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ORIGINAL RESEARCH

The local discontinuous Galerkin finite element method for a multiterm time-fractional initial-boundary value problem

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Abstract
In this paper, we study a multiterm time-fractional initial-boundary value problem, whose differential equation contains a sum of Caputo time fractional derivatives with orders in $(0, 1)$. In general, the solution of this kind of problem exhibits a weak regularity at the initial time. Based on the L1 formula on non-uniform meshes for time discretization and the local discontinuous Galerkin (LDG) method for space discretization, fully discrete numerical schemes for one and two space dimensions are constructed. The stability and convergence of the schemes are analyzed. It is shown that the error bounds are α_1-robust, that is, they remain valid as $\alpha_1 \to 1^-$, where α_1 is the biggest fractional order. Furthermore, a numerical experiment is given to verify the effectiveness of the current method.

Keywords Caputo fractional derivative · Local discontinuous Galerkin method · α_1-robust · Stability · Convergence

Mathematics Subject Classification 65M06 · 65M12 · 65M60

1 Introduction

Fractional calculus (fractional differentiation and integration) has received much attention in recent years due to its successful simulation of many phenomena in science and engineering [8,11–13,20]. Although the analytical solutions of some fractional differential equations can be obtained by means of some special transforms, the complexity involving special functions and infinite series are inconvenient for numerical evaluation. Hence, efficient and accurate numerical approaches are demanded.

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In this paper, we will investigate numerical methods and related numerical analysis of the following multiterm time-fractional initial-boundary value problem:

$$\sum_{i=1}^{l}\left[q_i\, {}_C\mathrm{D}_{0,t}^{\alpha_i}u(\mathbf{x},t)\right]-\Delta u(\mathbf{x},t)+c(\mathbf{x})u(\mathbf{x},t)=f(\mathbf{x},t),\ (\mathbf{x},t)\in\Omega\times(0,T],\tag{1.1}$$

with initial and boundary conditions:

$$u|_{t=0}=u_0(\mathbf{x}),\ \mathbf{x}\in\overline{\Omega},\tag{1.2}$$

$$u|_{\mathbf{x}\in\partial\Omega}=0,\ t\in(0,T].\tag{1.3}$$

Here, $\Omega\subseteq\mathbb{R}^d(d=1,2)$ is a bounded rectangular domain, l is a positive integer, $q_i>0,\ i=1,2,\dots,l,\ 0<\alpha_l<\dots<\alpha_2<\alpha_1<1$ are given constants, $c(\mathbf{x})\in C(\overline{\Omega})$ with $c(\mathbf{x})\geq 0$, source term $f(\mathbf{x},t)\in L^\infty(0,T;L^2(\Omega))$ and initial value $u_0(\mathbf{x})\in L^2(\Omega)$ are given functions, and ${}_C\mathrm{D}_{0,t}^{\alpha_i}$ is the α_ith-order left-sided Caputo derivative operator defined by [13]

$${}_C\mathrm{D}_{0,t}^{\alpha_i}u(\mathbf{x},t)=\frac{1}{\Gamma(1-\alpha_i)}\int_0^t(t-s)^{-\alpha_i}\frac{\partial u}{\partial s}\mathrm{d}s,\ 0<\alpha_i<1,\tag{1.4}$$

in which $\Gamma(\cdot)$ denotes the usual Gamma function.

The multiterm time-fractional initial-boundary value problem (1.1)–(1.3) has proved to be flexible to describe complex multirate physical processes [25]. So far, several different methods have been developed to solve this problem. Luchko [17] developed the maximum principle for a multiterm time-fractional diffusion equation and constructed a generalized solution by means of the multinomial Mittag-Leffler functions. Wei [22] proposed a fully discrete local discontinuous Galerkin (LDG) method for a class of multiterm time fractional diffusion equations. Zaky [26] used a Legendre spectral quadrature tau method solving the multiterm time-fractional diffusion equations. Very recently, Huang et al. [10] showed that, under proper regularity and compatibility assumptions, the system (1.1)–(1.3) has a unique solution u such that

$$\left|\frac{\partial^k u(\mathbf{x},t)}{\partial\mathbf{x}^k}\right|\leq C\ \text{ for }\ k=0,1,2,3,4,\tag{1.5}$$

$$\left|\frac{\partial^m u(\mathbf{x},t)}{\partial t^m}\right|\leq C(1+t^{\alpha_1-m})\ \text{ for }\ m=0,1,2,\tag{1.6}$$

where $C>0$ is a bounded constant independent of the variable t but dependent of T. Meng and Stynes [18] presented an L1 finite element method for a multiterm time-fractional initial-boundary value problem.

The result in (1.6) implies that the solution u of the multiterm time-fractional initial-boundary value problem (1.1)–(1.3) exhibits some weak regularity at the starting time and $\frac{\partial u(\cdot,t)}{\partial t}$ blows up as $t\to 0^+$. When seeking numerical solutions, the initial layer very likely leads to a loss of accuracy if uniform temporal meshes are used. To tackle such a problem, one can consider the numerical approaches on non-uniform meshes. This is also the topic of the present paper. In this work, we develop a non-uniform L1/LDG method for the multiterm time-fractional initial-boundary value problem (1.1)–(1.3) with weak regular solution. The Caputo time-fractional derivatives are discretized by the L1 finite difference method on non-uniform meshes, and the spatial discretization is performed by using the LDG finite element method. Then the stability and convergence analysis of the proposed numerical scheme are given. However, the obtained error bounds blow up if we consider the limit $\alpha_1 \to 1^-$. Such error bounds are called α_1-nonrobust [2]. On the contrary, if the error bound does not blow up as $\alpha_1 \to 1^-$, we call it α_1-robust. Therefore, we further investigate the α_1-robust stability and convergence of the proposed non-uniform L1/LDG scheme.

The LDG method is a special class of discontinuous Galerkin (DG) finite element methods, introduced first by Cockburn and Shu [4] and has been successful for solving fractional differential equations, e.g., [6,7,14–16,19,23]. The main technique of LDG method is to rewrite higher-order derivative equation into an equivalent system containing only the first derivative, and then discretize it by the standard DG method. More details about the LDG method for high-order time dependent partial differential equations can be found in the review paper [24].

The rest of the paper is organized as follows. In Sects. 2 and 3, we establish the fully discrete non-uniform L1/LDG schemes for the initial-boundary value problem in one and two space dimensions, respectively. The α_1-robust and α_1-nonrobust stability and convergence analysis are studied too. In Sect. 4, we provide a numerical example to verify the theoretical results. Concluding remarks are given in the last section.

$Notations$: Through out this paper we let C be a generic positive constant, which is independent of the mesh sizes and can take different values in different circumstances. We use $\|\cdot\|$ as the L^2-norm on domain Ω and define the $L^2(\Omega)$ inner product $(u, v) = \iint_\Omega uv\, \mathrm{d}\mathbf{x}$.

2 One-dimensional case

In the section, we will develop a non-uniform L1/LDG scheme for the one-dimensional multiterm time-fractional initial-boundary value problem (1.1)–(1.3). The scheme employs the L1 formula with non-uniform meshes for the time-fractional derivative and a LDG method in space. The usual notations are introduced below.

Let $\mathcal{T}_h = \left\{I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})\right\}_{j=1}^N$ be the partition of Ω, where $a = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N+\frac{1}{2}} = b$. The cell center and cell length are denoted by $x_j = (x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}})/2$ and $h_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$, respectively. Let $h = \max_{1\le j\le N} h_j$ be the length of the largest cell. We use $u^-_{j+\frac{1}{2}}$ and $u^+_{j+\frac{1}{2}}$ to represent the values of u at the discontinuity point $x_{j+\frac{1}{2}}$, from the left cell, I_j, and from the right cell, I_{j+1}, respectively. The jump value of u at each element boundary is denoted by $[\![u]\!]_{j+\frac{1}{2}} = u^+_{j+\frac{1}{2}} - u^-_{j+\frac{1}{2}}$. Associated with this mesh, we define the discontinuous finite element space

$$V_h = \left\{ v \in L^2(\Omega) : v|_{I_j} \in \mathcal{P}^k(I_j),\ v|_{\partial\Omega} = 0,\ j = 1, \ldots, N \right\},$$

where $\mathcal{P}^k(I_j)$ denotes the set of polynomials of degree up to $k \geq 0$ defined on the cell I_j. For any nonnegative integer m, $H^m(\Omega)$ denotes the usual Sobolev space. Then we define the broken Sobolev space on $\mathcal{T}_h$ by

$$H^m(\mathcal{T}_h) = \{v \in L^2(\Omega) :\ v|_{I_j} \in H^m(I_j),\ \forall j = 1, \ldots, N\},$$

which contains the discontinuous finite element space V_h.

