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ORIGINAL RESEARCH

Complex Pythagorean fuzzy threshold graphs with application in petroleum replenishment

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Abstract

The purpose of this article is to present a novel idea of complex Pythagorean fuzzy threshold graphs ($\mathcal{CPFTG}_s$). We introduce the relation between vertex cardinality and threshold values of a $\mathcal{CPFTG}$. We propose that $\mathcal{CPFTG}_s$ are free from alternating $4-cycle$ and these graphs can be built up repeatedly adding an isolated or a dominating vertex. We present that the crisp graph of $\mathcal{CPFTG}$ is a split graph ($\mathcal{SG}$). Further, the threshold dimension and threshold partition number of $\mathcal{CPFG}_s$ is defined. Some basic results on threshold dimension and threshold partition number also have been discussed. Finally, an application is presented on this developed concept. Due to the wide range of complex Pythagorean fuzzy sets ($\mathcal{CPFS}_s$), it is obvious that $\mathcal{CPFTG}_s$ are more helpful and beneficial in modeling a problem as compared to complex fuzzy threshold graphs ($\mathcal{CFTG}_s$) and complex intuitionistic fuzzy threshold graphs ($\mathcal{CIFTG}_s$).

Keywords Vertex cardinality of $\mathcal{CPFTG}_s$ · Complex Pythagorean fuzzy alternating $4-cycle$ · $\mathcal{SG}$ · Complex Pythagorean fuzzy threshold dimension and threshold partition number

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1 Introduction

Various theories including soft sets theory [19], rough sets theory [28] and fuzzy sets ($\mathcal{FS}_s$) theory [41] have been introduced to deal with vague and imprecise information. Among all of these theories, Zadeh's theory presents a framework which deals with fuzzy situations in a suitable manner. However, an $\mathcal{FS}$ provides the degree of membership grades only, but no information about the non-membership grades of an object. Atanassov [9] presented the idea of intuitionistic fuzzy sets ($\mathcal{IFS}_s$), which also provide information about non-membership grades such that the sum of membership grades and non-membership grades does not exceed 1. In real life, there occur situations, where the sum of membership grades and non-membership grades exceed 1. In order to cope with such situations, Yager [37] proposed the concept of a Pythagorean fuzzy set ($\mathcal{PFS}$) satisfying the square sum of membership grades and non-membership grades not exceeding 1.

All the models which were discussed above are incapable to describe the insensitivity of data and ignorance of time period, however, with the help of a complex data set the vagueness of data and periodicity of time can be dealt with. In 2002, Ramot et al. [31] proposed the idea of a complex fuzzy set ($\mathcal{CFS}$) to deal with such environments. In a $\mathcal{CFS}$ the range of membership grade lies in a unit disk. A model of $\mathcal{CFS}$ consisting of amplitude and phase term is remarkably important due to the representation of two-dimensional phenomena, which is prevalent in time-periodic phenomena. In a $\mathcal{CFS}$ due to the presence of phase term whole data is provided and no loss of information occurs. A systematic review of $\mathcal{CFS}_s$ was presented by Yazdanbakhsh and Dick [40]. Afterwards, many researchers have attracted towards $\mathcal{CFS}_s$ and extended this concept in different environments with applications. The concept of $\mathcal{CFS}_s$ was extended to complex intuitionistic fuzzy sets ($\mathcal{CIFS}_s$) by adding non-membership grades in $\mathcal{CFS}$ by Alkouri and Salleh [7] in 2012. The basic characteristics of $\mathcal{CIFS}_s$ and basic operations of union, complement and intersection were also discussed by them. However, $\mathcal{CIFS}_s$ are incapable of handling some situations. Due to some restrictions in $\mathcal{CIFS}_s$, the novel concept of $\mathcal{CPFS}_s$ was introduced by Ullah et al. [36].

A graph is a framework, which represents the relations among different objects, where objects are represented by vertices and relation between them is represented by edges. In 1736, Euler [12] discussed graphs in solving koñigsberg problem. By using Zadeh's fuzzy relation, Kaufmann [16] proposed the concept of fuzzy graphs ($\mathcal{FG}_s$) in 1973. Afterwards, the notion of $\mathcal{FG}_s$ and related concepts were introduced by Rosenfeld [32] in 1975. Later on, in 1999, intuitionistic fuzzy relations and intuitionistic graphs ($\mathcal{IFG}_s$) were introduced by Atanassov [34]. Naz et al. [25] proposed the idea of Pythagorean fuzzy graphs ($\mathcal{PFG}_s$).

$\mathcal{CFS}$ deals with imprecision whose range lies in a complex plane, hence handles the two dimensional information in a single set. To utilize this benefit, the concept of complex fuzzy graphs ($\mathcal{CFG}_s$) was introduced by Thirunavukarasu et al. [35]. The notion of complex intuitionistic fuzzy graphs ($\mathcal{CIFG}_s$) was put forward by Yaqoob

Table 1 The characteristic comparison of different graphs with existing graphs

Method	whether have ability to handle periodic problems	whether have ability to represent 2-D information	Whether have the characteristics of generalization
Samanta and Pal [33]	×	×	×
Pramanik et al [39]	×	×	✓
Mahapatra and Pal [24]	×	×	✓
Hameed et al. [13]	✓	✓	×
Hameed et al. [14]	✓	✓	✓
Akram et al. [6]	✓	×	✓
The proposed $\mathcal{CPFTG}$	✓	✓	✓

et al. [39] with its implementations in cellular network provider companies. In 2019, Akram et al. [3] introduced complex Pythagorean fuzzy graphs ($\mathcal{CPFG}_s$). After that, planarity of $\mathcal{CPFG}_s$ was discussed by Akram et al. [1]. In 2020, Akram and Sattar [4] presented the notion of competition graphs under complex Pythagorean fuzzy information. Afterwards, the concept of complex Pythagorean Dombi fuzzy graphs was introduced by Akram and Khan [2].

In this paper, we discuss the class of threshold graphs ($\mathcal{TG}_s$). In graph theory, these graphs play a vital role. $\mathcal{TG}_s$ were discovered in 1977 by Chavatal and Hammer [11] for their use in set packing problems. Henderson and Zalcstein [15] introduced the same graphs and called them PV-chunk definable graphs. $\mathcal{TG}_s$ have numerous implementations in numerous fields like graph labeling, cyclic scheduling, psychology, parallel processing, controlling traffic flow etc. Ordman [26] used these graphs in resource allocation problems. Koop [17] discussed threshold graphs in manpower allocation problems. Afterwards, Peled and Mahadev [29] discussed $\mathcal{TG}_s$ and related topics on them. Andelic and Simic [8] gave some characterizations of $\mathcal{TG}_s$. After that, fuzzy threshold graphs ($\mathcal{FTG}_s$) were first discovered by Smanta and Pal [33]. Later on, interval valued fuzzy threshold graphs were proposed by Paramanik et al. [27]. Yang and Mao [38] introduced intuitionistic fuzzy threshold graphs ($\mathcal{IFTG}_s$) in 2019. The concept of m-polar fuzzy threshold graphs was put forward by Mahapatra and Pal [24] in 2021. For more useful notions of fuzzy graphs, the reader are suggested to [4,5,10,18, 20–23,30]. In this paper, we introduce $\mathcal{CPFTG}_s$, which broad the concept of $\mathcal{FTG}_s$. $\mathcal{PFTG}_s$ are special $\mathcal{CPFTG}_s$. These graphs have a wide range of applications in many fields. It is interesting to study $\mathcal{CPFTG}_s$ since these graphs are more capable to handle uncertainty. The characteristics comparison of our proposed approach with different existing approaches is given in Table 1.

The structure of this paper is organized as follows: Section 2 provides basic definitions. Section 3 proposes the concept of $\mathcal{CPFTG}_s$ and related concepts of $\mathcal{CPFTG}_s$. Section 4 describes an algorithm for exploring $\mathcal{CPFTG}_s$ and provides an application of $\mathcal{CPFTG}_s$. Finally, Sect. 5 provides the conclusion.

2 Preliminaries

Definition 1 [36] A $\mathcal{CPFS}$ on a universe of discourse Z is an object of the form

$$S = \{(z, \xi_S(z)e^{i\psi_S(z)}, \eta_S(z)e^{i\phi_S(z)}) : z \in Z\},$$

where $i = \sqrt{-1}$. $\xi_S(z)$ and $\eta_S(z)$ are membership and non-membership amplitude terms such that $\xi_S(z), \eta_S(z) \in [0, 1]$ satisfying the condition $0 \leq \xi_S^2(z) + \eta_S^2(z) \leq 1$. $\psi_S(z)$ and $\phi_S(z)$ are phase terms associated with membership and non-membership grades, respectively, such that $\psi_S(z), \phi_S(z) \in [0, 2\pi]$ satisfying the condition $0 \leq \psi_S^2(z) + \phi_S^2(z) \leq 2\pi$.

Definition 2 [3] A $\mathcal{CPFG}$ on a non-empty set Z is a pair $G = (S, T)$ where S and T depict complex Pythagorean fuzzy set and complex Pythagorean fuzzy relation, respectively, on Z such that

$$\begin{aligned}
\xi_T(z_1z_2) &\leq \xi_S(z_1) \wedge \xi_S(z_2),\\
\eta_T(z_1z_2) &\leq \eta_S(z_1) \vee \eta_S(z_2),\\
\psi_T(z_1z_2) &\leq \psi_S(z_1) \wedge \psi_S(z_2),\\
\phi_T(z_1z_2) &\leq \phi_S(z_1) \vee \phi_S(z_2),
\end{aligned}$$

where $0 \leq \xi_T^2(z_1z_2) + \eta_T^2(z_1z_2) \leq 1$ and $0 \leq \psi_T^2(z_1z_2) + \phi_T^2(z_1z_2) \leq 2\pi$ for all $z_1z_2 \in Z \times Z$. Complex Pythagorean fuzzy vertex set and complex Pythagorean fuzzy edge set of G are denoted by S and T, respectively.

Throughout this paper, $G^* = (Z, E)$ denotes the crisp graph.

Example 1 Consider a graph $G^* = (Z, E)$, where $Z = \{z_1, z_2, z_3, z_4, z_5, z_6\}$ is the vertex set and $E = \{z_1z_2, z_1z_6, z_2z_6, z_2z_3, z_2z_4, z_3z_4, z_5z_6, z_4z_6, z_4z_5\}$ is the edge set of G^*. Let $G = (S, T)$ be a $\mathcal{CPFG}$ on Z as depicted in Fig. 1, defined by:

$$\begin{aligned}
S = \Big\langle &\Big(\frac{z_1}{0.7e^{i2\pi(0.01)}}, \frac{z_2}{0.3e^{i2\pi(0.01)}}, \frac{z_3}{0.02e^{i2\pi(0.03)}}, \frac{z_4}{0.4e^{i2\pi(0.1)}}, \frac{z_5}{0.6e^{i2\pi(0.04)}}, \frac{z_6}{0.1e^{i2\pi(0.03)}}\Big),\\
&\Big(\frac{z_1}{0.6e^{i2\pi(0.03)}}, \frac{z_2}{0.4e^{i2\pi(0.1)}}, \frac{z_3}{0.1e^{i2\pi(0.04)}}, \frac{z_4}{0.2e^{i2\pi(0.2)}}, \frac{z_5}{0.3e^{i2\pi(0.02)}}, \frac{z_6}{0.4e^{i2\pi(0.1)}}\Big)\Big\rangle,\\
T = \Big\langle &\Big(\frac{z_1z_2}{0.2e^{i2\pi(0.01)}}, \frac{z_1z_6}{0.1e^{i2\pi(0.01)}}, \frac{z_2z_6}{0.1e^{i2\pi(0.01)}}, \frac{z_2z_3}{0.01e^{i2\pi(0.01)}},\\
&\frac{z_2z_4}{0.2e^{i2\pi(0.01)}}, \frac{z_3z_4}{0.01e^{i2\pi(0.02)}}, \frac{z_5z_6}{0.1e^{i2\pi(0.02)}}, \frac{z_4z_6}{0.03e^{i2\pi(0.02)}}, \frac{z_4z_5}{0.5e^{i2\pi(0.03)}}\Big),\\
&\Big(\frac{z_1z_2}{0.5e^{i2\pi(0.02)}}, \frac{z_1z_6}{0.5e^{i2\pi(0.01)}}, \frac{z_2z_6}{0.3e^{i2\pi(0.02)}}, \frac{z_2z_3}{0.3e^{i2\pi(0.03)}}, \frac{z_2z_4}{0.3e^{i2\pi(0.1)}},\\
&\frac{z_3z_4}{0.2e^{i2\pi(0.03)}}, \frac{z_5z_6}{0.4e^{i2\pi(0.1)}}, \frac{z_4z_6}{0.4e^{i2\pi(0.2)}}, \frac{z_4z_5}{0.3e^{i2\pi(0.2)}}\Big)\Big\rangle.
\end{aligned}$$

