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ORIGINAL RESEARCH

$\mathbb{Z}_p\mathbb{Z}_p[v]$-additive cyclic codes are asymptotically good

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Abstract
We construct a class of $\mathbb{Z}_p\mathbb{Z}_p[v]$-additive cyclic codes, where p is a prime number and $v^2 = v$. We determine the asymptotic properties of the relative minimum distance and rate of this class of codes. We prove that, for any positive real number $0 < \delta < 1$ such that the p-ary entropy at $\frac{k+l}{2}\delta$ is less than $\frac{1}{2}$, the relative minimum distance of the random code is convergent to δ and the rate of the random code is convergent to $\frac{1}{k+l}$, where p, k, l are pairwise coprime positive integers.

Keywords $\mathbb{Z}_p\mathbb{Z}_p[v]$-additive cyclic codes · Relative minimum distance · Rate · Asymptotically good codes

Mathematics Subject Classification 94B05 · 94B65

1 Introduction

Additive codes are important error-correcting codes in coding theory. In 1998, Delsarte firstly gave the definition of additive codes in [9]. Afterwards, many coding scientists paid their attentions on additive codes. Recently, $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes were studied impressed [1,6–8] including generator matrix, minimum generating sets, codes construction and so on. From then on, there are many papers on additive codes. Aydogdu et al. studied properties of $\mathbb{Z}_2\mathbb{Z}_2[u]$-additive cyclic codes and $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-additive cyclic codes in [3,4], respectively. Diao et al. studied $\mathbb{Z}_p\mathbb{Z}_p[v]$-additive cyclic codes in [10]. Many good linear codes and quantum codes were constructed by $\mathbb{Z}_p\mathbb{Z}_p[v]$-additive cyclic codes.

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The asymptotic property is an important index of good codes. A class of codes is said to be asymptotically good if there exist a sequence of codes $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \ldots$ with length n_i, when $n_i \rightarrow \infty$, both the relative minimum distance and the rate of $\mathcal{C}_i$ are positively bounded from below. Assmus et al. had already studied the problem of the asymptotic property of cyclic codes in [2]. Afterwards, Kasami proved that quasi-cyclic codes of index 2 are asymptotically good in [15]. Bazzi et al. proved that random binary quasi-abelian codes of index 2 and random binary dihedral group codes are asymptotically good [5]. Martínez-Pérez et al. proved that self-dual doubly even 2-quasi-cyclic transitive codes are asymptotically good [16]. Fan and Liu proved that the quasi-cyclic codes of fractional index between 1 and 2 are asymptotically good in [12]. Mi et al. proved that quasi-cyclic codes of fractional index are also asymptotically good [17]. In [14], we proved that $\mathbb{Z}_4$-double cyclic codes are asymptotically good.

In recent years, the asymptotic property of additive cyclic codes has been studied more widely. In [18], Shi et al. proved the existence of asymptotically good additive cyclic codes. Fan and Liu proved that $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes are asymptotically good [11]. Following [11], Yao et al. proved that $\mathbb{Z}_p\mathbb{Z}_{p^s}$-additive cyclic codes and $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-additive cyclic codes with $1 \leq r < s$ are asymptotically good in [19,20], respectively. Note that all of the rings mentioned above are finite chain rings. To the best of our knowledge, there is no any study on asymptotic property of additive cyclic codes over the finite non-chain ring $\mathbb{Z}_p \times (\mathbb{Z}_p + v\mathbb{Z}_p)$ with $v^2 = v$. Moreover, the well known results on asymptotic property of additive cyclic codes are with the same component length. So in this paper, we will study the asymptotic property of $\mathbb{Z}_p\mathbb{Z}_p[v]$-additive cyclic codes with the different component length.

The rest of this paper is organized as follows. In Sect. 2, we firstly give some results on $\mathbb{Z}_p\mathbb{Z}_p[v]$-additive cyclic codes. In Sect. 3, we construct a class of $\mathbb{Z}_p\mathbb{Z}_p[v]$-additive cyclic codes. In Sect. 4, by the probabilistic method and the Chinese remainder theorem, we prove that constructed $\mathbb{Z}_p\mathbb{Z}_p[v]$-additive cyclic codes are asymptotically good.

2 $\mathbb{Z}_p\mathbb{Z}_p[v]$-additive cyclic codes

Let $\mathbb{Z}_p$ be the prime field of p elements, where p is a prime. Let

$$\mathbb{Z}_p[v] = \mathbb{Z}_p + v\mathbb{Z}_p = \{va + (1-v)b | a, b \in \mathbb{Z}_p\},$$

where $v^2 = v$. Clearly, $\mathbb{Z}_p$ is a subring of ring $\mathbb{Z}_p[v]$. For any element $d \in \mathbb{Z}_p[v]$, it can be expressed as $d = va + (1-v)b$, where $a, b \in \mathbb{Z}_p$. Define a map

$$\begin{aligned} \pi : \mathbb{Z}_p[v] &\longrightarrow \mathbb{Z}_p \\ d = va + (1-v)b &\longmapsto a. \end{aligned}$$

Obviously, π is a ring homomorphism.

Define $\mathbb{Z}_p^\alpha$ to be α-tuples over $\mathbb{Z}_p$ and $\mathbb{Z}_p[v]^\beta$ to be β-tuples over $\mathbb{Z}_p[v]$, where α and β are positive integers. Let $\varsigma = (c, c') \in \mathbb{Z}_p^\alpha \times \mathbb{Z}_p[v]^\beta$ be a vector, where

$c = (c_0, c_1, \ldots, c_{\alpha-1}) \in \mathbb{Z}_p^\alpha$ and $c' = (c'_0, c'_1, \ldots, c'_{\beta-1}) \in \mathbb{Z}_p[v]^\beta$. For any $d = va + (1-v)b \in \mathbb{Z}_p[v]$, define a $\mathbb{Z}_p[v]$-scalar multiplication on $\mathbb{Z}_p^\alpha \times \mathbb{Z}_p[v]^\beta$ as

$$\begin{aligned} d\varsigma &= (\pi(d)c_0, \pi(d)c_1, \ldots, \pi(d)c_{\alpha-1}, dc'_0, dc'_1, \ldots, dc'_{\beta-1}) \\ &= (ac_0, ac_1, \ldots, ac_{\alpha-1}, dc'_0, dc'_1, \ldots, dc'_{\beta-1}). \end{aligned}$$

One can verify that, under the above $\mathbb{Z}_p[v]$-scalar multiplication and the usual addition of vectors, the $\mathbb{Z}_p^\alpha \times \mathbb{Z}_p[v]^\beta$ forms a $\mathbb{Z}_p[v]$-module.

