Journal of Applied Mathematics and Computing (2021) 66:809–821
https://doi.org/10.1007/s12190-020-01462-0

ORIGINAL RESEARCH

# Toughness and isolated toughness conditions for $P_{\geq 3}$-factor uniform graphs

Hongbo Hua[1]

Received: 4 August 2020 / Revised: 8 November 2020 / Accepted: 9 November 2020 /
Published online: 19 November 2020


## Abstract

Given a graph $G$ and an integer $k \geq 2$. A spanning subgraph $F$ of a graph $G$ is said to be a $P_{\geq k}$-factor of $G$ if each component of $F$ is a path of order at least $k$. A graph $G$ is called a $P_{\geq k}$-factor uniform graph if for any two distinct edges $e_1$ and $e_2$ of $G$, $G$ admits a $P_{\geq k}$-factor including $e_1$ and excluding $e_2$. More recently, Zhou and Sun (Discret Math 343:111715, 2020) gave binding number conditions for a graph to be $P_{\geq 2}$-factor and $P_{\geq 3}$-factor uniform graphs, respectively. In this paper, we present toughness and isolated toughness conditions for a graph to be a $P_{\geq 3}$-factor uniform graph, respectively.

**Keywords** Graph · Path factor · $P_{\geq 3}$-factor uniform graph · Toughness · Isolated toughness

**Mathematics Subject Classification** 05C38 · 05C75

## 1 Introduction

Throughout this paper we consider only simple connected graphs. For a graph $G = (V, E)$ with vertex set $V = V(G)$ and edge set $E = E(G)$, the *degree* of a vertex $v$ in $G$, denoted by $d_G(v)$, is the number of edges incident with $v$. The *open neighborhood* of a vertex $v$, denoted by $N_G(v)$, is the set of vertices adjacent to $v$ in $G$. For a subset $S \subseteq V(G)$, we use $N_G(S)$ to denote $\bigcup_{v \in S} N_G(v)$. If $d_G(v) = 0$ for some vertex $v$ in $G$, then $v$ is said to be an *isolated vertex* in $G$. If $d_G(v) = 1$ for some vertex $v$ in $G$, then

This research was supported by National Natural Science Foundation of China under Grant Nos. 11971011, 11571135 and sponsored by Qing Lan Project of Jiangsu Province, P.R. China.

✉ Hongbo Hua
hongbo_hua@163.com

1 Faculty of Mathematics and Physics, Huaiyin Institute of Technology, 223003 Huai'an, People's Republic of China

$v$ is said to be a *leaf* in $G$. Let $ISO(G)$ be the set of isolated vertices of $G$, and let $i(G) = |ISO(G)|$. For $S \subseteq V(G)$, let $G[S]$ be the subgraph of $G$ induced by $S$, and write $G - S = G[V(G) \setminus S]$. For an edge subset $E_0$ of $G$, let $G - E_0$ be the subgraph of $G$ obtained by deleting all edges in $E_0$. The number of connected components of $G$ is denoted by $\omega(G)$.

We first introduce three parameters for a graph, namely, the binding number, the toughness and the isolated toughness. Let $G$ be a graph. The *binding number* of $G$ is defined as

$$bind(G) = \min\left\{\frac{|N_G(X)|}{|X|} : \emptyset \neq X \subseteq V(G), N_G(X) \neq V(G)\right\}.$$

The *toughness* of $G$ is defined by Chvátal in [4] as

$$t(G) = \min\left\{\frac{|X|}{\omega(G - X)} : X \subseteq V(G), \omega(G - X) \geq 2\right\},$$

if $G$ is not complete; otherwise, $t(G) = +\infty$.

The *isolated toughness* of $G$ is defined by Yang et al. in [7] as

$$I(G) = \min\left\{\frac{|X|}{i(G - X)} : X \subseteq V(G), i(G - X) \geq 2\right\},$$

if $G$ is not complete; otherwise, $I(G) = +\infty$.

An $\mathcal{H}$-factor is a spanning subgraph of a graph, whose connected components are isomorphic to graphs from the set $\mathcal{H}$. A *path-factor* is a spanning subgraph $F$ of $G$ such that each component of $F$ is a path of order at least two. This concept was introduced by Akiyama and Kano [2]. A $P_{\geq k}$-factor means a path factor in which each component has order at least $k$ ($k \geq 2$). To characterize those graphs having a $P_{\geq 3}$-factor, Kaneko [5] introduced the concept of a sun. If $H - v$ has a perfect matching for each $v \in V(H)$, then $H$ is called a *factor-critical graph*. Let $H$ be a factor-critical graph with vertex set $V(H) = \{v_1, v_2, \cdots, v_n\}$. By adding new vertices $\{u_1, u_2, \cdots, u_n\}$ together with new edges $\{v_i u_i | 1 \leq i \leq n\}$ to $H$, we obtain a new graph, which is called a *sun*. According to Kaneko, $K_1$ and $K_2$ are also suns. Usually, $K_1$ and $K_2$ are called a small sun and the others are called big suns (with order at least 6). If a component of $G - X$ is isomorphic to a sun, it is called a *sun component* of $G - X$. Let $Sun(G - X)$ be the set of sun components of $G - X$ and $sun(G - X)$ be the number of sun components of $G - X$.