Remark 1 Let $G = (S, T)$ be a $\mathcal{CPFG}$. If $\eta_S(z) = 0$ and $\phi_S(z) = 0$ for all $z \in Z$ also $\eta_T(z_1, z_2) = 0$ and $\phi T(z_1, z_2) = 0$ for all $(z_1, z_2) \in E$, then $G = (S, T)$ is a $\mathcal{CFG}$. Moreover, let $\xi_S(z) = 1$, $\eta_S(z) = 0$, $\phi_S(z) = 0$ for all $z \in Z$. According to the

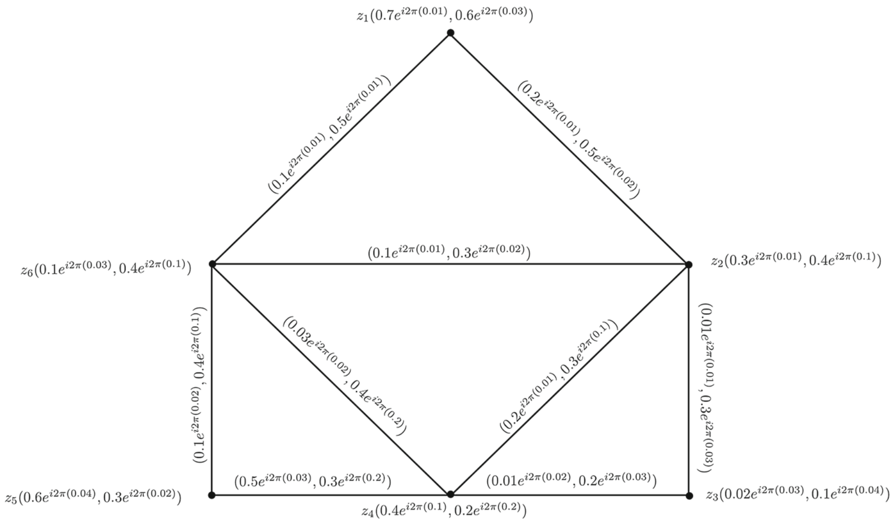

Fig. 1 Complex Pythagorean fuzzy graph

Definition 2, it is obvious that $G = (S, T)$ will be a $\mathcal{CPFG}$ only if $\eta_T(z_1, z_2) = 0$, $\phi_T(z_1, z_2) = 0$, for all $(z_1, z_2) \in E$.

3 Complex Pythagorean fuzzy threshold graphs

Definition 3 A $\mathcal{CPFG}$ $G = (S, T)$ on a non-empty set Z is called $\mathcal{CPFTG}$ if there exist $\hat{t_1} > 0,\ \hat{t_2} > 0,\ \theta_{t_1} > 0,\ \theta_{t_2} > 0$ such that

$$\sum_{z\,\in\,\mathcal{I}} \xi_S(z) \le \hat{t_1},\quad \sum_{z\,\in\,\mathcal{I}} (1-\eta_S(z)) \le \hat{t_2},\quad \sum_{z\,\in\,\mathcal{I}} \psi_S(z) \le \theta_{t_1},\quad \sum_{z\,\in\,\mathcal{I}} (2\pi - \phi_S(z)) \le \theta_{t_2}$$

if and only if $\mathcal{I} \subset Z$ is an independent set in G^*. We use the notation $G = (S, T; \hat{t_1}, \hat{t_2}, \theta_{t_1}, \theta_{t_2})$ for $\mathcal{CPFTG}$.

An independent set $\mathcal{I}$ in G^* is the set of vertices such that no two of which are connected by an edge.

Remark 2 If $G = (S, T; \hat{t_1}, \hat{t_2}, \theta_{t_1}, \theta_{t_2})$ is $CPFTG$ and $\mathcal{U} \subset Z$ is not an independent set in G, then $\sum_{z\,\in\,\mathcal{U}} \xi_S(z) > \hat{t_1}$ or $\sum_{z\,\in\,\mathcal{U}} (1-\eta_S(z)) > \hat{t_2}$ also $\sum_{z\,\in\,\mathcal{U}} \psi_S(z) > \theta_{t_1}$ or $\sum_{z\,\in\,\mathcal{I}} (2\pi - \phi_S(z)) > \theta_{t_2}$.

Here we describe an example of a $\mathcal{CPFTG}$ with four threshold values $\hat{t_1} = 0.04,\ \hat{t_2} = 1.3,\ \theta_{t_1} = 0.04\pi$ and $\theta_{t_2} = 3.55\pi$.

Example 2 Consider a $\mathcal{CPFTG}$ as displayed in Fig. 2, defined by:

$$S = \Big\langle \Big(\frac{z_1}{0.02e^{i2\pi(0.01)}}, \frac{z_2}{0.02e^{i2\pi(0.01)}}, \frac{z_3}{0.03e^{i2\pi(0.3)}}, \frac{z_4}{0.07e^{i2\pi(0.1)}} \Big),$$

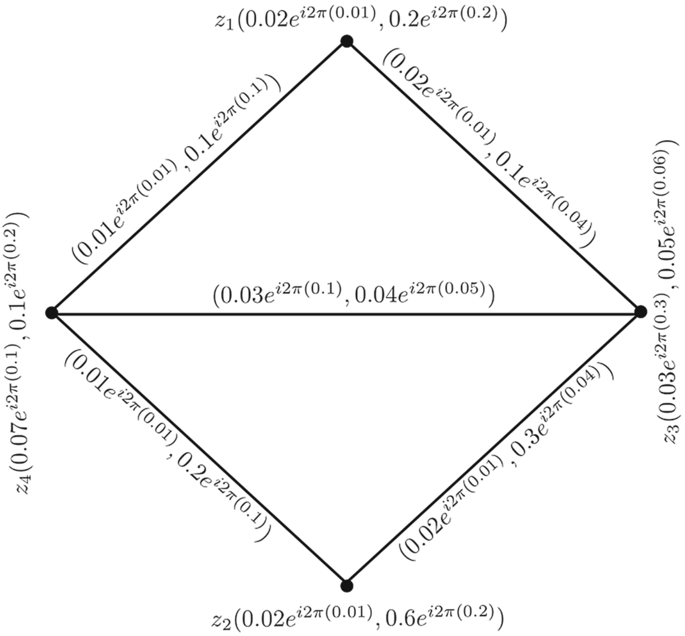

Fig. 2 Complex Pythagorean fuzzy threshold graph

$$T = \Big\langle \Big(\frac{z_1}{0.2e^{i2\pi(0.2)}}, \frac{z_2}{0.6e^{i2\pi(0.2)}}, \frac{z_3}{0.05e^{i2\pi(0.06)}}, \frac{z_4}{0.1e^{i2\pi(0.2)}}\Big)\Big\rangle,$$
$$\Big\langle\Big(\frac{z_1z_3}{0.02e^{i2\pi(0.01)}}, \frac{z_1z_4}{0.01e^{i2\pi(0.01)}}, \frac{z_3z_4}{0.03e^{i2\pi(0.1)}}, \frac{z_2z_3}{0.02e^{i2\pi(0.01)}}, \frac{z_2z_4}{0.01e^{i2\pi(0.01)}}\Big),$$
$$\Big(\frac{z_1z_3}{0.1e^{i2\pi(0.04)}}, \frac{z_1z_4}{0.1e^{i2\pi(0.1)}}, \frac{z_3z_4}{0.04e^{i2\pi(0.05)}}, \frac{z_2z_3}{0.3e^{i2\pi(0.04)}}, \frac{z_2z_4}{0.2e^{i2\pi(0.1)}}\Big)\Big\rangle.$$

In the following calculations, $\mathcal{K}_i's$ represent clique set (the set of vertices in which any two vertices are connected by an edge) and $\mathcal{I}_1$ represents an independent set in G^*.

We observe that:
for $\mathcal{I}_1 = \{z_1, z_2\}$,

$$\sum_{z_i\in\mathcal{I}_1} (\xi_S(z_i)) = 0.02 + 0.02 = 0.04 \leq 0.04,$$
$$\sum_{z_i\in\mathcal{I}_1} (\psi_S(z_i)) = 2\pi(0.01) + 2\pi(0.01) = 0.04\pi \leq 0.04\pi,$$
$$\sum_{z_i\in\mathcal{I}_1} (1 - \eta_S(z_i)) = (1 - 0.2) + (1 - 0.6) = 1.2 \leq 1.3,$$
$$\sum_{z_i\in\mathcal{I}_1} (2\pi - \phi_S(z_i)) = 2\pi - 2\pi(0.2) + 2\pi - 2\pi(0.2) = 3.2\pi < 3.55\pi.$$

for $\mathcal{K}_1 = \{z_1, z_3\}$,

$$\sum_{z_i \in \mathcal{K}_1} (\xi_S(z_i)) = 0.02 + 0.03 = 0.05 > 0.04,$$
$$\sum_{z_i \in \mathcal{K}_1} (\psi_S(z_i)) = 2\pi(0.3) + 2\pi(0.01) = 0.62\pi > 0.04\pi,$$
$$\sum_{z_i \in \mathcal{I}_1} (1 - \eta_S(z_i)) = (1 - 0.2) + (1 - 0.05) = 1.75 > 1.3,$$
$$\sum_{z_i \in \mathcal{I}_1} (2\pi - \phi_S(z_i)) = 2\pi - 2\pi(0.2) + 2\pi - 2\pi(0.06) = 3.48\pi < 3.55\pi.$$

for $\mathcal{K}_2 = \{z_1, z_4\}$,

$$\sum_{z_i \in \mathcal{K}_2} (\xi_S(z_i)) = 0.02 + 0.07 = 0.09 > 0.04,$$
$$\sum_{z_i \in \mathcal{K}_2} (\psi_S(z_i)) = 2\pi(0.01) + 2\pi(0.1) = 0.22\pi > 0.04\pi,$$
$$\sum_{z_i \in \mathcal{I}_1} (1 - \eta_S(z_i)) = (1 - 0.2) + (1 - 0.1) = 1.7 > 1.3,$$
$$\sum_{z_i \in \mathcal{I}_1} (2\pi - \phi_S(z_i)) = 2\pi - 2\pi(0.2) + 2\pi - 2\pi(0.2) = 3.2\pi < 3.55\pi.$$

for $\mathcal{K}_3 = \{z_3, z_4\}$,

$$\sum_{z_i \in \mathcal{K}_3} (\xi_S(z_i)) = 0.03 + 0.07 = 0.1 > 0.04,$$
$$\sum_{z_i \in \mathcal{K}_3} (\psi_S(z_i)) = 2\pi(0.3) + 2\pi(0.1) = 0.8\pi > 0.04\pi,$$
$$\sum_{z_i \in \mathcal{I}_1} (1 - \eta_S(z_i)) = (1 - 0.05) + (1 - 0.1) = 1.85 > 1.3,$$
$$\sum_{z_i \in \mathcal{I}_1} (2\pi - \phi_S(z_i)) = 2\pi - 2\pi(0.06) + 2\pi - 2\pi(0.2) = 3.48\pi < 3.55\pi.$$

