



Construction of operational matrices based on linear cardinal B-spline functions for solving fractional stochastic integro-differential equation

Somayeh Abdi-Mazraeh¹ · Hossein Kheiri¹ · Safar Irandoust-Pakchin¹ 

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Abstract

The main purpose of this paper is to develop a new method based on operational matrices of the linear cardinal B-spline (LCB-S) functions to numerically solve of the fractional stochastic integro-differential (FSI-D) equations. To reach this aim, LCB-S functions are introduced and their properties are considered, briefly. Then, the operational matrices based on LCB-S functions are constructed, for the first time, including the fractional Riemann-Liouville integral operational matrix, the stochastic integral operational matrix, and the integer integral operational matrix. The main characteristic of the new scheme is to convert the FSI-D equation into a linear system of algebraic equations which can be easily solved by applying a suitable method. Also, the convergence analysis and error estimate of the proposed method are studied and an upper bound of error is obtained. Numerical experiments are provided to show the potential and efficiency of the new method. Finally, some numerical results, for various values of perturbation in the parameters of the main problem are presented which can indicate the stability of the suggested method.

1 Introduction

The most physical phenomena can be modeled as different stochastic equations including stochastic integral equations or stochastic integro-differential equations.

✉ Safar Irandoust-Pakchin
s.irandoust@tabrizu.ac.ir

Somayeh Abdi-Mazraeh
s.abdi_m@tabrizu.ac.ir

Hossein Kheiri
h-kheiri@tabrizu.ac.ir

¹ Department of Applied Mathematics, Faculty of Mathematical Sciences, University of Tabriz, Tabriz, Iran

An integral equation is called a stochastic integral equation if it contains at least one term as a stochastic process, accordingly the solution of this equation is a stochastic process. In most cases, such equations can not be solved explicitly. So, the different numerical methods and different basis functions have been presented for solving of them such as cubic B-spline and bicubic B-spline collocation method [19], hybrid of block-pulse and parabolic functions [22], Bernoulli polynomials and Bernoulli wavelet methods [23,36], delta basis functions [29], quintic B-spline collocation method [21], Euler polynomial [27], wavelet-based computational method [30] and the other methods [5–7,12,13,16,18,26,31,34].

Fractional differential equations (FDEs) arise in the various sciences including finance, physics, signal processing and control theory [3,8–11,32,33]. Sometimes, these equations are combined with the stochastic integral equations. A stochastic integro–differential equation is called a FSI-D equation if the order of derivative is non–integer. Due to non–existing of the analytical solution for these type equations, it is clear that obtaining the numerical solutions of them can be interesting issue for the researchers.

The basis functions are one of the most common methods used to solve such equations [1,20,24,25,28]. When the basis functions are applied, the solution of the considering problem is approximated as the linear combination of the basis functions with unknown coefficients. The main characteristic of such methods is to convert the differential or integral equation to a algebraic system of equations. It not only simplified the problem but also speeds up the computation.

In this work, a new scheme based on the LCB-S basis functions is introduced for solving FSI-D equation as

$$\begin{aligned} {}_0^C D_\tau^\nu Y(\tau) &= U(\tau) + \int_0^\tau \lambda_1(\tau, s)Y(s)ds + \int_0^\tau \lambda_2(\tau, s)Y(s)d\mathcal{B}(s), \\ Y(0) &= 0, \end{aligned} \quad (1.1)$$

wherein ${}_0^C D_\tau^\nu$ is the Caputo fractional derivative of order $0 < \nu < 1$. Also, $U(t) \in C^2([0, 1])$, $\lambda_i(\tau, s) \in C^2([0, 1] \times [0, 1])$ for $i = 1, 2$ are stochastic processes defined on the probability space (Ω, F, P) and $\mathcal{B}(\tau)$ is Brownian motion process. In Eq. (1.1), $Y(\tau)$ is an unknown function that must be determined. Lakestani et al. [15] have applied the LCB-S functions for solving the fractional differential equations. They have constructed the operational matrix of derivative and fractional derivative. Here, the LCB-S functions are developed for solving the FSI-D Eq. (1.1). For this purpose, the operational matrices based on LCB-S functions are constructed, for the first time, including the fractional Riemann-Liouville integral operational matrix, the stochastic integral operational matrix, and the integer integral operational matrix. The mentioned matrices are all upper triangular which can be calculated simply. It is worth noting that the LCB-S functions have the interpolation properties. Therefore, the coefficients of each known function can be easily calculated without using any integration and this is a major advantage of the LCB-S functions respect to the other basis functions [4]. Moreover, these functions have cardinality properties and using this characteristic, the cost of calculation is decreased (see cpu times of proposed method in the tables of numerical verifications section).

The organization of this work is as follows: Some definitions, notations, the formulation of LCB-S functions on $[0, 1]$ and the derivative operational matrix are expressed in Sect. 2. The operational matrices of fractional Riemann-Liouville, stochastic and integer integrals are constructed in Sect. 3. The proposed method is used to the approximate solution of the FSI-D equation in Sect. 4. Convergence analysis and error estimate of the proposed method are studied in Sect. 5. In Sect. 6, numerical experiments are reported. Also in this section, some numerical results, for various values of perturbation in the parameters of the main problem are presented which can indicate the stability of the suggested method. A brief conclusion is presented in Sect. 7.

2 Some definitions and notations

Definition 2.1 [14] For the given function $Z(\tau)$

$$\begin{aligned} {}_{\tau_0} \mathcal{J}_\tau^\nu Z(\tau) &= \frac{1}{\Gamma(\nu)} \int_{\tau_0}^\tau (\tau - p)^{\nu-1} Z(p) dp, \quad \nu > 0, \\ {}_{\tau_0}^{RL} D_\tau^\nu Z(\tau) &= \frac{1}{\Gamma(q - \nu)} \left(\frac{d}{d\tau}\right)^q \int_{\tau_0}^\tau (\tau - p)^{q-\nu-1} Z(p) dp, \quad q - 1 \leq \nu < q, \\ {}_{\tau_0}^C D_\tau^\nu Z(\tau) &= \frac{1}{\Gamma(q - \nu)} \int_{\tau_0}^\tau (\tau - p)^{q-\nu-1} Z^{(q)}(p) dp, \quad q - 1 < \nu \leq q, \end{aligned} \tag{2.1}$$

are called the Riemann-Liouville integral, the Riemann-Liouville derivative and the Caputo derivative of order ν , respectively.

Definition 2.2 [14] If $Z \in C^q[\tau_0, \tau]$ then

$${}_{\tau_0}^{RL} D_\tau^\nu (Z(\tau) - \Phi(\tau)) = {}_{\tau_0}^C D_\tau^\nu Z(\tau), \quad q - 1 < \nu < q, \quad q \in \mathbb{Z}^+, \tag{2.2}$$

wherein $\Phi(\tau) = \sum_{j=0}^{q-1} \frac{Z^{(j)}(\tau_0)}{\Gamma(j + 1)} (\tau - \tau_0)^j$.

2.1 LCB-S functions on $[0, 1]$

In this part, a brief definition of the cardinal B-spline (CB-S) functions is expressed. In the sequel, the operational matrix of derivative based on the LCB-S functions is presented.

Definition 2.3 [17] Let the CB-S function of first order, represented by $\mathcal{N}_1(\tau) = \mathcal{X}_{[0,1)}(\tau)$, wherein $\mathcal{X}_{[0,1)}(\tau)$ is a characteristic function on the interval $[0, 1)$. Also, The CB-S function of order r , defined as

$$\mathcal{N}_r(\tau) = (\mathcal{N}_{r-1} * \mathcal{N}_1)(\tau) = \int_{-\infty}^\infty \mathcal{N}_{r-1}(\tau - s) \mathcal{N}_1(s) ds = \int_0^1 \mathcal{N}_{r-1}(\tau - s) ds, \tag{2.3}$$

recursively, and $\text{supp}[\mathcal{N}_r(\tau)] = [0, r]$.

In explicit form, the LCB-S function of 2 order is presented as

$$\mathcal{N}_2(\tau) = \begin{cases} \tau, & \tau \in [0, 1], \\ 2 - \tau, & \tau \in [1, 2], \\ 0, & \text{elsewhere.} \end{cases} \tag{2.4}$$

Let $\mathcal{N}_{r,q}(\tau) = \mathcal{N}_2(2^r \tau - q)$, $r, q \in \mathbb{Z}$. It can easily be shown that

$$\text{supp}[\mathcal{N}_{r,q}(\tau)] = [2^{-r}q, 2^{-r}(2 + q)], \quad r, q \in \mathbb{Z}.$$

Define $\mathcal{S}_r = \{q : [2^{-r}q, 2^{-r}(2 + q)] \cap (0, 1) \neq \emptyset\}$, $r \in \mathbb{Z}$. It can be derived that $\min\{\mathcal{S}_r\} = -1$ and $\max\{\mathcal{S}_r\} = 2^r - 1$, $r \in \mathbb{Z}$.