To obtain the optimal error estimate, we recall two kinds of Gauss-Radau projections $\mathscr{P}_h^{\pm} : H^1(\mathcal{T}_h) \to V_h$, which were introduced by Castillo et al. [1], i.e., for each j,

$$\int_{I_j} \left(\mathscr{P}_h^+ q(x) - q(x) \right) v_h \mathrm{d}x = 0,\ \forall v_h \in \mathcal{P}^{k-1}(I_j),\ (\mathscr{P}_h^+ q)^+_{j-\frac{1}{2}} = q(x^+_{j-\frac{1}{2}}),\quad (2.1)$$

and

$$\int_{I_j} \left(\mathscr{P}_h^- q(x) - q(x) \right) v_h \mathrm{d}x = 0,\ \forall v_h \in \mathcal{P}^{k-1}(I_j),\ \ (\mathscr{P}_h^- q)^-_{j+\frac{1}{2}} = q\left(x^-_{j+\frac{1}{2}} \right). \quad (2.2)$$

Denote by $\zeta = q(x) - \mathbb{P}_h q(x)$ ($\mathbb{P}_h = \mathscr{P}_h^+$ or $\mathscr{P}_h^-$) the projection error. Then a standard scaling argument as that in [3] yields

$$\|\zeta\| + h\|\zeta_x\| + h^{\frac{1}{2}}\|\zeta\|_{\Gamma_h} \leq C\|\zeta\|_{H^{k+1}(\mathcal{T}_h)} h^{k+1}, \quad (2.3)$$

where $\|\zeta\|^2_{\Gamma_h} = \sum_{j=1}^{N} \left((\zeta^+|_{j-\frac{1}{2}})^2 + (\zeta^-|_{j+\frac{1}{2}})^2 \right)$.

2.1 The fully discrete non-uniform L1/LDG scheme

For a given $T > 0$, let $t_n = T(n/M)^r$, $n = 0, 1, \ldots, M$ be the mesh points, $r \geq 1$. Denote $\tau_n = t_n - t_{n-1}$, $n = 1, \ldots, M$ be the time mesh sizes. If $r = 1$, then the mesh is just uniform. Throughout this paper, we denote $u^n = u(x, t_n)$ if no confusion appears.

The L1 approximation on the non-uniform meshes to the Caputo derivative is given by [10]

$$\begin{aligned} {}_C\mathrm{D}^{\alpha_i}_{0,t} u|_{t=t_n} &= \frac{1}{\Gamma(2-\alpha_i)} \left[d^i_{n,1} u^n + \sum_{k=1}^{n-1} (d^i_{n,k+1} - d^i_{n,k}) u^{n-k} - d^i_{n,n} u^0 \right] \\ &\quad + R^n_i \\ &:= \Upsilon^{\alpha_i}_t u^n + R^n_i \end{aligned} \quad (2.4)$$

for $i = 1, \dots, l$, where $d^i_{n,k} = \frac{(t_n - t_{n-k})^{1-\alpha_i} - (t_n - t_{n-k+1})^{1-\alpha_i}}{\tau_{n-k+1}}$ and R^n_i is the truncation error. The coefficients $d^i_{n,k}$ have the following properties

$$d^i_{n,k+1} \le d^i_{n,k}, \tag{2.5}$$

$$(1-\alpha_i)(t_n - t_{n-k})^{-\alpha_i} \le d^i_{n,k} \le (1-\alpha_i)(t_n - t_{n-k+1})^{-\alpha_i}. \tag{2.6}$$

Denote

$$\Upsilon^{\alpha_i}_t u^n = \frac{1}{\Gamma(2-\alpha_i)} \left[d^i_{n,1} u^n + \sum_{k=1}^{n-1} (d^i_{n,k+1} - d^i_{n,k}) u^{n-k} - d^i_{n,n} u^0 \right] \tag{2.7}$$

for $i = 1, \dots, l,\ n = 1, \dots, M$.

Let $p = u_x$, then we can get the weak form of system (1.1)–(1.3) at t_n as follows,

$$\begin{cases} \left(\sum_{i=1}^{l} q_i\, {}_C\mathrm{D}^{\alpha_i}_{0,t} u^n, v \right) + (c(x)u^n, v) + (p^n, v_x) \\ \quad - \sum_{j=1}^{N} \left(p^n v^- |_{j+\frac{1}{2}} - p^n v^+ |_{j-\frac{1}{2}} \right) = (f^n, v), \\ (p^n, w) + (u^n, w_x) - \sum_{j=1}^{N} \left(u^n w^- |_{j+\frac{1}{2}} - u^n w^+ |_{j-\frac{1}{2}} \right) = 0, \end{cases} \tag{2.8}$$

where $v, w \in H^1(\Omega)$ are test functions.

The fully discrete non-uniform L1/LDG scheme is defined as follows: find $U^n_h, P^n_h \in V_h$ such that for all test functions $v_h, w_h \in V_h$,

$$\begin{cases} \left(\sum_{i=1}^{l} q_i \Upsilon^{\alpha_i}_t U^n_h, v_h \right) + (c(x)U^n_h, v_h) + (P^n_h, (v_h)_x) \\ \quad - \sum_{j=1}^{N} \left(\widehat{P}^n_h v^-_h |_{j+\frac{1}{2}} - \widehat{P}^n_h v^+_h |_{j-\frac{1}{2}} \right) = (f^n, v_h), \\ (P^n_h, w_h) + (U^n_h, (w_h)_x) - \sum_{j=1}^{N} \left(\widehat{U}^n_h w^-_h |_{j+\frac{1}{2}} - \widehat{U}^n_h w^+_h |_{j-\frac{1}{2}} \right) = 0. \end{cases} \tag{2.9}$$

The “hat" terms in (2.9) are the boundary terms that emerge from the integration by parts. These are the so-called “numerical fluxes" that are yet to be determined. The freedom in choosing numerical fluxes can be utilized for designing a scheme that enjoys certain stability properties. It turns out that we can take the following choices simply

$$\widehat{U}^n_h = (U^n_h)^-, \ \widehat{P}^n_h = (P^n_h)^+. \tag{2.10}$$

2.2 α_1-Nonrobust error analysis of the non-uniform L1/LDG method

This subsection presents the α_1-nonrobust stability and convergence of the scheme (2.9). We give the following α_1-nonrobust stability result.

Lemma 2.1 *The solution U_h^n of the fully discrete scheme* (2.9) *satisfies*

$$\|U_h^n\| \leq \|U_h^0\| + \frac{1}{\sum_{i=1}^{l} \eta_{n,1}^i} \sum_{j=1}^{n} \theta_{n,j} \|f^j\|, \ n = 1, \ldots, M, \tag{2.11}$$

where

$$\eta_{n,j}^i = \frac{q_i d_{n,j}^i}{\Gamma(2-\alpha_i)}, \ i = 1, 2, \ldots, l, \ \ j = 1, 2, \ldots, M,$$

$$\theta_{n,n} = 1, \ \theta_{n,j} = \sum_{i=1}^{l} \sum_{k=1}^{n-j} \frac{1}{\sum_{i=1}^{l} \eta_{n-k,1}^i} (\eta_{n,k}^i - \eta_{n,k+1}^i)\theta_{n-k,j}, \ \ j = 1, 2, \ldots, n-1.$$

Proof Taking the test functions $(v_h, w_h) = (U_h^n, P_h^n)$, and adding the two equations in (2.9), we obtain

$$\begin{aligned}
&\left(\sum_{i=1}^{l} \frac{q_i d_{n,1}^i}{\Gamma(2-\alpha_i)} U_h^n, U_h^n\right) + \left(c(x)U_h^n, U_h^n\right) + \|P_h^n\|^2 \\
&= \left(\sum_{i=1}^{l} \frac{q_i d_{n,n}^i}{\Gamma(2-\alpha_i)} U_h^0, U_h^n\right) + \left(\sum_{i=1}^{l} \frac{q_i}{\Gamma(2-\alpha_i)} \sum_{k=1}^{n-1} (d_{n,k}^i - d_{n,k+1}^i) U_h^{n-k}, U_h^n\right) \\
&\quad + (f^n, U_h^n).
\end{aligned} \tag{2.12}$$

By using (2.5) and the Cauchy-Schwarz inequality, we have

$$\sum_{i=1}^{l} \eta_{n,1}^i \|U_h^n\| \leq \sum_{i=1}^{l} \eta_{n,n}^i \|U_h^0\| + \sum_{i=1}^{l} \sum_{k=1}^{n-1} (\eta_{n,k}^i - \eta_{n,k+1}^i) \|U_h^{n-k}\| + \|f^n\|. \tag{2.13}$$

Now, we prove this lemma by mathematical induction. When $n = 1$, (2.13) becomes

$$\|U_h^1\| \leq \|U_h^0\| + \frac{1}{\sum_{i=1}^{l} \eta_{1,1}^i} \|f^1\|, \tag{2.14}$$

which is identical to (2.11).