for $\mathcal{K}_4 = \{z_2, z_3\}$,

$$\sum_{z_i \in \mathcal{K}_4} (\xi_S(z_i)) = 0.02 + 0.03 = 0.05 > 0.04,$$
$$\sum_{z_i \in \mathcal{K}_4} (\psi_S(z_i)) = 2\pi(0.01) + 2\pi(0.3) = 0.62\pi > 0.04\pi,$$
$$\sum_{z_i \in \mathcal{I}_1} (1 - \eta_S(z_i)) = (1 - 0.6) + (1 - 0.05) = 1.35 > 1.3,$$

$$\sum_{z_i \in \mathcal{I}_1} (2\pi - \phi_S(z_i)) = 2\pi - 2\pi(0.2) + 2\pi - 2\pi(0.06) = 3.48\pi < 3.55\pi.$$

for $\mathcal{K}_5 = \{z_2, z_4\}$,

$$\sum_{z_i \in \mathcal{K}_5} (\xi_S(z_i)) = 0.02 + 0.07 = 0.09 > 0.04,$$
$$\sum_{z_i \in \mathcal{K}_5} (\psi_S(z_i)) = 2\pi(0.01) + 2\pi(0.1) = 0.22\pi > 0.04\pi,$$
$$\sum_{z_i \in \mathcal{I}_1} (1 - \eta_S(z_i)) = (1 - 0.6) + (1 - 0.1) = 1.3 = 1.3,$$
$$\sum_{z_i \in \mathcal{I}_1} (2\pi - \phi_S(z_i)) = 2\pi - 2\pi(0.2) + 2\pi - 2\pi(0.2) = 3.2\pi < 3.55\pi.$$

for $\mathcal{K}_6 = \{z_2, z_3, z_4\}$,

$$\sum_{z_i \in \mathcal{K}_6} (\xi_S(z_i)) = 0.02 + 0.03 + 0.07 = 0.12 > 0.04,$$
$$\sum_{z_i \in \mathcal{K}_6} (\psi_S(z_i)) = 2\pi(0.01) + 2\pi(0.3) + 2\pi(0.1) = 0.82\pi > 0.04\pi,$$
$$\sum_{z_i \in \mathcal{I}_1} (1 - \eta_S(z_i)) = (1 - 0.6) + (1 - 0.1) + (1 - 0.05) = 2.25 > 1.3,$$
$$\sum_{z_i \in \mathcal{I}_1} (2\pi - \phi_S(z_i)) = 2\pi - 2\pi(0.2) + 2\pi - 2\pi(0.06) + 2\pi - 2\pi(0.2) = 5.08\pi > 3.55\pi.$$

for $\mathcal{K}_7 = \{z_1, z_3, z_4\}$,

$$\sum_{z_i \in \mathcal{K}_7} (\xi_S(z_i)) = 0.02 + 0.03 + 0.07 = 0.12 > 0.04,$$
$$\sum_{z_i \in \mathcal{K}_7} (\psi_S(z_i)) = 2\pi(0.01) + 2\pi(0.3) + 2\pi(0.1) = 0.82\pi > 0.04\pi,$$
$$\sum_{z_i \in \mathcal{I}_1} (1 - \eta_S(z_i)) = (1 - 0.2) + (1 - 0.05) + (1 - 0.1) = 2.65 > 1.3,$$
$$\sum_{z_i \in \mathcal{I}_1} (2\pi - \phi_S(z_i)) = 2\pi - 2\pi(0.2) + 2\pi - 2\pi(0.06) + 2\pi - 2\pi(0.2) = 5.08\pi > 3.55\pi.$$

We can see from calculations that for a clique set, necessarily one of the four threshold condition does not hold, since $\mathcal{K}$ is not an independent set.

Definition 4 Consider a $\mathcal{CPFG}$ $G = (S, T)$. The *vertex cardinality* of Z for amplitude and phase terms is defined as:

$$|Z|_{CPF} = \sum_{z \in Z} \frac{1 + \xi_S(z) - \eta_S(z)}{2},$$

$$|Z|_{CPF} = \sum_{z \in Z} \frac{2\pi + \psi_S(z) - \phi_S(z)}{2}.$$

Proposition 1 *Consider a $\mathcal{CPFTG}$, $G = (S, T; \hat{t_1}, \hat{t_2}, \theta_{t_1}, \theta_{t_2})$. Let $\mathcal{I} \subset Z$ represents an independent set in the crisp graph of G, then the following inequalities hold*

$$|Z|_{CPF} \leq \frac{1}{2}(\hat{t_1} + \hat{t_2}),$$
$$|Z|_{CPF} \leq \frac{1}{2}(\theta_{t_1} + \theta_{t_2}).$$

Proof As $G = (S, T; \hat{t_1}, \hat{t_2}, \theta_{t_1}, \theta_{t_2})$ represents a $\mathcal{CPFTG}$ with four threshold values $\hat{t_1}, \hat{t_2}, \theta_{t_1}$ and θ_{t_2} satisfying

$$\sum_{z \in \mathcal{I}} \xi_S(z) \leq \hat{t_1} \;\; and \;\; \sum_{z \in \mathcal{I}} (1 - \eta_S(z)) = m - \sum_{z \in \mathcal{I}} \eta_S(z) \leq \hat{t_2},$$
$$\sum_{z \in \mathcal{I}} \psi_S(z) \leq \theta_{t_1} \;\; and \;\; \sum_{z \in \mathcal{I}} (2\pi - \phi_S(z)) = m(2\pi) - \sum_{z \in \mathcal{I}} \phi_S(z) \leq \theta_{t_2}.$$

So, $\sum_{z \in \mathcal{I}} \eta_S(z) \geq m - \hat{t_2}$ and $\sum_{z \in \mathcal{I}} \phi_S(z) \geq m(2\pi) - \theta_{t_2}$, where m denotes the cardinality of $\mathcal{I}$. Thus, by Definition 4, we get the result

$$|\mathcal{I}|_{CPF} = \sum_{z \in \mathcal{I}} \frac{1 + \xi_S(z) - \eta_S(z)}{2} = \sum_{z \in \mathcal{I}} \frac{\xi_S(z)}{2} + \sum_{z \in \mathcal{I}} \frac{1 - \eta_S(z)}{2}$$
$$|\mathcal{I}|_{CPF} \leq \frac{1}{2}(\hat{t_1} + \hat{t_2}).$$

Also

$$|\mathcal{I}|_{CPF} = \sum_{z \in \mathcal{I}} \frac{2\pi + \psi_S(z) - \phi_S(z)}{2} = \sum_{z \in \mathcal{I}} \frac{\psi_S(z)}{2} + \sum_{z \in \mathcal{I}} \frac{2\pi - \phi_S(z)}{2}$$
$$|\mathcal{I}|_{CPF} \leq \frac{1}{2}(\theta_{t_1} + \theta_{t_2}).$$

□

Definition 5 A $\mathcal{CFG}$ $G = (S, T)$ on a non-empty set Z is called complex fuzzy threshold graph ($\mathcal{CFTG}$) if there exist $\hat{t_1} > 0$, $\theta_{t_1} > 0$, such that

$$\sum_{z \in \mathcal{I}} \xi_S(z) \leq \hat{t_1}, \quad \sum_{z \in \mathcal{I}} \psi_S(z) \leq \theta_{t_1}$$

if and only if $\mathcal{I} \subset Z$ is an independent set in G^*. We use the notation $G = (S, T; \hat{t_1}, \theta_{t_1})$ for $\mathcal{CFTG}$.

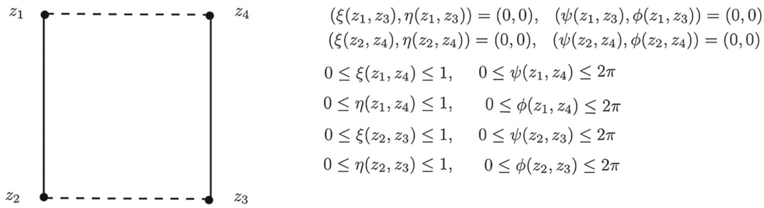

Fig. 3 Complex Pythagorean fuzzy alternating $4-\mathit{cycle}$

Proposition 2 *A $\mathcal{CFTG}$ is a special case of $\mathcal{CPFTG}$.*

Proof Let $G=(S,T;\hat{t_1},\theta_{t_1})$ be a $\mathcal{CFTG}$ on a non-empty set Z. According to the definition of $\mathcal{CFTG}$, it is clear that $\mathcal{I}\subset Z$ is an independent set in G^* if and only if there exist $\hat{t_1}>0,\ \theta_{t_1}>0$ such that

$$\sum_{z\in\mathcal{I}}\xi_S(z)\le\hat{t_1},\quad\sum_{z\in\mathcal{I}}\psi_S(z)\le\theta_{t_1}.$$

It is obvious that in the case of a $\mathcal{CFG}$ $\eta_S(z)=0$ and $\phi_S(z)=0$ for all $z\in\mathcal{I}$. Consider $\hat{t_2}=m$, then there exist $\hat{t_1}>0,\ \hat{t_2}>0,\ \theta_{t_1}>0$ and $\theta_{t_2}>0$ such that

$$\sum_{z\,\in\,\mathcal{I}}\xi_S(z)\le\hat{t_1},\quad\sum_{z\,\in\,\mathcal{I}}(1-\eta_S(z))\le\hat{t_2},$$
$$\sum_{z\,\in\,\mathcal{I}}\psi_S(z)\le\theta_{t_1},\quad\sum_{z\,\in\,\mathcal{I}}(2\pi-\phi_S(z))\le\theta_{t_2},$$

where m is the cardinality of an independent set $\mathcal{I}$. Therefore a $\mathcal{CFTG}$ $G=(S,T;\hat{t_1},\theta_{t_1})$ is a $\mathcal{CPFTG}$. □

Here we introduce the term complex Pythagorean fuzzy alternating $4-\mathit{cycle}$ as explained below.

Definition 6 Let $G=(S,T)$ be a $\mathcal{CPFG}$ on a non-empty set Z and $Z=\{z_1,z_2,z_3,z_4\}$. If the four conditions given below are satisfied:

(i) $(\xi_T(z_1,z_2),\eta_T(z_1,z_2))\ne(0,0),\quad(\psi_T(z_1,z_2),\phi_T(z_1,z_2))\ne(0,0)$
(ii) $(\xi_T(z_3,z_4),\eta_T(z_3,z_4))\ne(0,0),\quad(\psi_T(z_3,z_4),\phi_T(z_3,z_4))\ne(0,0)$
(iii) $(\xi_T(z_1,z_3),\eta_T(z_1,z_3))=(0,0),\quad(\psi_T(z_1,z_3),\phi_T(z_1,z_3))=(0,0)$
(iv) $(\xi_T(z_2,z_4),\eta_T(z_2,z_4))=(0,0),\quad(\psi_T(z_2,z_4),\phi_T(z_2,z_4))=(0,0)$,

then this construction by four vertices z_1,z_2,z_3,z_4 is called *complex Pythagorean fuzzy alternating* $4-\mathit{cycle}$ as displayed in Fig. 3.

