n \right). \tag{2}$$

Let $r \in \mathbb{C} \setminus \{0\}$. An $n \times n$ matrix $C_r = \left[c_{ij}\right]$ with entries

$$c_{ij} = \begin{cases} c_{j-i}, & j \geq i, \\ rc_{n+j-i}, & j < i, \end{cases}$$

is called an r-*circulant* matrix. In other words, C_r has the following form:

$$C_r = \begin{bmatrix} c_0 & c_1 & c_2 & \dots & c_{n-2} & c_{n-1} \\ rc_{n-1} & c_0 & c_1 & \dots & c_{n-3} & c_{n-2} \\ rc_{n-2} & rc_{n-1} & c_0 & \dots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ rc_2 & rc_3 & rc_4 & \dots & c_0 & c_1 \\ rc_1 & rc_2 & rc_3 & \dots & rc_{n-1} & c_0 \end{bmatrix}.$$

For simplicity, we denote C_r by $\text{circ}_r\left[c_0, c_1, \dots, c_{n-1}\right]$. Note that C_r is reduced to a circulant matrix for $r = 1$. The eigenvalues of C_r are given as

$$\lambda_j\left(C_r\right) = \sum_{k=0}^{n-1} c_k \left(\rho\omega^{-j}\right)^k \tag{3}$$

with $j = 0, 1, \dots, n-1$, where ρ is any nth root of r, and ω is any nth root of unity. For details, we refer to [6, Lemma 4]. An eigenvalue formula was provided in [23, Theorem 7] for r-circulant matrices with k-Horadam numbers. With suitable initial values and polynomials, this formula contains the eigenvalues of r-circulants with Fibonacci numbers and several other Fibonacci-like numbers as special cases.

Let $A = \left[a_{ij}\right]$ be an $m \times n$ matrix. The Frobenius norm (also known as Hilbert-Schmidt norm or Schur norm) $\|A\|_F$ of A is the square root of the sum of the squares of the absolute values of all entries of A. That is,

$$\|A\|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}\left|a_{ij}\right|^2}.$$

Another important norm of A is the spectral norm, defined as

$$\|A\|_2 = \sqrt{\lambda_{\max}\left(A^*A\right)},$$

where $\lambda_{\max}\left(A^*A\right)$ denotes the largest eigenvalue of A^*A. Here, A^* is the conjugate transpose of A. The following inequality by Stone [17] provides a relationship between Frobenius and spectral norms:

$$\frac{1}{\sqrt{n}}\|A\|_F \leq \|A\|_2 \leq \|A\|_F \leq \sqrt{n}\,\|A\|_2\,. \tag{4}$$

Note that the Frobenius norm is an upper bound on the spectral norm.

For $m \times n$ matrices $A = \left[a_{ij}\right]$ and $B = \left[b_{ij}\right]$, the Hadamard product of A and B is defined as $A \circ B = \left[a_{ij} \cdot b_{ij}\right]$. It is simply the entrywise multiplication of A and B. This product appears [8, Theorem 5.5.3] in

$$\|A \circ B\|_2 \leq r_1(A)c_1(B).$$

Here,

$$r_1(A) = \max_{1 \leq i \leq m} \sqrt{\sum_{j=1}^{n} \left|a_{ij}\right|^2} \quad \text{and} \quad c_1(B) = \max_{1 \leq j \leq n} \sqrt{\sum_{i=1}^{m} \left|b_{ij}\right|^2}.$$

Note that $r_1(A)$ is the maximum row length norm of A, and $c_1(B)$ is the maximum column length norm of B.

2 Main results

Throughout this section, we let a, b and w_1 be positive integers and let w_0 be a nonnegative integer unless otherwise is stated.