Supposing the following estimates hold

$$\|U_h^m\| \leq \|U_h^0\| + \frac{1}{\sum_{i=1}^{l} \eta_{m,1}^i} \sum_{j=1}^{m} \theta_{m,j} \|f^j\|, \ m = 2, \ldots, s, \tag{2.15}$$

we need prove

$$\|U_h^{s+1}\| \le \|U_h^0\| + \frac{1}{\sum_{i=1}^l \eta_{s+1,1}^i} \sum_{j=1}^{s+1} \theta_{s+1,j} \|f^j\|. \tag{2.16}$$

Letting $n = s+1$ in (2.13) and using (2.15), we have

$$\begin{aligned}
\|U_h^{s+1}\| \le& \frac{\sum_{i=1}^l \eta_{s+1,s+1}^i}{\sum_{i=1}^l \eta_{s+1,1}^i} \|U_h^0\| + \frac{1}{\sum_{i=1}^l \eta_{s+1,1}^i} \left[\sum_{i=1}^l \sum_{k=1}^s (\eta_{s+1,k}^i - \eta_{s+1,k+1}^i) \right.\\
&\left. \times \left(\|U_h^0\| + \frac{1}{\sum_{i=1}^l \eta_{s+1-k,1}^i} \sum_{j=1}^{s+1-k} \theta_{s+1-k,j} \|f^j\| \right) \right] + \frac{\|f^{s+1}\|}{\sum_{i=1}^l \eta_{s+1,1}^i}\\
=& \frac{1}{\sum_{i=1}^l \eta_{s+1,1}^i} \left\{ \|f^{s+1}\| + \left[\sum_{i=1}^l \sum_{k=1}^s \frac{\eta_{s+1,k}^i - \eta_{s+1,k+1}^i}{\sum_{i=1}^l \eta_{s+1-k,1}^i} \sum_{j=1}^{s+1-k} \theta_{s+1-k,j} \|f^j\| \right] \right\}\\
&+ \left\{ \frac{1}{\sum_{i=1}^l \eta_{s+1,1}^i} \left[\sum_{i=1}^l \sum_{k=1}^s (\eta_{s+1,k}^i - \eta_{s+1,k+1}^i) \right] + \frac{\sum_{i=1}^l \eta_{s+1,s+1}^i}{\sum_{i=1}^l \eta_{s+1,1}^i} \right\} \|U_h^0\|\\
=& \|f^{s+1}\| + \sum_{j=1}^s \left[\sum_{i=1}^l \sum_{k=1}^{s+1-j} \frac{\eta_{s+1,k}^i - \eta_{s+1,k+1}^i}{\sum_{i=1}^l \eta_{s+1-k,1}^i} \theta_{s+1-k,j} \right] \|f^j\|\\
&+ \frac{1}{\sum_{j=1}^l \eta_{s+1,1}^j} \left\{ \sum_{i=1}^l \sum_{k=1}^s (\eta_{s+1,k}^i - \eta_{s+1,k+1}^i) + \sum_{i=1}^l \eta_{s+1,s+1}^i \right\} \|U_h^0\|\\
=& \sum_{j=1}^{s+1} \theta_{s+1,j} \|f^j\| + \|U_h^0\|.
\end{aligned}$$

This completes the proof of this lemma. □

Lemma 2.2 [10] *Let $\beta \le r\alpha_1$, then for $n = 1, 2, \ldots, M$, one has*

$$\frac{1}{\sum_{i=1}^l \eta_{n,1}^i} \sum_{j=1}^n j^{-\beta} \theta_{n,j} \le \Gamma(1-\alpha_1) T^{\alpha_1} M^{-\beta}.$$

Theorem 2.1 (L^2-norm stability) *The solution U_h^n of the fully discrete scheme* (2.9) *satisfies*

$$\|U_h^n\| \le \|U_h^0\| + \Gamma(1-\alpha_1) T^{\alpha_1} \max_{1 \le j \le n} \left\| f^j \right\|, \; n = 1, \ldots, M. \tag{2.17}$$

Proof By Lemma 2.2, one has

$$\frac{1}{\sum_{i=1}^{l}\eta_{n,1}^{i}}\sum_{j=1}^{n}\theta_{n,j}\le\Gamma(1-\alpha_1)T^{\alpha_1}.$$

Combining the above estimate with Lemma 2.1 yields the assertion. □

Next, we present the α_1-nonrobust convergence analysis. Firstly, we introduce a lemma that will be used later on.

Lemma 2.3 [10] *Suppose that the solution $u(x,t)$ of problem (1.1)–(1.3) satisfies (1.6). Then there exists a constant C such that for all t_n one has*

$$|R_i^n|\le Cn^{-\min\{2-\alpha_1,r\alpha_1\}}$$

for $i=1,2,\ldots,l$, $n=1,2,\ldots,M$.

Theorem 2.2 (L^2-norm error estimate) *Let u be the exact solution of (1.1)–(1.3) and U_h^n be the numerical solution of the fully discrete non-uniform L1/LDG scheme (2.9). Suppose that u satisfies condition (1.6) and $u(\cdot,t)\in H^{k+1}(\mathcal{T}_h)$. Then, it holds that*

$$\|u^n-U_h^n\|\le C\sqrt{\Gamma(1-\alpha_1)}\left(M^{-\min\{2-\alpha_1,r\alpha_1\}}+h^{k+1}\right), \tag{2.18}$$

where C is a positive constant independent of M and h.

Proof As usual in the finite element analysis, we denote the error by $e_u^n=u^n-U_h^n$ and $e_p^n=p^n-P_h^n$, respectively, and decompose them into two parts, namely,

$$\begin{aligned}e_u^n&=u^n-U_h^n=\mathscr{P}_h^-e_u^n+\left(u^n-\mathscr{P}_h^-u^n\right):=\xi_u^n+\eta_u^n,\\ e_p^n&=p^n-P_h^n=\mathscr{P}_h^+e_p^n+\left(p^n-\mathscr{P}_h^+p^n\right):=\xi_p^n+\eta_p^n.\end{aligned} \tag{2.19}$$

Subtracting (2.9) from (2.8), and with the fluxes (2.10), we can obtain the error equation

$$\begin{cases}\left(\displaystyle\sum_{i=1}^{l}q_i\left({}_C\mathrm{D}_{0,t}^{\alpha_i}u^n-\Upsilon_t^{\alpha_i}U_h^n\right),v_h\right)+(c(x)e_u^n,v_h)+\left(e_p^n,(v_h)_x\right)\\ \quad-\displaystyle\sum_{j=1}^{N}\left((e_p^n)^+v_h^-|_{j+\frac12}-(e_p^n)^+v_h^+|_{j-\frac12}\right)=0,\\ (e_p^n,w_h)+\left(e_u^n,(w_h)_x\right)-\displaystyle\sum_{j=1}^{N}\left((e_u^n)^-w_h^-|_{j+\frac12}-(e_u^n)^-w_h^+|_{j-\frac12}\right)=0.\end{cases} \tag{2.20}$$

Substituting (2.19) into (2.20), we have

$$
\begin{cases}
\left(\sum_{i=1}^{l} q_i \Upsilon_t^{\alpha_i} \xi_u^n, v_h\right) + \left(c(x)\xi_u^n, v_h\right) + \left(\xi_p^n, (v_h)_x\right) \\
\quad - \sum_{j=1}^{N} \left((\xi_p^n)^+ v_h^- |_{j+\frac{1}{2}} - ((\xi_p^n)^+ v_h^+ |_{j-\frac{1}{2}}\right) \\
\quad = -\left(\sum_{i=1}^{l} q_i R_i^n, v_h\right) - \left(\sum_{i=1}^{l} q_i \Upsilon_t^{\alpha_i} \eta_u^n, v_h\right) - \left(c(x)\eta_u^n, v_h\right) \\
\qquad -\left(\eta_p^n, (v_h)_x\right) + \sum_{j=1}^{N} \left((\eta_p^n)^+ v_h^- |_{j+\frac{1}{2}} - (\eta_p^n)^+ v_h^+ |_{j-\frac{1}{2}}\right), \\
(\xi_p^n, w_h) + \left(\xi_u^n, (w_h)_x\right) - \sum_{j=1}^{N} \left((\xi_u^n)^- w_h^- |_{j+\frac{1}{2}} - (\xi_u^n)^- w_h^+ |_{j-\frac{1}{2}}\right) \\
\quad = -(\eta_p^n, w_h) - \left(\eta_u^n, (w_h)_x\right) + \sum_{j=1}^{N} \left((\eta_u^n)^- w_h^- |_{j+\frac{1}{2}} - (\eta_u^n)^- w_h^+ |_{j-\frac{1}{2}}\right).
\end{cases} \tag{2.21}
$$