Since it is required that the support of $\mathcal{N}_{r,q}(\tau)$ is restricted on $[0, 1]$, so put

$$\psi_q(\tau) = \psi_q^r(\tau) = \mathcal{N}_{r,q}(\tau)\chi_{[0,1]}(\tau), \quad r \in \mathbb{Z}, \quad q \in \mathcal{S}_r. \tag{2.5}$$

As a result, one can write

$$\psi_q(\tau) = \begin{cases} \begin{cases} 1 - 2^r \tau, & \tau \in [0, 2^{-r}), \\ 0, & \text{otherwise,} \end{cases} & \text{for } q = -1, \\ \begin{cases} 2^r \tau - q, & \tau \in [2^{-r}q, 2^{-r}(q + 1)), \\ 2 - (2^r \tau - q), & \tau \in [2^{-r}(q + 1), 2^{-r}(q + 2)), \\ 0, & \text{otherwise,} \end{cases} & \text{for } q = 0, 2^r - 2, \dots, \\ \begin{cases} 2^r \tau - 2^r + 1, & \tau \in [1 - 2^{-r}, 1), \\ 0, & \text{otherwise,} \end{cases} & \text{for } q = 2^r - 1, \end{cases} \tag{2.6}$$

2.2 Vector form and function approximation

Suppose

$$\Psi(\tau) = [\psi_{-1}(\tau), \psi_0(\tau), \dots, \psi_{2^r-1}(\tau)]^T, \tag{2.7}$$

where $r \in \mathbb{Z}$ is a fixed number. A function $U(\tau) \in L^2[0, 1]$ is approximated by the LCB-S functions as

$$U(\tau) \approx \sum_{q=-1}^{2^r-1} u_q \psi_q(\tau) = \mathcal{U}^T \Psi(\tau), \tag{2.8}$$

where

$$\mathcal{U} = [u_{-1}, u_0, \dots, u_{2^r-1}]^T, \quad u_q = U(2^{-r}(q + 1)), \quad q = -1, \dots, 2^r - 1. \tag{2.9}$$

Let e_q be the $(q + 2)$ -th column of unit matrix of order $2^r + 1$, so it is easy to verify that

$$\Psi(\xi_q) = e_q, \quad \xi_q = 2^{-r}(q + 1), \quad q = -1, \dots, 2^r - 1. \tag{2.10}$$

Also, a function $\lambda(\tau, s) \in L^2([0, 1] \times [0, 1])$ may be approximated by LCB-S functions as

$$\lambda(\tau, s) \approx \sum_{q=-1}^{2^r-1} \sum_{p=-1}^{2^r-1} u_{q,p} \psi_q(\tau) \psi_p(s) = \Psi(\tau)^T \mathbf{\Lambda} \Psi(s), \tag{2.11}$$

where $\mathbf{\Lambda}$ is the $(2^r + 1) \times (2^r + 1)$ coefficient matrix with entries $\Lambda_{q,p}$ as follows

$$\Lambda_{q,p} = U(2^{-r}(q + 1), 2^{-r}(p + 1)), \quad q, p = -1, \dots, 2^r - 1. \tag{2.12}$$

2.3 Operational matrix of the derivative

Suppose

$$\mathbf{P} = 2^{-r} \begin{bmatrix} \frac{1}{6} & \frac{1}{12} & & & & & \\ & \frac{1}{12} & \frac{1}{3} & & & & \\ & & \frac{1}{12} & & & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & \frac{1}{12} & \frac{1}{3} & \frac{1}{12} \\ & & & & & \frac{1}{12} & \frac{1}{6} \end{bmatrix}, \quad \mathbf{E} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & & & & & \\ \frac{1}{2} & \mathbf{0} & -\frac{1}{2} & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & & \frac{1}{2} & \mathbf{0} & -\frac{1}{2} & \\ & & & & \frac{1}{2} & & \frac{1}{2} \end{bmatrix}. \tag{2.13}$$

The differentiation of $\Psi(\tau)$ in (2.7) is calculated as

$$\Psi'(\tau) = \mathbf{D}\Psi(\tau), \quad \mathbf{D} = \mathbf{E}(\mathbf{P}^{-1}), \tag{2.14}$$

where \mathbf{D} recalls $(2^r + 1) \times (2^r + 1)$ operational matrix of derivative for the LB-S functions on $[0, 1]$ (for more details see [15]).

3 Operational matrices of the fractional integral, stochastic integral and integer integral

In this section, the fractional integral, stochastic integral and integer integral of the LCB-S functions on $[0, 1]$ are constructed.

3.1 Operational matrix of the fractional integral

Let $v \in (0, 1)$ then applying Definition 2.1, one can write

$${}_0\mathcal{J}_\tau^v \psi_q(\tau) = \frac{1}{\Gamma(v)} \int_0^\tau (\tau - s)^{v-1} \psi_q(s) ds. \tag{3.1}$$

To calculate the above integral, the following three cases are considered:

Case 1: $q = -1$.

In this case, if $\tau \in [0, 2^{-r}]$ then Eq. (3.1) can be calculated as follows

$$\begin{aligned} {}_0\mathcal{J}_\tau^\nu \psi_{-1}(\tau) &= \frac{1}{\Gamma(\nu)} \int_0^\tau (\tau - s)^{\nu-1} \psi_{-1}(\tau) ds, \\ &= \frac{1}{\Gamma(\nu)} \int_0^\tau (\tau - s)^{\nu-1} (1 - 2^r s) ds, \end{aligned}$$

where above integral can be solved using integration by parts. Therefore, one can write

$${}_0\mathcal{J}_\tau^\nu \psi_{-1}(\tau) = \frac{1}{\Gamma(\nu)} \left(\frac{\tau^\nu}{\nu} - \frac{2^r \tau^{\nu+1}}{\nu(\nu + 1)} \right). \tag{3.2}$$

If $\tau > 2^{-r}$ then Eq. (3.1) can be calculated as

$$\begin{aligned} {}_0\mathcal{J}_\tau^\nu \psi_{-1}(\tau) &= \frac{1}{\Gamma(\nu)} \int_0^{2^{-r}} (\tau - s)^{\nu-1} (1 - 2^r s) ds, \\ &= \frac{1}{\Gamma(\nu)} \left(\frac{\tau^\nu}{\nu} - \frac{2^r \tau^{\nu+1}}{\nu(\nu + 1)} + \frac{(2^r \tau - 1)^{\nu+1}}{\nu(\nu + 1)2^{r\nu}} \right). \end{aligned} \tag{3.3}$$

Case 2: $q = 0, \dots, 2^r - 2$.

In this case, if $\tau \in [0, 2^{-r}q]$ then it can be proven ${}_0\mathcal{J}_\tau^\nu \psi_q(\tau) = 0$.

If $\tau \in [2^{-r}q, 2^{-r}(q + 1)]$ then

$$\begin{aligned} {}_0\mathcal{J}_\tau^\nu \psi_q(\tau) &= \frac{1}{\Gamma(\nu)} \int_{2^{-r}q}^\tau (\tau - s)^{\nu-1} (2^r s - q) ds, \\ &= \frac{1}{2^{r\nu} \Gamma(\nu + 2)} [(2^r \tau - q)^{\nu+1}]. \end{aligned} \tag{3.4}$$

If $\tau \in [2^{-r}(q + 1), 2^{-r}(q + 2)]$ then

$$\begin{aligned} {}_0\mathcal{J}_\tau^\nu \psi_q(\tau) &= \frac{1}{\Gamma(\nu)} \left[\int_{2^{-r}q}^{2^{-r}(q+1)} (\tau - s)^{\nu-1} (2^r s - q) ds \right. \\ &\quad \left. + \int_{2^{-r}(q+1)}^\tau (\tau - s)^{\nu-1} (2 - (2^r s - q)) ds \right], \\ &= \frac{1}{2^{r\nu} \Gamma(\nu + 2)} [(2^r \tau - q)^{\nu+1} - 2(2^r \tau - q - 1)^{\nu+1}]. \end{aligned} \tag{3.5}$$

If $\tau > 2^{-r}(q + 2)$ then

$${}_0\mathcal{J}_\tau^\nu \psi_q(\tau) = \frac{1}{\Gamma(\nu)} \left[\int_{2^{-r}q}^{2^{-r}(q+1)} (\tau - s)^{\nu-1} (2^r s - q) ds \right.$$

$$\begin{aligned}
 & + \int_{2^{-r}(q+1)}^{2^{-r}(q+2)} (\tau - s)^{\nu-1} (2 - (2^r s - q)) ds, \\
 & = \frac{1}{2^{r\nu} \Gamma(\nu+2)} [(2^r \tau - q)^{\nu+1} - 2(2^r \tau - q - 1)^{\nu+1} + (2^r \tau - q - 2)^{\nu+1}]. \tag{3.6}
 \end{aligned}$$

Case 3: $q = 2^r - 1$.

In this case, if $\tau \in [0, 2^{-r}q]$ then it can be derived ${}_0\mathcal{J}_\tau^\nu \psi_{2^r-1}(\tau) = 0$.