We study the matrix

$$W_r = \operatorname{circ}_r\left[\left(\frac{a}{b}\right)^{\frac{\xi(0)}{2}} w_0, \left(\frac{a}{b}\right)^{\frac{\xi(1)}{2}} w_1, \ldots, \left(\frac{a}{b}\right)^{\frac{\xi(n-1)}{2}} w_{n-1}\right].$$

Lemma 1 *For $n > 1$, we have*

$$\sum_{k=1}^{n} \left(\frac{a}{b}\right)^{\xi(k)} w_k^2 = \frac{1}{b}\left(w_n w_{n+1} - w_0 w_1\right).$$

Proof Recall that the Binet formula of the sequence $\{w_k\}$ is given by

$$w_k = \frac{a^{\xi(k+1)}}{(ab)^{\left\lfloor \frac{k}{2} \right\rfloor}}\left(X\alpha^k - Y\beta^k\right).$$

Since $\xi(n) + \xi(n+1) = 1$ and $\lfloor n/2 \rfloor + \lfloor (n+1)/2 \rfloor = n$, we have

$$\begin{aligned} w_n w_{n+1} &= \frac{a^{\xi(n+1)+\xi(n)}}{(ab)^{\left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n+1}{2} \right\rfloor}}\left(X\alpha^n - Y\beta^n\right)\left(X\alpha^{n+1} - Y\beta^{n+1}\right) \\ &= \frac{a}{(ab)^n}\left[X^2\alpha^{2n+1} - XY(\alpha\beta)^n(\alpha+\beta) + Y^2\beta^{2n+1}\right] \\ &= \frac{a}{(ab)^n}\left[X^2\alpha^{2n+1} + Y^2\beta^{2n+1} - XY(-ab)^n(ab)\right]. \end{aligned} \tag{5}$$

On the other hand, we have

$$w_k^2 = \frac{a^{2\xi(k+1)}}{(ab)^{2\left\lfloor \frac{k}{2} \right\rfloor}} \left[X\alpha^k - Y\beta^k \right]^2$$
$$= \frac{a^{2\xi(k+1)}}{(ab)^{2\left\lfloor \frac{k}{2} \right\rfloor}} \left[X^2\alpha^{2k} + Y^2\beta^{2k} - 2XY(\alpha\beta)^k \right].$$

Now, if k is even, we get

$$w_k^2 = \frac{a^2}{(ab)^k} \left[X^2\alpha^{2k} + Y^2\beta^{2k} - 2XY(\alpha\beta)^k \right],$$

and if k is odd,

$$w_k^2 = \frac{ab}{(ab)^k} \left[X^2\alpha^{2k} + Y^2\beta^{2k} - 2XY(\alpha\beta)^k \right].$$

Since $\alpha\beta = -ab$ and

$$a^2 \left(\frac{b}{a}\right)^{\xi(k)} = \begin{cases} a^2, & \text{if } k \text{ is even,} \\ ab, & \text{if } k \text{ is odd,} \end{cases}$$

we can write

$$w_k^2 = a^2 \left(\frac{b}{a}\right)^{\xi(k)} \left[X^2 \left(\frac{\alpha^2}{ab}\right)^k + Y^2 \left(\frac{\beta^2}{ab}\right)^k - 2XY(-1)^k \right],$$

or equivalently,

$$a^{-2} \left(\frac{a}{b}\right)^{\xi(k)} w_k^2 = X^2 \left(\frac{\alpha^2}{ab}\right)^k + Y^2 \left(\frac{\beta^2}{ab}\right)^k - 2XY(-1)^k. \tag{6}$$

By using the geometric sum formula, it can be seen that

$$\sum_{k=1}^{n} \left(\frac{\alpha^2}{ab}\right)^k = \frac{\alpha^{2n+1}}{(ab)^{n+1}} - \frac{\alpha}{ab} \quad \text{and} \quad \sum_{k=1}^{n} \left(\frac{\beta^2}{ab}\right)^k = \frac{\beta^{2n+1}}{(ab)^{n+1}} - \frac{\beta}{ab}.$$