Akiyama et al. [1] provided a criterion for a graph having a $P_{\geq 2}$-factor, which reads as follows.

**Theorem 1.1** ([1]) *A graph $G$ admits a $P_{\geq 2}$-factor if and only if $i(G - X) \leq 2|X|$ for any $X \subseteq V(G)$.*

Kaneko [5] presented a criterion for a graph having a $P_{\geq 3}$-factor, which is stated as follows.

**Theorem 1.2** ([5]) *A graph $G$ admits a $P_{\geq 3}$-factor if and only if $sun(G-X) \leq 2|X|$ for any $X \subseteq V(G)$.*

Later, Kano et al. [6] gave a simple proof to Theorem 1.2. Kaneko [5] showed that a regular graph with degree greater than or equal to two has a $P_{\geq 3}$-factor. Bazgan et al. [3] proved that for a graph $G$, if $t(G) \geq 1$, then $G$ contains a $P_{\geq 3}$-factor.

Later, Zhang and Zhou [8] defined a graph $G$ to be a *$P_{\geq k}$-factor covered graph* if $G$ admits a $P_{\geq k}$-factor containing $e$ for any $e \in E(G)$. Furthermore, they gave a characterization for a graph to be a $P_{\geq 2}$-factor covered graph and $P_{\geq 3}$-factor covered graph, respectively. Their results are stated as follows.

**Theorem 1.3** ([10]) *Let $G$ be a connected graph. Then $G$ is a $P_{\geq 2}$-factor covered graph if and only if*

$$i(G-X) \leq 2|X| - \varepsilon_1(X)$$

*for any $X \subseteq V(G)$, where $\varepsilon_1(X)$ is defined as follows:*

$$\varepsilon_1(X) = \begin{cases} 2, & \text{if } X \text{ is not an independent set;} \\ 1, & \text{if } X \text{ is a nonempty independent set} \\ & \text{and } G-X \text{ admits a nontrivial component;} \\ 0, & \text{otherwise.} \end{cases}$$

**Theorem 1.4** ([10]) *Let $G$ be a connected graph. Then $G$ is a $P_{\geq 3}$-factor covered graph if and only if*

$$sun(G-X) \leq 2|X| - \varepsilon_2(X)$$

*for any $X \subseteq V(G)$, where $\varepsilon_2(X)$ is defined as follows:*

$$\varepsilon_2(X) = \begin{cases} 2, & \text{if } X \neq \emptyset \text{ and } X \text{ is not an independent set;} \\ 1, & \text{if } X \neq \emptyset \text{ and } X \text{ is a nonempty independent set} \\ & \text{and } G-X \text{ admits a non-sum component;} \\ 0, & \text{otherwise.} \end{cases}$$

Zhou et al. [9] showed that if $t(G) > \frac{2}{3}$ holds for a graph $G$, then $G$ is a $P_{\geq 3}$-factor covered graph. This result improved the result of Bazgan et al. [3].

More recently, Zhou and Sun [10] defined a graph $G$ to be a *$P_{\geq k}$-factor uniform graph* if for any two distinct edges $e_1$ and $e_2$ of $G$, $G$ admits a $P_{\geq k}$-factor including $e_1$ and excluding $e_2$. In the same paper, they gave binding number conditions for a graph to be $P_{\geq 2}$-factor and $P_{\geq 3}$-factor uniform graphs, respectively. Their results are stated as follows.