Definition 1 A non-empty subset $\mathcal{C}$ of $\mathbb{Z}_p^\alpha \times \mathbb{Z}_p[v]^\beta$ is called a $\mathbb{Z}_p\mathbb{Z}_p[v]$-additive code of length $n = \alpha + \beta$ if $\mathcal{C}$ is a $\mathbb{Z}_p[v]$-submodule of $\mathbb{Z}_p^\alpha \times \mathbb{Z}_p[v]^\beta$.

Definition 2 The $\mathbb{Z}_p[v]$-submodule $\mathcal{C}$ of $\mathbb{Z}_p^\alpha \times \mathbb{Z}_p[v]^\beta$ is called a $\mathbb{Z}_p\mathbb{Z}_p[v]$-additive cyclic code of length $n = \alpha + \beta$ if for any codeword

$$\varsigma = (c_0, c_1, \ldots, c_{\alpha-1}, c'_0, c'_1, \ldots, c'_{\beta-1}) \in \mathcal{C},$$

Then $(c_{\alpha-1}, c_0, \ldots, c_{\alpha-2}, c'_{\beta-1}, c'_0, \ldots, c'_{\beta-2})$ is also in $\mathcal{C}$.

Define a generalized Gray map

$$\begin{aligned} \Phi : \mathbb{Z}_p^\alpha \times \mathbb{Z}_p[v]^\beta &\longrightarrow \mathbb{Z}_p^{\alpha+2\beta} \\ \varsigma = (c, c') &\longmapsto (c, \phi(c')), \end{aligned}$$

where ϕ is a Gray map defined by

$$\begin{aligned} \phi : \mathbb{Z}_p[v] &\longrightarrow \mathbb{Z}_p^2 \\ d = va + (1-v)b &\longmapsto (a+b, a-b). \end{aligned}$$

Obviously, if $\mathcal{C}$ is a $\mathbb{Z}_p\mathbb{Z}_p[v]$-additive code of length $n = \alpha + \beta$, then the generalized Gray image $\Phi(\mathcal{C})$ is a linear code of length $\alpha + 2\beta$ over $\mathbb{Z}_p$.

Let $\varsigma = (c, c') = (c_0, c_1, \ldots, c_{\alpha-1}, c'_0, c'_1, \ldots, c'_{\beta-1}) \in \mathbb{Z}_p^\alpha \times \mathbb{Z}_p[v]^\beta$. The Gray weight of ς is defined as $wt_G(\varsigma) = wt_H(\Phi(\varsigma))$, where wt_H denotes the Hamming weight. Further, for any $\varsigma_1, \varsigma_2 \in \mathbb{Z}_p^\alpha \times \mathbb{Z}_p[v]^\beta$, the Gray distance between ς_1 and ς_2 is defined as $d_G(\varsigma_1, \varsigma_2) = wt_G(\varsigma_1 - \varsigma_2)$. Moreover, if $\mathcal{C}$ is a $\mathbb{Z}_p\mathbb{Z}_p[v]$-additive code, then the minimum Gray weight and the minimum Gray distance of $\mathcal{C}$ are defined to be $wt_G(\mathcal{C}) = \min\{wt_G(\varsigma) | \varsigma \in \mathcal{C}, \varsigma \neq 0\}$ and $d_G(\mathcal{C}) = \min\{wt_G(x-y)|, x, y \in \mathcal{C}, x \neq y\}$, respectively. Note that since $\mathbb{Z}_p\mathbb{Z}_p[v]$-additive code $\mathcal{C}$ is a $\mathbb{Z}_p[v]$-submodule, then $d_G(\mathcal{C}) = wt_G(\mathcal{C})$.

Let $\mathbb{R}_{\alpha,\beta} = \mathbb{Z}_p[x]/\langle x^\alpha - 1\rangle \times \mathbb{Z}_p[v][x]/\langle x^\beta - 1\rangle$. Define the following one-to-one correspondence

$$\begin{aligned} \Psi : \mathbb{Z}_p^\alpha \times \mathbb{Z}_p[v]^\beta &\longrightarrow \mathbb{R}_{\alpha,\beta} \\ \varsigma = (c_0, c_1, \ldots, c_{\alpha-1}, c'_0, c'_1, \ldots, c'_{\beta-1}) &\longmapsto \varsigma(x) = (c(x), c'(x)), \end{aligned}$$

where $c(x) = c_0 + c_1x + \cdots + c_{\alpha-1}x^{\alpha-1}$ and $c'(x) = c'_0 + c'_1x + \cdots + c'_{\beta-1}x^{\beta-1}$.

Let $d(x) = d_0 + d_1x + \cdots + d_tx^t \in \mathbb{Z}_p[v][x]$ and $\varsigma(x) = (c(x), c'(x)) \in \mathbb{R}_{\alpha,\beta}$. Define the following $\mathbb{Z}_p[v][x]$-scalar multiplication

$$d(x) * \varsigma(x) = d(x) * (c(x), c'(x)) = (\pi(d(x))c(x), d(x)c'(x)), \tag{1}$$

where $\pi(d(x)) = \pi(d_0) + \pi(d_1)x + \cdots + \pi(d_t)x^t$. Under the above $\mathbb{Z}_p[v][x]$-scalar multiplication and the usual addition of polynomials, $\mathbb{R}_{\alpha,\beta}$ forms a $\mathbb{Z}_p[v][x]$-module.

Theorem 1 *The code $\mathcal{C}$ is a $\mathbb{Z}_p\mathbb{Z}_p[v]$-additive cyclic code if and only if $\Psi(\mathcal{C})$ is a $\mathbb{Z}_p[v][x]$-submodule of $\mathbb{R}_{\alpha,\beta}$.*

Proof For any codeword $\varsigma = (c_0, c_1, \ldots, c_{\alpha-1}, c'_0, c'_1, \ldots, c'_{\beta-1}) \in \mathcal{C} \subseteq \mathbb{Z}_p^\alpha \times \mathbb{Z}_p[v]^\beta$, it can be viewed as a polynomial $\varsigma(x) = (c(x), c'(x)) \in \Psi(\mathcal{C}) \subseteq \mathbb{R}_{\alpha,\beta}$, where $c(x) = c_0 + c_1x + \cdots + c_{\alpha-1}x^{\alpha-1}$ and $c'(x) = c'_0 + c'_1x + \cdots + c'_{\beta-1}x^{\beta-1}$. From the Eq. (1), we have

$$x * \varsigma(x) = (c_{\alpha-1} + c_0x + \cdots + c_{\alpha-2}x^{\alpha-2}, c'_{\beta-1} + c'_0x + \cdots + c'_{\beta-2}x^{\beta-2}) \in \Psi(\mathcal{C}),$$

which implies that $(c_{\alpha-1}, c_0, \ldots, c_{\alpha-2}, c'_{\beta-1}, c'_0, \ldots, c'_{\beta-2}) \in \mathcal{C}$. Thus, $\mathcal{C}$ is a $\mathbb{Z}_p\mathbb{Z}_p[v]$-additive cyclic code.