Now we take the summation of both sides of Eq. (6) from 1 to n:

$$\sum_{k=1}^{n} a^{-2}\left(\frac{a}{b}\right)^{\xi(k)} w_k^2 = X^2\sum_{k=1}^{n}\left(\frac{\alpha^2}{ab}\right)^k + Y^2\sum_{k=1}^{n}\left(\frac{\beta^2}{ab}\right)^k - XY\sum_{k=1}^{n}2(-1)^k$$
$$= X^2\left[\frac{\alpha^{2n+1}}{(ab)^{n+1}} - \frac{\alpha}{ab}\right] + Y^2\left[\frac{\beta^{2n+1}}{(ab)^{n+1}} - \frac{\beta}{ab}\right] - XY\left[(-1)^n - 1\right]$$
$$= \frac{1}{ab}\left[X^2\alpha^{2n+1}\left(\frac{1}{ab}\right)^n + Y^2\beta^{2n+1}\left(\frac{1}{ab}\right)^n - X^2\alpha - Y^2\beta - XYab\left[(-1)^n - 1\right]\right].$$

By taking Equation (5) into account in the last line of the equation above, we get the desired result:

$$\sum_{k=1}^{n} a^{-2}\left(\frac{a}{b}\right)^{\xi(k)} w_k^2 = \frac{1}{ab}\frac{1}{a}(w_n w_{n+1}) + XY - \frac{X^2\alpha}{ab} - \frac{Y^2\beta}{ab}$$
$$= \frac{1}{ab}\left[\frac{1}{a}(w_n w_{n+1}) + XY(ab) - \left(X^2\alpha + Y^2\beta\right)\right]$$
$$= \frac{1}{ab}\left[\frac{1}{a}w_n w_{n+1} - \frac{1}{a}w_0 w_1\right]$$
$$= \frac{1}{a^2 b}\left[w_n w_{n+1} - w_0 w_1\right].$$

□

An immediate consequence of Lemma 1 is the following.

Corollary 1 *For $n > 0$,*

$$\sum_{k=0}^{n-1}\left(\frac{a}{b}\right)^{\xi(k)} w_k^2 = \frac{1}{b}\left(w_n w_{n-1} - w_0 w_1 + b w_0^2\right).$$

Remark 1 If we take the initial values $w_0 = 0$ and $w_1 = 1$, we get

$$\sum_{k=1}^{n}\left(\frac{a}{b}\right)^{\xi(k)} q_k^2 = \frac{1}{b} q_n q_{n+1}.$$

This identity was given in [22, Theorem 2.3]. Similarly, with the initial values $w_0 = 2$ and $w_1 = b$, we get [9, Theorem 2.1]:

$$\sum_{k=1}^{n}\left(\frac{a}{b}\right)^{\xi(k)} p_k^2 = \frac{1}{b} p_n p_{n+1} - 2.$$

Now we are ready to provide bounds for $\|W_r\|_2$. But let us first calculate $\|W_r\|_F$.

Lemma 2 *The Frobenius norm*

$$\|W_r\|_F = \sqrt{\sum_{k=0}^{n-1} \left(n + k\left(|r|^2 - 1\right)\right) \left(\frac{a}{b}\right)^{\xi(k)} w_k^2}.$$

Proof By using Lemma 1 and Corollary 1, it is clear that

$$\begin{aligned} \|W_r\|_F &= \sqrt{\sum_{k=0}^{n-1} (n-k) \left(\frac{a}{b}\right)^{\xi(k)} w_k^2 + \sum_{k=1}^{n-1} k\,|r|^2 \left(\frac{a}{b}\right)^{\xi(k)} w_k^2} \\ &= \sqrt{\sum_{k=0}^{n-1} \left(n + k\left(|r|^2 - 1\right)\right) \left(\frac{a}{b}\right)^{\xi(k)} w_k^2}. \end{aligned}$$

□

Theorem 1 *Let*

$$\Delta = w_{n-1} w_n - w_0 w_1 + b w_0^2.$$

(i) If $|r| \geq 1$, *then*

$$\sqrt{\tfrac{\Delta}{b}} \leq \|W_r\|_2 \leq \sqrt{\left((n-1)\,|r|^2 + 1\right) \tfrac{\Delta}{b}}.$$