**Theorem 1.5** ([10]) *Let $G$ be a 2-edge-connected graph. If $bind(G) > \frac{4}{3}$, then $G$ is a $P_{\geq 2}$-factor uniform graph.*

**Theorem 1.6** ([10]) *Let $G$ be a 2-edge-connected graph. If $bind(G) > \frac{9}{4}$, then $G$ is a $P_{\geq 3}$-factor uniform graph.*

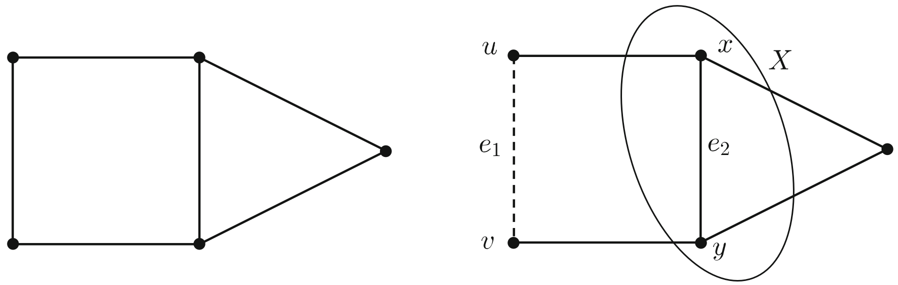


**Fig. 1** The graph $G$ with $t(G) = 1$ (left); The edge $e_1$ is an edge to be excluded from $G$ and $e_2$ is an edge not containing in any $P_{\geq 3}$-factor of $G^{'} = G - e_1$ (right)

In this paper, we present toughness and isolated toughness conditions for a graph to be a $P_{\geq 3}$-factor uniform graph, respectively. Our results are as follows.

**Theorem 1.7** *Let $G$ be a 2-edge-connected graph. If $t(G) > 1$, then $G$ is a $P_{\geq 3}$-factor uniform graph.*

**Theorem 1.8** *Let $G$ be a 2-edge-connected graph. If $I(G) > 2$, then $G$ is a $P_{\geq 3}$-factor uniform graph.*

We postpone the proofs of Theorems 1.7 and 1.8 to the subsequent two sections.

## 2 The proof of Theorem 1.7

***Remark 2.1*** We say that the condition that $t(G) > 1$ in Theorem 1.7 can not be replaced by $t(G) \geq 1$. To see this, we let $G$ be the graph as shown in Fig. 1. Clearly, $t(G) = 1$. Set $X = \{x, y\}$ and $G^{'} = G - e_1$. Then $\varepsilon_2(X) = 2$ and $sun(G^{'} - X) = 3 > 2 = 2|X| - 2 = 2|X| - \varepsilon_2(X)$. By Theorem 1.4, $G^{'}$ is not a $P_{\geq 3}$-factor covered graph. So, $G$ is not a $P_{\geq 3}$-factor uniform graph. In fact, the edge $e_2$ is not included into any $P_{\geq 3}$-factor of $G^{'}$.

**The proof of Theorem 1.7**

***Proof*** If $G$ is a complete graph, then $G$ is evidently a $P_{\geq 3}$-factor uniform graph. So, we may assume that $G$ is not a complete graph. Since $G$ is a 2-edge-connected graph, we have $|V(G)| \geq 4$.

We proceed by contradiction. Suppose that there exists an edge $e = uv$ in $G$ such that $G^{'} = G - e$ is not a $P_{\geq 3}$-factor covered graph. By Theorem 1.4, there exists a vertex subset $X$ of $V(G^{'})$ such that

$$sun(G^{'} - X) \geq 2|X| - \varepsilon_2(X) + 1. \tag{1}$$

We distinguish between the following three cases.

**Case 1.** $|X| = 0$.

In this case, we have $\varepsilon_2(X) = 0$. By (1), we have $sun(G^{'} - X) \geq 1$. Since $G^{'}$ is connected, we have $sun(G^{'} - X) \leq \omega(G^{'} - X) = \omega(G^{'}) = 1$. So, $sun(G^{'} - X) = 1$.

By our assumption that $|V(G^{'})| = |V(G)| \geq 4$ and the definition of sun, it is easy to see that $G^{'}$ is a big sun with at least six vertices. Also, $G^{'}$ has $\frac{|V(G)|}{2}(\geq 3)$ leaves. So, $G$ has at least one leaf. It is a contradiction to our assumption that $G$ is a 2-edge-connected graph.

**Case 2.** $|X| = 1$.

In this case, we have $\varepsilon_2(X) \leq 1$. By (1), we have

$$sun(G^{'} - X) \geq 2|X| = 2. \tag{2}$$

Since $\omega(G^{'} - X) \leq \omega(G - X) + 1$ and $\omega(G^{'} - X) \geq sun(G^{'} - X)$, by (2) and the definition of toughness, we have

$$\begin{aligned} 1 < t(G) &\leq \frac{|X|}{\omega(G - X)} \\ &\leq \frac{|X|}{\omega(G^{'} - X) - 1} \\ &\leq \frac{|X|}{sun(G^{'} - X) - 1} \\ &\leq \frac{|X|}{2|X| - 1} \\ &= 1, \end{aligned}$$

a contradiction.