Conversely, if $\mathcal{C}$ is a $\mathbb{Z}_p\mathbb{Z}_p[v]$-additive cyclic code, then by Definition 1, $\mathcal{C}$ is a $\mathbb{Z}_p[v]$-submodule of $\mathbb{Z}_p^\alpha \times \mathbb{Z}_p[v]^\beta$. Thus, by the definition of Ψ, $\Psi(\mathcal{C}) \subseteq \mathbb{R}_{\alpha,\beta}$ is a $\mathbb{Z}_p[v][x]$-submodule of $\mathbb{R}_{\alpha,\beta}$. □

In the following, we identify $\mathbb{Z}_p\mathbb{Z}_p[v]$-additive cyclic codes of length $n = \alpha + \beta$ with $\mathbb{Z}_p[v][x]$-submodules of $\mathbb{R}_{\alpha,\beta}$.

3 A class of $\mathbb{Z}_p\mathbb{Z}_p[v]$-additive cyclic codes

In this section, we will construct a new class of $\mathbb{Z}_p\mathbb{Z}_p[v]$-additive cyclic codes. We always assume that $\alpha = km$ and $\beta = lm$, where m is a positive integer such that $\gcd(m, p) = 1$ and p, k, l are pairwise coprime positive integers.

Define

$$\mathbb{R}_{km} = \mathbb{Z}_p[x]/\langle x^{km} - 1\rangle, \quad \mathbb{R}_{lm} = \mathbb{Z}_p[x]/\langle x^{lm} - 1\rangle, \quad \mathbb{R}'_{lm} = \mathbb{Z}_p[v][x]/\langle x^{lm} - 1\rangle.$$

By the Chinese remainder theorem, it is well known that

$$\mathbb{Z}_p[v] = v\mathbb{Z}_p[v] \oplus (1 - v)\mathbb{Z}_p[v] = v\mathbb{Z}_p \oplus (1 - v)\mathbb{Z}_p. \tag{2}$$

Therefore, we have $v\mathbb{Z}_p[v] = v\mathbb{Z}_p \subset \mathbb{Z}_p[v]$. Let

$$v\mathbb{R}'_{lm} = \left\{c'(x) = \sum_{i=0}^{lm-1} c'_ix^i \in \mathbb{R}'_{lm} \mid c'_i = va_i,\ a_i \in \mathbb{Z}_p\right\},$$

which is a $\mathbb{Z}_p[v][x]$-submodule of $\mathbb{R}'_{lm}$. Define the following map

$$\eta : \mathbb{R}_{lm} = \mathbb{Z}_p[x]/\langle x^{lm} - 1\rangle \rightarrow v\mathbb{R}'_{lm}$$
$$\sum_{i=0}^{lm-1} a_i x^i \mapsto \sum_{i=0}^{lm-1} v a_i x^i,$$

where $a_i \in \mathbb{Z}_p$. Clearly, η is a $\mathbb{Z}_p[x]$-module isomorphism.

Let $\mathbb{R}_{km} \times \mathbb{R}_{lm} = \mathbb{Z}_p[x]/\langle x^{km} - 1\rangle \times \mathbb{Z}_p[x]/\langle x^{lm} - 1\rangle$. The elements of $\mathbb{R}_{km} \times \mathbb{R}_{lm}$ can be uniquely expressed as $(a(x), b(x))$, where $a(x) = \sum_{i=0}^{km-1} a_i x_i,\ b(x) = \sum_{j=0}^{lm-1} b_j x_j \in \mathbb{Z}_p[x]$. For any $f(x) \in \mathbb{Z}_p[x]$ and any $(a(x), b(x)) \in \mathbb{R}_{km} \times \mathbb{R}_{lm}$, define the scalar multiplication on $\mathbb{R}_{km} \times \mathbb{R}_{lm}$ as

$$f(x)\big(a(x), b(x)\big) = \big(f(x)a(x)(\text{mod } x^{km} - 1),\ f(x)b(x)(\text{mod } x^{lm} - 1)\big),$$

which is abbreviated as $f(x)\big(a(x), b(x)\big) = \big(f(x)a(x),\ f(x)b(x)\big)$. The $\mathbb{R}_{km} \times \mathbb{R}_{lm}$ forms an $\mathbb{R}_{klm}$-module under the above scalar multiplication, where $\mathbb{R}_{klm} = \mathbb{Z}_p[x]/\langle x^{klm} - 1\rangle$. Since η is a $\mathbb{Z}_p[x]$-module isomorphism from $\mathbb{R}_{lm}$ to $v\mathbb{R}'_{lm}$, then $\mathbb{R}_{km} \times v\mathbb{R}'_{lm}$ forms an $\mathbb{R}_{klm}$-module.

For any $(a(x), b(x)) \in \mathbb{R}_{km} \times \mathbb{R}_{lm}$, let

$$\mathcal{C}_{a,b} = \left\{(f(x)a(x), vf(x)b(x)) \in \mathbb{R}_{km} \times v\mathbb{R}'_{lm} | f(x) \in \mathbb{R}_{klm}\right\}.$$

Then $\mathcal{C}_{a,b}$ can be viewed as an $\mathbb{R}_{klm}$-submodule of $\mathbb{R}_{km} \times v\mathbb{R}'_{lm}$ generated by $(a(x), vb(x))$. In other words, $\mathcal{C}_{a,b}$ is a $\mathbb{Z}_p\mathbb{Z}_p[v]$-additive cyclic code in $\mathbb{R}_{km} \times v\mathbb{R}'_{lm}$ generated by $(a(x), vb(x))$.

Let $\mathcal{C}_{a,b}$ be a $\mathbb{Z}_p\mathbb{Z}_p[v]$-additive cyclic code generated by $F(x)$, where $F(x) = (a(x), vb(x)) \in \mathbb{R}_{km} \times v\mathbb{R}'_{lm}$ and $a(x) \in \mathbb{R}_{km},\ b(x) \in \mathbb{R}_{lm}$ are monic polynomials. By the $\mathbb{Z}_p[x]$-module isomorphism η, $\mathcal{C}_{a,b}$ can also be viewed as a $\mathbb{Z}_p$-linear space. Let $g_1(x) = \gcd(a(x), x^{km} - 1)$, $g_2(x) = \gcd(b(x), x^{lm} - 1)$ and $h(x) = \text{lcm}\left\{\frac{x^{km}-1}{g_1(x)}, \frac{x^{lm}-1}{g_2(x)}\right\}$ with $\deg h(x) = h$. Then, as a $\mathbb{Z}_p$-linear space, the dimension of $\mathcal{C}_{a,b}$ is h.