Remark 3 This construction of complex Pythagorean fuzzy alternating $4-\mathit{cycle}$ can induce:

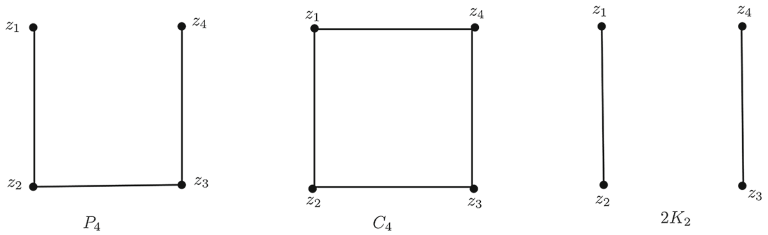

Fig. 4 Complex Pythagorean fuzzy induced subgraphs

1. a complex Pythagorean fuzzy path P_4 if

$$(\xi_T(z_1, z_4), \eta_T(z_1, z_4)) = (0, 0), \quad (\psi_T(z_1, z_4), \phi_T(z_1, z_4)) = (0, 0),$$
$$(\xi_T(z_2, z_3), \eta_T(z_2, z_3)) \neq (0, 0), \quad (\psi_T(z_2, z_3), \phi_T(z_2, z_3)) \neq (0, 0) \ \ or$$
$$(\xi_T(z_2, z_3), \eta_T(z_2, z_3)) = (0, 0), \quad (\psi_T(z_2, z_3), \phi_T(z_2, z_3)) = (0, 0),$$
$$(\xi_T(z_1, z_4), \eta_T(z_1, z_4)) \neq (0, 0), \quad (\psi_T(z_1, z_4), \phi_T(z_1, z_4)) \neq (0, 0).$$

2. a complex Pythagorean fuzzy square C_4 if

$$(\xi_T(z_1, z_4), \eta_T(z_1, z_4)) \neq (0, 0), \quad (\psi_T(z_1, z_4), \phi_T(z_1, z_4)) \neq (0, 0),$$
$$(\xi_T(z_2, z_3), \eta_T(z_2, z_3)) \neq (0, 0), \quad (\psi_T(z_2, z_3), \phi_T(z_2, z_3)) \neq (0, 0).$$

3. a complex Pythagorean fuzzy matching $2K_2$ if

$$(\xi_T(z_1, z_4), \eta_T(z_1, z_4)) = (0, 0) \quad (\psi_T(z_1, z_4), \phi_T(z_1, z_4)) = (0, 0),$$
$$(\xi_T(z_2, z_3), \eta_T(z_2, z_3)) = (0, 0) \quad (\psi_T(z_2, z_3), \phi_T(z_2, z_3)) = (0, 0).$$

All of these complex Pythagorean fuzzy induced subgraphs are depicted in Fig. 4.

Definition 7 A strong complex Pythagorean fuzzy alternating $4-cycle$ is a complex Pythagorean fuzzy alternating $4-cycle$ if complex Pythagorean fuzzy C_4 can be originated from it.

Theorem 1 *A* $\mathcal{CPFTG}$ $G = (S, T; \hat{t_1}, \hat{t_2}, \theta_{t_1}, \theta_{t_2})$ *does not contain a complex Pythagorean fuzzy alternating* $4-cycle$.

Proof Let $G = (S, T; \hat{t_1}, \hat{t_2}, \theta_{t_1}, \theta_{t_2})$ be a $\mathcal{CPFTG}$. Contrarily suppose that $\mathcal{CPFTG}$ contains a complex Pythagorean fuzzy alternating $4-cycle$. Then there exist four vertices z_1, z_2, z_3, and z_4 satisfying the conditions:

$$(\xi_T(z_1, z_2), \eta_T(z_1, z_2)) \neq (0, 0), \quad (\psi_T(z_1, z_2), \phi_T(z_1, z_2)) \neq (0, 0),$$
$$(\xi_T(z_3, z_4), \eta_T(z_3, z_4)) \neq (0, 0), \quad (\psi_T(z_3, z_4), \phi_T(z_3, z_4)) \neq (0, 0),$$
$$(\xi_T(z_1, z_3), \eta_T(z_1, z_3)) = (0, 0), \quad (\psi_T(z_1, z_3), \phi_T(z_1, z_3)) = (0, 0)$$

and

$$(\xi_T(z_2,z_4),\eta_T(z_2,z_4))=(0,0),\quad (\psi_T(z_2,z_4),\phi_T(z_2,z_4))=(0,0).$$

Since $G=(S,T;\hat{t_1},\hat{t_2},\theta_{t_1},\theta_{t_2})$, therefore

$$\xi_S(z_1)+\xi_S(z_2)>\hat{t_1},\quad \eta_S(z_1)+\eta_S(z_2)<2-\hat{t_2}, \tag{1}$$
$$\psi_S(z_1)+\psi_S(z_2)>\theta_{t_1},\quad \phi_S(z_1)+\phi_S(z_2)<4\pi-\theta_{t_2}, \tag{2}$$
$$\xi_S(z_1)+\xi_S(z_3)\leq\hat{t_1},\quad \eta_S(z_1)+\eta_S(z_3)\geq 2-\hat{t_2}, \tag{3}$$
$$\psi_S(z_1)+\psi_S(z_3)\leq\theta_{t_1},\quad \phi_S(z_1)+\phi_S(z_3)\geq 4\pi-\theta_{t_2}, \tag{4}$$
$$\xi_S(z_3)+\xi_S(z_4)>\hat{t_1},\quad \eta_S(z_3)+\eta_S(z_4)<2-\hat{t_2}, \tag{5}$$
$$\psi_S(z_3)+\psi_S(z_4)>\theta_{t_1},\quad \phi_S(z_3)+\phi_S(z_4)<4\pi-\theta_{t_2}, \tag{6}$$
$$\xi_S(z_2)+\xi_S(z_4)\leq\hat{t_1},\quad \eta_S(z_2)+\eta_S(z_4)\geq 2-\hat{t_2}, \tag{7}$$
$$\psi_S(z_2)+\psi_S(z_4)\leq\theta_{t_1},\quad \phi_S(z_2)+\phi_S(z_4)\geq 4\pi-\theta_{t_2}. \tag{8}$$

By solving Eqs. (1), (2), (3) and (4), we get the result

$$\xi_S(z_2)-\xi_S(z_3)>0,\quad \eta_S(z_2)-\eta_S(z_3)<0, \tag{9}$$
$$\psi_S(z_2)-\psi_S(z_3)>0,\quad \phi_S(z_2)-\phi_S(z_3)<0. \tag{10}$$

By solving Eqs. (5), (6), (7) and (8), we get the result

$$\xi_S(z_3)-\xi_S(z_2)>0,\quad \eta_S(z_3)-\eta_S(z_2)<0, \tag{11}$$
$$\psi_S(z_3)-\psi_S(z_2)>0,\quad \phi_S(z_3)-\phi_S(z_2)<0. \tag{12}$$

It is obvious that Eq. (9) contradicts Eq. 11 and Equation (10) contradicts Eq. (12). Hence we obtain the result. □

Definition 8 If a $\mathcal{CPFG}$ $G=(S,T)$ does not have induced cycles of length four or more, then it is called a *complex Pythagorean fuzzy triangulated graph*.

Remark 4 By Definition 8, $\mathcal{CPFTG}$, i.e., $G=(S,T;\hat{t_1},\hat{t_2},\theta_{t_1},\theta_{t_2})$ is a complex Pythagorean fuzzy triangulated graph.

Definition 9 A $\mathcal{CPFG}$ $G=(S,T)$ is called $\mathcal{SG}$ if it's crisp set of vertices can be partitioned into a clique set $\mathcal{K}$ and an independent set $\mathcal{I}$ in G^*.

Theorem 2 *The crisp graph $G^*=(Z,E)$ of a $\mathcal{CPFTG}$ $G=(S,T;\hat{t_1},\hat{t_2},\theta_{t_1},\theta_{t_2})$ is an $\mathcal{SG}$, i.e., $G^*=(\mathcal{K},\mathcal{I})$, where $\mathcal{K}$ represents a clique set and $\mathcal{I}$ represents an independent set in G^*.*

Proof Consider $G=(S,T;\hat{t_1},\hat{t_2},\theta_{t_1},\theta_{t_2})$ is a $\mathcal{CPFTG}$. Consider $\mathcal{K}$ denotes a maximum clique in G^*, then it is enough to show that $Z-\mathcal{K}$ forms an independent set in

G^*. Contrarily suppose that $Z - \mathcal{K}$ does not form an independent set, then there are two vertices which construct an arc (z_1, z_2) in $Z - \mathcal{K}$ such that

$$(\xi_T(z_1, z_2), \eta_T(z_1, z_2)) \neq (0, 0), \quad (\psi_T(z_1, z_2), \phi_T(z_1, z_2)) \neq (0, 0).$$

Since $\mathcal{K}$ represents a maximum clique set, therefore there are distinct nodes z_3, z_4 in $\mathcal{K}$ such that

$$(\xi_T(z_1, z_3), \eta_T(z_1, z_3)) = (0, 0), \quad (\psi_T(z_1, z_3), \phi_T(z_1, z_3)) = (0, 0)$$

also

$$(\xi_T(z_2, z_4), \eta_T(z_2, z_4)) = (0, 0), \quad (\psi_T(z_2, z_4), \phi_T(z_2, z_4)) = (0, 0).$$

Consequently the four vertices z_1, z_2, z_3, z_4 construct a complex Pythagorean fuzzy alternating $4 - cycle$ and by Theorem 1 contradiction holds. Therefore, $Z - \mathcal{K}$ constructs an independent set and G^* is an $\mathcal{SG}$, i.e., $G^* = (\mathcal{K}, \mathcal{I})$. □

Lemma 1 *The $\mathcal{SG}$ $G^* = (\mathcal{K}, \mathcal{I})$ of a $\mathcal{CPFTG}$ having cycles, contains an $m - cycle$ (of maximum length), which includes the whole $\mathcal{K}$. Here $m - cycle$ denotes the cycle of length m.*

Proof Consider $\mathcal{C}$ as an $m - cycle$. To prove the lemma, we consider two cases.

Case 1: If $\mathcal{C}$ contains no nodes of $\mathcal{I}$, then $\mathcal{C}$ necessarily have $\mathcal{K}$ since $\mathcal{K}$ represents a clique set and $\mathcal{C}$ represents a cycle of maximum length.

Case 2: Now, let $\mathcal{C}$ contains a node $z_i \in \mathcal{I}$ and let z_j, z_k be two nodes, which are connected to z_i in $\mathcal{C}$. As $\mathcal{I}$ is an independent set, so z_j and z_k lie in $\mathcal{K}$. z_j and z_k cannot lie in $\mathcal{I}$ since these vertices are connected to z_i, which lies in $\mathcal{I}$. If $\mathcal{C}$ prevents the occurrence of some nodes of $\mathcal{K}$, eliminate z_i and consider all which lie between z_j and z_k to get a cycle of length m, which is another $m - cycle$ including the whole $\mathcal{K}$.

□

Theorem 3 *If $G = (S, T; \hat{t_1}, \hat{t_2}, \theta_{t_1}, \theta_{t_2})$ is a $\mathcal{CPFTG}$, then its crisp graph $G^* = (Z, E)$ can be constructed from one vertex graph by repeatedly adding an isolated vertex (a vertex, which is not connected to any other vertex) or a dominating vertex (a vertex, which is connected to all other vertices).*

Proof By Theorem 2, the underlying graph of $\mathcal{CPFTG}$ $G = (S, T; \hat{t_1}, \hat{t_2}, \theta_{t_1}, \theta_{t_2})$ is a $\mathcal{SG}$, namely, $G^* = (\mathcal{K}, \mathcal{I})$. It is sufficient to prove that the underlying graph $G^* = (Z, E)$ of $G = (S, T; \hat{t_1}, \hat{t_2}, \theta_{t_1}, \theta_{t_2})$ contains an isolated node or a dominating node. Then G^* can be established by repeatedly adding an isolated node or dominating node. If a node of $G^* = (Z, E)$ is deleted, then the graph still remains an $\mathcal{SG}$, i.e., $G^* = (\mathcal{K}, \mathcal{I})$. We show that in an $\mathcal{SG}$, the clique set $\mathcal{K}$ contains dominating nodes and independent set $\mathcal{I}$ contains isolated nodes. Let $G^* = (\mathcal{K}, \mathcal{I})$ has an independent set denoted by $\mathcal{I}$. If $\mathcal{I}$ is non-empty and includes only isolated vertices, then the conclusion satisfies. If $\mathcal{I}$ have no isolated nodes, then the node $z \in \mathcal{I}$ with the smallest neighborhood has some neighbor $x \in \mathcal{K}$. Since $\mathcal{K}$ represents the largest clique, so the node x is dominating node of G. Hence G^* can be constructed by repeatedly adding an isolated node from $\mathcal{I}$ and dominating node from $\mathcal{K}$. □

Theorem 4 *The complement* $\bar{\mathbb{G}}^*$ *of a strong* $\mathcal{CPFTG}$ $G = (S, T; \hat{t_1}, \hat{t_2}, \theta_{t_1}, \theta_{t_2})$ *is also a strong* $\mathcal{CPFTG}$.