(ii) If $|r| < 1$, *then*

$$|r| \sqrt{\tfrac{\Delta}{b}} \leq \|W_r\|_2 \leq \sqrt{n \tfrac{\Delta}{b}}.$$

Proof (i) Let $|r| \geq 1$. From Corollary 1 and Lemma 2, we have

$$\|W_r\|_F \geq \sqrt{\sum_{k=0}^{n-1} n \left(\frac{a}{b}\right)^{\xi(k)} w_k^2} = \sqrt{n \tfrac{\Delta}{b}}.$$

Therefore, we can write

$$\frac{1}{\sqrt{n}} \|W_r\|_F \geq \sqrt{\frac{\Delta}{b}}.$$

From (4), we obtain

$$\sqrt{\frac{\Delta}{b}} \leq \|W_r\|_2 .$$

In order to provide an upper bound, let

$$U = \mathrm{circ}_r\,[1, 1, \ldots, 1] \quad \text{and} \quad W = W_1.$$

Then

$$W_r = U \circ W.$$

Since $|r| \geq 1$, we have

$$r_1\,(U) = \max_{1\leq i\leq n} \sqrt{\sum_{j=1}^{n} \left|u_{ij}\right|^2} = \sqrt{\sum_{j=1}^{n} \left|u_{nj}\right|^2} = \sqrt{(n-1)\,|r|^2 + 1}$$

and

$$c_1\,(W) = \max_{1\leq j\leq n} \sqrt{\sum_{i=1}^{n} \left|w_{ij}\right|^2} = \sqrt{\sum_{k=0}^{n-1} \left(\frac{a}{b}\right)^{\xi(k)} w_k^2} = \sqrt{\frac{\Delta}{b}}.$$

Using the above quantities, we obtain

$$\|W_r\|_2 = \|U \circ W\|_2 \leq r_1\,(U)\,c_1\,(W) = \sqrt{\left((n-1)|r|^2 + 1\right)\frac{\Delta}{b}}.$$

(ii) Let $|r| < 1$. Suppose k is an integer with $0 \leq k \leq n-1$. Since $|r|^2 - 1 < 0$, the minimum of $n + k(|r|^2 - 1)$ is achieved when $k = n-1$. So, for $k = n-1$ we have $n + k(|r|^2 - 1) = n|r|^2 - |r|^2 + 1 \geq n|r|^2$. Then

$$n + k(|r|^2 - 1) \geq n|r|^2$$

for each k with $0 \leq k \leq n-1$. Therefore, we can write

$$\begin{aligned}\|W_r\|_F &= \sqrt{\sum_{k=0}^{n-1} \left(n + k\left(|r|^2 - 1\right)\right)\left(\frac{a}{b}\right)^{\xi(k)} w_k^2} \\ &\geq \sqrt{\sum_{k=0}^{n-1} n\,|r|^2 \left(\frac{a}{b}\right)^{\xi(k)} w_k^2}.\end{aligned}$$

Then it follows that

$$\frac{1}{\sqrt{n}}\,\|W_r\|_F \geq |r|\sqrt{\frac{\Delta}{b}}.$$

By (4), we get

$$\|W_r\|_2 \geq |r|\sqrt{\frac{\Delta}{b}}.$$

In order to provide an upper bound, we have

$$r_1(U) = \max_{1\leq i\leq n}\sqrt{\sum_{j=1}^{n}\left|u_{ij}\right|^2} = \sqrt{\sum_{j=1}^{n}\left|u_{1j}\right|^2} = \sqrt{n}$$

and

$$c_1(W) = \max_{1\leq j\leq n}\sqrt{\sum_{i=1}^{n}\left|w_{ij}\right|^2} = \sqrt{\sum_{k=0}^{n-1}\left(\frac{a}{b}\right)^{\xi(k)} w_k^2} = \sqrt{\frac{\Delta}{b}}.$$

In conclusion,

$$\|W_r\|_2 = \|U\circ W\|_2 \leq r_1(U)c_1(W) = \sqrt{n\tfrac{\Delta}{b}}.$$

□

Remark 2 We can use Theorem 1 to provide bounds for special cases.