For any positive integer m with $\gcd(m, p) = 1$, by the Chinese remainder theorem, $\mathbb{R}_m = \mathbb{Z}_p[x]/\langle x^m - 1\rangle = \mathbb{Z}_p[x]/\langle x - 1\rangle \oplus \mathbb{Z}_p[x]/\langle x^{m-1} + x^{m-2} + \cdots + x + 1\rangle$. Note that $v\mathbb{R}'_m = v\mathbb{Z}_p[v][x]/\langle x^m - 1\rangle = v\mathbb{R}_m$. Define

$$\mathbb{J}_{km} = \left\langle \frac{x^{km} - 1}{x^m - 1}(x - 1)\right\rangle_{\mathbb{R}_{km}}, \quad \mathbb{J}_{lm} = \left\langle \frac{x^{lm} - 1}{x^m - 1}(x - 1)\right\rangle_{\mathbb{R}_{lm}},$$
$$\mathbb{J}_{klm} = \left\langle \frac{x^{klm} - 1}{x^m - 1}(x - 1)\right\rangle_{\mathbb{R}_{klm}}, \quad \mathbb{J}_m = \langle x - 1\rangle_{\mathbb{R}_m}.$$

If $(a(x), b(x)) \in \mathbb{J}_{km} \times \mathbb{J}_{lm}$, i.e. $(a(x), vb(x)) \in \mathbb{J}_{km} \times v\mathbb{J}'_{lm}$, where $v\mathbb{J}'_{lm} = \left\langle v\left(\frac{x^{lm}-1}{x^m-1}(x - 1)\right)\right\rangle_{\mathbb{R}'_{lm}}$, then the $\mathbb{Z}_p\mathbb{Z}_p[v]$-additive cyclic code $\mathcal{C}_{a,b}$ can be reformulated as

$$\mathcal{C}_{a,b} = \left\{(f(x)a(x), vf(x)b(x)) \in \mathbb{R}_{km} \times v\mathbb{R}'_{lm} | f(x) \in \mathbb{J}_{klm}\right\}.$$

Example 1 Let $p = 3$, $m = 2$, $k = 5$ and $l = 2$. Define

$$\mathbb{R}_{10} = \mathbb{Z}_3[x]/\langle x^{10} - 1\rangle, \qquad \mathbb{J}_{10} = \left\langle \frac{x^{10} - 1}{x^2 - 1}(x - 1)\right\rangle_{\mathbb{R}_{10}},$$
$$\mathbb{R}_4 = \mathbb{Z}_3[x]/\langle x^4 - 1\rangle, \qquad \mathbb{J}_4 = \left\langle \frac{x^4 - 1}{x^2 - 1}(x - 1)\right\rangle_{\mathbb{R}_4},$$
$$\mathbb{R}_{20} = \mathbb{Z}_3[x]/\langle x^{20} - 1\rangle, \qquad \mathbb{J}_{20} = \left\langle \frac{x^{20} - 1}{x^2 - 1}(x - 1)\right\rangle_{\mathbb{R}_{20}}.$$

Let $\mathcal{C}_{a,b} = \left\{(f(x)a(x), vf(x)b(x)) \in \mathbb{R}_{10} \times v\mathbb{R}'_4 | f(x) \in \mathbb{J}_{20}\right\}$ be a $\mathbb{Z}_3\mathbb{Z}_3[v]$-additive cyclic code generated by $(a(x), vb(x)) \in \mathbb{J}_{10} \times v\mathbb{J}'_4$, where $a(x) = x^9+2x^8+x^7+2x^6+x^5+2x^4+x^3+2x^2+x+2 \in \mathbb{J}_{10}, b(x) = x^3+2x^2+x+2 \in \mathbb{J}_4$.

Let $g_1(x) = \gcd(a(x), x^{10} - 1)$ and $g_2(x) = \gcd(b(x), x^4 - 1)$. Clearly, $g_1(x) = x^9+2x^8+x^7+2x^6+x^5+2x^4+x^3+2x^2+x+2$ and $g_2(x) = x^3+2x^2+x+2$. Since $h(x) = \mathrm{lcm}\left\{\frac{x^{10}-1}{g_1(x)}, \frac{x^4-1}{g_2(x)}\right\} = x^{11} + 2x^{10} + 2x^9 + x^8 + 2x^7 + x^6 + 2x^5 + x^4 + 2x^3 + x^2 + x + 2$, then the dimension of $\mathcal{C}_{a,b}$ is 11. Further, the generator matrix of $\mathcal{C}_{a,b}$ is

$$G = \left(\begin{array}{cccccccccc|cccc}
2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2v & v & 2v & v \\
1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & v & 2v & v & 2v \\
2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2v & v & 2v & v \\
1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & v & 2v & v & 2v \\
2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2v & v & 2v & v \\
1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & v & 2v & v & 2v \\
2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2v & v & 2v & v \\
1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & v & 2v & v & 2v \\
2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2v & v & 2v & v \\
1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & v & 2v & v & 2v \\
2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2v & v & 2v & v
\end{array}\right)_{11\times 14}.$$

4 Asymptotically good $\mathbb{Z}_p\mathbb{Z}_p[v]$-additive cyclic codes

In this section, we will consider the asymptotic property of $\mathbb{Z}_p\mathbb{Z}_p[v]$-additive cyclic codes, i.e. study the asymptotic property of the rate and the relative minimum distance of $\mathcal{C}_{a,b}$. The rate and the relative minimum distance of $\mathcal{C}_{a,b}$ is defined by $R(\mathcal{C}_{a,b}) = \frac{\dim(\mathcal{C}_{a,b})}{n}$ and $\Delta(\mathcal{C}_{a,b}) = \frac{d_G(\mathcal{C}_{a,b})}{n}$, respectively, where n is the length of $\mathcal{C}_{a,b}$ and $\dim(\mathcal{C}_{a,b})$ is the dimension of $\mathcal{C}_{a,b}$. So we need to study the asymptotic property of probabilities $\Pr(\Delta(\mathcal{C}_{a,b}) > \delta)$ and $\Pr(\dim(\mathcal{C}_{a,b}) = m-1)$. In Sect. 3, we have constructed a new class of $\mathbb{Z}_p\mathbb{Z}_p[v]$-additive cyclic codes

$$\mathcal{C}_{a,b} = \{(f(x)a(x), vf(x)b(x)) \in \mathbb{R}_{km} \times v\mathbb{R}'_{lm} | f(x) \in \mathbb{J}_{klm}\}.$$

However, it is not easy to study this class of codes directly. It is well known that the asymptotic property of codes

$$\mathcal{C}_{a',b'} = \{(f(x)a'(x), vf(x)b'(x)) \in \mathbb{R}_m \times v\mathbb{R}'_m | f(x) \in \mathbb{J}_m\}$$

can be determined easily. Therefore, we will consider whether we can find a relationship between these two class of codes.

