Journal of Applied Mathematics and Computing (2021) 66:823–834
https://doi.org/10.1007/s12190-020-01465-x

ORIGINAL RESEARCH

Further results on the signed Italian domination

A. Karamzadeh[1] · H. R. Maimani[1] · A. Zaeembashi[1]

Received: 28 July 2020 / Revised: 5 November 2020 / Accepted: 10 November 2020 / Published online: 19 November 2020

Abstract
A signed Italian dominating function on a graph $G = (V, E)$ is a function $f : V \to \{-1, 1, 2\}$ satisfying the condition that for every vertex u, $f[u] \geq 1$. The weight of the signed Italian dominating function, f, is the value $f(V) = \sum_{u \in V} f(u)$. The signed Italian dominating number of a graph G, denoted by $\gamma_{sI}(G)$, is the minimum weight of a signed Italian dominating function on a graph G. In this paper, we prove that for any tree T of order $n \geq 2$, $\gamma_{sI}(T) \geq \frac{-n+4}{2}$ and we characterize all trees attaining this bound. In addition, we obtain some results about the signed Italian domination number of some graph operations. Furthermore, we prove that the signed Italian domination problem is **NP**-Complete for bipartite graphs.

Keywords Domination · Signed Italian dominating function · Signed Italian dominating number

Mathematics Subject Classification 05C69

1 Introduction

In this paper, G is a simple graph with the vertex set $V = V(G)$ and the edge set $E = E(G)$. The order $|V|$ of G is denoted by n. For every vertex $v \in V$, the *open neighborhood* $N(v)$ is the set $\{u \in V(G) \mid uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. For every vertex $v \in V$ the number of neighbors of v is denoted by $deg_G(v)$. We denote $deg_G(v)$ by $d_G(v)$

✉ H. R. Maimani
maimani@ipm.ir

A. Karamzadeh
karamzadehmath@gmail.com

A. Zaeembashi
azaeembashi@sru.ac.ir

1 Mathematics Section, Department of Basic Sciences, Shahid Rajaee Teacher Training University, P.O. Box 16785-163, Tehran, Iran

for notational convenience. The *minimum* and *maximum* degree of a graph G are denoted by δ and Δ, respectively.

A *signed dominating function* (SDF) on a graph $G = (V, E)$ is a function $f : V \rightarrow \{-1, 1\}$ such that $f[v] \geq 1$ for every vertex $v \in V$. The signed domination number, denoted by $\gamma_s(G)$, is the minimum weight of a SDF in G; that is, $\gamma_s(G) = \min\{w(f) \mid f \text{ is a SDF in } G\}$.

An *Italian dominating function* (IDF) or a *Roman* $\{2\}$*-dominating function* on a graph $G = (V, E)$ is a function $f : V \rightarrow \{0, 1, 2\}$ with the property that for every vertex $v \in V$ with $f(v) = 0$, either v is adjacent to a vertex assigned 2 under f, or v is adjacent to at least two vertices assigned 1 under f. The *Italian domination function number* $\gamma_I(G)$ equals to the minimum weight of an Italian dominating function on G. An Italian domination has been introduced in 2015 by Chellali et al. [1], and studied further in [4,5].

In this paper, we continue the study of the signed Italian domination in graphs introduced in [6] as follows. A *signed Italian dominating function* (SIDF) on graph $G = (V, E)$ is a function $f : V \rightarrow \{-1, 1, 2\}$ which has the property that for every vertex $v \in V$ the sum of the values assigned to vertex v and its neighbors is at least 1. Thus a signed Italian dominating function combines the properties of both an Italian dominating function and a signed dominating function. The *signed Italian domination number*, denoted by $\gamma_{sI}(G)$, is the minimum weight of a SIDF in G; that is, $\gamma_{sI}(G) = \min\{w(f) \mid \text{f is a SIDF in } G\}$. A SIDF of weight $\gamma_{sI}(G)$ is called a $\gamma_{sI}(G)$-function. For a vertex $v \in V$, we denote $f(N[v])$ by $f[v]$ for notational convenience. For a SIDF f on G, let $V_i = \{v \in V(G) \mid f(v) = i\}$ for $i = -1, 1, 2$. In the context of a fixed SIDF, we suppress the argument and simply write V_{-1}, V_1 and V_2. Since this partition determines f, we can equivalently write $f = (V_{-1}, V_1, V_2)$.

A *tree* on n vertices is denoted by T_n. A *leaf* of T is a vertex with degree one and a *support vertex* is a vertex adjacent to a leaf. A tree T is a *double star* if it contains exactly two vertices that are not leaves. A double star with respectively p and q leaves attached at each support vertex is denoted by $DS_{p,q}$. The *distance* $d_G(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $u - v$ path in G. The *diameter* of a graph G, denoted by $diam(G)$, is the greatest distance between two vertices of G.

Recall that the *join* of two graphs G_1 and G_2, which is denoted by $G = G_1 \vee G_2$, has the vertex set $V(G) = V(G_1) \cup V(G_2)$ and the edge set $E(G) = E(G_1) \cup E(G_2) \cup \{uv|u \in V(G_1), v \in V(G_2)\}$. For example, $K_1 \vee P_n$ is the *fan* F_n, $K_1 \vee C_n$ is the *wheel* W_n, and the *friendship graph* Fr_n where $n = 2m + 1$, is the graph obtained by joining K_1 to the m disjoint copies of K_2. For two arbitrary graphs G and H, the *corona product* of G and H to be the graph $G \odot H$ is obtained by taking one copy of G and $|V(G)|$ copies of H by joining each vertex of i-th copy of H to the i-th vertex of G where $1 \leq i \leq |V(G)|$.