(i) If $w_0 = 0$ and $w_1 = 1$, then

$$\begin{aligned} \sqrt{\tfrac{q_{n-1}q_n}{b}} \leq \|W_r\|_2 \leq \sqrt{\left((n-1)|r|^2+1\right)\tfrac{q_{n-1}q_n}{b}}, \quad & |r|\geq 1,\\ |r|\sqrt{\tfrac{q_{n-1}q_n}{b}} \leq \|W_r\|_2 \leq \sqrt{n\tfrac{q_{n-1}q_n}{b}}, \quad & |r|<1. \end{aligned}$$

(ii) If $w_0 = 2$ and $w_1 = b$, then

$$\begin{aligned} \sqrt{\tfrac{p_{n-1}p_n}{b}+2} \leq \|W_r\|_2 \leq \sqrt{\left((n-1)|r|^2+1\right)\left(\tfrac{p_{n-1}p_n}{b}+2\right)}, \quad & |r|\geq 1,\\ |r|\sqrt{\tfrac{p_{n-1}p_n}{b}+2} \leq \|W_r\|_2 \leq \sqrt{n\left(\tfrac{p_{n-1}p_n}{b}+2\right)}, \quad & |r|<1. \end{aligned}$$

(iii) Finally, if $a = b = 1$, then

$$\begin{aligned} \sqrt{\Delta} \leq \|W_r\|_2 \leq \sqrt{\left((n-1)|r|^2+1\right)\Delta}, \quad & |r|\geq 1,\\ |r|\sqrt{\Delta} \leq \|W_r\|_2 \leq \sqrt{n\Delta}, \quad & |r|<1. \end{aligned}$$

Note that $\Delta = w_{n-1}w_n - w_0w_1 + w_0^2$ since $b = 1$.

Remark 3 The lower bounds in the first two parts of Remark 2 equal those of [9, Theorems 2.2–3]. However, the upper bounds are weaker except for the case that $|r| < 1$ in the second part. The weakness is caused by the choice of the matrices U and W in the Hadamard product. Special initial values in [9] allow a flexibility to make better choices for U and W in order to improve the bounds. So, Theorem 1 extends [9, Theorems 2.2–3] as to the lower bounds and one upper, but not as to the remaining upper bounds. The bounds in Remark 2(iii) can be found in [5].

Theorem 2 *The eigenvalues*

$$\lambda_j(W_r) = \frac{\left(\frac{a}{b}\right)^{\frac{\xi(n)}{2}} rw_n - w_0 + \rho\omega^{-j}\left[\left(\frac{a}{b}\right)^{\frac{\xi(n+1)}{2}} rw_{n-1} + \left(\frac{a}{b}\right)^{\frac{1}{2}}(bw_0 - w_1)\right]}{\rho^2\omega^{-2j} + (ab)^{\frac{1}{2}}\rho\omega^{-j} - 1},$$

for $j = 0, 1, \ldots, n-1$, *provided that*

$$r \neq \left(\frac{-\alpha}{\beta}\right)^{\pm\frac{n}{2}}.$$

Here, $\rho = r^{\frac{1}{n}}$ *and* ω *is any nth root of unity. For* α *and* β, *see (1).*