Proof As the crisp graph of $\mathcal{CPFTG}$ is $\mathcal{SG}$, that is, $G^* = (\mathcal{K}, \mathcal{I})$. By taking complement of $\mathcal{CPFTG}$, the independent set in G^* becomes clique set in $\bar{G}^*$ and clique set in G^* becomes an independent set in $\bar{G}^*$. So $\bar{G}^*$ is also an $\mathcal{SG}$, that is, $\bar{G}^* = (\bar{\mathcal{K}}, \bar{\mathcal{I}})$. To show that $\bar{G}^*$ is also a $\mathcal{CPFTG}$, it is sufficient to prove that it can be constructed by repeatedly adding an isolated node or a dominating node. If a node of $\bar{G}^*$ is deleted, then the graph still remains $\mathcal{SG}$, i.e., $\bar{G}^* = (\bar{\mathcal{K}}, \bar{\mathcal{I}})$. Let $\bar{G}^* = (\bar{\mathcal{K}}, \bar{\mathcal{I}})$ has an independent set denoted by $\bar{\mathcal{I}}$. If $\bar{\mathcal{I}}$ is non-empty and includes isolated nodes only, then the conclusion satisfies. If $\bar{\mathcal{I}}$ contains no isolated nodes, then the node $z \in \bar{\mathcal{I}}$ with the smallest neighborhood has some neighbor $x \in \bar{\mathcal{K}}$. Since $\bar{\mathcal{K}}$ represents a maximum clique, so the node x is dominating node of $\bar{G}^*$. This completes the proof. □

Partitioning of degrees is a very significant term. Here we introduce partition of degrees of an underlying graph $G^* = (Z, E)$ of a $\mathcal{CPFTG}$, i.e., $G = (S, T; \hat{t_1}, \hat{t_2}, \theta_{t_1}, \theta_{t_2})$.

Definition 10 Consider $G = (S, T; \hat{t_1}, \hat{t_2}, \theta_{t_1}, \theta_{t_2})$ as a $\mathcal{CPFTG}$, where $G^* = (Z, E)$ represents an underlying graph of $\mathcal{CPFTG}$. The distinct positive vertex degrees of $G^* = (Z, E)$ are $\gamma_1 < \gamma_2 < \ldots < \gamma_s$ and let $\gamma_0 = 0$ (even if there are no isolated nodes), $\gamma_{s+1} = |Z| - \gamma_1$. Let $\mathcal{D}_g = \{z_i \in Z : g \leq d(z_i) < g + 1\}$ for non negative integer $g \leq s$. The sequence $\mathcal{D}_0$, $\mathcal{D}_1$, ... ,$\mathcal{D}_s$ is known as partitioning of degrees of the graph $G^* = (Z, E)$.

For $z_1 \in \mathcal{D}_g$ and $z_2 \in \mathcal{D}_h$, z_1 is connected to z_2 if and only if $g + h > s$. A straight line between $\mathcal{D}_g$ and $\mathcal{D}_h$ shows that each node of $\mathcal{D}_g$ is connected to each node of $\mathcal{D}_h$. The nodes contained in an oval construct a clique. Fig. 5, explains this with $s = 7$, where s is the maximum degree of G^*, i.e., $s = \triangle(G^*)$. For each $z \in \mathcal{D}_i$,

$$\mathcal{N}(z) = \bigcup_{h=1}^{i} \mathcal{D}_{s+1-h} \qquad i = 1, \ldots, \left\lfloor \frac{s}{2} \right\rfloor$$

$$\mathcal{N}[z] = \bigcup_{h=1}^{i} \mathcal{D}_{s+1-h} \qquad i = \left\lfloor \frac{s}{2} \right\rfloor + 1, \ldots, s$$

Lemma 2 *Consider* $G^* = (Z, E)$ *represents an underlying graph of a* $\mathcal{CPFTG}$ $G = (S, T; \hat{t_1}, \hat{t_2}, \theta_{t_1}, \theta_{t_2})$ *with degree partition* $\mathcal{D}_0, \mathcal{D}_1, \ldots, \mathcal{D}_s$ *and let* $z_1 \in \mathcal{D}_g$, $z_2 \in \mathcal{D}_h$ *represent distinct nodes.*

1. *Let* $e = z_1z_2 \in E$, *then* $g + h = s + 1$ *if and only if the graph* $G^* - e$ *acquired by deleting e from* G^* *is a* $\mathcal{TG}$.

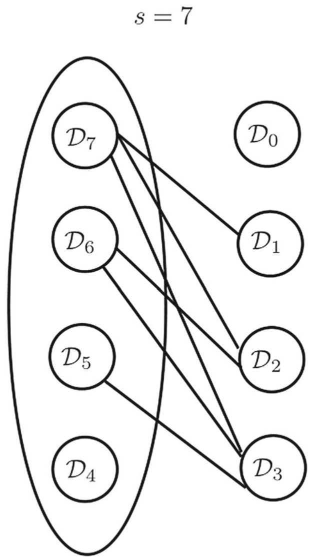

Fig. 5 Illustration of partition of degrees

2. *Let* $e = z_1 z_2 \in E$, *then* $g + h = s$ *if and only if the graph* $G^* + e$ *acquired by inserting* e *in* G^* *is a* $\mathcal{TG}$.

Remark 5 Adding and deleting an isolated or a dominating vertex from a $\mathcal{CPFTG}$ $G = (S, T; \hat{t_1}, \hat{t_2}, \theta_{t_1}, \theta_{t_2})$ results in a $\mathcal{CPFTG}$.

We explain the above mentioned remark with the help of an example.

Example 3 Take a $\mathcal{CPFTG}$ $G = (S, T; \hat{t_1}, \hat{t_2}, \theta_{t_1}, \theta_{t_2})$ as depicted in Fig. 6, defined by:

$$S = \Big\langle \Big(\frac{z_1}{0.8e^{i2\pi(0.2)}}, \frac{z_2}{0.01e^{i2\pi(0.01)}}, \frac{z_3}{0.7e^{i2\pi(0.05)}}, \frac{z_4}{0.2e^{i2\pi(0.03)}}, \frac{z_5}{0.5e^{i2\pi(0.02)}} \Big), \Big(\frac{z_1}{0.2e^{i2\pi(0.1)}}, \frac{z_2}{0.3e^{i2\pi(0.04)}}, \frac{z_3}{0.3e^{i2\pi(0.2)}}, \frac{z_4}{0.4e^{i2\pi(0.3)}}, \frac{y_5}{0.7e^{i2\pi(0.1)}} \Big) \Big\rangle,$$

$$T = \Big\langle \Big(\frac{z_1z_2}{0.01e^{i\pi(0.01)}}, \frac{z_1z_3}{0.1e^{i\pi(0.04)}}, \frac{z_1z_4}{0.01e^{i\pi(0.02)}}, \frac{z_1z_5}{0.3e^{i\pi(0.02)}}, \frac{z_3z_4}{0.01e^{i\pi(0.02)}}, \frac{z_3z_5}{0.1e^{i\pi(0.02)}} \Big), \Big(\frac{z_1z_2}{0.2e^{i\pi(0.1)}}, \frac{z_1z_3}{0.3e^{i\pi(0.1)}}, \frac{z_1z_4}{0.3e^{i\pi(0.2)}}, \frac{z_1z_5}{0.3e^{i\pi(0.1)}}, \frac{z_3z_4}{0.4e^{i\pi(0.3)}}, \frac{z_3z_5}{0.7e^{i\pi(0.2)}} \Big) \Big\rangle.$$

The following Fig. 7 explains that adding a vertex to a $\mathcal{CPFTG}$ displayed in Fig. 6, results in a $\mathcal{CPFTG}$.

The following Fig. 8 explains that removing a vertex from a $\mathcal{CPFTG}$ displayed in Fig. 6, results in a $\mathcal{CPFTG}$.

Definition 11 The minimum number l of complex Pythagorean fuzzy threshold subgraphs, say, $G_1, G_2, \ldots, G_l$ of a $\mathcal{CPFG}$ $G = (S, T)$, which cover arcs set of G is called *threshold dimension* of G, denoted by $\hat{t}(G)$. That is, if $\tilde{G} = G_1 \cup G_2 \cup, \ldots, \cup G_l$, then $G = \tilde{G}$.

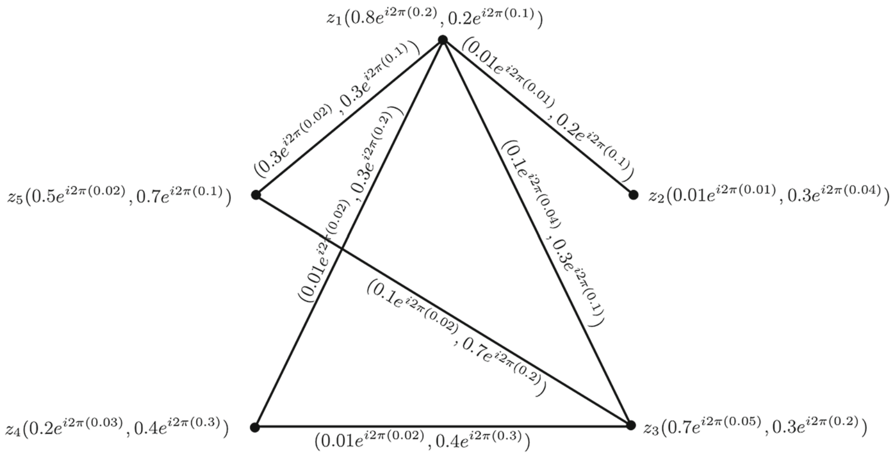

Fig. 6 A $\mathcal{CPFTG}$ having $\hat{t_1} = 0.71$, $\hat{t_2} = 1.6$, $\theta_{t_1} = 0.12\pi$, $\theta_{t_2} = 5.14\pi$

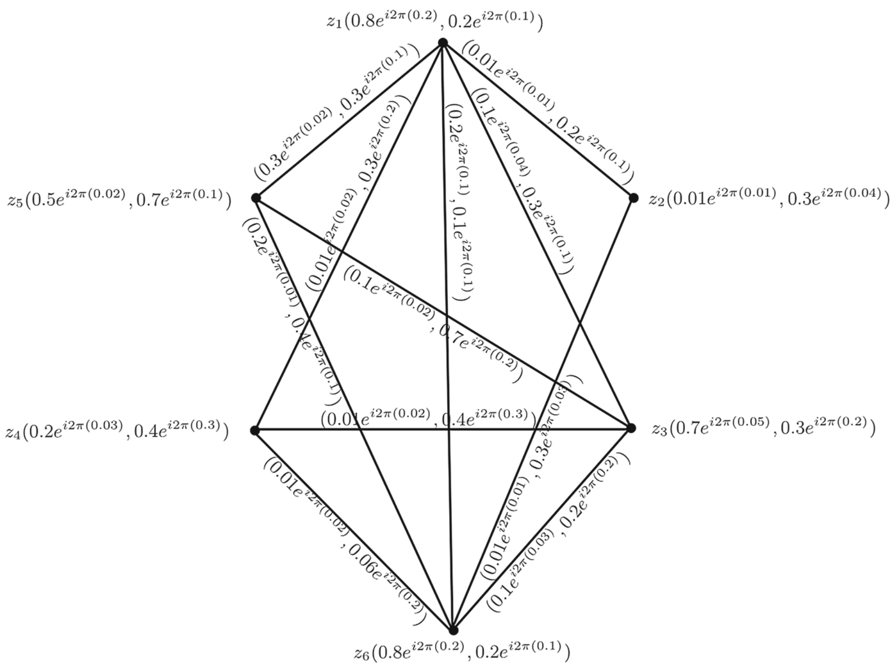

Fig. 7 $G + z_6$ having $\hat{t_1} = 0.71$, $\hat{t_2} = 1.6$, $\theta_{t_1} = 0.12\pi$, $\theta_{t_2} = 5.14\pi$

In the following, we discuss an example to explain the term of threshold dimension.