**Case 3.** $|X| \geq 2$.

In this case, we have $\varepsilon_2(X) \leq 2$. Also, by (1), we have

$$sun(G^{'} - X) \geq 2|X| - 1. \tag{3}$$

Since $\omega(G - X) \geq \omega(G^{'} - X) - 1$ and $\omega(G^{'} - X) \geq sun(G^{'} - X)$, by (3) and the definition of toughness, we have

$$\begin{aligned} 1 < t(G) &\leq \frac{|X|}{\omega(G - X)} \\ &\leq \frac{|X|}{\omega(G^{'} - X) - 1} \\ &\leq \frac{|X|}{sun(G^{'} - X) - 1} \\ &\leq \frac{|X|}{2(|X| - 1)}, \end{aligned}$$

which gives $|X| < 2$, a contradiction.

This completes the proof. □

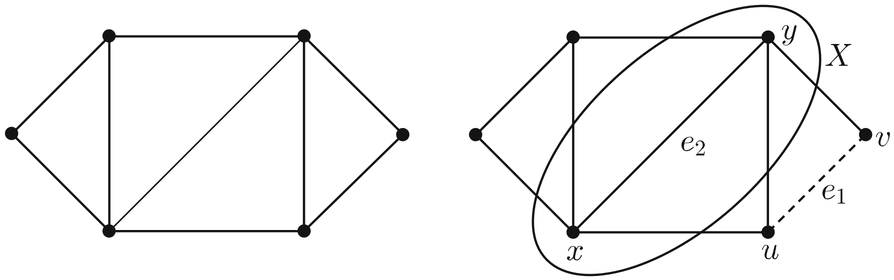


**Fig. 2** The graph $G$ with $I(G) = 2$ (left); The edge $e_1$ is an edge to be excluded from $G$ and $e_2$ is an edge not containing in any $P_{\geq 3}$-factor of $G^{'} = G - e_1$ (right)

## 3 The proof of Theorem 1.8

***Remark 3.1*** We say that the condition that $I(G) > 2$ in Theorem 1.8 can not be replaced by $I(G) \geq 2$. To see this, we let $G$ be the graph as shown in Fig. 2. Clearly, $I(G) = 2$. Set $X = \{x, y\}$ and $G^{'} = G - e_1$. Then $\varepsilon_2(X) = 2$ and $sun(G^{'} - X) = 3 > 2 = 2|X| - \varepsilon_2(X)$. By Theorem 1.4, $G^{'}$ is not a $P_{\geq 3}$-factor covered graph. So, $G$ is not a $P_{\geq 3}$-factor uniform graph. In fact, the edge $e_2$ is not contained in any $P_{\geq 3}$-factor of $G^{'}$.

**The proof of Theorem 1.8**

***Proof*** Since $G$ is a 2-edge-connected graph, we have $|V(G)| \geq 3$. If $G$ is the complete graph, then $G$ is obviously a $P_{\geq 3}$-factor uniform graph. Now, we suppose that $G$ is not the complete graph. So, $|V(G)| \geq 4$.

We proceed by contradiction. Suppose that there exists an edge $e = uv$ in $G$ such that $G^{'} = G - e$ is not a $P_{\geq 3}$-factor covered graph. By Theorem 1.4, there exists a vertex subset $X$ of $V(G^{'})$ such that

$$sun(G^{'} - X) \geq 2|X| - \varepsilon_2(X) + 1. \tag{4}$$

We suppose that there exist $a$ $K_1'$s, $b$ $K_2'$s, and $c$ big sun components $L_1, \ldots, L_c$ with $|V(L_i)| \geq 6$ in $G^{'} - X$ for each $i = 1, \ldots, c$. By the definition of sun, we have

$$sun(G^{'} - X) = a + b + c. \tag{5}$$

We consider the following three cases.

**Case 1.** $|X| = 0$.

In this case, we have $\varepsilon_2(X) = 0$. By (4), we have $sun(G^{'} - X) \geq 1$. Since $G$ is 2-edge-connected, $G^{'} - X = G^{'}$ is connected, and then $sun(G^{'} - X) \leq \omega(G^{'} - X) = \omega(G^{'}) = 1$. So, $sun(G^{'} - X) = 1$.