In this paper, we show that the associated decision problem for the signed Italian domination is **NP**-complete for the bipartite graphs. we obtain a probabilistic bound for the signed Italian domination number. Also we present sharp bounds on the signed Italian domination number of trees and in the end, we determine the signed Italian domination number of some graph operations. In addition, we determine the signed Italian

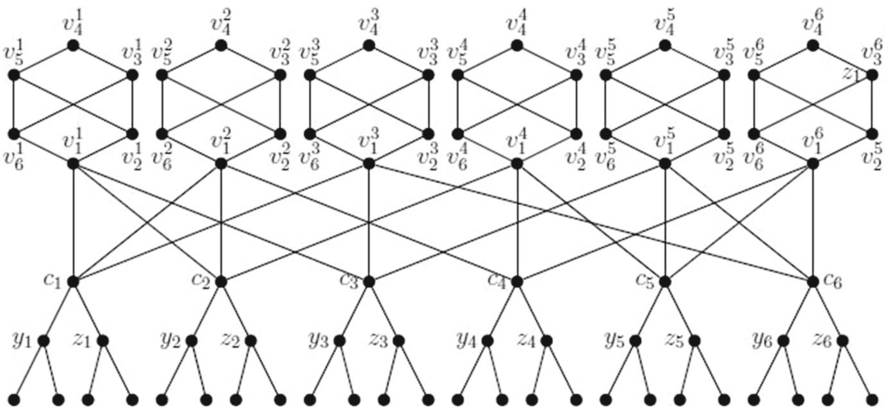

Fig. 1 NP-completeness of signed Italian for bipartite graphs

domination number special classes of graphs including fans, wheels, and friendship graphs.

2 Complexity result

In this section, we present the **NP**-complete result for the signed Italian domination problem in bipartite graphs.

SIGNED ITALIAN DOMINATING FUNCTION(SIDF)

Instance : Graph $G = (V, E)$, positive integer $k \leq |V|$.

Question : Does G have a signed Italian dominating function of weight at most k?

We will show that this problem is **NP**-complete by reducing the special case of Exact Cover by 3-sets (**X3C**) to which we refer as **X3C3**. The **NP**-completeness of **X3C3** was proven in 2008 by Hickey et al. [3].

X3C3

Instance : A set of elements X with $|X| = 3q$ and a collection C of $m = 3q$, 3-element subsets of X such that each element appears in exactly 3 elements of C.

Question : Is there a sub-collection $C^{'}$ of C such that every element of X appears in exactly one element of $C^{'}$?

Theorem 2.1 *SIDF is NP-complete for bipartite graphs.*

Proof Since we can check in polynomial time that a function $f : V \to \{-1, 1, 2\}$ has weight at most k and is a signed Italian dominating function, then SIDF is a member of $\mathcal{NP}$. Now let us show how to transform any instance of X3C3 into an instance G of SIDF, so that one of them has a solution if and only if the other one has a solution. Let $X = \{x_1, x_2, \ldots, x_{3q}\}$ and $C = \{C_1, C_2, \ldots, C_{3q}\}$ be an arbitrary instance of X3C3.

For each $x_i \in X$, we build a connected graph H_i obtained from a cycle $C_6 : v_1^i - v_2^i - v_3^i - v_4^i - v_5^i - v_6^i - v_1^i$ by adding edges $v_2^i v_5^i$ and $v_3^i v_6^i$. Let $W = \{v_1^1, v_1^2, \ldots, v_1^{3q}\}$. For each $C_j \in C$, we build a connected graph K_j obtained from two stars $K_{1,2}$ with centers y_j, z_j and a path $P_3 : y_j - c_j - z_j$. Let $A = \{c_1, c_2, \ldots, c_{3q}\}$. Now to obtain a graph G, we add edges $c_j v_1^i$ if $x_i \in C_j$ (see Fig. 1). Clearly, G is a bipartite graph and set $k = 10q$.

Suppose that the instance X, C of X3C3 has a solution $C^{'}$. We construct a signed Italian dominating function f on G of weight k. We assign the value 2 to all y_j's, z_j's, v_3^i's and v_5^i's, the value 1 to all v_4^i's and the value -1 to all v_1^i's, v_2^i's, v_6^i's and leaves of G. For every c_j, assign value 2 if $C_j \in C^{'}$ and value 1 if $C_j \notin C^{'}$. Note that since $C^{'}$ exists, its cardinality is precisely q and so the number of c_j's with weight 2 is q, having disjoint neighborhoods in W. Since $C^{'}$ is a solution for X3C3, every vertex in W is adjacent to vertex assigned a 2. Hence, it is straightforward to see that f is a signed Italian dominating function with weight $f(V) = 8(3q) + 3q + 2q + 2q - 7(3q) = 10q = k$.