Proof From (3), we have

$$\begin{aligned}
\lambda_j(W_r) &= \sum_{k=0}^{n-1}\left(\frac{a}{b}\right)^{\frac{\xi(k)}{2}} w_k\rho^k\omega^{-kj} \\
&= Xa\sum_{k=0}^{n-1}\left(\frac{\alpha\rho\omega^{-j}}{(ab)^{\frac{1}{2}}}\right)^k - Ya\sum_{k=0}^{n-1}\left(\frac{\beta\rho\omega^{-j}}{(ab)^{\frac{1}{2}}}\right)^k \qquad (7)\\
&= \frac{a}{(ab)^{\frac{n-1}{2}}}\left(X\frac{\alpha^n r - (ab)^{\frac{n}{2}}}{\alpha\rho\omega^{-j} - (ab)^{\frac{1}{2}}} - Y\frac{\beta^n r - (ab)^{\frac{n}{2}}}{\beta\rho\omega^{-j} - (ab)^{\frac{1}{2}}}\right) \\
&= \frac{a}{(ab)^{\frac{n-1}{2}}\left(\alpha\rho\omega^{-j} - (ab)^{\frac{1}{2}}\right)\left(\beta\rho\omega^{-j} - (ab)^{\frac{1}{2}}\right)} \\
&\quad\times\Big[r\rho\omega^{-j}\alpha\beta\left(X\alpha^{n-1} - Y\beta^{n-1}\right) - r(ab)^{\frac{1}{2}}\left(X\alpha^n - Y\beta^n\right) \\
&\quad -\rho\omega^{-j}(ab)^{\frac{n}{2}}(X\beta - Y\alpha) + (ab)^{\frac{n+1}{2}}(X - Y)\Big].
\end{aligned}$$

Therefore, after lengthy calculations,

$$\lambda_j(W_r) = \frac{rw_n - w_0 + \left(\frac{a}{b}\right)^{\frac{1}{2}}\rho\omega^{-j}\left[rw_{n-1} + (bw_0 - w_1)\right]}{\rho^2\omega^{-2j} + (ab)^{\frac{1}{2}}\rho\omega^{-j} - 1},$$

when n is even, and

$$\lambda_j(W_r) = \frac{\left(\frac{a}{b}\right)^{\frac{1}{2}} rw_n - w_0 + \rho\omega^{-j}\left[rw_{n-1} + \left(\frac{a}{b}\right)^{\frac{1}{2}} (bw_0 - w_1)\right]}{\rho^2\omega^{-2j} + (ab)^{\frac{1}{2}}\rho\omega^{-j} - 1},$$

when n is odd. This completes the proof. □

Remark 4 The geometric sum formula fails in (7) when $\alpha\rho\omega^{-j}/(ab)^{\frac{1}{2}} = 1$ or $\beta\rho\omega^{-j}/(ab)^{\frac{1}{2}} = 1$. Therefore, the assumption $r \neq (-\alpha/\beta)^{\pm\frac{n}{2}}$ is needed for it to be valid. This also guarantees that the denominator $\rho^2\omega^{-2j} + (ab)^{\frac{1}{2}}\rho\omega^{-j} - 1 \neq 0$.

Theorem 3 *The determinant*

$$\det(W_r) = \frac{\left[w_0 - \left(\frac{a}{b}\right)^{\frac{\xi(n)}{2}} rw_n\right]^n - r\left[\left(\frac{a}{b}\right)^{\frac{\xi(n+1)}{2}} rw_{n-1} + \left(\frac{a}{b}\right)^{\frac{1}{2}} (bw_0 - w_1)\right]^n}{1 - \left(\frac{a}{b}\right)^{\frac{\xi(n)}{2}} p_n r + (-1)^n r^2},$$

provided that

$$r \neq \left(\frac{-\alpha}{\beta}\right)^{\pm\frac{n}{2}}.$$

Here, p_n is as in (2).

Proof The formula follows from the fact that $\det(W_r) = \prod_{j=0}^{n-1} \lambda_j(W_r)$. □

3 Conclusion

In this paper, we obtained bounds on the spectral norm of r-circulant matrices whose entries are generalized bi-periodic Fibonacci numbers. We also calculated the eigenvalues and determinants of these matrices explicitly. By means of this work, we have a unified approach for dealing with many r-circulant matrices with special entries such as Fibonacci, Lucas, Pell, Pell-Lucas, generalized Fibonacci, bi-periodic Fibonacci, and bi-periodic Lucas numbers. We note that our bounds can be improved for specified initial values by choosing suitable matrices in the Hadamard product.