Example 4 Let's take a $\mathcal{CPFG}$ shown in Fig. 9, defined by:

$$S = \Big\langle \Big(\frac{z_1}{0.7e^{i2\pi(0.3)}}, \frac{z_2}{0.3e^{i2\pi(0.02)}}, \frac{z_3}{0.4e^{i2\pi(0.2)}}, \frac{z_4}{0.2e^{i2\pi(0.01)}},$$

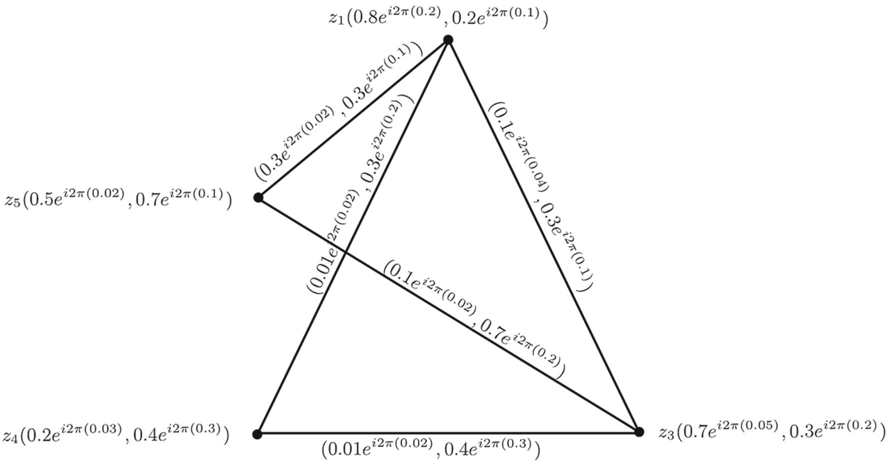

Fig. 8 $G - z_2$ having $\hat{t_1} = 0.7$, $\hat{t_2} = 0.9$, $\theta_{t_1} = 0.1\pi$, $\theta_{t_2} = 3.21\pi$

$$
\begin{aligned}
&\frac{z_5}{0.3e^{i2\pi(0.02)}}, \frac{z_1'}{0.7e^{i2\pi(0.3)}}, \frac{z_2'}{0.3e^{i2\pi(0.02)}}, \frac{z_5'}{0.3e^{i2\pi(0.02)}}\Bigg), \\
&\Bigg(\frac{z_1}{0.2e^{i2\pi(0.04)}}, \frac{z_2}{0.2e^{i2\pi(0.2)}}, \frac{z_3}{0.4e^{i2\pi(0.1)}}, \frac{z_4}{0.5e^{i2\pi(0.3)}}, \\
&\frac{z_5}{0.6e^{i2\pi(0.2)}}, \frac{z_1'}{0.2e^{i2\pi(0.04)}}, \frac{z_2'}{0.2e^{i2\pi(0.2)}}, \frac{z_5'}{0.6e^{i2\pi(0.2)}}\Bigg)\Bigg\rangle, \\
T = \Bigg\langle\Bigg(&\frac{z_1z_2}{0.2e^{i\pi(0.02)}}, \frac{z_1z_3}{0.3e^{i\pi(0.1)}}, \frac{z_1z_4}{0.2e^{i\pi(0.01)}}, \frac{z_1z_5}{0.2e^{i\pi(0.01)}}, \\
&\frac{z_2z_3}{0.2e^{i\pi(0.01)}}, \frac{z_2z_5}{0.1e^{i\pi(0.01)}}, \frac{z_3z_4}{0.1e^{i\pi(0.01)}}, \frac{z_3z_5}{0.3e^{i\pi(0.01)}}, \\
&\frac{z_1'z_2'}{0.2e^{i\pi(0.02)}}, \frac{z_1'z_3}{0.3e^{i\pi(0.1)}}, \frac{z_1'z_4}{0.2e^{i\pi(0.01)}}, \frac{z_1'z_5'}{0.2e^{i\pi(0.01)}}, \\
&\frac{z_2'z_3}{0.2e^{i\pi(0.01)}}, \frac{z_2'z_5'}{0.1e^{i\pi(0.01)}}, \frac{z_3z_5'}{0.3e^{i\pi(0.01)}}\Bigg), \\
&\Bigg(\frac{z_1z_2}{0.1e^{i\pi(0.1)}}, \frac{z_1z_3}{0.3e^{i\pi(0.03)}}, \frac{z_1z_4}{0.4e^{i\pi(0.2)}}, \frac{z_1z_5}{0.4e^{i\pi(0.03)}}, \\
&\frac{z_2z_3}{0.4e^{i\pi(0.2)}}, \frac{z_2z_5}{0.5e^{i\pi(0.1)}}, \frac{z_3z_4}{0.4e^{i\pi(0.2)}}, \\
&\frac{z_3z_5}{0.4e^{i\pi(0.2)}}, \frac{z_1'z_2'}{0.1e^{i\pi(0.1)}}, \frac{z_1'z_3}{0.3e^{i\pi(0.03)}}, \frac{z_1'z_4}{0.4e^{i\pi(0.2)}}, \\
&\frac{z_1'z_5'}{0.4e^{i\pi(0.03)}}, \frac{z_2'z_3}{0.4e^{i\pi(0.2)}}, \frac{z_2'z_5'}{0.5e^{i\pi(0.1)}}, \frac{z_3z_5'}{0.4e^{i\pi(0.2)}}\Bigg)\Bigg\rangle.
\end{aligned}
$$

Two complex Pythagorean fuzzy threshold subgraphs with common edge are depicted in Fig. 10.

Theorem 5 *For each* $\mathcal{CPFTG}$ $G = (S, T)$ *on* n *vertices, we have* $\hat{t}(G) \leq (n - \alpha(G^*))$. *Further,* $\hat{t}(G) = (n- \mid supp(\mathcal{I}) \mid)$ *if* $G = (S, T)$ *does not contain a triangle.*

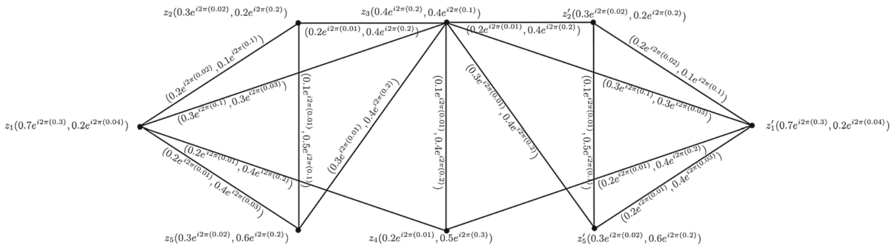

Fig. 9 A $\mathcal{CPFG}$ having threshold dimension 2

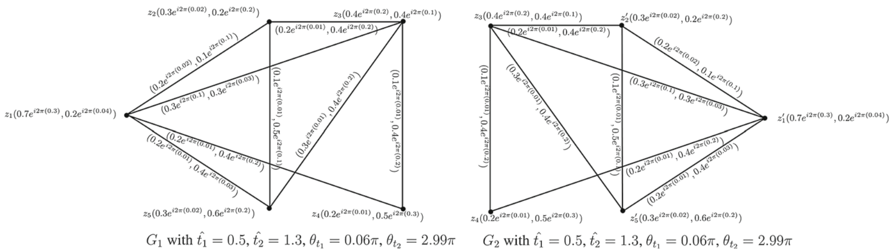

Fig. 10 Complex Pythagorean fuzzy threshold subgraphs

Here $\mathcal{I}$ represents an independent set with the largest size and $\alpha(G^*)$ represents the cardinality of an independent set.

Proof Let $\mathcal{I}$ be an independent set with the largest cardinality in G^* and $G^* = (Z, E)$ with n nodes. Consider a star which is centered at each z, where $z \in Z - \mathcal{I}$. Every star can be considered as a complex Pythagorean fuzzy threshold subgraph. If one or more weak fuzzy arcs are added to the independent set of stars then they still construct a $\mathcal{CPFTG}$. The edge set of G is covered by all of these stars along with weak arcs of independent sets. Therefore $\hat{t}(G) \leq |Z - \mathcal{I}|$. As we notice that $|Z| = n$ and $\alpha(G^*) \leq |\mathcal{I}|$, Z, $\mathcal{I}$ being the crisp sets. So $\hat{t}(G) \leq (n - \alpha(G^*))$.
we observe that $|\mathcal{I}| = |supp(\mathcal{I})|$. Thus $\hat{t}(\mathcal{G}) \leq (n - |supp(\mathcal{I})|)$. If in addition, $G = (S, T)$ does not contain complex Pythagorean fuzzy triangle, then each $\mathcal{CPFTG}$ constructs a star or a star along with weak arcs. Therefore, $\hat{t}(G) \geq (n - |supp(\mathcal{I})|)$. So proved that $\hat{t}(G) = (n - |supp(\mathcal{I})|)$. □

Definition 12 The minimum number ρ of complex Pythagorean fuzzy threshold subgraphs, say, $G_1, G_2, \ldots, G_\rho$ of a $\mathcal{CPFG}$ $G = (S, T)$, covering arcs set of G and do not have common arcs is called *threshold partition number* of G, denoted by $\hat{t}_p(G)$.

Remark 6 For a $\mathcal{CPFG}$, $\hat{t}(G) \leq \hat{t}_p(G) \leq |E(G)|$. $E(G)$ represents the edges of G.

In the following, we give an example of a $\mathcal{CPFG}$ whose threshold partition number is 3.