Conversely, suppose that G has a signed Italian dominating function with weight at most k. Among all such functions, let g be one that assigns small values to the leaves of G and v_1^i's. If $g(y_j) = -1$ for some $1 \leq j \leq 3q$, then every leaves adjacent to y_j have label 2. Now we define $g^{'} : V(G) \to \{-1, 1, 2\}$ such that $g^{'}(y_j) = 2$, $g^{'}(x_1) = -1$ for a leaf x_1 adjacent to y_j, $g^{'}(x_2) = 1$ for another leaf x_2 adjacent to y_j and $g^{'} = g$ for remaining vertices of G. Thus $g^{'}$ is a signed Italian dominating function such that $w(g^{'}) \lneq w(g)$, which is a contradiction and so $g(y_j) > 0$ for $1 \leq j \leq 3q$. With the similar reasoning, we conclude that $g(z_j) > 0$ for $1 \leq j \leq 3q$.

Next we shall show that $g(c_j) > 0$ for $1 \leq j \leq 3q$. If $g(c_j) = -1$ for some j, then $g(y_j) \geq 0$. If $g(y_j) = 1$. then every two leaves adjacent to y_j have label 1 and if $g(y_j) = 2$, then at least one of leaf adjacent to y_j has label 1 under g. In any cases we conclude that there exists a leaf in neighbor set of y_j with label 1 under g. This fact is true for the vertex z_j. Now we define $g^{'} : V(G) \to \{-1, 1, 2\}$ such that $g^{'}(c_j) = 1$, $g^{'}(y_j) = g^{'}(z_j) = 2$, $g^{'}(x) = -1$ for any leaf x of y_j and z_j, and $g^{'} = g$ for remaining vertices of G. Thus $g^{'}$ is a signed Italian dominating function such that $w(g^{'}) \leq w(g)$, and the weight of leaves under $g^{'}$ is less than the weight of leaves under g, which is a contradiction with choosing the function g. So $g(c_j) > 0$. Therefore every support vertex of G is assigned a 2 and every leaf of G is assigned a -1.

Finally, we shall show that $g(v_1^i) = -1$ for $1 \leq i \leq 3q$ and conclude that $g(V(H_i)) \geq 2$. If $g(v_1^i) > 0$ for some $1 \leq i \leq 3q$, then define $g^{'} : V(G) \to \{-1, 1, 2\}$ such that $g(v_1^i) = g(v_2^i) = g(v_6^i) = -1$, $g(v_3^i) = g(v_5^i) = 2$, $g(v_4^i) = 1$, $g(c_j) = 2$ for q of j's where $1 \leq j \leq 3q$ and $g^{'} = g$ for remaining vertices of G. Thus $g^{'}$ is a signed Italian dominating function such that $w(g^{'}) \lneq w(g)$, which is a contradiction. In another case, we conclude that $g(v_1^i) = -1$ for $1 \leq i \leq 3q$ and $g(V(H_i)) \geq 2$. Now assume that $r = |V_2 \cap c_j|$, then $2r + 3q - r + 2(3q) \leq k = 10q$ and so $r \leq q$. On the other hand, if define $T = \{(v_1^i, c_j) \mid v_1^i \sim c_j \text{ and } g(c_j) = 2\}$, then

$$\underbrace{3 + 3 + \cdots + 3}_{\text{r}} \geq |T| \geq \underbrace{1 + 1 + \cdots + 1}_{\text{3q}},$$

hence $r \geq q$ and consequently $r = q$. Now since each c_j has exactly three neighbors in W, we conclude that $C^{'} = \{C_j : g(c_j) = 2\}$ is an exact cover for C. □

3 Probabilistic bound

For the probabilistic methods, we follow [7]. Analogously to some results of [7], we obtain a probabilistic bound for the signed Italian domination number.

Theorem 3.1 *For any graph G with $\delta \geq 1$,*

$$\gamma_{sI}(G) \leq n\left(2 - \frac{2\hat{\delta}\tilde{d}_{0.5}}{(1+\hat{\delta})^{1+\frac{1}{\hat{\delta}}}\tilde{d}_{0.5}^{1+\frac{1}{\hat{\delta}}}}\right),$$

where $\hat{\delta} = \lfloor 0.5\delta \rfloor$ and $\tilde{d}_{0.5} = \binom{\delta'+1}{\lceil 0.5\delta' \rceil}$ such that $\delta' = \delta$ if δ is odd and $\delta' = \delta + 1$ if δ is even.

Proof Let A be a set formed by independent choice of vertices of G, where each vertex is selected with the probability

$$p = 1 - \frac{1}{(1+\hat{\delta})^{\frac{1}{\hat{\delta}}}\tilde{d}_{0.5}^{\frac{1}{\hat{\delta}}}}.$$