Acknowledgements We are very grateful to anonymous referees for various comments that have led to a number of improvements in the paper.

References

1. Alptekin, E.G., Mansour, T., Tuglu, N.: Norms of circulant and semicirculant matrices with Horadam's numbers. ARS Combinatoria **85**, 353–359 (2007)

2. Bahsi, M.: On the norms of circulant matrices with the generalized Fibonacci and Lucas numbers. TWMS J. Pure Appl. Math. **6**(1), 84–92 (2015)
3. Belbachir, H., Belkhir, A.: On some generalizations of Horadam's numbers. Filomat **32**(14), 5037–5052 (2018)
4. Bilgici, G.: Two generalizations of Lucas sequence. Appl. Math. Comput. **245**, 526–538 (2014)
5. Chandoul, A.: On the norms of r-circulant matrices with generalized Fibonacci numbers. J. Algebra Comb. Discrete Appl. **4**, 13–21 (2017)
6. Cline, R.E., Plemmons, R.J., Worm, G.: Generalized inverses of certain Toeplitz matrices. Linear Algebr. Appl. **8**, 25–33 (1974)
7. Edson, M., Yayenie, O.: A new generalization of Fibonacci sequences and extended Binet's Formula. Integers **9**(A48), 639–654 (2009)
8. Horn, R.A., Johnson, C.R.: Topics in Matrix Analysis. Cambridge University Press, Cambridge (1991)
9. Köme, C., Yazlik, Y.: On the spectral norms of r-circulant matrices with the biperiodic Fibonacci and Lucas numbers. J. Inequal. Appl. **2017**, 192 (2017)
10. Lind, D.A.: A Fibonacci circulant. Fibonacci Quart. **8**(5), 449–455 (1970)
11. Merikoski, J.K., Haukkanen, P., Mattila, M., Tossavainen, T.: On the spectral and Frobenius norm of a generalized Fibonacci r-circulant matrix. Spec. Matrices **6**, 23–36 (2018)
12. Nalli, A., Şen, M.: On the norms of circulant matrices with generalized Fibonacci numbers. Selçuk J. Appl. Math. **11**(1), 107–116 (2010)
13. Panario, D., Sahin, M., Wang, Q.: A family of Fibonacci-like conditional sequences. Integers **13**, A78 (2013)
14. Shen, S., Cen, J.: On the bounds for the norms of r-circulant matrices with the Fibonacci and Lucas numbers. Appl. Math. Comput. **216**, 2891–2897 (2010)
15. Solak, S.: On the norms of circulant matrices with the Fibonacci and Lucas numbers. Appl. Math. Comput. **160**(1), 125–132 (2005)
16. Solak, S.: Erratum to "On the norms of circulant matrices with the Fibonacci and Lucas numbers" [Appl. Math. Comput. 160(1) 125–132 (2005)]. Appl. Math. Comput. **190**(2), 1855–1856 (2007)
17. Stone, B.J.: Best possible ratios of certain matrix norms. Numer. Math. **4**, 114–116 (1962)
18. Tan, E.: General sum formula for the bi-periodic Fibonacci and Lucas numbers. Integers **17**, A42 (2017)
19. Tan, E.: Some properties of bi-periodic Horadam sequences. Notes Number Theory Discrete Math. **23**(4), 56–65 (2017)
20. Tan, E., Leung, H.-H.: Some basic properties of the generalized bi-periodic Fibonacci and Lucas sequences. Adv. Differ. Equ. **2020**, 26 (2020)
21. Yayenie, O.: A note on generalized Fibonacci sequence. Appl. Math. Comput. **217**, 5603–5611 (2011)
22. Yayenie, O.: New identities for generalized Fibonacci sequences and new generalization of Lucas sequences. Southeast Asian Bull. Math. **36**, 739–752 (2012)
23. Yazlik, Y., Taskara, N.: On the norms of an r-circulant matrix with the generalized k-Horadam numbers. J. Inequal. Appl. **2013**, 394 (2013)