If $\tau \in [1 - 2^{-r}, 1]$ then

$$\begin{aligned}
 {}_0\mathcal{J}_\tau^\nu \psi_{2^r-1}(\tau) & = \frac{1}{\Gamma(\nu)} \int_{1-2^{-r}}^\tau (\tau - s)^{\nu-1} (2^r s - 2^r + 1) ds, \\
 & = \frac{1}{2^{r\nu} \Gamma(\nu+2)} [(2^r \tau - 2^r + 1)^{\nu+1}]. \tag{3.7}
 \end{aligned}$$

If $\tau > 1$ then

$$\begin{aligned}
 {}_0\mathcal{J}_\tau^\nu \psi_{2^r-1}(\tau) & = \frac{1}{\Gamma(\nu)} \int_{1-2^{-r}}^1 (\tau - s)^{\nu-1} (2^r s - 2^r + 1) ds, \\
 & = \frac{-(2^r \tau - 2^r)^\nu (\nu + 1) - (2^r \tau - 2^r)^{\nu+1} + (2^r \tau - 2^r + 1)^{\nu+1}}{2^{r\nu} \Gamma(\nu+2)}. \tag{3.8}
 \end{aligned}$$

Using Eqs. (3.2)–(3.8), one can write

$${}_0\mathcal{J}_\tau^\nu \psi_q(\tau) = \begin{cases} \left\{ \begin{aligned} & \frac{(\frac{\tau}{\nu} - \frac{2^r \tau^{\nu+1}}{\nu(\nu+1)})}{\Gamma(\nu)}, & \tau \in [0, \frac{1}{2^r}), \\ & \frac{(\frac{\tau}{\nu} - \frac{2^r \tau^{\nu+1}}{\nu(\nu+1)} + \frac{(2^r \tau - 1)^{\nu+1}}{\nu(\nu+1)2^{r\nu}})}{\Gamma(\nu)}, & \tau \geq \frac{1}{2^r}, \\ & 0, & \text{otherwise,} \end{aligned} \right. & \text{for } q = -1, \\ \left\{ \begin{aligned} & \frac{(2^r \tau - q)^{\nu+1}}{2^{r\nu} \Gamma(\nu+2)}, & \tau \in [\frac{q}{2^r}, \frac{q+1}{2^r}), \\ & \frac{(2^r \tau - q)^{\nu+1} - 2(2^r \tau - q - 1)^{\nu+1}}{2^{r\nu} \Gamma(\nu+2)}, & \tau \in [\frac{q+1}{2^r}, \frac{q+2}{2^r}), \\ & \frac{(2^r \tau - q)^{\nu+1} - 2(2^r \tau - q - 1)^{\nu+1} + (2^r \tau - q - 2)^{\nu+1}}{2^{r\nu} \Gamma(\nu+2)}, & \tau \geq \frac{q+2}{2^r}, \\ & 0, & \text{otherwise,} \end{aligned} \right. & \text{for } q = 0, 2^r - 2, \dots \\ \left\{ \begin{aligned} & \frac{(2^r \tau - 2^r + 1)^{\nu+1}}{2^{r\nu} \Gamma(\nu+2)}, & \tau \in [1 - \frac{1}{2^r}, 1), \\ & \frac{-(2^r \tau - 2^r)^\nu - (2^r \tau - 2^r)^{\nu+1} + (2^r \tau - 2^r + 1)^{\nu+1}}{2^{r\nu} \Gamma(\nu+2)}, & \tau \geq 1, \\ & 0, & \text{otherwise,} \end{aligned} \right. & \text{for } q = 2^r - 1, \end{cases} \tag{3.9}$$

Applying Eqs. (2.7)–(2.8) on (3.9), it can be obtained

$${}_0\mathcal{J}_\tau^\nu \Psi(\tau) = \mathbf{J}_\nu \Psi(\tau), \quad (3.10)$$

wherein

$$(\mathbf{J}_\nu)_{p,q} = {}_0\mathcal{J}_\tau^\nu \psi_q(\xi_p), \quad \xi_p = \frac{p+1}{2^r}, \quad p, q = -1, \dots, 2^r - 1. \quad (3.11)$$

It can be proven that \mathbf{J}_ν is a $(2^r + 1) \times (2^r + 1)$ matrix as follows

$$\mathbf{J}_\nu = \begin{bmatrix} 0 & \omega_0 & \omega_1 & \omega_2 & \dots & \omega_{2^r-1} \\ \gamma_0 & \gamma_1 & \gamma_2 & \dots & \gamma_{2^r-1} \\ & \gamma_0 & \gamma_1 & \dots & \gamma_{2^r-2} \\ & & \ddots & \ddots & \vdots \\ & & & \gamma_0 & \gamma_1 \\ & & & & \gamma_0 \end{bmatrix}, \quad (3.12)$$

where

$$\begin{aligned} \omega_0 &= \nu\lambda, \\ \omega_q &= [(\nu+1)(q+1)^\nu - (q+1)^{\nu+1} + q^{\nu+1}]\lambda, \quad q = 1, \dots, 2^r - 1, \\ \gamma_0 &= \lambda, \\ \gamma_1 &= [2^{\nu+1} - 2]\lambda, \\ \gamma_q &= [(1+q)^{\nu+1} - 2q^{\nu+1} + (q-1)^{\nu+1}]\lambda \end{aligned} \quad (3.13)$$

$$\text{and } \lambda = \frac{1}{2^{\nu}\Gamma(\nu+2)}.$$

3.2 Operational matrix of the stochastic integral

Suppose

$$\mathcal{I}^s \psi_q(\tau) = \int_0^\tau \psi_q(s) dB(s). \quad (3.14)$$

To calculate the above integral, the following three cases are considered:

Case 1: $q = -1$.

In this case, if $\tau \in [0, 2^{-r}]$ then Eq. (3.14) is calculated as

$$\mathcal{I}^s \psi_{-1}(\tau) = \int_0^\tau (1 - 2^r s) dB(s) = \mathcal{B}(t) - 2^r \int_0^\tau s dB(s), \quad (3.15)$$

where the above integral can be solved using integration by parts. Therefore, it can be written

$$\mathcal{I}^s \psi_{-1}(\tau) = \mathcal{B}(\tau) - 2^r (\tau \mathcal{B}(\tau) - \int_0^\tau \mathcal{B}(s) ds). \tag{3.16}$$

In Eq. (3.16), the integral can be approximated by Simpson’s rule. So it can be obtained that

$$\mathcal{I}^s \psi_{-1}(\tau) = \mathcal{B}(\tau) \left(1 - \frac{2^{r+1}\tau}{3}\right) + \frac{2^{r+2}\tau}{3} \mathcal{B}\left(\frac{\tau}{2}\right). \tag{3.17}$$

If $\tau > 2^{-r}$ then Eq. (3.17) can be written as

$$\begin{aligned} \mathcal{I}^s \psi_{-1}(\tau) &= \int_0^{2^{-r}} (1 - 2^r s) d\mathcal{B}(s) = \mathcal{B}(2^{-r}) - 2^r \int_0^{2^{-r}} s d\mathcal{B}(s), \\ &= \mathcal{B}(2^{-r}) - 2^r (s\mathcal{B}(s)|_0^{2^{-r}} - \int_0^{2^{-r}} \mathcal{B}(s) ds). \end{aligned} \tag{3.18}$$

Applying Simpson’s rule on the above integral, one can derive that

$$\mathcal{I}^s \psi_{-1}(\tau) = \frac{4}{3} \mathcal{B}(2^{-r-1}) + \frac{1}{3} \mathcal{B}(2^{-r}). \tag{3.19}$$

Case 2: $q = 0, \dots, 2^r - 2$.

In this case, if $\tau \in [0, 2^{-r}q]$ then it can be proven $\mathcal{I}^s \psi_q(\tau) = 0$.

If $\tau \in [2^{-r}q, 2^{-r}(q+1)]$ then using integral by parts and Simpson’s rule, it is obtained that

$$\begin{aligned} \mathcal{I}^s \psi_q(\tau) &= \int_{q2^{-r}}^\tau (2^r s - q) d\mathcal{B}(s), \\ &= \frac{2^r \tau - q}{3} [2\mathcal{B}(\tau) - \mathcal{B}\left(\frac{q}{2^r}\right) - 4\mathcal{B}\left(\frac{2^r \tau + q}{2^{r+1}}\right)]. \end{aligned} \tag{3.20}$$

If $\tau \in [2^{-r}(q+1), 2^{-r}(q+2)]$ then

$$\begin{aligned} \mathcal{I}^s \psi_q(\tau) &= \int_{2^{-r}q}^{2^{-r}(q+1)} (2^r s - q) d\mathcal{B}(s) + \int_{2^{-r}(q+1)}^\tau (2 - (2^r s - q)) d\mathcal{B}(s), \\ &= \frac{5 + 2q - 2^{r+1}\tau}{3} \mathcal{B}(\tau) - \frac{1}{3} \mathcal{B}(q2^{-r}) + \frac{-2 + 2^r \tau - q}{3} \mathcal{B}((q+1)2^{-r}) \\ &\quad - \frac{4}{3} \mathcal{B}((2q+1)2^{-r-1}) + \frac{2^{r+2}}{3} [\tau - 2^{-r}(q+1)] \mathcal{B}\left(\frac{\tau + 2^{-r}(q+1)}{2}\right). \end{aligned} \tag{3.21}$$

If $\tau > 2^{-r}(q + 2)$ then

$$\begin{aligned} \mathcal{I}^s \psi_q(\tau) &= \int_{2^{-r}q}^{2^{-r}(q+1)} (2^r s - q) d\mathcal{B}(s) + \int_{2^{-r}(q+1)}^{2^{-r}(q+2)} (2 - (2^r s - q)) d\mathcal{B}(s), \\ &= \frac{-1}{3} \mathcal{B}(q2^{-r}) + \frac{1}{3} \mathcal{B}((q + 2)2^{-r}) - \frac{4}{3} \mathcal{B}((2q + 1)2^{-r-1}) + \frac{4}{3} \mathcal{B}((2q + 3)2^{-r-1}). \end{aligned} \tag{3.22}$$

Case 3: $q = 2^r - 1$.