For $m \geq 0$, let B_m be the set of vertices $v \in V(G)$ dominated by exactly m vertices of A such that $N[v] \cap A = m \leq \lceil 0.5 d_v \rceil$. Now form the set B by selecting $\lceil 0.5 d_v \rceil - m + 1$ vertices from $N[v]$ that are not in A for each vertex $v \in B_m$. We set $D = A \cup B$. Assume that $f : V(G) \rightarrow \{-1, 1, 2\}$ is a function such that all vertices in A and B are labeled by 2 and 1, respectively and all other vertices by -1. It is obvious that $f(V(G)) = 3|A| - 2|B| - n$ and f is a signed Italian domination function. The expectation of $f(V(G))$ is

$$\begin{aligned}
E[f(V(G))] &= 3E[|A|] + 2E[|B|] - n \\
&\leq 3\sum_{i=1}^{n} p(v_i \in A) + 2\sum_{i=1}^{n}\sum_{m=0}^{\lceil 0.5d_i \rceil}(\lceil 0.5 d_i \rceil - m + 1)p(v_i \in B_m) - n \\
&= 3pn + 2\sum_{i=1}^{n}\sum_{m=0}^{\lceil 0.5d_i \rceil}(\lceil 0.5 d_i \rceil - m + 1)\binom{d_i+1}{m}p^m(1-p)^{d_i+1-m} - n \\
&\leq 3pn + 2\sum_{i=1}^{n}\max_{d_i \geq \delta} f(d_i, p) - n,
\end{aligned}$$

where $f(d, p) = \sum_{m=0}^{\lceil 0.5d \rceil}(\lceil 0.5d \rceil - m + 1)\binom{d+1}{m}p^m(1-p)^{d+1-m}$ for any $d \geq \delta \geq 2$. By Lemma 1 [2], $\max_{d \geq \delta} f(d, p) \in \{f(\delta, p), f(\delta+1, p)\}$ and so $\max_{d \geq \delta} f(d, p) = f(\delta', p)$. Therefore

$$E[f(V(G))] \leq 3np + 2n\sum_{m=0}^{\lceil 0.5\delta' \rceil}(\lceil 0.5\delta' \rceil - m + 1)\binom{\delta'+1}{m}p^m(1-p)^{\delta'+1-m} - n.$$

Since $(\lceil 0.5\delta^{'}\rceil - m + 1)\binom{\delta'+1}{m} \leq \binom{\delta'+1}{m}\binom{\lceil 0.5\delta'\rceil}{m}$, then we obtain

$$\begin{aligned} E[f(V(G))] &\leq 3np + 2n\sum_{m=0}^{\lceil 0.5\delta^{'}\rceil}(\lceil 0.5\delta^{'}\rceil - m + 1)\binom{\delta' + 1}{m}p^m(1-p)^{\delta^{'}+1-m} - n \\ &= 3np + 2n\binom{\delta' + 1}{\lceil 0.5\delta'\rceil}(1-p)^{\delta^{'}-\lceil 0.5\delta'\rceil+1}\sum_{m=0}^{\lceil 0.5\delta^{'}\rceil}\binom{\lceil 0.5\delta^{'}\rceil}{m}p^m(1-p)^{\lceil 0.5\delta^{'}\rceil-m} - n \\ &= 3np + 2n\tilde{d}_{0.5}(1-p)^{\delta^{'}-\lceil 0.5\delta^{'}\rceil+1} - n. \end{aligned}$$

Taking into account that $\delta' - \lceil 0.5\delta'\rceil = \lfloor 0.5\delta'\rfloor = \lfloor 0.5\delta\rfloor = \hat{\delta}$, we have

$$\begin{aligned} E[f(V(G))] &\leq 3np + 2n\tilde{d}_{0.5}(1-p)^{\hat{\delta}+1} - n \\ &\leq 2n\left(1 - \frac{\hat{\delta}\tilde{d}_{0.5}}{(1+\hat{\delta})^{1+\frac{1}{\hat{\delta}}}\tilde{d}_{0.5}^{1+\frac{1}{\hat{\delta}}}}\right). \end{aligned}$$

□

4 Tree

In this section, we present sharp bounds on the signed Italian domination number of trees. In first, we show that there exist graphs with signed Italian domination number which are negative.

Proposition 4.1 *For every integer $k \geq 1$, there exists a tree T of order $4k + 3$ such that $\gamma_{sI}(T) \leq -k$.*

Proof For every integer $k \geq 1$, let T be the tree obtained from $K_{1,2}$ with center vertex x_0 and k copies of graph $K_{1,3}$ with centers $x_1, x_2, \cdots, x_k$, where x_i is adjacent to x_{i+1} for $0 \leq i \leq k-1$. Define the function $f : V(T) \to \{-1, 1, 2\}$ such that $f(x_i) = 2$ for each $0 \leq i \leq k$ and $f(y) = -1$ for each leaf of T. It is straightforward to check that f is a SIDF on T of weight $2(k+1) - 3k - 2 = -k$. Therefore $\gamma_{sI}(T_k) \leq -k$. □

Remark 4.2 Let G be a graph and f be a signed Italian domination. Hence there exists a signed Italian domination g with $w(g) \leq w(f)$ such that for any support vertex v, we have $v \notin V^g_{-1}$. Since if v is support vertex with $f(v) = -1$, then for any u_i adjacent to v, we have $f(u_i) = 2$. Consider the leaf u adjacent to v and define $g : V(G) \to \{-1, 1, 2\}$ by $g(v) = 2$, $g(u) = -1$ and $g(x) = f(x)$ for each vertex $x \in V(G) \setminus \{v, u\}$. Clearly, again g is a signed Italian domination function and $w(g) \leq w(f)$.

Now based on the above remark, the following result is obvious.