By the fact that $|V(G^{'})| = |V(G)| \geq 4$ and the definition of sun, we conclude that $G^{'}$ is a big sun. Thus, $|V(G^{'})| = |V(G)| \geq 6$. Denote by $N$ the factor-critical subgraph of $G^{'}$. According to the definition of factor-critical subgraph, we have $|V(N)| \geq 3$. So,

$G^{'}$ has $|V(N)|$ leaves. Note that $G^{'} = G - e$. Now, $G$ must have at least $|V(N)| - 2$ ($\geq 1$) leaf. It is a contradiction to the fact $G$ is a 2-edge-connected graph.

**Case 2.** $|X| = 1$.

In this case, we have $\varepsilon_2(X) \leq 1$. So, by (4), we have

$$sun(G^{'} - X) \geq 2|X|. \tag{6}$$

First, we assume that $c \geq 1$. We consider any one big sun component, say $L_1$, in $G^{'}$ and let $N_1$ be its factor-critical subgraph. Set $Z = V(N_1)$, by the definition of big sun and factor-critical subgraph, we have $|Z| \geq 3$. Then, $L_1$ has at least three leaves in $G^{'} - X$. So, we can always choose one vertex, say $w$, in $Z$ such that $G - (X \cup \{w\})$ has an isolated vertex, no matter whether $u$ and $v$ belong to $L_1$ or not. This means that $i\Big(G - (X \cup \{w\})\Big) \geq 1$.

By the definition of isolated toughness, we have

$$\begin{aligned} 2 < I(G) &\leq \frac{|X \cup \{w\}|}{i\Big(G - (X \cup \{w\})\Big)} \\ &= 2, \end{aligned}$$

a contradiction.

Second, we assume that $c = 0$. We first assume that $a \neq 0$. In this case, we claim that $u \in ISO(G^{'} - X)$ or $v \in ISO(G^{'} - X)$. To see this, we first show that $X \neq \{u\}$ and $X \neq \{v\}$. Suppose, to the contrary, that $X = \{u\}$. Then $v \notin IOS(G^{'} - X)$. Otherwise, $d_G(v) = 1$, a contradiction to the fact that $G$ is a 2-edge-connected graph. Also, for any other vertex $w \in V(G) \setminus \{u, v\}$, $w \notin IOS(G^{'} - X)$. Otherwise, $d_G(w) = 1$, a contradiction. So, $u \in V(G) \setminus X$ and $v \in V(G) \setminus X$. Now, let $z$ be a vertex in $V(G) \setminus \{u, v\}$ such that $X = \{z\}$. Then, for any one vertex $w$ in $V(G) \setminus \{u, v, z\}$, $w$ can not be an isolated vertex in $G^{'} - X$, for otherwise, $d_G(w) = 1$, a contradiction to the fact that $G$ is a 2-edge-connected. Since $a \geq 1$, we must have $|\{u, v\} \cap IOS(G^{'} - X)| \geq 1$. Assume that $v \in IOS(G^{'} - X)$. Then $d_G(v) = 2$.

Let $S = X \cup \{u\}$. Then $v$ is an isolated vertex in $G - S$ and $|S| = 2$. So,

$$\begin{aligned} 2 < I(G) &\leq \frac{|S|}{i(G - S)} \\ &= 2, \end{aligned}$$

a contradiction.

Now, we assume that $a = 0$ and $c = 0$. Then, by (5) and (6), we have $b = sun(G^{'} - X) \geq 2|X| = 2$. Let $W$ be the vertex set composed of all $2b$ end-vertices of $b$ $K_2^{'}s$, and let $T = V(G^{'}) \setminus (X \cup W)$. Then we have

- $X = \{u\}$ and $v \in W \cup T$ (or $X = \{v\}$ and $u \in W \cup T$), or
- $u \in W$ and $v \in W$, or
- $u \in W$ and $v \in T$ (or $v \in W$ and $u \in T$), or
- $u \in T$ and $v \in T$.

First, we assume that $X = \{u\}$ and $v \in W \cup T$ (or $X = \{v\}$ and $u \in W \cup T$). If $X = \{u\}$ and $v \in W$, there is an independent edge $vx$ in $G^{'} - X$. So, $x$ is an isolated vertex in $G - \{u, v\}$. Then

$$2 < I(G) \leq \frac{|\{u, v\}|}{i(G - \{u, v\})} = 2,$$

a contradiction. If $X = \{u\}$ and $v \in T$, there is an independent edge $yz$ in $G^{'} - X$. So, $z$ is an isolated vertex in $G - \{u, y\}$. Then

$$2 < I(G) \leq \frac{|\{u, y\}|}{i(G - \{u, y\})} = 2,$$

a contradiction.

Second, we assume that $u \in W$ and $v \in W$.