In this case, if $\tau \in [0, 2^{-r}q]$ then it can be derived $\mathcal{I}^s \psi_{2^r-1}(\tau) = 0$.

If $\tau \in [1 - 2^{-r}, 1]$ then

$$\begin{aligned} \mathcal{I}^s \psi_{2^r-1}(\tau) &= \int_{1-2^{-r}}^{\tau} (2^r s - 2^r + 1) d\mathcal{B}(s), \\ &= \frac{2^r \tau - q}{3} [2\mathcal{B}(\tau) - \mathcal{B}(q2^{-r}) - 4\mathcal{B}(\frac{\tau + q2^{-r}}{2})]. \end{aligned} \tag{3.23}$$

If $\tau > 1$ then

$$\begin{aligned} \mathcal{I}^s \psi_{2^r-1}(\tau) &= \int_{1-2^{-r}}^1 (2^r s - 2^r + 1) d\mathcal{B}(s), \\ &= \frac{1}{3} [2\mathcal{B}(2^{-r}(q + 1)) - \mathcal{B}(2^{-r}q) - 4\mathcal{B}(2^{-r-1}(2q + 1))]. \end{aligned} \tag{3.24}$$

Using Eqs. (3.15)–(3.24), one can write

$$\mathcal{I}^s \psi_q(\tau) = \begin{cases} \begin{cases} \mathcal{B}(\tau)(1 - \frac{2^{r+1}\tau}{3}) + \frac{2^{r+2}\tau}{3} \mathcal{B}(\frac{\tau}{2}), & \tau \in [0, \frac{1}{2^r}), \\ \frac{4}{3} \mathcal{B}(2^{-r-1}) + \frac{1}{3} \mathcal{B}(2^{-r}), & \tau \geq \frac{1}{2^r}, \\ 0, & \text{otherwise,} \end{cases} & \text{for } q = -1, \\ \begin{cases} \frac{2^r \tau - q}{3} [2\mathcal{B}(\tau) - \mathcal{B}(\frac{q}{2^r}) - 4\mathcal{B}(\frac{2^r \tau + q}{2^{r+1}})], & \tau \in [\frac{q}{2^r}, \frac{q+1}{2^r}), \\ \frac{5+2q-2^{r+1}\tau}{3} \mathcal{B}(\tau) - \frac{1}{3} \mathcal{B}(q2^{-r}) + \frac{-2+2^r \tau - q}{3} \mathcal{B}((q + 1)2^{-r}) \\ - \frac{4}{3} \mathcal{B}((2q + 1)2^{-r-1}) + \frac{2^{r+2}}{3} [\tau - 2^{-r}(q + 1)] \\ \times \mathcal{B}(\frac{\tau + 2^{-r}(q+1)}{2}), & \tau \in [\frac{q+1}{2^r}, \frac{q+2}{2^r}), \\ \frac{-1}{3} \mathcal{B}(q2^{-r}) + \frac{1}{3} \mathcal{B}((q + 2)2^{-r}) - \frac{4}{3} \mathcal{B}((2q + 1)2^{-r-1}) \\ + \frac{4}{3} \mathcal{B}((2q + 3)2^{-r-1}), & \tau \geq \frac{q+2}{2^r}, \\ 0, & \text{otherwise,} \end{cases} & \text{for } q = 0, 2^r - 2, \dots \\ \begin{cases} \frac{2^r \tau - 2^r + 1}{3} [2\mathcal{B}(\tau) - \mathcal{B}(1 - 2^{-r}) - 4\mathcal{B}(\frac{\tau + 1 - 2^{-r}}{2})], & \tau \in [1 - \frac{1}{2^r}, 1), \\ \frac{1}{3} [2\mathcal{B}(1) - \mathcal{B}(1 - 2^{-r}) - 4\mathcal{B}(1 - 2^{-r-1})], & \tau \geq 1, \\ 0, & \text{otherwise,} \end{cases} & \text{for } q = 2^r - 1, \end{cases} \tag{3.25}$$

Applying Eqs. (2.7)–(2.8) on (3.25), it can be achieved as

$$\mathcal{I}^s \Psi(\tau) = \mathbf{I}_s \Psi(\tau), \tag{3.26}$$

wherein

$$(\mathbf{I}_s)_{p,q} = \mathcal{I}^s \psi_q(\xi_p), \quad \xi_p = \frac{p+1}{2^r}, \quad p, q = -1, \dots, 2^r - 1. \tag{3.27}$$

It can be demonstrated that \mathbf{I}_s is a $(2^r + 1) \times (2^r + 1)$ matrix as follows

$$\mathbf{I}_s = \begin{bmatrix} 0 & \mu & \mu & \mu & \dots & \mu \\ \kappa_0 & \eta_0 & \eta_0 & \dots & \dots & \eta_0 \\ & \kappa_1 & \eta_1 & \dots & \dots & \eta_1 \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & & \kappa_{2^r-2} & \eta_{2^r-2} \\ & & & & & \gamma \end{bmatrix}, \tag{3.28}$$

where

$$\begin{aligned} \mu &= \frac{1}{3} \left[4\mathcal{B}\left(\frac{1}{2^{r+1}}\right) + \mathcal{B}\left(\frac{1}{2^r}\right) \right], \\ \kappa_q &= \frac{1}{3} \left[2\mathcal{B}\left(\frac{q+1}{2^r}\right) - \mathcal{B}\left(\frac{q}{2^r}\right) - 4\mathcal{B}\left(\frac{2q+1}{2^{r+1}}\right) \right], \quad q = 0, \dots, 2^r - 2, \\ \eta_q &= \frac{1}{3} \left[\mathcal{B}\left(\frac{q+2}{2^r}\right) - \mathcal{B}\left(\frac{q}{2^r}\right) - 4\mathcal{B}\left(\frac{2q+1}{2^{r+1}}\right) + 4\mathcal{B}\left(\frac{2q+3}{2^{r+1}}\right) \right], \quad q = 0, \dots, 2^r - 2, \\ \gamma &= \frac{1}{3} \left[2\mathcal{B}(1) - \mathcal{B}\left(\frac{2^r-1}{2^r}\right) - 4\mathcal{B}\left(\frac{2^{r+1}-1}{2^{r+1}}\right) \right]. \end{aligned} \tag{3.29}$$

3.3 Operational matrix of the integer integral

Define

$$\mathcal{I} \psi_q(\tau) = \int_0^\tau \psi_q(\tau) ds. \tag{3.30}$$

To calculate the above integral, the following three cases are investigated:

Case 1: $q = -1$.

In this case, if $\tau \in [0, 2^{-r}]$ then Eq. (3.1) is calculated as follows

$$\mathcal{I} \psi_{-1}(\tau) = \int_0^\tau \psi_{-1}(\tau) ds = \int_0^\tau (1 - 2^r s) ds = \tau - 2^{r-1} \tau^2. \tag{3.31}$$

If $\tau > 2^{-r}$ then Eq. (3.32) can be obtained as

$$\mathcal{I} \psi_{-1}(\tau) = \int_0^{2^{-r}} \psi_{-1}(\tau) ds + \int_{2^{-r}}^{\tau} (1 - 2^r s) ds = 2^{-r-1}. \tag{3.32}$$

Case 2: $q = 0, \dots, 2^r - 2$.

In this case, if $\tau \in [0, 2^{-r}q]$ then it can be proven $\mathcal{I}\psi_q(\tau) = 0$.

If $\tau \in [2^{-r}q, 2^{-r}(q + 1)]$ then

$$\mathcal{I}\psi_q(\tau) = \int_{2^{-r}q}^{\tau} (2^r s - q) ds = 2^{r-1}\tau^2 - q\tau + 2^{-r-1}q^2. \tag{3.33}$$

If $\tau \in [2^{-r}(q + 1), 2^{-r}(q + 2)]$ then

$$\begin{aligned} \mathcal{I}\psi_q(\tau) &= \int_{2^{-r}q}^{2^{-r}(q+1)} (2^r s - q) ds + \int_{2^{-r}(q+1)}^{\tau} (2 - (2^r s - q)) ds \\ &= (2 + q)\tau - 2^{r-1}\tau^2 - 2^{-r-1}(q^2 + 4q + 2). \end{aligned} \tag{3.34}$$

If $\tau > 2^{-r}(q + 2)$ then

$$\mathcal{I}\psi_q(\tau) = \int_{2^{-r}q}^{2^{-r}(q+1)} (2^r s - q) ds + \int_{2^{-r}(q+1)}^{2^{-r}(q+2)} (2 - (2^r s - q)) ds = 2^{-r} \tag{3.35}$$

Case 3: $q = 2^r - 1$.

In this case, if $\tau \in [0, 2^{-r}q]$ then it can be derived $\mathcal{I}\psi_{2^r-1}(\tau) = 0$.