Proposition 4.3 *For $r \geq s \geq 1$,*

$$\gamma_{sI}(DS_{r,s}) = \begin{cases} 2 & \textit{if } s = r = 1, \\ 1 & \textit{if } s = 1\ r > 1 \textit{ is even}, \\ 0 & \textit{if } s = 1\ r > 1 \textit{ is odd}, \\ 0 & \textit{if } s > 1\ s, r \textit{ are even}, \\ -1 & \textit{if } s > 1\ s \textit{ is odd and } r \textit{ is even}, \\ -2 & \textit{if } s > 1\ s, r \textit{ are odd}. \end{cases}$$

Now we present a sharp bound on the signed Italian domination number in Trees. In first, we introduce some notation for convenience.

Let $V_l = \{v \in V(T) \mid v\ is\ a\ leaf\}$, $V_s = \{v \in V(T) \mid v\ is\ a\ support\ vertex\}$ and $V_w = V(T) \setminus (V_l \cup V_s)$.

For any tree T, let F_T be the tree obtained from T by adding $2deg(v) + 1$ pendant edges to each vertex $v \in V(T)$. Assume that $\mathcal{T} = \{F_T \mid T\ is\ tree\}$.

Theorem 4.4 *If T be a tree of order $n \geq 2$, then $\gamma_{sI}(T) \geq \frac{-n+4}{2}$ with equality holds if and only if $T \in \mathcal{T}$.*

Proof We proceed by induction on $n \geq 2$. If $n = 2$, then $T = K_2$, $\gamma_{sI}(K_2) = 1 = \frac{-n+4}{2}$. If $n = 3$, then $T = P_3$, $\gamma_{sI}(T) = 2 > \frac{-n+4}{2}$. Let $n \geq 4$ and suppose that the statement holds for all trees of order less than n. Let T be tree of order n. If $diam(T) = 2$, then T is a star and by Proposition 4 in [6], we have $\gamma_{sI}(T) \geq \frac{-n+4}{2}$. If $diam(T) = 3$, then T is a double star $DS_{r,s}$ with $r \geq s \geq 1$ and by Proposition 4.3, we have $\gamma_{sI}(T) \geq \frac{-n+4}{2}$ with equality if and only if $T = DS_{3,3}$. Therefore assume that $diam(T) \geq 4$. Let $f = (V_{-1}, V_1, V_2)$ be a γ_{sI}-function of T.

At first, suppose that V_{-1} is not an independent set. If $u, v \in V_{-1}$ are adjacent vertices, then u and v are not leaves. Hence if T_1 and T_2 are two trees obtained from T by removing the edge uv, then T_1 and T_2 are nontrivial and the function $f_i = f|_{T_i}$ is a SIDF of T_i for each $i \in \{1, 2\}$. Using the inductive hypothesis on T_1 and T_2 with the fact that $w(f) = w(f_1) + w(f_2)$ we obtain

$$\gamma_{sI}(T) = w(f_1) + w(f_2) \geq \frac{-|V(T_1)| + 4}{2} + \frac{-|V(T_2| + 4}{2} > \frac{-n+4}{2},$$

and result is obtained. Suppose that V_{-1} is an independent set. Consider the following cases:

***Case* 1**. $(V_w \cup V_s) \cap V_{-1} \neq \emptyset$.

By Remark 4.2, we can assume that v is not support vertex i.e $v \in V_w$. Suppose that $T_1, T_2, \cdots, T_r$ are the components of $T \setminus v$ and let f_i be the restriction of f on T_i for $1 \leq i \leq r$. Since T_i's are nontrivial for any $1 \leq i \leq r$, then by inductive hypothesis we have

$$\begin{aligned} \gamma_{sI}(T) &= \sum_{i=1}^{r} w(f_i) + f(v) \geq \sum_{i=1}^{r} \frac{-|V(T_i)| + 4}{2} + f(v) \\ &\geq \frac{-n+4}{2} + \frac{1 + 4(r-1)}{2} - 1 > \frac{-n+4}{2}. \end{aligned}$$

According to Case 1, we may assume that all non-leaf vertex of T have positive weight under f.
***Case* 2**. $(V_w \cup V_s) \cap V_{-1} = \emptyset$.
***Subcase* 1**. $(V_w \cup V_s) \cap V_1 \neq \emptyset$.

Let $v \in V_s$ with $f(v) = 1$. Hence any leaf adjacent to v must be assigned 1 or 2 under f. Let $T_1, T_2, \cdots, T_r$ be the components of $T \setminus v$ of order at least two and let $v_i \in V(T_i)$ be a vertex adjacent to v for each i. Then we have $f(v_i) \geq 1$ for each i (by Case 1). If $f(v_i) = 1$ for some i, say $i = 1$, then let F_1 and F_2 be the components of $T \setminus vv_1$ containing v_1 and v, respectively. Define $g : V(F_1) \to \{-1, 1, 2\}$ by $g(v_1) = f(v_1) + 1$ and $g(x) = f(x)$ for $x \in V(F_1) - \{v_1\}$. Clearly, g is a SIDF of F_1 and the function $f_2 = f|_{F_2}$ is a SIDF of F_2. By inductive hypothesis we deduce that

$$\begin{aligned}\gamma_{sI}(T) &= w(g) + w(f_2) - 1\\ &\geq \frac{-|V(F_1)| + 4}{2} + \frac{-|V(F_2)| + 4}{2} - 1\\ &\geq \frac{-n + 4}{2} + \frac{4}{2} - 1 > \frac{-n + 4}{2}.\end{aligned}$$