Clearly, we can view the sets $\mathbb{J}_m \times v\mathbb{J}'_m$ and $\mathbb{J}_{km} \times v\mathbb{J}'_{lm}$ as a probability space of $\mathbb{R}_m \times v\mathbb{R}'_m$ and $\mathbb{R}_{km} \times v\mathbb{R}'_{lm}$ respectively, whose samples are afforded with equal probability. Moreover, $\mathcal{C}_{a,b}$ is a random code over the probability space $\mathbb{J}_{km} \times v\mathbb{J}'_{lm}$, the $R(\mathcal{C}_{a,b})$ and $\triangle(\mathcal{C}_{a,b})$ are random variables over the probability space. Similarly, $\mathcal{C}_{a',b'}$ is a random code over the probability space $\mathbb{J}_m \times v\mathbb{J}'_m$, the $R(\mathcal{C}_{a',b'})$ and $\triangle(\mathcal{C}_{a',b'})$ are random variables over the probability space.

Define a map

$$\begin{aligned} \Omega: \mathbb{J}_m \times v\mathbb{J}'_m &\rightarrow \mathbb{J}_{km} \times v\mathbb{J}'_{lm}, \\ \left(a'(x), vb'(x)\right) &\mapsto \left(a'(x)\frac{x^{km}-1}{x^m-1}, vb'(x)\frac{x^{lm}-1}{x^m-1}\right). \end{aligned} \tag{3}$$

Clearly, Ω is an $\mathbb{R}_{klm}$-isomorphism. For simplicity, we write $(a(x), vb(x)) = \Omega(a'(x), vb'(x))$ and $\mathcal{C}_{a,b} = \Omega(\mathcal{C}_{a',b'})$. For our purpose, we need two concepts: *p-ary entropy* and *Bernoulli variable*.

For $0 < x < 1$, let $h_p(x) = x\log_p(p-1) - x\log_p x - (1-x)\log_p(1-x)$, then the function $h_p(x)$ is called a *p-ary entropy*. In addition, let δ be a real number such that $0 < \delta < 1$ and $h_p(\delta) < \frac{1}{2}$.

For any $f(x) \in \mathbb{J}_m$, $(a'(x), vb'(x)) \in \mathbb{J}_m \times v\mathbb{J}'_m$, define a *Bernoulli variable* Y_f over the probability space $\mathbb{J}_m \times v\mathbb{J}'_m$

$$Y_f = \begin{cases} 1, & 1 \le wt_G(f(x)a'(x), vf(x)b'(x)) \le 2m\delta, \\ 0, & \text{otherwise}. \end{cases} \tag{4}$$

Since $f(x) \in \mathbb{J}_m$, then the set $\{f(x)a'(x) \in \mathbb{R}_m | a'(x) \in \mathbb{J}_m\}$ can be viewed as an ideal of $\mathbb{R}_m$ generated by $f(x)$. Let $\mathbb{I}_f = \langle f(x)\rangle_{\mathbb{R}_m} \subseteq \mathbb{J}_m$ and $d_f = \dim\mathbb{I}_f$. Moreover, the set $\{vf(x)b'(x) \in v\mathbb{R}'_m | b'(x) \in \mathbb{J}_m\}$ can be viewed as an ideal of $\mathbb{R}'_m$ generated by $vf(x)$. Let $\mathbb{I}'_f = \langle vf(x)\rangle_{\mathbb{R}'_m} \subseteq v\mathbb{J}'_m$. Clearly, as a $\mathbb{Z}_p$-linear space, $\dim\mathbb{I}'_f = d_f$.

In the following, we firstly consider the asymptotic property of $\mathcal{C}_{a',b'}$.

Lemma 1 *Let* $(a'(x), b'(x)) \in \mathbb{R}_m \times \mathbb{R}_m$ *and*

$$\mathcal{C}_{a',b'} = \left\{(f(x)a'(x), vf(x)b'(x)) \in \mathbb{R}_m \times v\mathbb{R}'_m\right\}.$$

Let

$$g_{a',b'}(x) = \gcd(a'(x), b'(x), x^m - 1) \text{ and } h_{a',b'}(x) = \frac{x^m - 1}{g_{a',b'}(x)}.$$

Define $\langle g_{a',b'}(x)\rangle_{\mathbb{R}_m}$ *as the ideal of* $\mathbb{R}_m$ *generated by* $g_{a',b'}(x)$. *Then* $\dim\mathcal{C}_{a',b'} = \deg h_{a',b'}(x)$. *Moreover, there is an* $\mathbb{R}_m$*-module isomorphism* $\langle g_{a',b'}(x)\rangle_{\mathbb{R}_m} \cong \mathcal{C}_{a',b'}$, *which maps* $c(x) \in \langle g_{a',b'}(x)\rangle_{\mathbb{R}_m}$ *to* $(c(x)a'(x), vc(x)b'(x)) \in \mathcal{C}_{a',b'}$.

Proof Define a map

$$\begin{aligned} \rho :\ \mathbb{R}_m &\rightarrow \mathbb{R}_m \times v\mathbb{R}'_m \\ f(x) &\mapsto (f(x)a'(x), vf(x)b'(x)). \end{aligned}$$

Obviously, the map ρ is an $\mathbb{R}_m$-module homomorphism, and the image $\mathrm{im}(\rho) = \mathcal{C}_{a',b'}$. In the following, we consider the kernel $\ker(\rho)$. For $f(x) \in \mathbb{R}_m$, $f(x) \in \ker(\rho)$ if and only if in $\mathbb{Z}_p[x]$ we have $f(x)a'(x) \equiv 0 \pmod{x^m - 1}$ and in $v\mathbb{Z}_p[v][x]$ we have $vf(x)b'(x) \equiv 0 \pmod{x^m - 1}$. Since $v\mathbb{Z}_p[v][x] = v\mathbb{Z}_p[x]$, so we can turn the second half of the sentence to be in $v\mathbb{Z}_p[x]$ we have $vf(x)b'(x) \equiv 0 \pmod{x^m - 1}$, i.e. in $\mathbb{Z}_p[x]$ we have $f(x)b'(x) \equiv 0 \pmod{x^m - 1}$. It means that $f(x) \in \ker(\rho)$ if and only if in $\mathbb{Z}_p[x]$ we have

$$\begin{cases} f(x)a'(x) \equiv 0 \pmod{x^m - 1}, \\ f(x)b'(x) \equiv 0 \pmod{x^m - 1}. \end{cases}$$

Therefore, $f(x)\gcd(a'(x), b'(x)) \equiv 0 \pmod{x^m - 1}$, which implies that $f(x) \equiv 0 \left(\bmod \frac{x^m-1}{\gcd(a'(x),b'(x),x^m-1)}\right)$. Thus, $\ker(\rho) = \langle h_{a',b'}(x)\rangle_{\mathbb{R}_m}$. Since $\gcd(m, p) = 1$, then $x^m - 1$ has no multiple roots in any extension of $\mathbb{Z}_p$. Therefore, we can obtain

$$\mathbb{R}_m = \langle g_{a',b'}(x)\rangle_{\mathbb{R}_m} \bigoplus \langle h_{a',b'}(x)\rangle_{\mathbb{R}_m}.$$

