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ORIGINAL RESEARCH

Bounded positive solutions of an iterative three-point boundary-value problem with integral boundary condtions

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Abstract
The aim of this paper is to investigate the existence, uniqueness and continuous dependence of solutions for a class of third order iterative differential equations with integral boundary conditions. The method applied here is based on Schauder's fixed point theorem. The main idea consists to convert the considered equation into an integral one before using the fixed point theorem. Moreover, an example is given to illustrate our main results.

Keywords Positive solutions · Third order iterative differential equations · Schauder's fixed point theorem

Mathematics Subject Classification 39B12 · 39B82

1 Introduction

This work establishes some suitable criteria that ensure the existence, uniqueness and continuous dependence of solutions for the following third-order iterative boundary-value problem:

$$x''' (t) + f \left(x^{[0]}(t), x^{[1]}(t), x^{[2]}(t), \ldots, x^{[n]}(t)\right) = 0, \tag{1.1}$$

$$x (0) = x'' (0) = 0,\ \alpha \int_0^\eta x (t)\, dt = x (T) \text{ with } \eta \in (0, T)\,, \tag{1.2}$$

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where $x^{[0]}(t) = t,\ x^{[1]}(t) = x(t),\ x^{[2]}(t) = x(x(t)),\ x^{[3]}(t) = x^{[2]}(x(t)), \ldots,$ $x^{[n]}(t) = x^{[n-1]}(x(t))$ are the iterates of the state $x(t)$ and f is a continuous function with respect to its arguments and satisfies some other conditions that will be specified later.

This problem includes many important models. For instance, it arises in the modeling of draining or coating fluid flow problems, electromagnetic waves, thin film flow and gravity-driven flows (see [10,11]).

In recent years, the existence of positive solutions for third-order ordinary differential equations has been the subject of several investigations (see [1,6,7] and the references cited therein). For example, by using a priori bounds and fixed point theory, it was shown in [4] that the following problem:

$$\begin{cases} y'''(t) = f\left(t, y(t), y'(t), y''(t)\right), \ 0 < t < 1, \\ y(0) = 0 \\ y'(0) - ay''(0) = \displaystyle\int_0^1 h_1\left(y(s), y'(s)\right) ds, \\ y'(1) + by''(1) = \displaystyle\int_0^1 h_2\left(y(s), y'(s)\right) ds, \end{cases}$$

where $a, b \geq 0,\ f : [0, 1] \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ and $h_1, h_2 : \mathbb{R}^2 \longrightarrow [0, +\infty)$ are continuous functions has at least one solution.

In [5], the authors converted the following problem:

$$\begin{cases} y'''(t) = \lambda^2 y(t) + \dfrac{\lambda}{2\sqrt{\pi}}\left(\dfrac{1}{t^{3/2}}\right), \ 0 < t < 1, \\ y(0) = 1 \\ y'(0) = 0 \\ y''(1) = \dfrac{-\lambda}{\sqrt{\pi}} \displaystyle\int_0^1 y(t)\, dt, \end{cases}$$

into a Volterra integral equation of the second kind before using the Adomian method for solving this last one.

Iterative differential equation often arises in the modeling of a wide range of natural phenomena such as disease transmission models in epidemiology, two-body problem of classical electrodynamics, population models, physical models, mechanical models and other numerous models. This kind of equations which relates an unknown function, its derivatives and its iterates, is a special type of the so-called differential equations with state-dependent delays. More precisely, the following class of functional differential equations with state and time-varying delays:

$$x'''(t) + f(t, x(t), x(t - \tau_1(t, x(t))), \ldots, x(t - \tau_n(t, x(t)))) = 0,$$

can be considered as the mother of Eq. (1.1) where the complicated lags $\tau_i(t, x(t))$ give rise to the appearance of the iterates.

Almost the literature related to this type of equations that presently exists includes first order equations. Unfortunately, very few papers have been devoted to study iterative differential equations of order greater than one (see [2,3,8,9,14]). The difficulty in studying these equations which have distinctive characteristics lies in the iterative terms that often hinder the application of usual methods.