Example 5 Consider a $\mathcal{CPFG}$ displayed in Fig. 11, defined by:

$$S = \Big\langle \Big(\frac{z_1}{0.1e^{i2\pi(0.01)}}, \frac{z_2}{0.3e^{i2\pi(0.04)}}, \frac{z_3}{0.3e^{i2\pi(0.04)}}, \frac{z_4}{0.6e^{i2\pi(0.07)}},$$

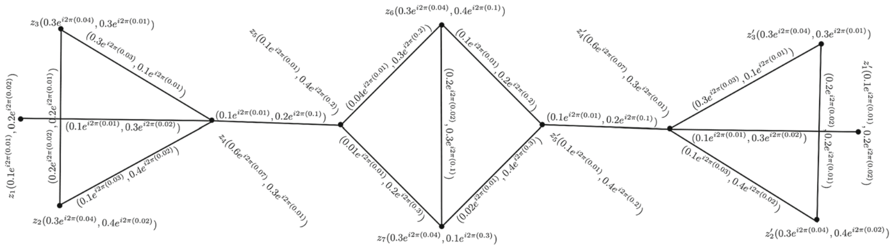

Fig. 11 A $\mathcal{CPFG}$ having threshold partition number 3

$$\frac{z_5}{0.1e^{i2\pi(0.01)}}, \frac{z_6}{0.3e^{i2\pi(0.04)}}, \frac{z_7}{0.3e^{i2\pi(0.04)}},$$
$$\frac{z'_5}{0.1e^{i2\pi(0.01)}}, \frac{z'_4}{0.6e^{i2\pi(0.07)}}, \frac{z'_3}{0.3e^{i2\pi(0.04)}}, \frac{z'_2}{0.3e^{i2\pi(0.04)}}, \frac{z'_1}{0.1e^{i2\pi(0.01)}}\Bigg),$$
$$\Bigg(\frac{z_1}{0.2e^{i2\pi(0.02)}}, \frac{z_2}{0.4e^{i2\pi(0.02)}}, \frac{z_3}{0.3e^{i2\pi(0.01)}}, \frac{z_4}{0.3e^{i2\pi(0.01)}},$$
$$\frac{z_5}{0.4e^{i2\pi(0.2)}}, \frac{z_6}{0.4e^{i2\pi(0.1)}}, \frac{z_7}{0.1e^{i2\pi(0.3)}},$$
$$\frac{z'_5}{0.4e^{i2\pi(0.2)}}, \frac{z'_4}{0.3e^{i2\pi(0.01)}}, \frac{z'_3}{0.3e^{i2\pi(0.01)}}, \frac{z'_2}{0.4e^{i2\pi(0.02)}}, \frac{z'_1}{0.2e^{i2\pi(0.02)}}\Bigg)\Bigg\rangle,$$
$$T = \Bigg\langle\Bigg(\Bigg(\frac{z_1z_4}{0.1e^{i\pi(0.01)}}, \frac{z_2z_3}{0.2e^{i\pi(0.02)}}, \frac{z_2z_4}{0.1e^{i\pi(0.03)}}, \frac{z_3z_4}{0.3e^{i\pi(0.03)}}, \frac{z_4z_5}{0.1e^{i\pi(0.01)}},$$
$$\frac{z_5z_6}{0.04e^{i\pi(0.01)}}, \frac{z_5z_7}{0.1e^{i\pi(0.01)}}, \frac{z_6z_7}{0.2e^{i\pi(0.02)}},$$
$$\frac{z_6z'_5}{0.1e^{i\pi(0.01)}}, \frac{z_7z'_5}{0.02e^{i\pi(0.01)}}, \frac{z'_4z'_5}{0.1e^{i\pi(0.01)}}, \frac{z'_2z'_4}{0.1e^{i\pi(0.03)}},$$
$$\frac{z'_3z'_4}{0.3e^{i\pi(0.03)}}, \frac{z'_2z'_3}{0.2e^{i\pi(0.02)}}, \frac{z'_1z'_4}{0.1e^{i\pi(0.01)}}\Bigg),$$
$$\Bigg(\frac{z_1z_4}{0.3e^{i\pi(0.02)}}, \frac{z_2z_3}{0.2e^{i\pi(0.01)}}, \frac{z_2z_4}{0.4e^{i\pi(0.02)}}, \frac{z_3z_4}{0.1e^{i\pi(0.01)}},$$
$$\frac{z_4z_5}{0.2e^{i\pi(0.1)}}, \frac{z_5z_6}{0.3e^{i\pi(0.2)}}, \frac{z_5z_7}{0.2e^{i\pi(0.3)}}, \frac{z_6z_7}{0.3e^{i\pi(0.1)}},$$
$$\frac{z_6z'_5}{0.2e^{i\pi(0.2)}}, \frac{z_7z'_5}{0.4e^{i\pi(0.3)}}, \frac{z'_4z'_5}{0.2e^{i\pi(0.1)}}, \frac{z'_2z'_4}{0.4e^{i\pi(0.02)}},$$
$$\frac{z'_3z'_4}{0.1e^{i\pi(0.01)}}, \frac{z'_2z'_3}{0.2e^{i\pi(0.01)}}, \frac{z'_1z'_4}{0.3e^{i\pi(0.02)}}\Bigg)\Bigg\rangle.$$

Complex Pythagorean fuzzy threshold subgraphs of $\mathcal{CPFG}$ depicted in Fig. 11, are shown in Fig. 12.

Theorem 6 *If $G = (S, T)$ is a $\mathcal{CPFG}$, which does not contain triangle, then $\hat{t}(G) = \hat{t}_p(G)$.*

Proof Suppose a $\mathcal{CPFG}$ $G = (S, T)$. We show that the edge set of $G = (S, T)$ can be covered by $\hat{t}(G)$ number of stars. Since $G = (S, T)$ is triangle free and according

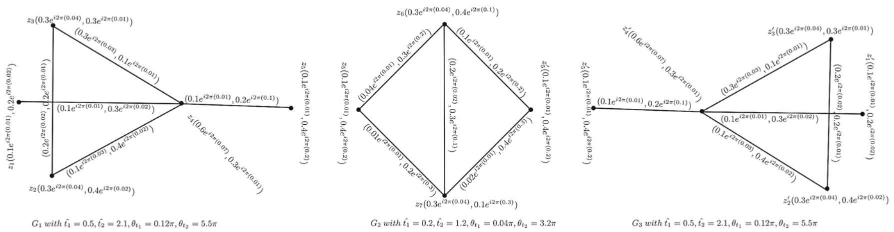

Fig. 12 Complex Pythagorean fuzzy threshold subgraphs

to the Definition 11, complex Pythagorean fuzzy threshold subgraphs cover the edge set of $G = (S, T)$. Therefore, edge set of $G = (S, T)$ can be covered by $\hat{t}(G)$ number of stars. If an edge is connected to more than one star, then remove it from all of these stars except one. So, this provides threshold partition number of size $\hat{t}(G)$ of a $\mathcal{CPFG}$. Hence proved that $\hat{t}(G) = \hat{t}_p(G)$. □

4 Application

$\mathcal{CPFTG}_s$ play a vital role in solving resource allocation problems. In solving resource allocation problem, $\mathcal{CPFG}$ is partitioned into complex Pythagorean fuzzy threshold subgraphs such that complex Pythagorean fuzzy threshold subgraphs cover the edge set of $\mathcal{CPFG}$ and threshold partition number of $\mathcal{CPFG}$ is equal to the number of resources in the complex Pythagorean fuzzy model if $\mathcal{CPFG}$ is free from triangle. To explore complex Pythagorean fuzzy threshold subgraphs from a $\mathcal{CPFG}$ is not easy. To avoid this difficulty, we apply Theorem 6 to a $\mathcal{CPFG}$ if it does not contain a triangle. According to Theorem 6 $\hat{t}(G) = \hat{t}_p(G) = n - \alpha(G^*)$, where $\alpha(G^*)$ represents cardinality of maximum independent set and n represents cardinality of vertex set of G^*.

We present an Algorithm 1 to find complex Pythagorean fuzzy threshold subgraphs from a $\mathcal{CPFG}$, which does not contain triangle.

4.1 An application of complex Pythagorean fuzzy threshold graphs in petroleum replenishment problem

In modern day life, petroleum is very necessary for many human needs. Petroleum is the world's most important source of energy. In the production of everyday essentials, petroleum plays an important role. Many products of petroleum are used in our daily life like shopping bags, fabrics, credit cards, roofing tiles, insulating materials, petrol and diesel for running vehicles, fertilizers, pesticides, lubricating oil, gasoline etc. In this modern era, people are addicted to many petroleum products. To fulfill people's need, it is necessary to provide enough petrol in cities.

Consider there are two resources of petroleum say, P_1 and P_2, and suppose there are six towns z_1, z_2, z_3, z_4, z_5 and z_6, which are connected to these resources for getting enough petroleum. Everyday the quantity of petroleum varies. Petroleum is provided

Algorithm 1 An algorithm for searching complex Pythagorean fuzzy threshold subgraphs

Input: A $\mathcal{CPFG}$ $G = (S, T)$, which does not contain triangle and it's crisp graph $G^* = (Z, E)$.
Output: Complex Pythagorean fuzzy threshold subgraphs $G_1, G_2, \ldots, G_u$, where $u = \hat{t}_p(G) = n - \alpha(G^*)$, $\alpha(G^*)$ denotes the number of vertices in maximum independent set of G^* and n represents cardinality of vertex set of G^*.
Step 1: Initialize $Z_1 = Z$, $\mathbb{V} = \phi$,
$\dot{G} = G$, $\grave{Z} = \phi$, $k = 1$, $l = 1$;
Step 2: Select any node $z_k \in Z_k$, and explore a maximal $\mathcal{CPFTG}$ G_l having z_k from $\dot{G}$;
Step 3: Calculate $\mathbb{V} = Z \cup \{z_1\}$, $\dot{G} = G - \bigcup_{l=1}^{l} E(G_l)$ and explore all isolated nodes $\grave{Z}$ from $\dot{G}$; $E(G_l)$ is the set of arcs of G_l;
Step 4: $k = k + 1$, $Z_k = Z - \grave{Z}$;
Step 5: If $E(\dot{G}) = \phi$, and $k = u$, then move to step 6;
Else if $E(\dot{G}) \neq \phi$, then $l = l + 1$, and move to step 2;
Else if $E(\dot{G}) = \phi$, and $k \neq u$, then $k = 1$, $Z_k = Z - \mathbb{V}$ and move to step 2.
End if
Step 6: Output $G_1, G_2, \ldots, G_u$.

by pipelines to all these towns. There exist some problems about pipeline corrosion, weld material failure, equipment failure, evaporation, leakage, sizing , pressure of pipelines etc. All of these issues affect the quantity of petroleum needed in each town. In 1920, when first pipeline construction began, pipelines were not so modern and many above mentioned issues were associated with them. With the passage of time, a lot of development has been made in pipeline construction. Many technologies are used to test leakage, safety early warning loop, ultrasonic optical fibre, internal pipeline testing technology, pipeline corrosion prevention technology, flow assurance etc. In earlier days, pipelines were not so developed and technologies based. The framework of petroleum resources and towns is modeled as a $\mathcal{CPFG}$ $G = (S, T)$ as shown in Fig. 13, where six towns and two resources are represented by vertices and pipelines are represented by edges, which connect towns and resources. Each vertex has membership grades and non-membership grades.

$$S = \Big\langle \Big(\frac{z_1}{0.05e^{i2\pi(0.02)}}, \frac{z_2}{0.1e^{i2\pi(0.02)}}, \frac{z_3}{0.3e^{i2\pi(0.05)}}, \frac{z_4}{0.2e^{i2\pi(0.02)}},
\frac{z_5}{0.05e^{i2\pi(0.01)}}, \frac{z_6}{0.1e^{i2\pi(0.04)}}, \frac{P_1}{0.91e^{i2\pi(0.3)}},
\frac{P_2}{0.85e^{i2\pi(0.35)}} \Big), \Big(\frac{z_1}{0.3e^{i2\pi(0.1)}}, \frac{z_2}{0.4e^{i2\pi(0.04)}}, \frac{z_3}{0.4e^{i2\pi(0.1)}}, \frac{z_4}{0.4e^{i2\pi(0.2)}},
\frac{z_5}{0.2e^{i2\pi(0.2)}}, \frac{z_6}{0.5e^{i2\pi(0.3)}},
\frac{P_1}{0.03e^{i2\pi(0.1)}}, \frac{P_2}{0.04e^{i2\pi(0.02)}} \Big) \Big\rangle,$$

$$T = \Big\langle \Big(\frac{P_1 z_6}{0.1e^{i\pi(0.03)}}, \frac{P_1 z_1}{0.04e^{i\pi(0.01)}}, \frac{P_1 z_2}{0.05e^{i\pi(0.01)}}, \frac{P_1 z_3}{0.2e^{i\pi(0.03)}},
\frac{P_2 z_2}{0.1e^{i\pi(0.01)}}, \frac{P_2 z_3}{0.2e^{i\pi(0.02)}}, \frac{P_2 z_4}{0.1e^{i\pi(0.02)}},$$