Thus, the above $\mathbb{R}_m$-module homomorphism ρ induces an $\mathbb{R}_m$-module isomorphism

$$\begin{aligned} \bar{\rho} : \langle g_{a',b'}(x)\rangle_{\mathbb{R}_m} &\rightarrow \mathcal{C}_{a',b'} \\ c(x) &\mapsto (c(x)a'(x), vc(x)b'(x)), \end{aligned}$$

which implies that

$$\dim\mathcal{C}_{a',b'} = \dim\langle g_{a',b'}(x)\rangle_{\mathbb{R}_m} = m - \deg g_{a',b'}(x) = \deg h_{a',b'}(x).$$

□

Lemma 2 [11] *Let* $\mathcal{C}_{a',b'} = \{(f(x)a'(x), vf(x)b'(x)) \in \mathbb{R}_m \times v\mathbb{R}'_m | f(x) \in \mathbb{J}_m\}$, *where* $(a'(x), b'(x)) \in \mathbb{J}_m \times \mathbb{J}_m$. *Then* $\dim(\mathcal{C}_{a',b'}) \leq m - 1$, *and* $\dim(\mathcal{C}_{a',b'}) = m - 1$ *if and only if there is no irreducible factor* $q(x)$ *of* $\frac{x^m-1}{x-1}$ *in* $\mathbb{Z}_p[x]$ *such that* $q(x)|a'(x)$ *and* $q(x)|b'(x)$.

Lemma 3 [11] *Let $q_k(x)$ be the lowest degree polynomial in the irreducible factors of $\frac{x^m-1}{x-1} = 1 + x + \cdots + x^{m-1}$ in $\mathbb{Z}_p[x]$. Let $k_m = \deg(q_k(x))$ and d be an integer with $k_m \leq d \leq m-1$. For any non-zero ideal $\mathbb{I}$ of $\mathbb{R}_m$, if $\mathbb{I}$ satisfies $\mathbb{I} \subseteq \mathbb{J}_m$, then $\dim\mathbb{I} \geq k_m$ and the number of ideals contained in $\mathbb{J}_m$ of dimension d is at most $m^{\frac{d}{k_m}}$.*

By Lemma 2.6 in [5], there exist odd positive integers $m_1, m_2, m_3, \ldots$ such that

$$\gcd(m_i, p) = 1, \quad m_i \to \infty, \ \lim_{i\to\infty} \frac{\log_p m_i}{k_{m_i}} = 0, \tag{5}$$

where k_{m_i} is defined as in Lemma 3. For each m_i and $(a'(x), b'(x)) \in \mathbb{J}_{m_i} \times \mathbb{J}_{m_i}$, let

$$\mathcal{C}^{(i)}_{a',b'} = \{(f(x)a'(x), vf(x)b'(x)) \in \mathbb{R}_{m_i} \times v\mathbb{R}'_{m_i} | f(x) \in \mathbb{J}_{m_i}\} \tag{6}$$

be a random $\mathbb{Z}_p\mathbb{Z}_p[v]$-additive cyclic code of length $2m_i$

Proposition 1 *Let $m_1, m_2, \ldots$ be positive integers satisfying Eq. (5) and $\mathcal{C}^{(i)}_{a',b'}$ be given as in Eq. (6). Then*

$$\lim_{i\to\infty} \Pr\left(\dim(\mathcal{C}^{(i)}_{a',b'}) = m_i - 1\right) = 1.$$

Proof Let $\frac{x^{m_i}-1}{x-1} = q_{i1}(x)q_{i2}(x)\cdots q_{ir}(x)$ be an irreducible decomposition in $\mathbb{Z}_p[x]$. By Chinese remainder theorem,

$$\mathbb{J}_{m_i} = \langle x-1 \rangle_{\mathbb{R}_{m_i}} \cong \mathbb{Z}_p[x]/\langle q_{i1}(x)\rangle \times \mathbb{Z}_p[x]/\langle q_{i2}(x)\rangle \times \cdots \times \mathbb{Z}_p[x]/\langle q_{ir}(x)\rangle,$$

which is given by

$$\begin{aligned} \mu^{(ij)}_{m_i} : \mathbb{J}_{m_i} &\to \mathbb{Z}_p[x]/\langle q_{ij}(x)\rangle \\ f(x) &\mapsto f(x) \pmod{q_{ij}(x)}, \end{aligned}$$

where $j = 1, 2, \ldots, r$. Therefore, for any $f(x) \in \mathbb{J}_{m_i}$, there is a unique

$$\left(\mu^{(i1)}_{m_i}(f(x)), \ldots, \mu^{(ir)}_{m_i}(f(x))\right) \in \mathbb{Z}_p[x]/\langle q_{i1}(x)\rangle \times \cdots \times \mathbb{Z}_p[x]/\langle q_{ir}(x)\rangle.$$

Let $(a'(x), vb'(x)) \in \mathbb{J}_{m_i} \times v\mathbb{J}'_{m_i}$, where $a'(x), b'(x) \in \mathbb{J}_{m_i}$. From Lemma 2, $\dim\left(\mathcal{C}^{(i)}_{a',b'}\right) = m_i - 1$ if and only if for any $j = 1, 2, \ldots, r$, there is no irreducible factor $q_{ij}(x)$ of $\frac{x^{m_i}-1}{x-1}$ such that $q_{ij}(x)|a'(x)$ and $q_{ij}(x)|b'(x)$, which implies that $\dim\left(\mathcal{C}^{(i)}_{a',b'}\right) = m_i - 1$ if and only if

$$\left(\mu^{(ij)}_{m_i}(a'(x)), \mu^{(ij)}_{m_i}(b'(x))\right) \neq (0, 0).$$

Let $\deg(q_{ij}(x)) = d_{ij}$. Then $|\mathbb{Z}_p[x]/\langle q_{ij}(x)\rangle| = p^{d_{ij}}$. Therefore the probability of $\left(\mu_{m_i}^{(ij)}(a'(x)), \mu_{m_i}^{(ij)}(b'(x))\right) \neq (0,0)$ is $\frac{p^{2d_{ij}-1}}{p^{2d_{ij}}} = 1 - p^{-2d_{ij}}$. Thus,

$$\Pr\left(\dim\left(\mathcal{C}^{(i)}_{a',b'}\right) = m_i - 1\right) = \prod_{j=1}^{r}(1 - p^{-2d_{ij}}).$$