Assume that $f(v_i) = 2$ for each i. Let F_1 and F_2 be the components of $T \setminus vv_1$ containing v_1 and v respectively. Let F' be the tree obtained from F_1 by adding a new vertex v' attached at v_1 by an edge v_1v'. Define $g : V(F_1') \to \{-1, 1, 2\}$ by $g(v') = 1$ and $g(x) = f(x)$ for $x \in V(F_1') \setminus \{v'\}$. Clearly, g is a SIDF of F_1' and the function $f_2 = f|_{F_2}$ is a SIDF of F_2. Again by inductive hypothesis for trees F_1' and F_2 we conclude that

$$\begin{aligned}\gamma_{sI}(T) &= w(g) + w(f_2) - 1\\ &\geq \frac{-|V(F_1')| + 4}{2} + \frac{-|V(F_2)| + 4}{2} - 1\\ &\geq \frac{-(n + 1) + 8}{2} - 1 > \frac{-n + 4}{2}.\end{aligned}$$

Regarding above cases, we may assume that $f(v) = 2$ for each non-leaf vertex, v, of T.
***Subcase* 2**. $(V_w \cup V_s) \cap V_2 \neq \emptyset$
At first, suppose that $v \in V_w$. Let us recall from the foregoing that all neighbors of v belong to V_2. Consider the forest $T \setminus v$ by adding $deg(v) - 1$ edges between vertices of $N(v)$ in $T \setminus v$ so that the resulting graph T^* is a tree. Clearly $f^* = f|_{T\setminus v}$ is a SIDF on T^*. Using the fact that $n = |V(T^*)| + 1$ and $w(f) = w(f^*) + 2$ and by applying inductive hypothesis on T^*, we have

$$\gamma_{sI}(T) = w(f^*) + 2 \geq \frac{-|V(T^*)| + 4}{2} + 2 > \frac{-n + 4}{2}.$$

Obviously, no support vertex can have all its leaves in $V \setminus V_{-1}$. Now let v be a support vertex and $u_1, u_2, \cdots, u_s$ its leaves.

Let $f(u_i) = 1$, $f(u_j) = -1$ for some i, j and $T^{'} = T \setminus \{u_i, u_j\}$. The function f restricted to $T^{'}$ is a SIDF of $T^{'}$, and inductive hypothesis implies that

$$\gamma_{sI}(T) = w(f|_{T'}) \geq \frac{-(n-2)+4}{2} > \frac{-n+4}{2}.$$

Let $f(u_i) = 2$, $f(u_j) = -1$ for some i, j, and $T^{'}$ be a tree obtained from $T \setminus u_j$. Clearly the function g defined on $T^{'}$ by $g(u_i) = 1$ and $g(x) = f(x)$ otherwise is a SIDF of $T^{'}$. By the inductive hypothesis we have

$$\gamma_{sI}(T) = w(f) = w(g) \geq \frac{-(n-1)+4}{2} > \frac{-n+4}{2}.$$

Finally, by above cases, we may assume that every vertex of T is either a leaf or a support vertex and all leaves of T belong to V_{-1}. Recall that all support vertices belong to V_2. For every support vertex v, let l_v be the number of leaves in $N_T(v)$. Let $T^{'}$ be the tree obtained from T by removing all leaves of T and let $n^{'} = |V(T^{'})|$. Since for every support vertex v, $f[v] \geq 1$ we must have $l_v \leq 2deg_{T'}(v) + 1$. Note that $\sum_{v \in V(T')} l_v \leq \sum_{v \in V(T')}(2deg_{T'}(v) + 1) = 5n^{'} - 4$. Using the facts that $n = n^{'} + \sum_{v \in V(T')} l_v$, and $\gamma_{sI}(T) = 2n^{'} - \sum_{v \in V(T')} l_v$, one can easily check that

$$\gamma_{sI}(T) = 2n' - \sum_{v \in V(T')} l_v \geq \frac{-(n' + \sum_{v \in V(T')} l_v) + 4}{2} = \frac{-n+4}{2}.$$

If further $\gamma_{sI}(T) = \frac{-n+4}{2}$, then we must have equality throughout the previous inequality chain which implies in particular that $l_v = 2deg(v) + 1$ for every $v \in V(T^{'})$ and therefore $T \in \mathcal{T}$.

Conversely, if $T \in \mathcal{T}$, then define the function $g : V(T) \to \{-1, 1, 2\}$ such that assigns -1 to all leaves and 2 to all support vertices. Thus g is a SIDF of T and so $\gamma_{sI}(T) \leq \frac{-n+4}{2}$, this implies that $\gamma_{sI}(T) = \frac{-n+4}{2}$, hence the proof is complete. □

5 Operations on graphs

In this section, we express signed Italian domination number of some graph operations.