Since $b \geq 2$, we let $us$ and $vt$ be two $K_2^{'}s$ in $W$ of $G^{'}$. Let $S = X \cup \{u, v\}$. Then $s$ and $t$ are isolated vertices in $G - S$. Moreover, $|S| = 3$. Hence,

$$2 < I(G) \leq \frac{|S|}{i(G - S)} = \frac{3}{2},$$

a contradiction.

Third, we assume that $u \in W$ and $v \in T$ (or $v \in W$ and $u \in T$).

Note that $v \in T$. Since $b \geq 2$, there must exist a $K_2$ edge, say $xy$, in $G^{'} - X$ such that $x \neq u$ and $y \neq u$. Let $S = X \cup \{x\}$. Then $|S| = 2$ and $G - S$ has an isolated vertex $y$. Therefore,

$$2 < I(G) \leq \frac{|S|}{i(G - S)} = 2,$$

a contradiction.

Finally, we assume that $u \in T$ and $v \in T$.

Let $xy$ be any one $K_2$ edge in $G^{'} - X$. Set $S = X \cup \{x\}$. Then $|S| = 2$ and $G - S$ has an isolated vertex $y$. Therefore,

$$2 < I(G) \leq \frac{|S|}{i(G - S)} = 2,$$

a contradiction.

**Case 3.** $|X| \geq 2$.

In this case, we have $\varepsilon_2(X) \leq 2$. Hence, by (4), we have

$$sun(G^{'} - X) \geq 2|X| - 1. \tag{7}$$

We consider the following two subcases.

**Subcase 3.1.** $|X| \leq a + b$.

Note that $i(G-X) \leq i(G^{'}-X) \leq i(G-X)+2$. We further consider the following three subcases.

**Subcase 3.1.1.** $i(G^{'} - X) = i(G - X)$.

In this subcase, we have $u \notin ISO(G^{'} - X)$ and $v \notin ISO(G^{'} - X)$. Otherwise, $i(G^{'} - X) > i(G - X)$, a contradiction.

Let $W$ be the vertex subset composed of all $2b$ end-vertices of $b$ independent edges in $G^{'} - X$. We choose the vertex subset $Y$ by the following procedure.

If $u \in W$ and $v \in W$, then $u$ and $v$ belong to two distinct $K_2^{'}s$ in $G^{'} - X$, and we let $Y$ be a $b$-element vertex subset by choosing one vertex from each of $b$ $K_2^{'}s$ such that $u \in Y$ or $v \in Y$; if one vertex of $u$ and $v$, say $u$, is an element in $W$, and $v \in V(G) \setminus (X \cup ISO(G^{'} - X) \cup W)$, then we let $Y$ be a $b$-element vertex subset by choosing one vertex from each of $b$ $K_2^{'}s$ such that $u \in Y$; Otherwise, we let $Y$ be a $b$-element vertex subset obtained by taking one vertex from each of $b$ $K_2^{'}s$.

Thus, $G - (X \cup Y)$ has $a + b$ isolated vertices, that is, $i(G - (X \cup Y)) = a + b$.

Since $|X \cup Y| = |X| + b$, by the definition of isolated toughness, we have

$$\begin{aligned} 2 < I(G) &\leq \frac{|X \cup Y|}{i\Big(G - (X \cup Y)\Big)} \\ &= \frac{|X| + b}{a + b} \\ &\leq \frac{a + 2b}{a + b}, \end{aligned}$$

resulting in $a < 0$, which is impossible.

**Subcase 3.1.2.** $i(G^{'} - X) = i(G - X) + 1$.

In this subcase, we have $u \in ISO(G^{'} - X)$ and $v \notin ISO(G^{'} - X)$, or $u \notin ISO(G^{'} - X)$ and $v \in ISO(G^{'} - X)$. Suppose without loss of generality that $u \in ISO(G^{'} - X)$ and $v \notin ISO(G^{'} - X)$. Then, $v \notin X$, for otherwise, $i(G^{'} - X) = i(G - X)$, a contradiction to our assumption.

Let $W$ be the same vertex subset defined as in Subcase 3.1.1. If $v \in W$, we let $Y$ be a $b$-element vertex subset obtained by taking one vertex from each of $b$ $K_2^{'}s$ such that $v \in Y$. Then $u$ is also an isolated vertex in $G - (X \cup Y)$. Thus, $i(G - (X \cup Y)) = a + b$ and $|X \cup Y| = |X| + b$. Similar to Subcase 3.1.1, we obtain a contradiction.

If $v \in V(G^{'}) \setminus (X \cup W \cup ISO(G^{'} - X))$, then we choose $Y$ to be a $b$-element vertex subset obtained by taking one vertex from each of $b$ $K_2^{'}s$. By our choice of $Y$, $u$ is still an isolated vertex in $G - (X \cup Y \cup \{v\})$. Thus, $i(G - (X \cup Y \cup \{v\})) \geq a + b$.