Let k_{m_i} be defined as in Lemma 3. Then, for any $j = 1, 2, \ldots, r$, $d_{ij} \geq k_{m_i}$ and $r \leq \frac{m_i - 1}{k_{m_i}} \leq \frac{m_i}{k_{m_i}}$. Therefore,

$$\Pr\left(\dim\left(\mathcal{C}^{(i)}_{a',b'}\right) = m_i - 1\right) = \prod_{j=1}^{r}(1 - p^{-2d_{ij}}) \geq \left(1 - p^{-2k_{m_i}}\right)^{\frac{m_i}{k_{m_i}}}$$
$$= \left(1 - p^{-2k_{m_i}}\right)^{p^{2k_{m_i}} \frac{m_i}{k_{m_i} p^{2k_{m_i}}}}.$$

Thus,

$$\lim_{i\to\infty}\Pr\left(\dim\left(\mathcal{C}^{(i)}_{a',b'}\right) = m_i - 1\right) \geq \lim_{i\to\infty}\left(1 - p^{-2k_{m_i}}\right)^{p^{2k_{m_i}} \frac{m_i}{k_{m_i} p^{2k_{m_i}}}} = 1.$$

□

Lemma 4 *Let* $\mathbb{I}_f \times \mathbb{I}_f \subseteq \mathbb{R}_m \times \mathbb{R}_m$ *and* $(\mathbb{I}_f \times \mathbb{I}_f)^{\leq 2m\delta} = \{(f_1(x), f_2(x)) \in \mathbb{I}_f \times \mathbb{I}_f | wt_H(f_1(x), f_2(x)) \leq 2m\delta\}$. *Then*

$$\left|(\mathbb{I}_f \times \mathbb{I}_f)^{\leq 2m\delta}\right| \leq p^{2d_f h_p(\delta)}.$$

Proof Since $|\mathbb{R}_m \times \mathbb{R}_m| = p^{2m}$ and $|\mathbb{I}_f \times \mathbb{I}_f| = p^{2d_f}$, then the fraction of $2m\delta$ over the length is $\frac{2m\delta}{2m} = \delta$. Moreover, since $0 < \delta < 1$, then, by Remark 3.2 and Corollary 3.5 in [13], the result follows directly. □

Lemma 5 $E(Y_f) \leq p^{-2d_f + 2d_f h_p(\delta)}$.

Proof In $v\mathbb{J}'_m \subset v\mathbb{R}'_m$, we have an ideal

$$\mathbb{I}'_f = \langle vf(x)\rangle_{\mathbb{R}'_m} = \{vf(x)b(x) \in v\mathbb{R}'_m | b(x) \in \mathbb{J}_m\} \subseteq v\mathbb{J}'_m.$$

For $\mathbb{I}_f \times \mathbb{I}'_f \subseteq \mathbb{R}_m \times v\mathbb{R}'_m$, let $(\mathbb{I}_f \times \mathbb{I}'_f)^{\leq 2m\delta} = \{(f_1(x), vf_2(x)) \in \mathbb{I}_f \times \mathbb{I}'_f | wt_G(f_1(x), vf_2(x)) \leq 2m\delta\}$. Since Y_f is a 0-1 variable, then the expectation of Y_f is only the probability of $Y_f = 1$. So we have

$$E(Y_f) = \Pr(Y_f = 1) = \frac{|(\mathbb{I}_f \times \mathbb{I}'_f)^{\leq 2m\delta}| - 1}{\mathbb{I}_f \times \mathbb{I}'_f}. \quad (7)$$

For $f_1(x), f_2(x) \in \mathbb{R}_m$, by Gray map ϕ, we have that

$$wt_G(vf_2(x)) = wt_H(\phi(vf_2(x))) = wt_H(f_2(x), f_2(x)) = 2wt_H(f_2(x)).$$

Therefore, by the generalized Gray map Φ, we have

$$\begin{aligned} wt_G(f_1(x), vf_2(x)) &= wt_H(f_1(x)) + wt_H(\phi(vf_2(x))) \\ &= wt_H(f_1(x)) + 2wt_H(f_2(x)) \\ &\geq wt_H(f_1(x), f_2(x)). \end{aligned}$$

Thus,

$$\left|(\mathbb{I}_f \times \mathbb{I}'_f)^{\leq 2m\delta}\right| \leq \left|(\mathbb{I}_f \times \mathbb{I}_f)^{\leq 2m\delta}\right|. \quad (8)$$

Moreover, we know that $\dim \mathbb{I}'_f = \dim \mathbb{I}_f = d_f$. Therefore, by Lemma 4, Eqs. (7) and (8), we have

$$E(Y_f) = \frac{|(\mathbb{I}_f \times \mathbb{I}'_f)^{\leq 2m\delta}| - 1}{|\mathbb{I}_f \times \mathbb{I}'_f|} \leq \frac{|(\mathbb{I}_f \times \mathbb{I}_f)^{\leq 2m\delta}|}{|\mathbb{I}_f \times \mathbb{I}_f|} \leq \frac{p^{2d_f h_p(\delta)}}{p^{2d_f}} = p^{-2d_f + 2d_f h_p(\delta)}.$$

□

Lemma 6 [11] *Let δ be a real number such that $0 < \delta < 1$ and $h_p(\delta) < \frac{1}{2}$. Then*

$$\Pr(\Delta(\mathcal{C}_{a',b'}) \leq \delta) \leq \sum_{j=k_m}^{m-1} p^{-2j\left(\frac{1}{2} - h_p(\delta) - \frac{\log_p m}{2k_m}\right)}.$$

Proposition 2 *Let $0 < \delta < 1$ and $h_p(\delta) < \frac{1}{2}$. Then*

$$\lim_{i\to\infty} \Pr\left(\Delta\left(\mathcal{C}^{(i)}_{a',b'}\right) \geq \delta\right) = 1.$$

Proof Clearly, $\frac{1}{2} - h_p(\delta) > 0$. Since $m_i \to \infty$, $\lim_{i\to\infty} \frac{\log_p m_i}{k_{m_i}} = 0$, then $\lim_{i\to\infty} \frac{\log_p m_i}{2k_{m_i}} = 0$, which implies that there are a positive real number ε and an integer N such that when $i > N$,

$$\frac{1}{2} - h_p(\delta) - \frac{\log_p m_i}{2k_{m_i}} \geq \varepsilon.$$

By Lemma 6,

$$\lim_{i\to\infty}\Pr\left(\Delta\left(\mathcal{C}^{(i)}_{a',b'}\right)\le\delta\right)\le \lim_{i\to\infty}\sum_{j=k_m}^{m_i-1} p^{-2j\left(\frac{1}{2}-h_p(\delta)-\frac{\log_p m_i}{2k_{m_i}}\right)}$$
$$\le \lim_{i\to\infty}\sum_{j=k_{m_i}}^{m_i-1} p^{-2j\varepsilon}$$
$$\le \lim_{i\to\infty}\sum_{j=k_{m_i}}^{m_i-1} p^{-2k_{m_i}\varepsilon}$$
$$\le \lim_{i\to\infty} m_i p^{-2k_{m_i}\varepsilon}$$
$$= \lim_{i\to\infty} p^{-2k_{m_i}\left(\varepsilon-\frac{\log_p m_i}{2k_{m_i}}\right)}=0.$$