Taking the test functions $(v_h, w_h) = (\xi_u^n, \xi_p^n)$ in (2.21) and using the properties (2.1) and (2.2), we get

$$
\begin{aligned}
&\left(\sum_{i=1}^{l} q_i \Upsilon_t^{\alpha_i} \xi_u^n, \xi_u^n\right) + \left(c(x)\xi_u^n, \xi_u^n\right) + \|\xi_p^n\|^2 \\
&= -\left(\sum_{i=1}^{l} q_i R_i^n, \xi_u^n\right) - \left(\sum_{i=1}^{l} q_i \Upsilon_t^{\alpha_i} \eta_u^n, \xi_u^n\right) - \left(c(x)\eta_u^n, \xi_u^n\right) - (\eta_p^n, \xi_p^n),
\end{aligned} \tag{2.22}
$$

which is equivalent to

$$
\begin{aligned}
&\left(\sum_{i=1}^{l} \frac{q_i d_{n,1}^i}{\Gamma(2-\alpha_i)} \xi_u^n, \xi_u^n\right) + \left(c(x)\xi_u^n, \xi_u^n\right) + \|\xi_p^n\|^2 \\
&= \left(\sum_{i=1}^{l} \frac{q_i d_{n,n}^i}{\Gamma(2-\alpha_i)} \xi_u^0, \xi_u^n\right) + \left(\sum_{i=1}^{l} \frac{q_i}{\Gamma(2-\alpha_i)} \sum_{k=1}^{n-1} (d_{n,k}^i - d_{n,k+1}^i) \xi_u^{n-k}, \xi_u^n\right) \\
&\quad - \left(\sum_{i=1}^{l} q_i R_i^n, \xi_u^n\right) - \left(\sum_{i=1}^{l} q_i \Upsilon_t^{\alpha_i} \eta_u^n, \xi_u^n\right) - \left(c(x)\eta_u^n, \xi_u^n\right) - (\eta_p^n, \xi_p^n).
\end{aligned} \tag{2.23}
$$

Thus, by using Lemma 2.3, we obtain

$$\begin{aligned}
&\sum_{i=1}^{l}\eta_{n,1}^{i}\|\xi_u^n\|^2+\left(c(x)\xi_u^n,\xi_u^n\right)+\|\xi_p^n\|^2\\
&\quad\le\sum_{i=1}^{l}\eta_{n,n}^{i}\|\xi_u^0\|\|\xi_u^n\|+\sum_{i=1}^{l}\sum_{k=1}^{n-1}(\eta_{n,k}^{i}-\eta_{n,k+1}^{i})\|\xi_u^{n-k}\|\|\xi_u^n\|+\sum_{i=1}^{l}q_i\|R_i^n\|\|\xi_u^n\|\\
&\qquad+\sum_{i=1}^{l}q_i\|\Upsilon_t^{\alpha_i}\eta_u^n\|\|\xi_u^n\|+\left\|\sqrt{c(x)}\eta_u^n\right\|\left\|\sqrt{c(x)}\xi_u^n\right\|+\|\eta_p^n\|\|\xi_p^n\|\\
&\quad\le\sum_{i=1}^{l}\sum_{k=1}^{n-1}\frac{1}{2}(\eta_{n,k}^{i}-\eta_{n,k+1}^{i})\|\xi_u^{n-k}\|^2+\frac{1}{2}\sum_{i=1}^{l}(\eta_{n,1}^{i}-\eta_{n,n}^{i})\|\xi_u^n\|^2\\
&\qquad+\frac{C}{\sum_{i=1}^{l}\eta_{n,n}^{i}}\left(n^{-2\min\{2-\alpha_1,r\alpha_1\}}+h^{2k+2}\right)+\frac{\sum_{i=1}^{l}\eta_{n,n}^{i}}{2}\|\xi_u^n\|^2+Ch^{2k+2}\\
&\qquad+\left(c(x)\xi_u^n,\xi_u^n\right)+\|\xi_p^n\|^2. \qquad (2.24)
\end{aligned}$$

From (2.6), it is easy to get that

$$\frac{1}{d_{n,n}^{i}}\le\frac{T^{\alpha_i}}{1-\alpha_i}n^{r\alpha_i}M^{-r\alpha_i}, \qquad (2.25)$$

which leads to

$$\frac{1}{\sum_{i=1}^{l}\eta_{n,n}^{i}}\le\frac{1}{\eta_{n,n}^{1}}=\frac{\Gamma(2-\alpha_1)}{q_1 d_{n,n}^{1}}\le\frac{\Gamma(1-\alpha_1)}{q_1}T^{\alpha_1}n^{r\alpha_1}M^{-r\alpha_1}. \qquad (2.26)$$

Combining (2.24) and (2.26), we can derive

$$\begin{aligned}
\sum_{i=1}^{l}\eta_{n,1}^{i}\|\xi_u^n\|^2\le&\sum_{i=1}^{l}\sum_{k=1}^{n-1}(\eta_{n,k}^{i}-\eta_{n,k+1}^{i})\|\xi_u^{n-k}\|^2\\
&+C\left(h^{2k+2}+M^{-r\alpha_1}n^{-(2\min\{2-\alpha_1,r\alpha_1\}-r\alpha_1)}\right). \qquad (2.27)
\end{aligned}$$

Now we prove that the truncation error ξ_u^n satisfies

$$\|\xi_u^n\|^2\le\frac{C}{\sum_{i=1}^{l}\eta_{n,1}^{i}}\sum_{j=1}^{n}\theta_{n,j}\left(h^{2k+2}+M^{-r\alpha_1}j^{-(2\min\{2-\alpha_1,r\alpha_1\}-r\alpha_1)}\right) \qquad (2.28)$$

for $n = 1, \ldots, M$, where C is the constant in (2.27). We prove (2.28) by mathematical induction. When $n = 1$, (2.28) becomes

$$\|\xi_u^1\|^2 \le \frac{C}{\sum_{i=1}^l \eta_{1,1}^i}(h^{2k+2+M^{-r\alpha_1}}). \tag{2.29}$$

Supposing the following estimates hold

$$\|\xi_u^m\|^2 \le \frac{C}{\sum_{i=1}^l \eta_{m,1}^i}\sum_{j=1}^{n}\theta_{m,j}\left(h^{2k+2} + M^{-r\alpha_1} j^{-(2\min\{2-\alpha_1, r\alpha_1\}-r\alpha_1)}\right) \tag{2.30}$$

for $m = 2, 3, \ldots, s$, we only need to prove

$$\|\xi_u^{s+1}\|^2 \le \frac{C}{\sum_{i=1}^l \eta_{s+1,1}^i}\sum_{j=1}^{s+1}\theta_{s+1,j}\left(h^{2k+2} + M^{-r\alpha_1} j^{-(2\min\{2-\alpha_1, r\alpha_1\}-r\alpha_1)}\right). \tag{2.31}$$

Letting $n = s + 1$ in (2.28) and using the induction hypothesis (2.30), we have

$$\begin{aligned}
\|\xi_u^{s+1}\|^2 &\le \frac{1}{\sum_{i=1}^l \eta_{s+1,1}^i}\Bigg[\sum_{i=1}^{l}\sum_{k=1}^{s}(\eta_{s+1,k}^i - \eta_{s+1,k+1}^i)\|\xi_u^{s+1-k}\|^2 \\
&\quad + C\left(h^{2k+2} + M^{-r\alpha_1}(s+1)^{(-2\min\{2-\alpha_1, r\alpha_1\}-r\alpha_1)}\right)\Bigg] \\
&\le \frac{1}{\sum_{i=1}^l \eta_{s+1,1}^i}\Bigg[\sum_{i=1}^{l}\sum_{k=1}^{s}(\eta_{s+1,k}^i - \eta_{s+1,k+1}^i)\Bigg(\frac{C}{\sum_{i=1}^l \eta_{s+1-k,1}^i} \\
&\quad \times \sum_{j=1}^{s+1-k}\theta_{s+1-k,j}\left(h^{2k+2} + M^{-r\alpha_1} j^{-(2\min\{2-\alpha_1, r\alpha_1\}-r\alpha_1)}\right)\Bigg) \\
&\quad + C\left(h^{2k+2} + M^{-r\alpha_1}(s+1)^{(-2\min\{2-\alpha_1, r\alpha_1\}-r\alpha_1)}\right)\Bigg] \\
&= \frac{1}{\sum_{i=1}^l \eta_{s+1,1}^i}\Bigg[\sum_{j=1}^{s}\Bigg(\sum_{i=1}^{l}\sum_{k=1}^{s+1-j}\frac{C(\eta_{s+1,k}^i - \eta_{s+1,k+1}^i)}{\sum_{i=1}^l \eta_{s+1-k,1}^i}\theta_{s+1-k,j} \\
&\quad \times\left(h^{2k+2} + M^{-r\alpha_1} j^{-(2\min\{2-\alpha_1, r\alpha_1\}-r\alpha_1)}\right)\Bigg) \\
&\quad + C\left(h^{2k+2} + M^{-r\alpha_1}(s+1)^{(-2\min\{2-\alpha_1, r\alpha_1\}-r\alpha_1)}\right)\Bigg] \\
&= \frac{C}{\sum_{i=1}^l \eta_{s+1,1}^i}\sum_{j=1}^{s+1}\theta_{s+1,j}\left(h^{2k+2} + M^{-r\alpha_1} j^{(-2\min\{2-\alpha_1, r\alpha_1\}-r\alpha_1)}\right).
\end{aligned}$$

Therefore, the estimate (2.28) holds.