To the best of the authors' knowledge, there is no work dedicated to study third order iterative differential equations except our two works [3,9]. The present paper is a continuation of these last papers since it is devoted to investigate the existence, uniqueness and continuous dependence of bounded positive solutions for a third order iterative differential equation with integral boundary conditions. Generally, the presence of iterative terms in the differential equation leads to use very difficult techniques, however, the advantage of our approach is that it combines Arzela–Ascoli theorem together with Schauder's fixed point theorem without using the Green's function technique as used in the most recent works.

Our motivation comes from the fact that very little is known about third order iterative problems and that several studies indicated that the delays in real phenomena are generally depending on both the state and the time and usually cannot be set aside or ignored especially in life sciences. Furthermore, as far as we know, this kind of problems with integral boundary conditions has not been studied till now. Thus, it is worthwhile to contribute in filling this gap by continuing the investigation in this direction in order to enrich and complement the available works in the literature.

The content of this paper is arranged as follows. We will give some basic definitions and some lemmas in the next section. Then, by virtue of Schauder's fixed point theorem, we will study the existence, uniqueness and continuous dependence of positive solutions for problem (1.1)–(1.2). As an illustration, we provide an example to show the feasibility of our main results in the fourth section. Finally, a conclusion and some remarks are given in the last section.

2 Preliminaries

Let $[0, T]$ denote a real interval and

$$\mathcal{CB}_{Int} = \Big\{x \in \mathcal{C}\left([0, T], \mathbb{R}\right) : x\left(0\right) = x''\left(0\right) = 0, \\ \alpha \int_0^\eta x\left(t\right) dt = x\left(T\right), \ \alpha \in \mathbb{R}^*, \ \eta \in (0, T)\Big\},$$

equipped with norm

$$\|x\| = \sup_{t\in[0,T]} |x\left(t\right)|,$$

the Banach space of all continuous functions on $[0, T]$ with the boundary conditions $x\left(0\right) = x''\left(0\right) = 0$ and $\alpha \int_0^\eta x\left(t\right) dt = x\left(T\right), \alpha \in \mathbb{R}^*, \eta \in (0, T)$.
For $0 \leq L \leq T$ and $M \geq 0$, let

$$\mathcal{CB}_{Int}\left(L, M\right) = \{x \in \mathcal{CB}_{Int}, \ 0 \leq x \leq L, \\ |x\left(t_2\right) - x\left(t_1\right)| \leq M\left|t_2 - t_1\right|, \ \forall t_1, t_2 \in [0, T]\}.$$

Then $\mathcal{CB}_{Int}(L, M)$ is a closed convex and bounded subset of $\mathcal{CB}_{Int}$.

Furthermore, we suppose that the nonlinearity f is globally Lipschitz in x, that is, there exists a positive constants $c_1, c_2, \ldots, c_n$ such that

$$|f(t, x_1, \ldots, x_n) - f(t, y_1, \ldots, y_2)| \leq \sum_{i=1}^{n} c_i \|x_i - y_i\|. \tag{2.1}$$

We introduce the constants

$$\rho = \sup_{s\in[0,T]} |f(s, 0, 0, \ldots, 0)|,$$
$$\zeta = \rho + L \sum_{i=1}^{n} c_i \sum_{j=0}^{j=i-1} M^j.$$

Lemma 1 [6] *Let* $2T \neq \alpha\eta^2$*, then for* $y \in \mathcal{C}([0, T], \mathbb{R})$*, the problem*

$$x'''(t) + y(t) = 0, \tag{2.2}$$
$$x(0) = x''(0) = 0, \ \alpha \int_0^\eta x(t)\,dt = x(T), \ \ \eta \in (0, T), \ \ \alpha \neq 0, \tag{2.3}$$

has a unique solution given by

$$x(t) = \frac{t}{2T - \alpha\eta^2} \int_0^T (T - s)^2 y(s)\,ds - \frac{\alpha t}{3\left(2T - \alpha\eta^2\right)} \int_0^\eta (\eta - s)^3 y(s)\,ds$$
$$- \frac{1}{2} \int_0^t (t - s)^2 y(s)\,ds.$$

Lemma 2 [6] *Let* $0 < \alpha \leq \dfrac{2T}{\eta^2}$*. If* $y \in \mathcal{C}[0, T]$ *and* $y(t) \geq 0$ *on* $(0, T)$*, then the unique solution of the problem* (2.2)–(2.3) *satisfies* $x(t) \geq 0$ *for* $t \in [0, T]$.