If $\tau \in [1 - 2^{-r}, 1]$ then

$$\mathcal{I}\psi_{2^r-1}(\tau) = \int_{1-2^{-r}}^{\tau} (2^r s - 2^r + 1) ds = 2^{r-1}\tau^2 - q\tau + 2^{-r-1}q^2. \tag{3.36}$$

If $\tau > 2^{-r}(q + 1)$ then

$$\mathcal{I}\psi_{2^r-1}(\tau) = \int_{1-2^{-r}}^1 (2^r s - 2^r + 1) ds = 2^{-r-1}. \tag{3.37}$$

Using Eqs. (3.31)–(3.37), one can write

$$\mathcal{I}\psi_q(\tau) = \begin{cases} \begin{cases} \tau - 2^{r-1}\tau^2, & \tau \in [0, \frac{1}{2^r}), \\ 2^{-r-1}, & \tau \geq \frac{1}{2^r}, \\ 0, & \text{otherwise,} \end{cases} & , \text{ for } q = -1 \\ \begin{cases} 2^{r-1}\tau^2 - q\tau + 2^{-r-1}q^2, & \tau \in [\frac{q}{2^r}, \frac{q+1}{2^r}), \\ 2^{-r-1}, & \tau \in [\frac{q+1}{2^r}, \frac{q+2}{2^r}), \\ 2^{-r}, & \tau \geq \frac{q+2}{2^r}, \\ 0, & \text{otherwise,} \end{cases} & , \text{ for } q = 0, 2^r - 2, \dots \\ \begin{cases} 2^{r-1}\tau^2 - (2^r - 1)\tau + 2^{-r-1}(2^r - 1)^2, & \tau \in [1 - \frac{1}{2^r}, 1), \\ 2^{-r-1}, & \tau \geq 1, \\ 0, & \text{otherwise,} \end{cases} & , \text{ for } q = 2^r - 1, \end{cases} \tag{3.38}$$

Using Eqs. (2.7)–(2.8) on (3.38), it can be obtained

$$\mathcal{I}\Psi(\tau) = \mathbf{I}\Psi(\tau), \tag{3.39}$$

wherein

$$\mathbf{I}_{p,q} = \mathcal{I}(\psi_q(\xi_p)), \quad \xi_p = \frac{p+1}{2^r}, \quad p, q = -1, \dots, 2^r - 1. \tag{3.40}$$

It can be proven that \mathbf{I} is a $(2^r + 1) \times (2^r + 1)$ matrix as follows

$$\mathbf{I} = \begin{bmatrix} 0 & \varrho & \varrho & \varrho & \dots & \varrho \\ & \varrho & 2\varrho & 2\varrho & \dots & 2\varrho \\ & & \varrho & 2\varrho & \dots & 2\varrho \\ & & & \ddots & \ddots & \vdots \\ & & & & \varrho & 2\varrho \\ & & & & & \varrho \end{bmatrix}, \tag{3.41}$$

where $\varrho = 2^{-r-1}$.

4 Numerical solution of FSI-D equation

In this section, a numerical scheme is presented for solving FSI-D Eq. (1.1).

From Definition 2.1, one can derive

$${}^C_{\tau_0} \mathcal{D}_\tau^\nu Y(\tau) = {}_{\tau_0} \mathcal{J}_\tau^{1-\nu} Y'(\tau), \quad 0 < \nu < 1. \tag{4.1}$$

Therefore Eq. (1.1) is written as follows

$${}_{\tau_0} \mathcal{J}_\tau^{1-\nu} Y'(\tau) = U(\tau) + \int_0^\tau \lambda_1(\tau, s) Y(s) ds + \int_0^\tau \lambda_2(\tau, s) Y(s) dB(s), \quad 0 < \nu < 1. \tag{4.2}$$

Now functions $Y(\tau)$, $U(\tau)$, $\lambda_1(\tau, s)$ and $\lambda_2(\tau, s)$ can be approximated by using LCB-S functions as the following form

$$\begin{aligned} Y(\tau) &\approx y^T \Psi(\tau) = \Psi^T(\tau) y, \\ U(\tau) &\approx u^T \Psi(\tau) = \Psi^T(\tau) u, \\ \lambda_1(\tau, s) &\approx \Psi^T(\tau) \lambda_1 \Psi(s) = \Psi^T(s) \lambda_1^T \Psi(\tau), \\ \lambda_2(\tau, s) &\approx \Psi^T(\tau) \lambda_2 \Psi(s) = \Psi^T(s) \lambda_2^T \Psi(\tau), \end{aligned} \tag{4.3}$$

wherein vector u from Eq. (2.9) and matrices λ_1, λ_2 from Eq. (2.12) can be obtained. Moreover, y is an unknown vector that must be found.

Considering Eq. (4.2) and using the presented operational matrices in the previous section, one can write Eq. (4.2) as follows

$$\begin{aligned}
 y^T \mathbf{D}\mathbf{J}_{1-\nu} \Psi(\tau) &= u^T \Psi(\tau) + \int_0^\tau \underbrace{\Psi^T(\tau)\lambda_1 \Psi(s)\Psi^T(s)}_{\tilde{y}^T \Psi(s)} ds + \int_0^\tau \underbrace{\Psi^T(\tau)\lambda_2 \Psi(s)\Psi^T(s)}_{\tilde{y}^T \Psi(s)} y d\mathcal{B}(s) \\
 &= u^T \Psi(\tau) + \Psi^T(\tau)\lambda_1 \tilde{y}^T \mathbf{I} \Psi(\tau) + \Psi^T(\tau)\lambda_2 \tilde{y}^T \mathbf{I}_s \Psi(\tau).
 \end{aligned}
 \tag{4.4}$$

Also, using the initial condition of problem (1.1), one can write

$$y^T \Psi(0) = 0. \tag{4.5}$$

Collocating Eq. (4.4) at the point $\xi_j = (j + 1)2^{-r}$, $j = 0, \dots, 2^r - 1$ and using Eq. (2.10) it can be obtained as

$$y^T \mathbf{D}\mathbf{J}_{1-\nu} e_j = u^T e_j + e_j^T \lambda_1 \tilde{y}^T \mathbf{I} e_j + e_j^T \lambda_2 \tilde{y}^T \mathbf{I}_s e_j. \tag{4.6}$$

Moreover from Eq. (4.5) it can be derived

$$y^T e_{-1} = 0. \tag{4.7}$$

Now, from Eq. (4.6) and Eq. (4.7), a linear system of equations is obtained which is calculated to find the unknown function $Y(\tau)$ in Eq. (4.3).

5 Convergence and error analysis

In this section, the error analysis of the proposed method is studied.

Theorem 5.1 Suppose $U(t) \in C^2[0, 1]$ and $\tilde{U}(t) = \sum_{q=-1}^{2^r-1} U(2^{-r}(q + 1))\psi_q(t)$ be the LCB-S approximation of $U(t)$, then the upper bound of error for the LCB-S method is obtained as

$$E(t) = \|U(t) - \tilde{U}(t)\| \leq \frac{1}{2 \times 2^{2r}} \|U''(t)\|, \tag{5.1}$$

eventually, it is derived that

$$E(t) = O\left(\frac{1}{2^{2r}}\right). \tag{5.2}$$

Proof Let

$$D_q = \left\{ t \mid \frac{q}{2^r} \leq t \leq \frac{q + 1}{2^r} \right\}, \quad q = 0, \dots, 2^r - 1$$

and define

$$E_q(t) = \begin{cases} U(t) - \tilde{U}(t), & t \in D_q, \\ 0, & \text{otherwise.} \end{cases} \tag{5.3}$$

Then it can be written that

$$\begin{aligned}
 E_q(t) &= U(t) - \sum_{z=-1}^{2^r-1} U(2^{-r}(z+1))\psi_z(t) \\
 &= U(t) - \left[U\left(\frac{q}{2^r}\right)\psi_{q-1}(t) + U\left(\frac{q+1}{2^r}\right)\psi_q(t) \right] \\
 &= U(t) - \left[U\left(\frac{q}{2^r}\right)(-2^r t + q + 1) + U\left(\frac{q+1}{2^r}\right)(2^r t - q) \right] \\
 &= U(t) - \left[U\left(\frac{q}{2^r}\right) - U\left(\frac{q}{2^r}\right)(2^r t - q) + U\left(\frac{q+1}{2^r}\right)(2^r t - q) \right] \\
 &= U(t) - \left[U\left(\frac{q}{2^r}\right) + 2^r\left(t - \frac{q}{2^r}\right)\left(U\left(\frac{q+1}{2^r}\right) - U\left(\frac{q}{2^r}\right)\right) \right] \\
 &= U(t) - \left[U\left(\frac{q}{2^r}\right) + \left(t - \frac{q}{2^r}\right)\frac{U\left(\frac{q+1}{2^r}\right) - U\left(\frac{q}{2^r}\right)}{2^r} \right].
 \end{aligned}$$

It is clear that when $\frac{1}{2^r} \rightarrow 0$, one can write

$$E_q(t) \simeq U(t) - \left[U\left(\frac{q}{2^r}\right) + \left(t - \frac{q}{2^r}\right)U'\left(\frac{q}{2^r}\right) \right]. \tag{5.4}$$

By replacing the Taylor expansion of $U(t)$ in terms of $\left(t - \frac{q}{2^r}\right)$ in Eq. (5.4), one can derive that

$$E_q(t) \simeq \frac{\left(t - \frac{q}{2^r}\right)^2}{2} U''(\zeta_q),$$

wherein $\zeta_q \in \left(\frac{q}{2^r}, \frac{q+1}{2^r}\right)$. Since $\left|t - \frac{q}{2^r}\right| < \frac{1}{2^r}$, then it is concluded that

$$|E_q(t)| \leq \frac{1}{2 \times 2^{2r}} |U''(\zeta_q)|.$$

Thus

$$\|E(t)\| = \sup_{t \in D_q} |E_q| \leq \frac{1}{2 \times 2^{2r}} \|U''(\zeta_q)\|. \tag{5.5}$$