Exploiting (2.28) and Lemma 2.2 directly, we obtain

$$
\begin{aligned}
\|\xi_u^n\|^2 &\le \frac{C}{\sum_{i=1}^{l}\eta_{n,1}^{i}}\sum_{j=1}^{n}\theta_{n,j}h^{2k+2} + \frac{CM^{-r\alpha_1}}{\sum_{i=1}^{l}\eta_{n,1}^{i}}\sum_{j=1}^{n}\theta_{n,j}\,j^{-(2\min\{2-\alpha_1,r\alpha_1\}-r\alpha_1)}\\
&\le C\Gamma(1-\alpha_1)T^{\alpha_1}h^{2k+2} + CM^{-r\alpha_1}\Gamma(1-\alpha_1)T^{\alpha_1}M^{-(2\min\{2-\alpha_1,r\alpha_1\}-r\alpha_1)}\\
&\le C\Gamma(1-\alpha_1)\left(h^{2k+2} + M^{-2\min\{2-\alpha_1,r\alpha_1\}}\right),
\end{aligned}
$$

which, together with the interpolation property(2.3) and the triangle inequality, completes the proof of this theorem. □

It is clear that the results derived in Theorems 2.1 and 2.2 are α_1-nonrobust, i.e., the bounds blow up as $\alpha_1 \to 1^-$. In the following subsection, we present the improved stability and convergence analysis for the scheme (2.9).

2.3 α_1-Robust error analysis of the non-uniform L1/LDG method

This subsection is devoted to the α_1-robust stability and convergence analysis of non-uniform L1/LDG scheme (2.9) for system (1.1)–(1.3). Let us start by introducing the following lemmas.

Lemma 2.4 ([9], Lemma 2) *Suppose that the solution $u(x,t)$ of problem (1.1)–(1.3) satisfies (1.6). Then there exists a constant C such that for all t_n one has*

$$|R_i^n| \le Ct_n^{-\alpha_i}M^{-\min\{2-\alpha_i,r\alpha_1\}}$$

for $i = 1, 2, \dots, l$, $n = 1, 2, \dots, M$.

Lemma 2.5 ([9], Corollary 1) *For $n = 1, 2, \dots, M$, one has*

$$\frac{1}{d_{n,1}}\sum_{j=1}^{n}\theta_{n,j} \le \sum_{i=1}^{l}\frac{t_n^{\alpha_i}}{q_i\Gamma(1+\alpha_i)}.$$

Lemma 2.6 ([9], Corollary 2) *Set $l_M = \frac{1}{\ln M}$. Assume that $M \ge 3$ so $0 < l_M < 1$. Then*

$$\frac{1}{d_{n,1}}\sum_{j=1}^{n}\left(\sum_{i=1}^{l}q_i t_j^{-\alpha_i}\right)\theta_{n,j} \le \frac{le^r\max_{1\le i\le l}\Gamma(1+l_M-\alpha_i)}{\Gamma(1+l_M)}.$$

Lemma 2.7 ([9], Lemma 6) *Assume that the sequences $\{\xi^n\}_{n=1}^{\infty}$, $\{\eta^n\}_{n=1}^{\infty}$ are nonnegative and the grid function $\{v^n : n = 0, 1, \dots, M\}$ satisfies $v^0 \ge 0$ and*

$$v^n\sum_{i=1}^{l}q_i\Upsilon_t^{\alpha_i}v^n \le \xi^n v^n + (\eta^n)^2,\ n = 1, 2, \dots, M.$$

Then

$$v^n \leq v^0 + \frac{1}{d_{n,1}} \sum_{j=1}^{n} \theta_{n,j} (\xi^j + \eta^j) + \max_{1 \leq j \leq n} \left\{ \eta^j \right\}, \; n = 1, 2, \ldots, M.$$

We shall now improve Theorem 2.1 by replacing (2.32) with a bound that is α_1-robust.

Theorem 2.3 (Improved L^2-norm stability) *The solution U_h^n of the fully discrete scheme* (2.9) *satisfies*

$$\|U_h^n\| \leq \|U_h^0\| + \left(\sum_{i=1}^{l} \frac{t_n^{\alpha_i}}{q_i \Gamma(1+\alpha_i)} \right) \max_{1 \leq j \leq n} \|f^j\|, \; n = 1, \ldots, M. \tag{2.32}$$

Proof Taking the test functions $(v_h, w_h) = (U_h^n, P_h^n)$, and adding the two equations in (2.9), we have

$$\left(\sum_{i=1}^{l} q_i \Upsilon_t^{\alpha_i} U_h^n, U_h^n \right) + (c(x)U_h^n, U_h^n) + (P_h^n, P_h^n) = (f^n, U_h^n). \tag{2.33}$$

It follows from [9, Lemma 3] that

$$\left(\sum_{i=1}^{l} q_i \Upsilon_t^{\alpha_i} v^n, v^n \right) \geq \left(\sum_{i=1}^{l} q_i \Upsilon_t^{\alpha_i} \|v^n\| \right) \|v^n\| \tag{2.34}$$

for $n = 1, 2, \ldots, M$.

Applying (2.33) and (2.34), as well as Cauchy–Schwarz inequality, we obtain

$$\left(\sum_{i=1}^{l} q_i \Upsilon_t^{\alpha_i} \|U_h^n\| \right) \|U_h^n\| \leq \|f^n\| \|U_h^n\|. \tag{2.35}$$

Then an application of Lemmas 2.5 and 2.7 immediately yields

$$\begin{aligned} \|U_h^n\| &\leq \|U_h^0\| + \frac{1}{d_{n,1}} \sum_{j=1}^{n} \theta_{n,j} \max_{1 \leq j \leq n} \|f^j\| \\ &\leq \|U_h^0\| + \left(\sum_{i=1}^{l} \frac{t_n^{\alpha_i}}{q_i \Gamma(1+\alpha_i)} \right) \max_{1 \leq j \leq n} \|f^j\|, \end{aligned}$$

which completes the proof. □

We also give an α_1-robust convergence result of the fully discrete non-uniform L1/LDG scheme (2.9) for (1.1)–(1.3). The conclusion is stated as follows.

Theorem 2.4 (Improved L^2-norm error estimate) *Let u be the exact solution of* (1.1)–(1.3) *and U_h^n be the numerical solution of the fully discrete non-uniform L1/LDG scheme* (2.9)*. Suppose that u satisfies condition* (1.6) *and $u(\cdot,t) \in H^{k+1}(\mathcal{T}_h)$. Then, it holds that*

$$\|u^n - U_h^n\| \le \frac{C \max_{1\le i\le l} \Gamma(1+l_M-\alpha_i)}{\Gamma(1+l_M)} M^{-\min\{2-\alpha_1, r\alpha_1\}} + Ch^{k+1}, \quad (2.36)$$

where C is a positive constant independent of M and h.

Proof As shown in Theorem 2.2, one has

$$\begin{aligned}
&\left(\sum_{i=1}^{l} q_i \Upsilon_t^{\alpha_i} \xi_u^n, \xi_u^n\right) + \left(c(x)\xi_u^n, \xi_u^n\right) + \|\xi_p^n\|^2 \\
&= -\left(\sum_{i=1}^{l} q_i R_i^n, \xi_u^n\right) - \left(\sum_{i=1}^{l} q_i \Upsilon_t^{\alpha_i} \eta_u^n, \xi_u^n\right) - \left(c(x)\eta_u^n, \xi_u^n\right) - (\eta_p^n, \xi_p^n).
\end{aligned}$$

By using (2.34), we obtain that

$$\begin{aligned}
&\left(\sum_{i=1}^{l} q_i \Upsilon_t^{\alpha_i} \|\xi_u^n\|\right) \|\xi_u^n\| + \left(c(x)\xi_u^n, \xi_u^n\right) + \|\xi_p^n\|^2 \\
&= -\left(\sum_{i=1}^{l} q_i R_i^n, \xi_u^n\right) - \left(\sum_{i=1}^{l} q_i \Upsilon_t^{\alpha_i} \eta_u^n, \xi_u^n\right) - \left(c(x)\eta_u^n, \xi_u^n\right) - (\eta_p^n, \xi_p^n) \\
&\le \sum_{i=1}^{l} q_i \|R_i^n\| \|\xi_u^n\| + \sum_{i=1}^{l} q_i \|\Upsilon_t^{\alpha_i} \eta_u^n\| \|\xi_u^n\| + \left\|\sqrt{c(x)}\eta_u^n\right\| \left\|\sqrt{c(x)}\xi_u^n\right\| \\
&\quad + \|\eta_p^n\| \|\xi_p^n\| \\
&\le \left(C \sum_{i=1}^{l} q_i t_n^{-\alpha_i} M^{-\min\{2-\alpha_i, r\alpha_1\}} + C \sum_{i=1}^{l} q_i h^{k+1}\right) \|\xi_u^n\| + Ch^{2k+2} \\
&\quad + \left\|\sqrt{c(x)}\xi_u^n\right\|^2 + \|\xi_p^n\|^2, \qquad (2.37)
\end{aligned}$$

where we invoked (2.3) and Lemma 2.4 for the last inequality.