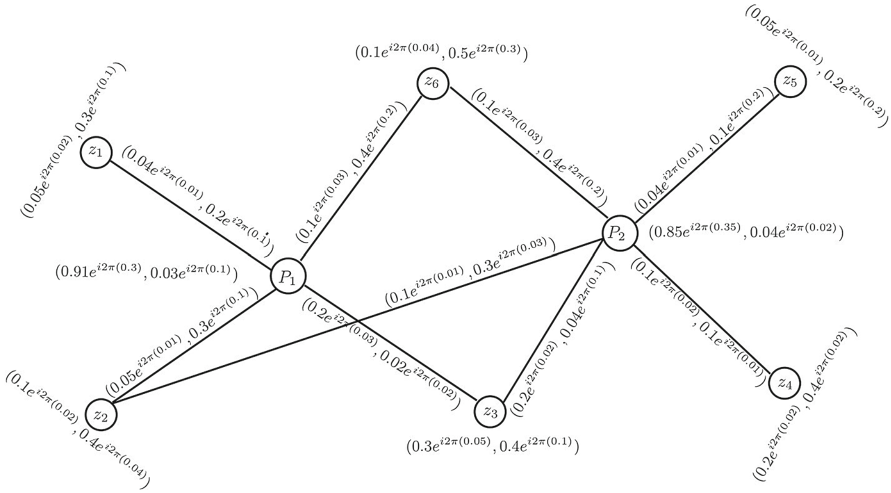

Fig. 13 A $\mathscr{CPFG}$ representing petroleum network

$$\frac{P_2z_5}{0.04e^{i\pi(0.01)}}, \frac{P_2z_6}{0.1e^{i\pi(0.03)}}\Bigg), \Bigg(\frac{P_1z_6}{0.4e^{i\pi(0.2)}}, \frac{P_1z_1}{0.2e^{i\pi(0.1)}}, \frac{P_1z_2}{0.3e^{i\pi(0.1)}},$$
$$\frac{P_1z_3}{0.02e^{i\pi(0.02)}}, \frac{P_2z_2}{0.3e^{i\pi(0.03)}},$$
$$\frac{P_2z_3}{0.04e^{i\pi(0.1)}}, \frac{P_2z_4}{0.1e^{i\pi(0.01)}}, \frac{P_2z_5}{0.1e^{i\pi(0.2)}}, \frac{P_2z_6}{0.4e^{i\pi(0.2)}}\Bigg)\Bigg\rangle.$$

Consider the town z_1 and resource P_1 as representatives. In order to control the petroleum resource of each town, we should know about the actual quantity of petroleum consumption, petroleum leakage and unforeseen petroleum consumption. Moreover, the resources of petroleum should provide a required quantity of petroleum to each town for fulfillment of people's basic needs. The resources also have minimum storage capacity of petroleum to keep petroleum level normal. In order to control petroleum resource, minimum quantity of petroleum replenishment should be retained for cities so that at any stage of time, the two resources could provide enough quantity of petroleum to assure the basic city petroleum consumption.

Here we describe the meanings of $\xi_S(z_1)$, $\eta_S(z_1)$, $\xi_S(P_1)$, $\eta_S(P_1)$, $\xi_T(z_1, P_1)$, $\eta_T(z_1, P_1)$, $\psi_S(z_1)$, $\phi_S(z_1)$, $\psi_S(P_1)$, $\phi_S(P_1)$, $\psi_T(z_1, P_1)$ and $\phi_T(z_1, P_1)$.

1. $\xi_S(z_1)$ represents the actual quantity of petroleum consumption of town z_1.
2. $\eta_S(z_1)$ represents petroleum leakage of town z_1. $\sqrt{1-\xi_S^2(z_1)-\eta_S^2(z_1)}$ denotes unforeseen petroleum consumption.
3. $\psi_S(z_1)$ represents the cost of petroleum consumption in the town z_1.
4. $\phi_S(z_1)$ represents the cost of leakage petroleum.
5. $\eta_S(P_1)$ represents the quantity of petroleum provided by the resource P_1.

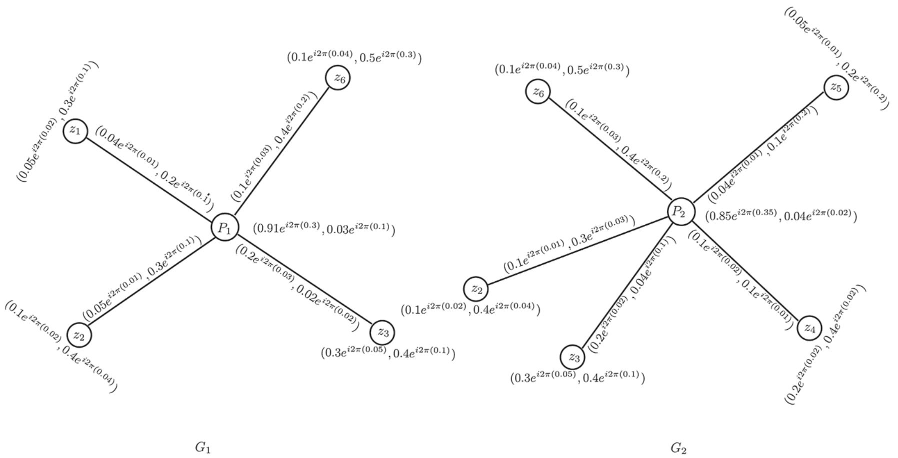

Fig. 14 Complex Pythagorean fuzzy threshold subgraphs

6. $\eta_S(P_1)$ represents the minimum quantity of petroleum storage of resource P_1. $\sqrt{1-\xi_S^2(P_1)-\eta_S^2(P_1)}$ represents unforeseen petroleum consumption of resource $P1$.
7. $\psi_S(P_1)$ represents the cost of petroleum at P_1.
8. $\phi_S(P_1)$ represents the cost of leaked quantity of petroleum of resource P_1.
9. $\xi_T(z_1, P_1)$ represents the flow assurance of petroleum from resource P_1 to town z_1.
10. $\xi_T(z_1, P_1)$ represents the non-flow degree of petroleum between resource P_1 and town z_1.
11. $\psi_T(z_1, P_1)$ represents the cost of petroleum, which flows between resource P_1 and town z_1.
12. $\phi_T(z_1, P_1)$ represents the cost of petroleum, which is leaked during flow between resource P_1 and town z_1.

As the quantity of petroleum consumption of each town is dominated by resources, thus by using Algorithm 1, we can easily find the threshold dimension by the number of resources. Consider the $\mathcal{CPFG}$ displayed in Fig. 13, which has threshold dimension and threshold partition number 2. Since $\mathcal{CPFG}$ does not contain a triangle, therefore two complex Pythagorean fuzzy threshold subgraphs can be induced from it, which are shown in Fig. 14.

It is worth noting that the threshold value $\hat{t_1}$ represents the limit quantity of petroleum replenishment and $m - \hat{t_2}$ represents the limit amount petroleum leakage, evaporation and unforeseen petroleum consumption, where m is the cardinality of maximum independent set of crisp graph of $\mathcal{CPFG}$. θ_{t_1} represents the limit amount of cost of petroleum and $2\pi - \theta_{t_2}$ represents the limit amount of cost of petroleum, which is evaporated, unforeseen and leaked.

By calculations, we observe that $\hat{t_1} = 0.55$, $\hat{t_2} = 2.5$, $\theta_{t_1} = 0.26\pi$ and $\theta_{t_2} = 6.9\pi$ for first subgraph of $\mathcal{CPFG}$. Also $\hat{t_1} = 0.75$, $\hat{t_2} = 3.1$, $\theta_{t_1} = 0.28\pi$ and $\theta_{t_2} = 8.68\pi$

for second subgraph of $\mathcal{CPFG}$. In the first $\mathcal{CPFTG}$, the maximum independent set is $\{z_1, z_2, z_3, z_6\}$ and

$$\xi_S(z_1) + \xi_S(z_2) + \xi_S(z_3) + \xi_S(z_6) = 0.55,$$
$$(1 - \eta_S(z_1)) + (1 - \eta_S(z_2)) + (1 - \eta_S(z_3)) + (1 - \eta_S(z_6)) = 2.5.$$

The resource P_1 can supply 0.91 quantity of petroleum of basic needs for four towns and the four towns require at least 0.55 quantity of petroleum consumption and

$$\psi_S(z_1) + \psi_S(z_2) + \psi_S(z_3) + \psi_S(z_6) = 0.26\pi,$$
$$(1 - \phi_S(z_1)) + (1 - \phi_S(z_2)) + (1 - \phi_S(z_3)) + (1 - \phi_S(z_6)) = 6.9\pi.$$

So the cost of petroleum provided by resource P_1 for four towns is 0.6π, which requires at least 0.26π cost for petroleum consumption. In the second $\mathcal{CPFTG}$, the maximum independent set is $\{z_2, z_3, z_4, z_5, z_6\}$ and

$$\xi_S(z_2) + \xi_S(z_3) + \xi_S(z_4) + \xi_S(z_5) + \xi_S(z_6) = 0.75,$$
$$(1 - \eta_S(z_2)) + (1 - \eta_S(z_3)) + (1 - \eta_S(z_4)) + (1 - \eta_S(z_5)) + (1 - \eta_S(z_6) = 3.1.$$

The resource P_2 can supply 0.85 quantity of petroleum of basic needs for five towns and the five towns require at least 0.75 quantity of petroleum consumption and

$$\psi_S(z_2) + \psi_S(z_3) + \psi_S(z_4) + \psi_S(z_5) + \psi_S(z_6) = 0.28\pi.$$
$$(1 - \phi_S(z_2)) + (1 - \phi_S(z_3)) + (1 - \phi_S(z_4)) + (1 - \phi_S(z_5)) + (1 - \phi_S(z_6)) = 8.68\pi.$$

So, the cost of petroleum provided by resource P_2 for five towns is 0.7π, which require at least 0.28π cost for petroleum consumption. By applying $\mathcal{CPFTG}_s$ to regulate petroleum resources, we estimate that the two resources can individually provide enough quantity of petroleum according to the required consumption of petroleum for each town.

- Based on the above discussion, we conclude that our proposed model of $\mathcal{CPFTG}_s$ is more suitable, since $\mathcal{CPFS}_s$ are more appropriate to deal with uncertainty and fuzziness of the system.
- The framework of $\mathcal{CPFTG}_s$ is very crucial to regulate petroleum resources rather than the model of $\mathcal{IFTG}_s$ and $\mathcal{FTG}_s$.
- The presented model of $\mathcal{CPFTG}_s$ will be more competent and applicable in managing power resources, since it handles two dimensional information on the basis of $\mathcal{CPFS}_s$.

5 Conclusion

A complex Pythagorean fuzzy set is a useful mathematical model which is used to handle the vagueness with the degrees whose ranges are enlarged from real to complex subset with unit disc. In this paper, a novel idea of $\mathcal{CPFTG}$ has been presented. $\mathcal{CPFTG}_s$ represent an extended model under complex Pythagorean fuzzy information, which provides a compatible and flexible structure to tackle the uncertainty and vagueness more precisely than $\mathcal{IFTG}_s$. A novel idea of $\mathcal{CPFTG}_s$ has been defined. We describe some elementary characteristics and related theorems of $\mathcal{CPFTG}_s$. The relation between vertex cardinality and threshold values has been discussed. We have proposed the concept of threshold dimension and threshold partition number for $\mathcal{CPFTG}_s$. We have provided an application of $\mathcal{CPFTG}_s$. We plan to extend our work to: (1) Fuzzy soft threshold graphs; (2) Rough fuzzy threshold graphs; (3) Pythagorean fuzzy soft threshold graphs, and (4) Single-valued neutrosophic soft threshold graphs.

Declarations

Conflict of interest The authors declare that they have no conflict of interest regarding the publication of this article.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

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