Also from (5.5), it can be derived

$$\|E(t)\| = O\left(\frac{1}{2^{2r}}\right).$$

□

Theorem 5.2 Suppose $\lambda(t, s) \in C^2([0, 1] \times [0, 1])$ and

$$\tilde{\lambda}(t, s) = \sum_{q=-1}^{2^r-1} \sum_{p=-1}^{2^r-1} U(2^{-r}(q+1), 2^{-r}(p+1))\psi_q(t)\psi_p(s),$$

be the LCB-S approximation of $\lambda(t, s)$, then

$$E(t, s) = \|\lambda(t, s) - \tilde{\lambda}(t, s)\| \leq \frac{1}{2 \times 2^{2r}} \left(\left\| \frac{\partial^2 \lambda(t, s)}{\partial t^2} \right\| + 2 \left\| \frac{\partial^2 \lambda(t, s)}{\partial t \partial s} \right\| + \left\| \frac{\partial^2 \lambda(t, s)}{\partial s^2} \right\| \right), \tag{5.6}$$

finally, it is derived that

$$E(t, s) = O\left(\frac{1}{2^{2r}}\right). \tag{5.7}$$

Proof Let

$$D_{q,p} = \left\{ (t, s) \mid \frac{q}{2^r} \leq t \leq \frac{q+1}{2^r}, \frac{p}{2^r} \leq s \leq \frac{p+1}{2^r} \right\}, \quad q, p = 0, \dots, 2^r - 1.$$

It can be defined

$$E_{q,p}(t, s) = \begin{cases} \lambda(t, s) - \tilde{\lambda}(t, s), & t \in D_{q,p}, \\ 0, & \text{otherwise.} \end{cases} \tag{5.8}$$

Then one can write that

$$\begin{aligned} E_{q,p}(t, s) &= \lambda(t, s) - \sum_{z=-1}^{2^r-1} \sum_{l=-1}^{2^r-1} \lambda(2^{-r}(z+1), 2^{-r}(l+1)) \psi_z(t) \psi_l(s) \\ &= \lambda(t, s) - \left[\lambda\left(\frac{q}{2^r}, \frac{p}{2^r}\right) \psi_{q-1}(t) \psi_{p-1}(s) + \lambda\left(\frac{q+1}{2^r}, \frac{p}{2^r}\right) \psi_q(t) \psi_{p-1}(s) \right. \\ &\quad \left. + \lambda\left(\frac{q}{2^r}, \frac{p+1}{2^r}\right) \psi_{q-1}(t) \psi_p(s) + \lambda\left(\frac{q+1}{2^r}, \frac{p+1}{2^r}\right) \psi_q(t) \psi_p(s) \right]. \end{aligned} \tag{5.9}$$

In Eq. (5.9) after using two-dimensional Taylor expansion of $\lambda(t, s)$ in terms of $(t - \frac{q}{2^r})$ and $(s - \frac{p}{2^r})$, when $\frac{1}{2^r} \rightarrow 0$, as a result, it can be written that

$$E_{q,p}(t, s) \simeq \frac{1}{2!} \left(\left(t - \frac{q}{2^r}\right) \frac{\partial}{\partial t} + \left(t - \frac{p}{2^r}\right) \frac{\partial}{\partial s} \right)^2 \lambda(\zeta_q, \eta_p),$$

wherein $\zeta_q \in (\frac{q}{2^r}, \frac{q+1}{2^r})$ and $\eta_p \in (\frac{p}{2^r}, \frac{p+1}{2^r})$. Since $|t - \frac{q}{2^r}|, |t - \frac{p}{2^r}| < \frac{1}{2^r}$, then it can be concluded that

$$|E_{q,p}(t, s)| \leq \frac{1}{2 \times 2^{2r}} \left(\left| \frac{\partial^2 \lambda(\zeta_q, \eta_p)}{\partial t^2} \right| + 2 \left| \frac{\partial^2 \lambda(\zeta_q, \eta_p)}{\partial t \partial s} \right| + \left| \frac{\partial^2 \lambda(\zeta_q, \eta_p)}{\partial s^2} \right| \right).$$

Therefore

$$\begin{aligned} \|E(t, s)\| &= \sup_{(t,s) \in D_{q,p}} |E_{q,p}(t, s)| \\ &\leq \frac{1}{2 \times 2^{2r}} \sup_{(t,s) \in ([0,1] \times [0,1])} \left(\left| \frac{\partial^2 \lambda(t, s)}{\partial t^2} \right| + 2 \left| \frac{\partial^2 \lambda(t, s)}{\partial t \partial s} \right| + \left| \frac{\partial^2 \lambda(t, s)}{\partial s^2} \right| \right). \end{aligned}$$

As a result

$$\|E(t, s)\| \leq \frac{1}{2 \times 2^{2r}} \left(\left\| \frac{\partial^2 \lambda(t, s)}{\partial t^2} \right\| + 2 \left\| \frac{\partial^2 \lambda(t, s)}{\partial t \partial s} \right\| + \left\| \frac{\partial^2 \lambda(t, s)}{\partial s^2} \right\| \right). \tag{5.10}$$

Also from Eq. (5.10), it can be derived

$$\|E(t, s)\| = O\left(\frac{1}{2^{2r}}\right).$$

□

Theorem 5.3 Suppose $Y(\tau)$ and $\tilde{Y}(\tau)$ are the analytical solution and the numerical solution of Eq. (1.1), respectively. Also let

- (a) $\|Y(t)\| \leq \vartheta$,
- (b) $\|\lambda_j(\tau, s)\| \leq H_j, \quad j = 1, 2,$
- (c) $H_1 + S_1 + \vartheta S_2 \|B(t)\| < 1,$

then

$$\|Y(\tau) - \tilde{Y}(\tau)\| \leq \frac{\Upsilon + \vartheta S_1 + \vartheta S_2 \|B(t)\|}{1 - (H_1 + S_1 + \|B(t)\|(H_2 + S_2))},$$

wherein

$$\begin{aligned} \Upsilon &= \frac{1}{2 \times 2^{2r}} \|U''(\tau)\| \\ S_j &= \frac{1}{2 \times 2^{2r}} \left(\left\| \frac{\partial^2 \lambda_j(\tau, s)}{\partial \tau^2} \right\| + 2 \left\| \frac{\partial^2 \lambda_j(\tau, s)}{\partial \tau \partial s} \right\| + \left\| \frac{\partial^2 \lambda_j(\tau, s)}{\partial s^2} \right\| \right), \quad j = 1, 2. \end{aligned}$$

Proof Consider Eq. (1.1) in special case $\nu = 0$. It is clear that

$$\begin{aligned} Y(\tau) - \tilde{Y}(\tau) &= U(\tau) - \tilde{U}(\tau) + \int_0^\tau (\lambda_1(\tau, s)Y(s) - \tilde{\lambda}_1(\tau, s)\tilde{Y}(s))ds \\ &\quad + \int_0^\tau (\lambda_2(\tau, s)Y(s) - \tilde{\lambda}_2(\tau, s)\tilde{Y}(s))dB(s), \end{aligned}$$

consequently

$$\|Y - \tilde{Y}\| \leq \|U - \tilde{U}\| + \tau \|\lambda_1 Y - \tilde{\lambda}_1 \tilde{Y}\| + \|B(\tau)\| \|\lambda_2 Y - \tilde{\lambda}_2 \tilde{Y}\|. \tag{5.11}$$

Now, using Theorems (5.1)–(5.2) and assumptions (a) and (b), one can get

$$\begin{aligned} \|\lambda_j Y - \tilde{\lambda}_j \tilde{Y}\| &= \|\lambda_j Y - \lambda_j \tilde{Y} + \lambda_j \tilde{Y} - \tilde{\lambda}_j \tilde{Y}\| \\ &= \|\lambda_j(Y - \tilde{Y}) + \lambda_j(\tilde{Y} - Y) - \tilde{\lambda}_j(\tilde{Y} - Y) + \lambda_j Y - \tilde{\lambda}_j Y\| \\ &\leq \|\lambda_j\| \|(Y - \tilde{Y})\| + \|\lambda_j - \tilde{\lambda}_j\| (\|Y - \tilde{Y}\| + \|Y\|) \\ &\leq (H_j + S_j)\|Y - \tilde{Y}\| + \vartheta S_j, \quad j = 1, 2. \end{aligned} \tag{5.12}$$

Replacing Eq. (5.12) in Eq. (5.11), one can obtain

$$\|Y(\tau) - \tilde{Y}(\tau)\| \leq \frac{\Upsilon + \vartheta S_1 + \vartheta S_2 \|\mathcal{B}(t)\|}{1 - (H_1 + S_1 + \|\mathcal{B}(t)\|(H_2 + S_2))}. \tag{5.13}$$

Using assumption (c) in Eq. (5.13), it can be derived as

$$\|Y(\tau) - \tilde{Y}(\tau)\| = O\left(\frac{1}{2^{2r}}\right). \tag{5.14}$$

□

6 Numerical verifications

In this section first, two numerical examples are reported in order to indicate the effectiveness of the proposed method. It is worth noting that Example 1 has the exact solution in a particular case, so the main focus is on this example to show the potential and efficiency of the proposed method. Next, numerical simulations and results of perturbation on the main problem are reported. In the presented tables the values of maximum absolute error and RMS-error are computed as

$$\begin{aligned} \|e\|_\infty &= \max_{t \in [0,1]} |Y(\tau) - Y_N(\tau)|, \\ \text{RMS-error} &= \sqrt{\frac{1}{N} \sum_{j=1}^N (Y(\tau) - Y_N(\tau))^2}, \end{aligned}$$

wherein $Y(\tau)$ and $Y_N(\tau)$ represent the exact solution and the numerical solution, respectively. Also, in these tables N indicates the number of the interpolation points used to approximate the solution function and it is as $N = 2^r + 1$ for proposed method. The numerical experiments are performed on a computer Intel(R) Pentium(R) CPU G2030 @ 3.00GHz 3.00GHz 4.00GB RAM by running some codes written in Maple 18 software.