Consequently, applying Lemmas 2.5–2.7, we can get

$$\begin{aligned}
\|\xi_u^n\| &\le \|\xi_u^0\| + \frac{C}{d_{n,1}} \sum_{j=1}^{n} \left(\sum_{i=1}^{l} q_i t_j^{-\alpha_i} M^{-\min\{2-\alpha_i, r\alpha_1\}} + h^{k+1}\right) \theta_{n,j} + Ch^{k+1} \\
&\le \frac{Ce^r \max_{1\le i\le l} \Gamma(1+l_M-\alpha_i)}{\Gamma(1+l_M)} M^{-\min\{2-\alpha_1, r\alpha_1\}} + Ch^{k+1}.
\end{aligned}$$

Finally, by using the triangle inequality and the interpolation property (2.3) again, we can complete the proof of Theorem 2.4. □

3 Two-dimensional case

For a bounded rectangular domain $\Omega = (a_1, b_1) \times (a_2, b_2) \subset \mathbb{R}^2$, we divide it into a Cartesian grid $\mathcal{T}_h = \{K\}$ consisting of $N_x \times N_y$ rectangular elements $K := I_i \times J_j = (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \times (y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}})$, $i = 1, \ldots, N_x$, $j = 1, \ldots, N_y$, where $a_1 = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N_x+\frac{1}{2}} = b_1$ and $a_2 = y_{\frac{1}{2}} < y_{\frac{3}{2}} < \cdots < y_{N_y+\frac{1}{2}} = b_2$. Denoting $\Delta x_i = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}$ and $\Delta y_j = y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}}$, respectively. Then the maximal length of all edges is defined by $h = \max_{1 \le i \le N_x, 1 \le j \le N_y} (\Delta x_i, \Delta y_j)$. We assume that the mesh $\mathcal{T}_h$ is quasi-uniform in the sense that there exist constants $C_1, C_2 > 0$ such that $h \le C_1 \Delta x_i$ and $h \le C_2 \Delta y_j$ for all $K \in \mathcal{T}_h$. Then the finite element space is defined by

$$\begin{aligned} V_h &= \{v_h \in L^2(\Omega) : v_h|_K \in \mathcal{Q}^k(K),\ v_h|_{\partial\Omega} = 0,\ \forall K \in \mathcal{T}_h\}, \\ \mathbf{V}_h &= \left\{\mathbf{w}_h \in L^2(\Omega)^2 : \mathbf{w}_h|_K \in \mathcal{Q}^k(K)^2,\ \mathbf{w}_h|_{\partial\Omega} = \mathbf{0},\ \forall K \in \mathcal{T}_h\right\}, \end{aligned} \tag{3.1}$$

where $\mathcal{Q}^k(K)$ denotes the space of polynomials of degrees at most k defined on K.

We use a fixed vector $\mathbf{I} = (1, 1)^\top$ to uniquely define the inflow and outflow boundaries of Ω, namely,

$$\partial\Omega^- = \{(x, y) \in \partial\Omega : \mathbf{I} \cdot \mathbf{n} < 0\},\ \partial\Omega^+ = \{(x, y) \in \partial\Omega : \mathbf{I} \cdot \mathbf{n} > 0\},$$

where $\mathbf{n}$ is the outward unit normal vector of Ω. Similarly, we denote ∂K^- and ∂K^+ the inflow and outflow boundaries of K, respectively, i.e.,

$$\partial K^- = \{(x, y) \in \partial K : \mathbf{I} \cdot \mathbf{n} < 0\},\ \partial K^+ = \{(x, y) \in \partial K : \mathbf{I} \cdot \mathbf{n} > 0\}.$$

If two elements K_1 and K_2 are neighbours and share one common side e, i.e., $e = \partial K_1 \cap \partial K_2$, then there are two traces for any function defined on e. We denote

$$u^+ = u|_{\partial K_2^- \cap e},\ u^- = u|_{\partial K_1^+ \cap e},\ [\![u]\!]_e = u^+ - u^-,\ [\![u]\!]_{\partial\Omega} = u|_{\partial\Omega}.$$

For each $h > 0$, $\mathcal{E}_B$ denotes the set of all boundary edges of the mesh $\mathcal{T}_h$ on $\partial\Omega$, $\mathcal{E}_I$ denotes the set of all interior edges of the mesh $\mathcal{T}_h$ in Ω, and $\mathcal{E}$ denotes the union of all edges, i.e., $\mathcal{E} = \mathcal{E}_B \cup \mathcal{E}_I$. The L^2 norm and L^2 inner product on the edges $\partial K^\pm$ are given by

$$\|u\|_{\partial K^\pm}^2 = (u, u)_{\partial K^\pm},\ (u, v)_{\partial K^\pm} = \int_{\partial K^\pm} u^\mp(s)\, v^\mp(s) \mathrm{d}s.$$

The norms on the whole outflow and inflow boundaries $\mathcal{E}$ are denoted by

$$\|u\|_{\mathcal{E}}^2 = \sum_{e \in \mathcal{E}} \|u\|_e^2.$$

We define the broken Sobolev space V on $\mathcal{T}_h$ by

$$V = \left\{ v \in L^2(\Omega) : v|_K \in H^1(K),\ \forall K \in \mathcal{T}_h \right\},$$

and denote by $\mathbf{V} = V \times V$.

3.1 The fully discrete non-uniform L1/LDG scheme

We still use the L1 method on non-uniform meshes (see (2.4)) as time discretization and LDG method as space discretization. As the usual treatment, we firstly rewrite (1.1)–(1.3) into a system of the first order derivatives

$$\begin{cases} \sum_{i=1}^{l} \left[q_i \, {}_C\mathrm{D}_{0,t}^{\alpha_i} u \right] - \nabla \cdot \mathbf{p} + cu = f(x, y, t),\ (x, y, t) \in \Omega \times (0, T], \\ \mathbf{p} = \nabla u,\ (x, y, t) \in \Omega \times (0, T], \\ u|_{t=0} = u_0(x, y),\ (x, y) \in \overline{\Omega}, \\ u|_{(x,y)\in\partial\Omega} = 0,\ t \in (0, T], \end{cases} \tag{3.2}$$

Then we can define the semi-discrete LDG scheme for (1.1)–(1.3) as follows: find $(U_h, \mathbf{P}_h) \in V_h \times \mathbf{V}_h$ such that for all test functions $(v_h, \mathbf{w}_h) \in V_h \times \mathbf{V}_h$, we have

$$\begin{cases} \left(\sum_{i=1}^{l} q_i \, {}_C\mathrm{D}_{0,t}^{\alpha_i} U_h, v_h \right) + (cU_h, v_h) = \mathcal{L}(\mathbf{P}_h, v_h) + (f, v_h), \\ (\mathbf{P}_h, \mathbf{w}_h) = \mathcal{K}(U_h, \mathbf{w}_h), \end{cases} \tag{3.3}$$

where

$$\mathcal{L}(\mathbf{P}_h, v_h) = -(\mathbf{P}_h, \nabla v_h) + \sum_{K \in \mathcal{T}_h} (\widehat{\mathbf{P}_h} \cdot \mathbf{n}, v_h)_{\partial K}, \tag{3.4}$$

$$\mathcal{K}(U_h, \mathbf{w}_h) = -(U_h, \nabla \cdot \mathbf{w}_h) + \sum_{K \in \mathcal{T}_h} (\widehat{U_h}, \mathbf{w}_h \cdot \mathbf{n})_{\partial K}. \tag{3.5}$$

Similar to the one-dimensional case, the numerical fluxes $\widehat{U_h}$, $\widehat{\mathbf{P}_h}$ can be chosen as

$$\widehat{U_h} = U_h^-,\ \widehat{\mathbf{P}_h} = \mathbf{P}_h^+. \tag{3.6}$$

Let $(U_h^n, \mathbf{P}_h^n) \in V_h \times \mathbf{V}_h$ be the approximation of $(u(x, y, t_n), \mathbf{p}(x, y, t_n))$. Then we define the fully discrete non-uniform L1/LDG scheme as follows: find $(U_h^n, \mathbf{P}_h^n) \in V_h \times \mathbf{V}_h$ such that for all test functions $(v_h, \mathbf{w}_h) \in V_h \times \mathbf{V}_h$,

$$\begin{cases} \left(\sum_{i=1}^{l} q_i \Upsilon_t^{\alpha_i} U_h^n, v_h \right) + (cU_h^n, v_h) = \mathcal{L}(\mathbf{P}_h^n, v_h) + (f^n, v_h), \\ (\mathbf{P}_h^n, \mathbf{w}_h) = \mathcal{K}(U_h^n, \mathbf{w}_h). \end{cases} \tag{3.7}$$

Here the notation $\Upsilon_t^{\alpha_i} U_h^n$ is defined in (2.4).

3.2 α_1-Nonrobust error analysis of the non-uniform L1/LDG method

The fully discrete non-uniform L1/LDG scheme (3.7) for the two-dimensional multiterm time-fractional initial-boundary value problem satisfies the following α_1-nonrobust stability. First of all, we introduce a lemma that will be used later on.