Also, we have $|X \cup Y \cup \{v\}| = |X| + b + 1$. Therefore, we obtain

$$\begin{aligned}2 < I(G) &\le \frac{|X \cup Y \cup \{v\}|}{i\Big(G - (X \cup Y \cup \{v\})\Big)}\\ &\le \frac{|X| + b + 1}{a + b}\\ &\le \frac{a + 2b + 1}{a + b},\end{aligned}$$

yielding that $a < 1$, which is a contradiction to our assumption that $u \in IOS(G^{'} - X)$.

**Subcase 3.1.3.** $i(G^{'} - X) = i(G - X) + 2$.

In this subcase, we must have $\{u, v\} \subseteq ISO(G^{'} - X)$. So, $a = i(G^{'} - X) \ge 2$.

First, we assume that $a \ge 3$. Let $Y$ be a $b$-element vertex subset obtained by taking one vertex from each of $b$ $K_2^{'}s$. Then $v$ is also an isolated vertex in $G - (X \cup Y \cup \{u\})$. Therefore, $G - (X \cup Y \cup \{u\})$ has $a + b - 1$ isolated vertices. Then

$$\begin{aligned}2 < I(G) &\le \frac{|X \cup Y \cup \{u\}|}{i\Big(G - (X \cup Y \cup \{u\})\Big)}\\ &= \frac{|X| + b + 1}{a + b - 1}\\ &\le \frac{a + 2b + 1}{a + b - 1},\end{aligned}$$

resulting in $a < 3$, which is a contradiction to our assumption.

Now, we assume that $a = 2$.

First, we suppose that $c = 0$. Since $b + 2 = a + b + c = sun(G^{'} - X) \ge 2|X| - 1$ by (5) and (7), we have $b \ge 2|X| - 3$. Let $Y$ be the $b$-element vertex subset obtained by taking one vertex from each of $b$ $K_2^{'}s$. It is easy to see that neither $u$ nor $v$ is an isolated vertex in $G - (X \cup Y)$. So,

$$\begin{aligned}2 < I(G) &\le \frac{|X \cup Y|}{i\Big(G - (X \cup Y)\Big)}\\ &= \frac{|X| + b}{a + b - 2}\\ &= \frac{|X| + b}{b},\end{aligned}$$

resulting in $b < |X|$. Thus, we have $|X| > 2|X| - 3$, that is, $|X| < 3$. By our assumption that $|X| \ge 2$, we have $|X| = 2$. So, $b = 1$. Let $w$ be one end-vertex of the unique $K_2$ in $G^{'} - X$. Note that $a = 2$, $b = 1$, $c = 0$ and $|X| = 2$. Then $G - (X \cup \{u, w\})$ has two isolated vertices, and thus

$$2 < I(G) \le \frac{|X \cup \{u,\ w\}|}{i\Big(G - (X \cup \{u,\ w\})\Big)} = 2,$$

a contradiction.

Second, we suppose that $c \ge 1$. For each $i = 1, \ldots, c$, we let $N_i$ be the factor-critical subgraph of big sun component $L_i$. According to the definition of big sun and factor-critical subgraph, for each $i = 1, \ldots, c$, we have $|V(N_i)| \ge 3$. For each $i = 1, \ldots, c$, we let $Z_i = V(N_i)$. So, for each $i = 1, \ldots, c$, $L_i - Z_i \cong |V(N_i)|K_1$. Let $Z = Z_1 \cup Z_2 \cup \cdots \cup Z_c$. Let $Y$ be the $b$-element vertex subset obtained by taking one vertex from each of $b$ $K_2^{'}s$. Since $a = 2$, both $u$ and $v$ are not isolated vertices in $G - (X \cup Y \cup Z)$, then $G - (X \cup Y \cup Z)$ has $b + \sum_{i=1}^{c} |V(N_i)|$ isolated vertices. Hence,

$$2 < I(G) \le \frac{|X \cup Y \cup Z|}{i\Big(G - (X \cup Y \cup Z)\Big)} = \frac{|X| + b + \sum_{i=1}^{c} |V(N_i)|}{b + \sum_{i=1}^{c} |V(N_i)|},$$

from which it follows that

$$|X| > b + \sum_{i=1}^{c} |V(N_i)| \ge b + 3c.$$

So, $2 + b + c = a + b + c = sun(G^{'} - X) \ge 2|X| - 1 > 2b + 6c - 1$, that is, $b + 5c < 3$, a contradiction to our assumption that $c \ge 1$.