6.1 Numerical example

Example 1. Consider the FSI-D equation as

$${}_0^C D_t^\nu Y(\tau) = -\frac{\tau^5 e^\tau}{5} + \frac{6\tau^{3-\nu}}{\Gamma(4-\nu)} + \int_0^\tau e^\tau {}_s Y(s) ds + \sigma \int_0^\tau e^\tau {}_s Y(s) d\mathcal{B}(s), \quad s, \tau \in [0, 1], \tag{6.1}$$

with the initial condition $Y(0) = 0$. Eq. (6.1) does not have an analytical solution. However, in the case $\sigma = 0$ the function $Y(t) = t^3$ is the analytical solution for problem (6.1). This example has been used in [20,25].

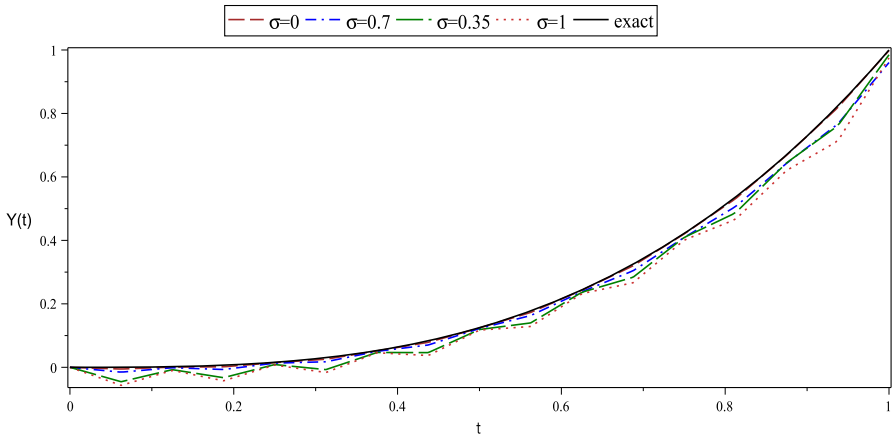


Fig. 1 The comparison of analytical solution in case $\sigma = 0$ and the numerical solutions for case $r = 4, v = 0.85$ and $\sigma = 0, 0.35, 0.70, 1$, for Example 1

In order to compare the numerical solution with the exact solution of Eq. (6.1), suppose $\sigma = 0$. In Table 1, the results of this experiment are listed for $r = 5, v = 0.35, 0.65, 0.95$. According to Table 1, it is clear that the proposed method has the considerable accuracy. In Table 2, the new scheme compared with the two different methods based on radial basis functions in [28]. The results show that the run times for the proposed method are lower than the presented methods in [28]. Furthermore, in last column of Table 2, the values of $\frac{1}{2^{2r}}$ are listed for different values of r . Comparing this column with the column $\|e\|_\infty$, one can result that $\|e\|_\infty < \frac{1}{2^{2r}}$. So, the numerical results in this table confirm the presented theoretical in the previous section. Therefore, the new scheme can be used as the practical and efficient method for solving the fractional stochastic integro-differential equation. In Table 3 the LCB-S method is compared with the Bernstein [25] and the Cubic B-spline methods [20]. As a result of this table, it can be derived that the Bernstein and the Cubic B-spline methods have only good accuracy near the initial condition $Y(0) = 0$. In other words, the accuracy of these methods is greatly reduced by moving away from $\tau = 0$ and approaching to $\tau = 1$. Unlike these two methods, the proposed method has good accuracy over the whole interval $[0, 1]$. In Table 4, the numerical solutions are reported for case $\sigma = 1$ with $r = 4$ and various values of v . In Fig. 1, the exact solution (in case $\sigma = 0$) is compared with the numerical solutions for case $r = 4, v = 0.85$ and $\sigma = 0, 0.35, 0.70, 1$.

Example 2. Consider FSI-D equation as follows [20,25,28]

$${}^C_0 D_\tau^\nu Y(\tau) = \frac{7}{12}\tau^4 - \frac{5}{6}\tau^3 + \frac{2\tau^{2-\nu}}{\Gamma(3-\nu)} + \frac{\tau^{1-\nu}}{\Gamma(2-\nu)} + \int_0^\tau (s + \tau)Y(s)ds + \int_0^\tau sY(s)d\mathcal{B}(s),$$

$s, \tau \in [0, 1],$ (6.2)

with the initial condition $Y(0) = 0$. Equation (6.2) does not have any analytical solution. Therefore, a numerical solution as an efficient approximation would be useful

Table 1 Analytical solutions of Example 1 for $\sigma = 0$, $r = 5$ and various values of ν

τ_i	$\nu = 0.35$	$\nu = 0.65$	$\nu = 0.95$	$exact(\sigma = 0)$
0.0	0.00000	0.00000	0.00000	0.00000
0.1	0.000709	0.00047	0.00041	0.00100
0.2	0.00812	0.00797	0.00709	0.00800
0.3	0.02715	0.02702	0.02617	0.02700
0.4	0.06369	0.06357	0.06275	0.06400
0.5	0.06369	0.06357	0.06275	0.12500
0.6	0.21525	0.21561	0.21481	0.21600
0.7	0.34330	0.34318	0.34237	0.34300
0.8	0.51237	0.51223	0.51139	0.51200
0.9	0.72887	0.72869	0.72779	0.72900
1.0	1.00031	1.00006	0.99907	1.00000

Table 2 Comparison the results of the proposed method with presented methods in [28] for Example 1 in cases $\sigma = 0$, $\nu = 0.75$

Method	N	RMS-error	$\ e\ _\infty$	CPU time(s)	$(r, \frac{1}{2^r})$
GA RBF [28]	10	$7.6250e-2$	$8.6021e-2$	041.24	-
	20	$7.2965e-3$	$4.1085e-3$	072.68	-
	30	$1.2593e-4$	$5.2760e-4$	125.45	-
	40	$8.6523e-4$	$3.5988e-4$	150.78	-
TPS RBF [28]	10	$4.6257e-2$	$3.7045e-2$	046.75	-
	20	$4.8967e-3$	$5.6048e-3$	078.87	-
	30	$5.6321e-4$	$8.3263e-4$	128.78	-
	40	$2.3569e-4$	$4.3341e-4$	157.31	-
proposed method	5	$3.2818e-2$	$5.8406e-2$	000.65	(2, $6.2500e-2$)
	9	$7.1475e-3$	$1.4517e-2$	000.76	(3, $1.5625e-2$)
	17	$1.5487e-3$	$2.7413e-3$	001.85	(4, $3.9062e-3$)
	33	$3.4746e-4$	$7.0342e-4$	009.65	(5, $9.7656e-4$)
	65	$7.9048e-5$	$1.4240e-4$	106.72	(6, $2.4414e-4$)

and interesting. Here, the numerical solutions for case $r = 3$ and various values of ν together with the numerical solutions of [20,25] are listed in Table 5. Also, in Table 6, the numerical solutions of proposed method are compared with the presented methods in [28] at $\tau = 0.9$. Furthermore, in this table run times for the mention methods are listed. According to the Table 6, it is clear that the computational speed of the proposed method is higher than the compared methods.

6.2 Perturbation on the main problem

In this part, firstly, the initial condition is prturbed as following form.