Lemma 3.1 [21] *For any $v_h \in V_h$ and $\mathbf{w}_h \in \mathbf{V}_h$, there holds the equality*

$$\mathcal{L}(\mathbf{w}_h, v_h) = -\mathcal{K}(v_h, \mathbf{w}_h).$$

Theorem 3.1 (L^2-norm stability) *The solution U_h^n of the fully discrete scheme (3.7) satisfies*

$$\|U_h^n\| \leq \|U_h^0\| + \Gamma(1-\alpha_1)T^{\alpha_1} \max_{1\leq j\leq n} \left\| f^j \right\|, \ n = 1, \ldots, M. \tag{3.8}$$

Proof Taking the test function $v_h = U_h^n$ and $w_h = \mathbf{P}_h^n$ in (3.7) and using Lemma 3.1, we obtain

$$\begin{aligned}
&\left(\sum_{i=1}^{l} \frac{q_i d_{n,1}^i}{\Gamma(2-\alpha_i)} U_h^n, U_h^n\right) + \left(cU_h^n, U_h^n\right) + \|\mathbf{P}_h^n\|^2 \\
&= \left(\sum_{i=1}^{l} \frac{q_i d_{n,n}^i}{\Gamma(2-\alpha_i)} U_h^0, U_h^n\right) + \left(\sum_{i=1}^{l} \frac{q_i}{\Gamma(2-\alpha_i)} \sum_{k=1}^{n-1} (d_{n,k}^i - d_{n,k+1}^i) U_h^{n-k}, U_h^n\right) \\
&\quad + (f^n, U_h^n).
\end{aligned} \tag{3.9}$$

By using an analysis similar to that in (2.11) and in Theorem 2.1, we can complete the proof of this theorem. □

Now we present the α_1-nonrobust convergence analysis and give the detailed proof. To obtain the optimal error estimate for the non-uniform L1/LDG scheme (3.7), we would like to use the elliptic projection introduced in [5] to eliminate the element boundary errors. Let $u \in V$ and $\mathbf{q} = \nabla u$, define the elliptic projection $(\mathcal{P}_h u, \mathcal{P}_h \mathbf{q}) \in V_h \times \mathbf{V}_h$ as: for any $(v_h, \mathbf{w}_h) \in V_h \times \mathbf{V}_h$, it holds that

$$\mathcal{L}(\mathbf{q}, v_h) = \mathcal{L}(\mathcal{P}_h \mathbf{q}, v_h), \tag{3.10}$$

$$(\mathcal{P}_h \mathbf{q}, \mathbf{w}_h) = \mathcal{K}(\mathcal{P}_h u, \mathbf{w}_h), \tag{3.11}$$

$$(u - \mathcal{P}_h u, 1) = 0. \tag{3.12}$$

The elliptic projection defined above uniquely exists and satisfies the following approximation properties.

Lemma 3.2 [21] *Assume $u \in H^{k+2}(\Omega)$, then there exists a constant C depending on the regularity of u such that*

$$\|u-\mathcal{P}_h u\| + h^{\frac{1}{2}}\|u-\mathcal{P}_h u\|_{\mathcal{E}} \le Ch^{k+1}. \tag{3.13}$$

Theorem 3.2 (L^2-norm error estimate) *Let u be the exact solution of (1.1)–(1.3) and U_h^n be the numerical solution of the fully discrete non-uniform L1/LDG scheme (3.7). Suppose that u satisfies condition (1.6) and $u(\cdot,t)\in H^{k+2}(\Omega)$. Then, it holds that*

$$\|u^n - U_h^n\| \le C\sqrt{\Gamma(1-\alpha_1)}\left(M^{-\min\{2-\alpha_1, r\alpha_1\}} + h^{k+1}\right), \tag{3.14}$$

where C is a positive constant independent of M and h.

Proof Denote

$$\begin{aligned} e_u^n &= u^n - U_h^n = \mathcal{P}_h u^n - U_h^n + (u^n - \mathcal{P}_h u^n) = \xi_u^n + \eta_u^n,\\ \mathbf{e}_{\mathbf{p}}^n &= \mathbf{p}^n - \mathbf{P}_h^n = \mathcal{P}_h \mathbf{p}^n - \mathbf{P}_h^n + (\mathbf{p}^n - \mathcal{P}_h \mathbf{p}^n) = \boldsymbol{\xi}_{\mathbf{p}}^n + \boldsymbol{\eta}_{\mathbf{p}}^n. \end{aligned} \tag{3.15}$$

From (3.2), we can get the weak form of (1.1)–(1.3) at t_n as follows,

$$\begin{cases} \left(\sum\limits_{i=1}^{l} q_i\, {}_C\mathrm{D}_{0,t}^{\alpha_i} u^n, v_h\right) + (cu^n, v_h) = \mathcal{L}(\mathbf{p}^n, v_h) + (f^n, v_h),\\ (\mathbf{p}^n, \mathbf{w}_h) = \mathcal{K}(u^n, \mathbf{w}_h). \end{cases} \tag{3.16}$$

Applying the property of elliptic projection (3.10)–(3.12), it holds that

$$\begin{cases} \mathcal{L}(\boldsymbol{\eta}_{\mathbf{p}}^n, v_h) = \mathcal{L}(\mathbf{p}^n - \mathcal{P}_h\mathbf{p}^n, v_h),\\ (\boldsymbol{\eta}_{\mathbf{p}}^n, \mathbf{w}_h) = \mathcal{K}(\eta_u^n, \mathbf{w}_h). \end{cases} \tag{3.17}$$

Subtracting (3.7) from (3.16) and noticing (3.17), we obtain the error equation

$$\begin{cases} \left(\sum\limits_{i=1}^{l} q_i\left({}_C\mathrm{D}_{0,t}^{\alpha_i} u^n - \Upsilon_t^{\alpha_i} U_h^n\right), v_h\right) + (ce_u^n, v_h) = \mathcal{L}(\boldsymbol{\xi}_{\mathbf{p}}^n, v_h),\\ (\boldsymbol{\xi}_{\mathbf{p}}^n, \mathbf{w}_h) = \mathcal{K}(\xi_u^n, \mathbf{w}_h). \end{cases} \tag{3.18}$$

Taking the test function $(v_h, \mathbf{w}_h) = (\xi_u^n, \boldsymbol{\xi}_{\mathbf{p}}^n)$ in (3.18) and using Lemma 3.1, we get

$$\begin{aligned} &\left(\sum_{i=1}^{l} q_i \Upsilon_t^{\alpha_i}\xi_u^n, \xi_u^n\right) + \left(c\xi_u^n, \xi_u^n\right) + \|\boldsymbol{\xi}_{\mathbf{p}}^n\|^2\\ &\quad = -\left(\sum_{i=1}^{l} q_i R_i^n, \xi_u^n\right) - \left(\sum_{i=1}^{l} q_i \Upsilon_t^{\alpha_i}\eta_u^n, \xi_u^n\right) - \left(c\eta_u^n, \xi_u^n\right). \end{aligned} \tag{3.19}$$

Repeating similar arguments as Theorem 2.2 (see the proof of (2.29) and (2.31)), we can use mathematical induction to obtain the error estimate

$$\|u^n - U_h^n\| \leq C\sqrt{\Gamma(1-\alpha_1)}\left(M^{-\min\{2-\alpha_1, r\alpha_1\}} + h^{k+1}\right).$$

The proof is thus completed. □

3.3 α_1-Robust error analysis of the non-uniform L1/LDG method

We are now ready to state the α_1-robust stability and convergence analysis of non-uniform L1/LDG scheme (3.7) for system (1.1)–(1.3).

Theorem 3.3 (Improved L^2-norm stability) *The solution U_h^n of the fully discrete scheme* (3.7) *satisfies*

$$\|U_h^n\| \leq \|U_h^0\| + \left(\sum_{i=1}^{l} \frac{t_n^{\alpha_i}}{q_i\Gamma(1+\alpha_i)}\right) \max_{1\leq j\leq n} \|f^j\|, \; n = 1, \ldots, M. \tag{3.20}$$

Proof Taking the test function $(v_h, w_h) = (U_h^n, \mathbf{P}_h^n)$ in (3.7) and applying Lemma 3.1, we can get

$$\left(\sum_{i=1}^{l} q_i \Upsilon_t^{\alpha_i} U_h^n, U_h^n\right) + (cU_h^n, U_h^n) + (\mathbf{P}_h^n, \mathbf{P}_h^n) = (f^n, U_h^n). \tag{3.21}$$

Then, similar to the proof of Theorem 2.3, the L^2-norm stability (3.20) can be obtained immediately. This finishes the proof. □

Next, we state the α_1-robust convergence result of the fully discrete non-uniform L1/LDG scheme (3.7).

Theorem 3.4 (Improved L^2-norm error estimate) *Let u be the exact solution of* (1.1)–(1.3) *and U_h^n be the numerical solution of the fully discrete non-uniform L1/LDG scheme* (3.7). *Suppose that u satisfies condition* (1.6) *and $u(\cdot, t) \in H^{k+2}(\Omega)$. Then, it holds that*

$$\|u^n - U_h^n\| \leq \frac{C\max_{1\leq i\leq l}\Gamma(1+l_M-\alpha_i)}{\Gamma(1+l_M)} M^{-\min\{2-\alpha_1, r\alpha_1\}} + Ch^{k+1}, \tag{3.22}$$

where C is a positive constant independent of M and h.