Proposition 5.1 *If G_1 and G_2 are two graphs such that $\gamma_{sI}(G_1) \geq 0$ and $\gamma_{sI}(G_2) \geq 0$, then $\gamma_{sI}(G_1 \vee G_2) \leq \gamma_{sI}(G_1) + \gamma_{sI}(G_2)$.*

Proof Let f_1 be a γ_{sI}-function on G_1 and f_2 be a γ_{sI}-function on G_2. Define the function $f : V(G_1 \vee G_2) \to \{-1, 1, 2\}$ by $f(v) = f_1(v)$ for each $v \in V(G_1)$ and $f(v) = f_2(v)$ for each $v \in V(G_2)$. Hence for each $v \in V(G_1)$, $f(N_{G_1 \vee G_2}[v]) = f(N_{G_1}[v]) + w(f_2) \geq 1$. Similarly, for each $v \in V(G_2)$, $f(N_{G_1 \vee G_2}[v]) = f(N_{G_2}[v]) + w(f_1) \geq 1$. Thus f is a SIDF on $(G_1 \vee G_2)$, and $\gamma_{sI}(G_1 \vee G_2) \leq w(f) = w(f_1) + w(f_2) = \gamma_{sI}(G_1) + \gamma_{sI}(G_2)$. □

Proposition 5.2 *Let $m \geq 2$ be an integer and $n = 2m+1$. If $G = Fr_n = K_1 \vee (mK_2)$, then $\gamma_{sI}(Fr_n) = 1$.*

Proof Suppose that $V(Fr_n) = \{x\} \cup \{y_i, z_i \mid 1 \leq i \leq m\}$ and $E(Fr_n) = \{xy_i, xz_i \mid 1 \leq i \leq m\} \cup \{y_i z_i \mid 1 \leq i \leq m\}$. Since $\Delta(Fr_n) = n - 1$, then Theorem 1 in [6] implies that $\gamma_{sI}(Fr_n) \geq 1$.

Now define the function $f : V(Fr_n) \to \{-1, 1, 2\}$ by $f(x) = 1$, $f(x_i) = 1$ for $1 \leq i \leq m$, and $f(y_i) = -1$ for $1 \leq i \leq m$. It is clear that f be a SIDF on Fr_n of weight 1. Hence $\gamma_{sI}(Fr_n) \leq 1$ and consequently, $\gamma_{sI}(Fr_n) = 1$. □

Proposition 5.3 *Let $W_n = K_1 \vee C_n$ be a wheel of order $n + 1$. Then $\gamma_{sI}(W_4) = 2$ and $\gamma_{sI}(W_n) = 1$ for each $n \neq 4$.*

Proof Suppose that $V(W_n) = \{v_0, \cdots, v_n\}$ and $E(W_n) = \{v_0 v_i \mid 1 \leq i \leq n\} \cup \{v_1 v_2, \cdots, v_n v_1\}$. The result is trivial to check for $n \leq 4$. Assume that $n \geq 5$, since $\Delta(W_n) = n$, then Theorem 1 in [6] implies that $\gamma_{sI}(W_n) \geq 1$.

To complete the proof, it is sufficient to provide a signed Italian domination function of weight 1 on W_n for each $n \neq 4$. First, assume that n is odd. Then define the function $f : V(W_n) \to \{-1, 1, 2\}$ by

$$f(v_i) = \begin{cases} 2 & \text{if } i = 0, \\ 1 & \text{if } i \equiv 0 \pmod 2, \\ -1 & \text{if } i \equiv 1 \pmod 2. \end{cases}$$

Now assume that n is even such that $n \equiv 0 \pmod 3$. Then define the function $f : V(W_n) \to \{-1, 1, 2\}$ by

$$f(v_i) = \begin{cases} 1 & \text{if } i = 0, \\ 2 & \text{if } i \geq 1,\ i \equiv 0 \pmod 3, \\ -1 & \text{o.w.} \end{cases}$$

Next assume that n is even such that $n \equiv 1 \pmod 3$. Then define the function $f : V(W_n) \to \{-1, 1, 2\}$ by

$$f(v_i) = \begin{cases} 2 & \text{if } i = 0, \\ 1 & \quad i \in \{n-4, n-1, n\} \\ 2 & \text{if } 1 \leq i \leq n-7,\ i \equiv 0 \pmod 3, \\ -1 & \text{o.w.} \end{cases}$$

Finally, assume that n is even such that $n \equiv 2 \pmod 3$. Then define the function $f : V(W_n) \to \{-1, 1, 2\}$ by

$$f(v_i) = \begin{cases} 2 & \text{if } i = 0, \\ 1 & \quad i \in \{n-2, n\}, \\ 2 & \text{if } 1 \leq i \leq n-5,\ i \equiv 0 \pmod 3, \\ -1 & \text{o.w.} \end{cases}$$

In any cases, one can easily to check that f is a SIDF on W_n of weight 1. Therefore in any cases, we have a SIDF on W_n of weight 1 and so $\gamma_{sI}(W_n) \leq 1$ for each $n \neq 4$ which proof is complete. □

Proposition 5.4 *Let $F_n = K_1 \vee P_n$ be a fan of order $n+1$. Then $\gamma_{sI}(F_n) = 1$.*

Proof Let $V(F_n) = \{v_0, v_1, \cdots, v_n\}$ and $E(F_n) = \{v_0 v_i \mid 1 \leq i \leq n\} \cup \{v_1 v_2, \cdots, v_{n-1} v_n\}$. The result is trivial to check for $n \leq 4$. Since $\Delta(F_n) = n$, then Theorem 1 in [6] implies that $\gamma_{sI}(F_n) \geq 1$.