Table 3 Comparison absolute errors of the proposed method with presented methods in [20,25] for Example 1 in cases $\sigma = 0, \nu = 0.75$

τ_i	Bernstein method [25]	Cubic B-spline [20]	proposed method	
	$n = 10$	$N = 10$	$r = 3$	$r = 4$
0.0	$5.7393e - 06$	$1.3724e - 41$	0.0000	0.0000
0.1	$3.0450e - 05$	$1.9763e - 08$	$1.4517e - 02$	$1.6408e - 03$
0.2	$3.7718e - 04$	$1.3097e - 06$	$5.9860e - 03$	$2.7413e - 03$
0.3	$1.4855e - 03$	$1.5733e - 05$	$3.8506e - 03$	$3.5668e - 03$
0.4	$3.3888e - 03$	$9.5404e - 05$	$1.0513e - 02$	$4.4285e - 04$
0.5	$4.7520e - 03$	$3.9813e - 04$	$1.3340e - 03$	$9.9904e - 05$
0.6	$1.2900e - 03$	$1.3125e - 03$	$9.9830e - 03$	$9.4177e - 05$
0.7	$1.6763e - 02$	$3.6800e - 03$	$6.7825e - 04$	$2.1132e - 04$
0.8	$6.9431e - 02$	$91738e - 03$	$1.7282e - 04$	$2.0850e - 05$
0.9	$1.9292e - 01$	$2.0890e - 02$	$8.7125e - 03$	$4.4967e - 04$
1.0	Not reported	Not reported	$3.4921e - 02$	$5.9473e - 04$

Table 4 Numerical solutions of Example 1 for $\sigma = 1, r = 4$ and various values of ν

τ_i	$\nu = 0.1$	$\nu = 0.25$	$\nu = 0.50$	$\nu = 0.75$	$\nu = 0.90$
0.0	0.00000	0.00000	0.00000	0.00000	0.00000
0.1	-0.00384	-0.00151	-0.00296	-0.02946	-0.03984
0.2	-0.01821	-0.00673	-0.04381	-0.04131	-0.04589
0.3	0.04375	0.02493	-0.02182	-0.01966	-0.02535
0.4	0.05603	0.06202	0.07437	0.04568	0.03004
0.5	0.06586	0.11607	0.19664	0.13281	0.10680
0.6	0.20348	0.19519	0.23838	0.19759	0.18030
0.7	0.33637	0.30873	0.29948	0.28757	0.28281
0.8	0.41840	0.37093	0.43000	0.44057	0.43616
0.9	0.58229	0.59884	0.66054	0.68105	0.64872
1.0	1.35244	1.15167	1.26888	1.01847	0.93240

Consider the FSI-D equation of Example 1 as

$$\begin{aligned}
 {}_0^C D_\tau^\nu Y(\tau) &= -\frac{\tau^5 e^\tau}{5} + \frac{6\tau^{3-\nu}}{\Gamma(4-\nu)} + \int_0^\tau e^\tau s Y(s) ds + \sigma \int_0^\tau e^\tau s Y(s) dB(s), \quad s, \tau \in [0, 1], \\
 Y(0) &= \varepsilon.
 \end{aligned}
 \tag{6.3}$$

Some numerical results, for different values of perturbation in the initial condition are presented in Table 7. This table shows that the new scheme preserves the stability of the initial value problem, with respect to small perturbations in the initial condition.

Secondly, the function $U(t)$ of the main problem is perturbed as following form.

Table 5 Numerical solutions of Example 2 for case $r = 3$ and different values of ν

τ_i	Bernstein method [25]		Cubic B-spline [20]		proposed method	
	$\nu = 0.25$	$\nu = 0.5$	$\nu = 0.25$	$\nu = 0.5$	$\nu = 0.25$	$\nu = 0.5$
0.0	0.35736	0.44108	0.13954	0.13574	0.00000	0.00000
0.1	0.44779	0.52717	0.12724	0.12496	0.10795	0.09904
0.2	0.56918	0.64472	0.26169	0.25122	0.23819	0.23833
0.3	0.72154	0.79375	0.40975	0.38139	0.37969	0.38199
0.4	0.90486	0.97424	0.55885	0.54822	0.53859	0.53156
0.5	1.11915	1.18621	0.67747	0.81460	0.71458	0.72597
0.6	1.36441	1.42964	0.76271	1.12205	0.85422	0.91911
0.7	1.64064	1.70455	0.92948	1.72096	1.11064	1.13434
0.8	1.94784	2.01092	1.31664	1.74094	1.25665	1.35202
0.9	2.28600	2.34877	1.93675	1.74917	1.41272	1.59635

Table 6 Comparison the results of the proposed method with presented methods in [28] for Example 2

Method	N	Numerical solution at $\tau = 0.9$		CPU time(s)	
		$\nu = 0.25$	$\nu = 0.5$	$\nu = 0.25$	$\nu = 0.5$
GA RBF [28]	10	0.6452	0.8456	36.54	38.71
	20	2.2058	1.1548	68.81	69.54
TPS RBF [28]	10	0.6328	0.8542	39.25	40.25
	20	2.2385	1.1521	71.38	73.15
proposed method	9	1.4341	1.5110	01.05	01.04
	17	1.5348	1.5859	01.90	01.68
	33	1.5492	1.6466	09.26	09.15

Table 7 Numerical solutions of Example 1, for $\sigma = 1, r = 4$ and $\nu = 0.4$, after perturbing the initial condition

τ_i	$\varepsilon = 0.00$	$\varepsilon = 0.01$	$\varepsilon = 0.05$	$\varepsilon = 0.10$	$\varepsilon = 0.20$
0.0	0.00000	0.01000	0.05000	0.10000	0.20000
0.1	-0.00414	0.01198	0.06028	0.10581	0.21090
0.2	-0.01372	0.01098	-0.00829	0.07333	0.13899
0.3	0.00325	0.02886	0.00811	0.09090	0.14981
0.4	0.06612	0.07255	0.12962	0.17555	0.27427
0.5	0.15169	0.14043	0.27992	0.28520	0.44387
0.6	0.20206	0.21691	0.26980	0.34105	0.44451
0.7	0.28542	0.22303	0.24691	0.41074	0.50161
0.8	0.40408	0.47298	0.31739	0.55251	0.68552
0.9	.628197	0.64423	0.61392	0.79393	0.94067
1.0	1.34837	0.86176	1.27730	1.19337	1.87645

Table 8 Numerical solutions of Example 1, for $\sigma = 0.6, r = 5$ and $\nu = 0.25$, after perturbing the function $U(\tau) = -\frac{\tau^5 e^\tau}{5} + \frac{6\tau^{3-\nu}}{\Gamma(4-\nu)}$

τ_i	$\varepsilon = 0.00$	$\varepsilon = 0.01$	$\varepsilon = 0.05$	$\varepsilon = 0.10$	$\varepsilon = 0.20$
0.0	0.00000	0.00000	0.00000	0.00000	0.00000
0.1	-0.00034	0.04339	-0.00488	-0.02273	-0.00670
0.2	0.00849	0.02345	0.01040	0.01705	0.01244
0.3	0.02725	0.04749	0.03034	0.04038	0.03590
0.4	0.06038	-0.00154	0.05610	0.03408	0.06315
0.5	0.12356	0.23236	0.14310	0.19376	0.17134
0.6	0.20623	0.12753	0.20154	0.18839	0.23327
0.7	0.32368	0.33987	0.35055	0.37812	0.40606
0.8	0.47848	0.47849	0.50037	0.55662	0.60518
0.9	0.67925	0.52789	0.69108	0.62306	0.77484
1.0	0.99761	1.04386	1.06075	1.63274	1.12484

Table 9 Numerical solutions of Example 1, for $\sigma = 0.85, r = 4$ and $\nu = 0.60$, after perturbing the function $\lambda_2(\tau, s) = se^\tau$

τ_i	$\varepsilon = 0.00$	$\varepsilon = 0.01$	$\varepsilon = 0.05$	$\varepsilon = 0.10$	$\varepsilon = 0.20$
0.0	0.00000	0.00000	0.00000	0.00000	0.00000
0.1	-0.02027	0.00393	0.00013	-0.00350	-0.00448
0.2	-0.05849	0.01668	0.00498	-0.00641	0.00928
0.3	-0.03546	0.03493	0.02409	0.01269	0.01059
0.4	0.05920	0.06387	0.06378	0.06134	0.06173
0.5	0.17799	0.11399	0.12494	0.13120	0.13456
0.6	0.21367	0.20903	0.20930	0.21008	0.20788
0.7	0.26974	0.23694	0.32088	0.31265	0.30412
0.8	0.41531	0.48355	0.46340	0.46370	0.45981
0.9	0.68571	0.67971	0.66341	0.65253	0.66512
1.0	1.10678	0.94218	0.95018	0.97472	1.07146

Consider the FSI-D equation of Example 1 as

$${}_0^C D_\tau^\nu Y(\tau) = \left(-\frac{\tau^5 e^\tau}{5} + \frac{6\tau^{3-\nu}}{\Gamma(4-\nu)}\right)(1 + \varepsilon) + \int_0^\tau e^\tau s Y(s) ds + \sigma \int_0^\tau e^\tau s Y(s) dB(s), \quad s, \tau \in [0, 1],$$

$Y(0) = 0$.

The numerical solutions, for different values of perturbation of the function $U(t)$ are listed in Table 8.

Finally, the function $\lambda_2(\tau, s)$ of the main problem is perturbed as following form. Consider the FSI-D equation of Example 1 as

$${}^C_0 D_\tau^\nu Y(\tau) = -\frac{\tau^5 e^\tau}{5} + \frac{6\tau^{3-\nu}}{\Gamma(4-\nu)} + \int_0^\tau e^\tau s Y(s) ds + \sigma \int_0^\tau (e^\tau s Y(s))(1+\varepsilon) dB(s), \quad s, \tau \in [0, 1],$$

$$Y(0) = 0.$$

The numerical results, for different values of perturbation of the function $\lambda_2(\tau, s)$ are reported in Table 9. From Tables 7, 8 and 9, it is clear that a small error in the input data leads to a small error in the solution of the problem, and this can confirm the stability of the proposed method.

7 Conclusion

In this study, a numerical scheme for solving the stochastic integro-differential equation of fractional order was developed. For this purpose, linear cardinal B-spline functions were applied. Also, the convergence and error analysis of the proposed method is studied. The reported results indicated that the problem can be solved effectively by the proposed approach.

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