Proof Set

$$e_u^n = u^n - U_h^n = \mathcal{P}_h u^n - U_h^n + (u^n - \mathcal{P}_h u^n) = \xi_u^n + \eta_u^n,$$
$$\mathbf{e}_{\mathbf{p}}^n = \mathbf{p}^n - \mathbf{P}_h^n = \mathcal{P}_h \mathbf{p}^n - \mathbf{P}_h^n + (\mathbf{p}^n - \mathcal{P}_h \mathbf{p}^n) = \boldsymbol{\xi}_{\mathbf{p}}^n + \boldsymbol{\eta}_{\mathbf{p}}^n.$$

By the similar techniques used in the proof of Theorem 3.2, it holds that

$$\begin{aligned}&\left(\sum_{i=1}^{l} q_i \Upsilon_t^{\alpha_i} \xi_u^n, \xi_u^n\right) + \left(c\xi_u^n, \xi_u^n\right) + \|\boldsymbol{\xi}_{\mathbf{p}}^n\|^2 \\ &= -\left(\sum_{i=1}^{l} q_i R_i^n, \xi_u^n\right) - \left(\sum_{i=1}^{l} q_i \Upsilon_t^{\alpha_i} \eta_u^n, \xi_u^n\right) - \left(c\eta_u^n, \xi_u^n\right).\end{aligned}$$

Then, repeating similar arguments as Theorem 2.4, we can obtain (3.22). The proof is thus completed. □

4 Numerical examples

In this section, we present a numerical example to validate our theoretical results.

Example 4.1 Consider the following three-term time-fractional diffusion equation

$$\begin{cases} {}_C\mathrm{D}_{0,t}^{\alpha_1} u + 0.1 {}_C\mathrm{D}_{0,t}^{0.1} u + 0.1 {}_C\mathrm{D}_{0,t}^{0.2} u - u_{xx} + u = f(x,t), \ (x,t) \in (0,1) \times (0,1], \\ u(x,0) = 0, \ x \in (0,1), \\ u(0,t) = u(1,t) = 0, \ t \in (0,1], \end{cases} \tag{4.1}$$

where $0 < \alpha_1 < 1$. The source term $f(x,t)$ is chosen such that the exact solution of the problem is $u = (t^{\alpha_1} + t^3)\sin(2\pi x)$.

The L^2 and L^∞ numerical errors and orders with different α_1 at $T = 1$ are given in Tables 1–5. From these results, we conclude that the non-uniform L1/LDG scheme (2.9) for the three-term time-fractional diffusion equation in Example 4.1 can achieve $\min\{2-\alpha_1, r\alpha_1\}$-th order convergence in time and $(k+1)$-th order convergence in space, which are in line with the theoretical rate established in Theorem 2.4.

5 Concluding remarks

In this paper, we have studied the multiterm time-fractional initial-boundary value problem. Considering the weak regularity of the solution at the starting time, we use the L1 scheme with non-uniform meshes to discretize the time fractional derivative, and the classical LDG method for the space derivative. Numerical stability and convergence of the established schemes are analyzed. Such stability and convergence results are proved to be α_1-robust. Finally, a numerical example is given to confirm the theoretical results.

Table 1 The time convergence results for Example 4.1 at $T = 1$ with $k = 1$, $M = N$, and $r = 1$

		$\alpha = 0.4$		$\alpha = 0.6$		$\alpha = 0.8$	
	M	Error	Order	Error	Order	Error	Order
L^2-norm	64	7.5891e−3	–	4.5890e−3	–	1.9342e−3	–
	128	6.4698e−3	0.2302	3.6048e−3	0.3483	1.3360e−3	0.5339
	256	5.5204e−3	0.2290	2.7908e−3	0.3692	8.8759e−4	0.5899
	512	4.6978e−3	0.2328	2.1179e−3	0.3980	5.6750e−4	0.6453
	1024	3.9801e−3	0.2392	1.5722e−3	0.4299	3.5119e−4	0.6924
L^∞-norm	64	1.2178e−2	–	7.0474e−3	–	2.8077e−3	–
	128	1.0244e−2	0.2496	5.3672e−3	0.3929	1.8828e−3	0.5765
	256	8.5889e−3	0.2542	4.0354e−3	0.4115	1.2344e−3	0.6092
	512	7.1656e−3	0.2614	2.9949e−3	0.4302	7.8866e−4	0.6463
	1024	5.9466e−3	0.2690	2.1941e−3	0.4488	4.9043e−4	0.6854

Table 2 The time convergence results for Example 4.1 at $T = 1$ with $k = 1$, $M = N$, and $r = \frac{1}{\alpha}$

		$\alpha = 0.4$		$\alpha = 0.6$		$\alpha = 0.8$	
	M	Error	Order	Error	Order	Error	Order
L^2-norm	64	1.5633e−3	–	1.5761e−3	–	1.0979e−3	–
	128	9.0085e−4	0.7953	9.1225e−4	0.7888	6.4693e−4	0.5339
	256	4.9551e−4	0.8624	5.0207e−4	0.8615	3.8137e−4	0.7624
	512	2.6295e−4	0.9142	2.6868e−4	0.9020	2.1418e−4	0.8324
	1024	1.3620e−4	0.9490	1.4341e−4	0.9057	1.1729e−4	0.8687
L^∞-norm	64	2.1788e−3	–	2.1995e−3	–	1.5342e−3	–
	128	1.2514e−3	0.8000	1.2669e−3	0.7958	8.9899e−4	0.7711
	256	6.9253e−4	0.8535	7.0146e−4	0.8529	5.2710e−4	0.6092
	512	3.6973e−4	0.9054	3.7444e−4	0.9056	2.9635e−4	0.8308
	1024	1.9217e−4	0.9441	2.0144e−4	0.8944	1.6348e−4	0.8582

Table 3 The time convergence results for Example 4.1 at $T = 1$ with $k = 1$, $M = N$, and $r = \frac{2-\alpha}{\alpha}$

		$\alpha = 0.4$		$\alpha = 0.6$		$\alpha = 0.8$	
	M	Error	Order	Error	Order	Error	Order
L^2-norm	64	4.4124e−4	–	5.1457e−4	–	7.5006e−4	–
	128	1.4416e−4	1.6138	2.3772e−4	1.1141	3.6715e−4	1.0306
	256	5.0778e−5	1.5054	1.0414e−4	1.1908	1.8235e−4	1.0097
	512	1.8014e−5	1.4951	4.3681e−5	1.2534	9.1984e−5	0.9873
L^∞-norm	64	9.6818e−4	–	1.0772e−3	–	1.7074e−3	–
	128	3.3720e−4	1.5217	4.1602e−4	1.3725	7.4786e−4	1.1909
	256	1.1435e−4	1.5601	1.5850e−4	1.3922	3.2565e−4	1.1994
	512	3.8197e−5	1.5820	6.1226e−5	1.3722	1.4156e−4	1.2019

Table 4 The time convergence results for Example 4.1 at $T = 1$ with $k = 1$, $M = N$, and $r = \frac{2(2-\alpha)}{\alpha}$

		$\alpha = 0.4$		$\alpha = 0.6$		$\alpha = 0.8$	
	$M = N$	Error	Order	Error	Order	Error	Order
L^2-norm	64	1.2935e−3	–	1.3101e−3	–	1.9171e−3	–
	128	4.9617e−4	1.3823	5.5501e−4	1.2391	9.0946e−4	1.0758
	256	1.7986e−4	1.4639	2.2503e−4	1.3024	4.1520e−4	1.1312
	512	6.2910e−5	1.5155	8.8735e−5	1.3426	1.8547e−4	1.1626
L^∞-norm	64	2.6759e−3	–	2.7129e−3	–	3.7454e−3	–
	128	9.6381e−4	1.4732	1.0638e−3	1.3507	1.6520e−3	1.1809
	256	3.3411e−4	1.5284	4.0912e−4	1.3786	7.2244e−4	1.1933
	512	1.1321e−4	1.5613	1.5580e−4	1.3929	3.1479e−4	1.1985

Table 5 The spatial convergence results for Example 4.1 at $T = 1$ with $M = 500$, $r = \frac{2-\alpha}{\alpha}$, and $k = 1$

		$\alpha = 0.4$		$\alpha = 0.6$		$\alpha = 0.8$	
	N	Error	Order	Error	Order	Error	Order
L^2-norm	4	1.6776e−1	–	1.6774e−1	–	1.6768e−1	–
	8	3.7964e−2	2.1437	3.7956e−2	2.1438	3.7923e−2	2.1446
	16	9.1764e−3	2.0487	9.1668e−3	2.0499	9.1312e−3	2.0542
	32	2.2614e−3	2.0207	2.2516e−3	2.0255	2.2159e−3	2.0429
L^∞-norm	4	2.2354e−1	–	2.2350e−1	–	2.2344e−1	–
	8	5.3252e−2	2.0697	5.3251e−2	2.0694	5.3240e−2	2.0693
	16	1.2972e−2	2.0374	1.2968e−2	2.0379	1.2952e−2	2.0395
	32	3.2134e−3	2.0132	3.2086e−3	2.0149	3.1918e−3	2.0206

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