To complete the proof, it is sufficient to provide a signed Italian domination function of weight 1 on F_n. First, assume that n is odd. Then define the function $f : V(F_n) \to \{-1, 1, 2\}$ by

$$f(v_i) = \begin{cases} 2 & \text{if } i = 0, \\ 1 & \text{if } i \equiv 0 \pmod 2, \\ -1 & \text{if } i \equiv 1 \pmod 2. \end{cases}$$

Now assume that n is even such that $n \equiv 0 \pmod 3$. Then define the function $f : V(F_n) \to \{-1, 1, 2\}$ by

$$f(v_i) = \begin{cases} 1 & \text{if } i = 0, \\ 2 & \text{if } i \geq 1,\ i \equiv 2 \pmod 3, \\ -1 & \text{o.w.} \end{cases}$$

Next assume that n is even such that $n \equiv 1 \pmod 3$. Then define the function $f : V(F_n) \to \{-1, 1, 2\}$ by

$$f(v_i) = \begin{cases} 2 & \text{if } i = 0, \\ 1 & \ i \in \{n-5, n-2, n\}, \\ 2 & \text{if } 2 \leq i \leq 3k+2,\ 0 \leq k \leq \frac{n-10}{3}, \\ -1 & \text{o.w.} \end{cases}$$

Finally, assume that n is even such that $n \equiv 2 \pmod 3$. Then define the function $f : V(F_n) \to \{-1, 1, 2\}$ by

$$f(v_i) = \begin{cases} 2 & \text{if } i = 0, \\ 1 & \ i \in \{n-3, n-1\}, \\ 2 & \text{if } 2 \leq i \leq 3k+2,\ 0 \leq k \leq \frac{n-8}{3}, \\ -1 & \text{o.w.} \end{cases}$$

In any cases, one can easily to check that f is a SIDF on F_n of weight 1. Therefore in any cases, we have a SIDF on F_n of weight 1 and so $\gamma_{sI}(F_n) \leq 1$. This completes the proof. □

Now we present the signed Italian dominating function of the corona product graph of some special graphs.

Proposition 5.5 *For two integer numbers $n, m > 2$, $\gamma_{sI}(C_n \odot K_m) = n$.*

Proof Let $G = C_n \odot K_m$ be a graph with vertices set V. We consider two following cases:

Case 1. Let m be even. Define the function $f : V \to \{-1, 1, 2\}$ by $f(v) = 2$ for one vertex $v \in K_m$, $f(v) = -1$ for $\frac{m+2}{2}$ vertices of K_m, $f(v) = 1$ for the remaining vertices of K_m and $f(v) = 2$ for each vertex of C_n. In this case, produce a SIDF of weight n and so $\gamma_{sI}(C_n \odot K_m) \le n$.

Case 2. Let m be odd. Define the function $f : V \to \{-1, 1, 2\}$ by $f(v) = -1$ for $\frac{m+1}{2}$ vertices of K_m, $f(v) = 1$ for the remaining vertices of K_m and $f(v) = 2$ for each vertex of C_n. In this case, produce a SIDF of weight n and so $\gamma_{sI}(C_n \odot K_m) \le n$.

Now let f be a signed Italian domination function of G and ${K_m}^i$ be copy corresponding vertex c_i. Since $f[x] = w(K_m^i) + f(c_i) \ge 1$ for each vertex $x \in K_m^i$ and $c_i \in V(C)$, then $w(K_m^i) \ge 1 - f(c_i) \ge -1$. If $w(K_m^i) = -1$, then $f(c_i) = 2$. If $w(K_m^i) = 0$, then $f(c_i) \ge 1$. If $w(K_m^i) = 1$, then $f(c_i) \ge 1$. If $w(K_m^i) \ge 2$, then $f(c_i) \ge -1$. In any case, $w(f) \ge n$. This is true for any signed Italian dominating function of G. Therefore $\gamma_{sI}(G) = \min\{f \mid f\ is\ a\ SIDF\ of\ G\} \ge n$. Hence $\gamma_{sI}(C_n \odot K_m) = n$. □

Acknowledgements The authors would like to thank the referees for their helpful remarks which have contributed to improve the presentation of the article.

References

1. Chellali, M., Haynes, T.W., Hedetniemi, S.T., McRae, A.A.: Roman {2}-domination. Discrete Appl. Math. **204**, 22–28 (2016)
2. Füredi, Z., Mubayi, D.: Signed domination in regular graphs and set-systems. J. Combin. Theory-Ser. B **76**(2), 223–239 (1999)
3. Hickey, G., Dehne, F., Rau-Chaplin, A., Blouin, C.: Spr distance computation for unrooted trees. Evolut. Bioinform. **4**, EBO-S419 (2008)
4. Henning, M.A., Klostermeyer, W.F.: Italian domination in trees. Discrete Appl. Math. **217**, 557–564 (2017)
5. Hajibaba, M., Rad, N.J.: Some notes on the roman domination number and Italian domination number in graphs. J. Phys. Conf. Ser. **890**, 012123 (2017)
6. Karamzadeh, A., Maimani, H.R., Zaeembashi, A.: On the signed italian domination of graphs. Comput. Sci. J. Moldova **27**(2), 204–229 (2019)
7. Poghosyan, A., Zverovich, V.: Discrepancy and signed domination in graphs and hypergraphs. Discrete Math. **310**(15–16), 2091–2099 (2010)