Thus, $\lim_{i\to\infty}\Pr\left(\Delta\left(\mathcal{C}^{(i)}_{a',b'}\right)>\delta\right)=1$. □

In the following, we will consider the asymptotic property of $\mathcal{C}^{(i)}_{a,b}$.
By the Eq. (3), we have

$$wt_G(a(x),vb(x))=wt_G(a(x))+wt_G(vb(x))=kwt_G(a'(x))+lwt_G(vb'(x))$$
$$\ge wt_G(a'(x),vb'(x)),$$

i.e.

$$wt_G(\mathcal{C}_{a,b})\ge wt_G(\mathcal{C}_{a',b'}).$$

By the definition of the relative minimum distance of $\mathcal{C}_{a,b}$ and $\mathcal{C}_{a',b'}$, we have $\triangle(\mathcal{C}_{a,b})=\frac{d_G(\mathcal{C}_{a,b})}{(k+l)m}=\frac{wt_G(\mathcal{C}_{a,b})}{(k+l)m}$ and $\triangle(\mathcal{C}_{a',b'})=\frac{d_G(\mathcal{C}_{a',b'})}{2m}=\frac{wt_G(\mathcal{C}_{a',b'})}{2m}$. Since $wt_G(\mathcal{C}_{a,b})\ge wt_G(\mathcal{C}_{a',b'})$, then $(k+l)m\triangle(\mathcal{C}_{a,b})\ge 2m\triangle(\mathcal{C}_{a',b'})$, which implies that $\triangle(\mathcal{C}_{a,b})\ge\frac{2}{k+l}\triangle(\mathcal{C}_{a',b'})$. Further, by Lemma 1 in [17], we have

$$\Pr\left(\Delta(\mathcal{C}_{a,b})>\delta\right)\ge\Pr\left(\Delta(\mathcal{C}_{a',b'})>\frac{k+l}{2}\delta\right).$$

Thus, by Propositions 1 and 2, we obtain the asymptotic property of $\mathcal{C}_{a,b}$ as follows.

Corollary 1 *Let* $\mathcal{C}_i=\{(f(x)a(x),vf(x)b(x))\in\mathbb{R}_{km_i}\times v\mathbb{R}'_{lm_i}\,|\,f(x)\in\mathbb{J}_{klm_i}\}$ *and* $m_1,m_2,\ldots$ *satisfy* $\gcd(m_i,p)=1$ *and when* $m_i\to\infty$, $\lim_{i\to\infty}\frac{\log_p m_i}{k_{m_i}}=0$, *where* k_{m_i} *is defined as in Lemma 3. Then we have*
(a) $\lim_{i\to\infty}\Pr(\dim(\mathcal{C}_i)=m_i-1)=1$*;*
(b) if $h_p(\frac{k+l}{2}\delta)<\frac{1}{2}$*, then* $\lim_{i\to\infty}\Pr(\Delta(\mathcal{C}_i)>\delta)=1$.

Proof (a) From the definition of map Ω, we know $\Omega(\mathcal{C}^{(i)}_{a',b'}) = \mathcal{C}^{(i)}_{a,b}$ and Ω is an isomorphism, so $\dim\left(\mathcal{C}^{(i)}_{a,b}\right) = \dim\left(\Omega\left(\mathcal{C}^{(i)}_{a',b'}\right)\right) = \dim\left(\mathcal{C}^{(i)}_{a',b'}\right)$. Therefore, by Proposition 1, we have

$$\lim_{i\to\infty}\Pr\left(\dim\left(\mathcal{C}^{(i)}_{a,b}\right) = m_i - 1\right) = 1.$$

(b) From Proposition 2, we know that if $h_p\left(\frac{k+l}{2}\delta\right) < \frac{1}{2}$, then we have

$$\lim_{i\to\infty}\Pr\left(\Delta\left(\mathcal{C}^{(i)}_{a',b'}\right) > \frac{k+l}{2}\delta\right) = 1.$$

Moreover, since $\Pr\left(\Delta(\mathcal{C}_{a,b}) > \delta\right) \geq \Pr\left(\Delta(\mathcal{C}_{a',b'}) > \frac{k+l}{2}\delta\right)$, then

$$\lim_{i\to\infty}\Pr\left(\Delta\left(\mathcal{C}^{(i)}_{a,b}\right) > \delta\right) \geq \lim_{i\to\infty}\Pr\left(\Delta(\mathcal{C}_{a',b'}) > \frac{k+l}{2}\delta\right) = 1.$$

□

According to Corollary 1, we get the main result in this paper as follows.

Theorem 2 *Let δ be a real number such that $0 < \delta < 1$ and $h_p(\frac{k+l}{2}\delta) < \frac{1}{2}$. Then, for $i = 1, 2, \ldots$, when $m_i \to \infty$, there exist a series of $\mathbb{Z}_p\mathbb{Z}_p[v]$-additive cyclic codes $\mathcal{C}_i$ of block length (km_i, lm_i) such that*
(a) $\lim_{i\to\infty}R(\mathcal{C}_i) = \frac{1}{k+l}$;
(b) $\Delta(\mathcal{C}_i) > \delta$.

Proof (a) By the definition of the rate of $\mathcal{C}_i$, we have $R(\mathcal{C}_i) = \frac{\dim(\mathcal{C}_i)}{km_i+lm_i}$. From Corollary 1, there exists a positive integer N such that, when $i > N$, we have $\dim(\mathcal{C}_i) = m_i - 1$. Thus,

$$\lim_{i\to\infty}R(\mathcal{C}_i) = \lim_{i\to\infty}\frac{\dim(\mathcal{C}_i)}{km_i + lm_i} = \lim_{i\to\infty}\frac{m_i - 1}{km_i + lm_i} = \frac{1}{k+l}.$$

(b) From Corollary 1, if $h_p\left(\frac{k+l}{2}\delta\right) < \frac{1}{2}$ then $\lim_{i\to\infty}\Pr(\Delta(\mathcal{C}_i) > \delta) = 1$. Therefore there exists a positive integer N such that, when $i > N$, we have $\Delta(\mathcal{C}_i) > \delta$. Thus, after deleting the first N codes and renumbering the remaining codes, we get the result.

□

From Theorem 2, we can conclude that $\mathbb{Z}_p\mathbb{Z}_p[v]$-additive cyclic codes are asymptotically good.

5 Conclusion

In this paper, we firstly constructed a class of $\mathbb{Z}_p\mathbb{Z}_p[v]$-additive cyclic codes with different component length. Then, based on the probabilistic method and the Chinese remainder theorem, we proved that these codes are asymptotically good. In the future, researching on the asymptotic property of some other classes of linear codes over finite non-chain rings may be an interesting open problem.

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