**Subcase 3.2.** $|X| > a + b$.

In this case, we have $a + b \le |X| - 1$. Since $a + b + c \ge 2|X| - 1$, we must have $|c| \ge |X|$.

For each $i = 1, \ldots, c$, we let $N_i$ be the factor-critical subgraph of $L_i$. According to the definition of factor-critical subgraph, we have $|V(N_i)| \ge 3$ for each $i = 1, \ldots, c$. For each $i = 1, \ldots, c$, we let $Z_i = V(N_i)$. So, for each $i = 1, \ldots, c$, $L_i - Z_i \cong |V(N_i)|K_1$. Let $Z = Z_1 \cup Z_2 \cup \cdots \cup Z_c$. Also, we let $Y$ be the $b$-element vertex subset obtained by taking one vertex from each of $b$ $K_2^{'}s$.

By above analysis, we have

$$i\Big(G^{'} - (X \cup Y \cup Z)\Big) = a + b + \sum_{i=1}^{c} |V(N_i)|. \tag{8}$$

Also, we have

$$|X \cup Y \cup Z| = |X| + b + \sum_{i=1}^{c} |V(N_i)|. \tag{9}$$

Since

$$i\Big(G - (X \cup Y \cup Z)\Big) \geq i\Big(G' - (X \cup Y \cup Z)\Big) - 2,$$

by (8), (9) and the definition of isolated toughness, we have

$$\begin{aligned} 2 < I(G) &\leq \frac{|X \cup Y \cup Z|}{i\Big(G - (X \cup Y \cup Z)\Big)} \\ &\leq \frac{|X \cup Y \cup Z|}{i\Big(G' - (X \cup Y \cup Z)\Big) - 2} \\ &= \frac{|X| + b + \sum_{i=1}^{c} |V(N_i)|}{a + b + \sum_{i=1}^{c} |V(N_i)| - 2}. \end{aligned} \tag{10}$$

By (10) and $|V(N_i)| \geq 3$ for each $i = 1, \ldots, c$, we arrive at

$$|X| > 2a + b + \sum_{i=1}^{c} |V(N_i)| - 4 \geq 2a + b + 3c - 4. \tag{11}$$

By (5) and (7), we have $a + b + c = sun(G' - X) \geq 2|X| - 1$. This, in joint with (11), gives $a + b + c > 4a + 2b + 6c - 9$, that is, $3a + b + 5c < 9$.

But, by our assumption that $|c| \geq |X|$ and $|X| \geq 2$, we have $10 \leq 5|X| \leq 5c \leq 3a + b + 5c < 9$, a contradiction.

This completes the proof. □

## Compliance with ethical standards

**Conflict of interest** The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

## References


1. Akiyama, J., Avis, D., Era, H.: On a {1, 2}-factor of a graph. TRU Math. **16**, 97–102 (1980)
2. Akiyama, J., Kano, M.: Path factors of a graph. In: F. Harary JS, Maybee (Eds.) Graphs and Applications, Proc. 1st Colorado Symposium on Graph Theory, Wiley, New York, 1982, pp. 1-21. (1985)

3. Bazgan, C., Benhamdine, A.H., Li, H., Woźniak, M.: Partitioning vertices of 1- tough graph into paths. Theoret. Comput. Sci. **263**, 255–261 (2001)
4. Chvátal, V.: Tough graphs and Hamiltonian circuits. Discret. Math. **5**, 215–228 (1973)
5. Kaneko, A.: A necessary and sufficient condition for the existence of a path factor every component of which is a path of length at least two. J. Combin. Theory Ser. B **88**, 195–218 (2003)
6. Kano, M., Katona, G.Y., Király, Z.: Packing paths of length at least two. Discret. Math. **283**, 129–135 (2004)
7. Yang, J., Ma, Y., Liu, G.: Fractional $(g, f)$-factors in graphs. Appl. Math. J. Chin. Univ. Ser. A **16**, 385–390 (2001)
8. Zhang, H., Zhou, S.: Characterizations for $P_{\geq 2}$-factor and $P_{\geq 2}$-factor covered graphs. Discret. Math. **309**, 2067–2076 (2009)
9. Zhou, S., Wu, J., Zhang, T.: The existence of $P_{\geq 3}$-factor covered graphs. Discuss. Math. Graph Theory **37**, 1055–1065 (2017)
10. Zhou, S., Sun, Z.: Binding number conditions for $P_{\geq 2}$-factor and $P_{\geq 3}$-factor uniform graphs. Discret. Math. **343**, 111715 (2020)