Lemma 3 [13] *If* $\varphi, \psi \in \mathcal{CB}_{Int}(L, M)$*, then*

$$\left\|\varphi^{[m]} - \psi^{[m]}\right\| \leq \sum_{j=0}^{m-1} M^j \|\varphi - \psi\|, \ \ m = 1, 2, \ldots$$

Theorem 1 [12] (Schauder's fixed point theorem) *Let* $\mathbb{M}$ *a non-empty compact convex subset of a Banach space* $(\mathbb{X}, \|.\|)$ *and* $\mathcal{A} : \mathbb{M} \longrightarrow \mathbb{M}$ *is a continuous mapping. Then* $\mathcal{A}$ *has a fixed point.*

3 Main results

3.1 Existence results

The main tool that we will use to prove the existence of positive bounded solutions of the problem (1.1)–(1.2) is the Schauder's fixed point theorem. From Lemma 1, we define the operator $\mathcal{A} : \mathcal{CB}_{Int}(L, M) \longrightarrow \mathcal{CB}_{Int}$ by

$$\begin{aligned}(\mathcal{A}\varphi)(t) = & \frac{t}{2T - \alpha\eta^2}\int_0^T (T-s)^2 f\left(\varphi^{[0]}(s), \varphi^{[1]}(s), \varphi^{[2]}(s), \ldots, \varphi^{[n]}(s)\right) ds \\ & - \frac{\alpha t}{3\left(2T - \alpha\eta^2\right)}\int_0^\eta (\eta-s)^3 f\left(\varphi^{[0]}(s), \varphi^{[1]}(s), \varphi^{[2]}(s), \ldots, \varphi^{[n]}(s)\right) ds \\ & - \frac{1}{2}\int_0^t (t-s)^2 f\left(\varphi^{[0]}(s), \varphi^{[1]}(s), \varphi^{[2]}(s), \ldots, \varphi^{[n]}(s)\right) ds. \qquad (3.1)\end{aligned}$$

Since $\mathcal{CB}_{Int}(L, M)$ is a compact set as a uniformly bounded, equicontinuous and closed subset of the space $\mathcal{CB}_{Int}$. To prove that operator $\mathcal{A}$ has at least one fixed point, we will show that: $\mathcal{A}$ is well defined, $\mathcal{A}$ is continuous and $\mathcal{A}(\mathcal{CB}_{Int}(L, M)) \subset \mathcal{CB}_{Int}(L, M)$, i.e.

$$(\mathcal{A}\varphi)(t) \in \mathcal{CB}_{Int}(L, M) \text{ for all } \varphi \in \mathcal{CB}_{Int}(L, M).$$

Lemma 4 *The operator $\mathcal{A} : \mathcal{CB}_{Int}(L, M) \longrightarrow \mathcal{CB}_{Int}$ given by the expression (3.1) is well defined.*

Proof To show that $\mathcal{A}$ is well defined it suffuses to show that $(\mathcal{A}\varphi)(0) = (\mathcal{A}\varphi)''(0) = 0$ and $\alpha\int_0^\eta (\mathcal{A}\varphi)(t)\,dt = (\mathcal{A}\varphi)(T)$ for all $\varphi \in \mathcal{CB}_{Int}(L, M)$. Clearly $(\mathcal{A}\varphi)(0) = 0$ and

$$(\mathcal{A}\varphi)''(t) = 2\int_0^t f\left(\varphi^{[0]}(s), \varphi^{[1]}(s), \varphi^{[2]}(s), \ldots, \varphi^{[n]}(s)\right) ds,$$

so

$$(\mathcal{A}\varphi)''(0) = 0.$$

Let $\varphi \in \mathcal{CB}_{Int}(L, M)$, we have

$$\begin{aligned}(\mathcal{A}\varphi)(T) = & \frac{T}{2T - \alpha\eta^2}\int_0^T (T-s)^2 f\left(\varphi^{[0]}(s), \varphi^{[1]}(s), \varphi^{[2]}(s), \ldots, \varphi^{[n]}(s)\right) ds \\ & - \frac{\alpha T}{3\left(2T - \alpha\eta^2\right)}\int_0^\eta (\eta-s)^3 f\left(\varphi^{[0]}(s), \varphi^{[1]}(s), \varphi^{[2]}(s), \ldots, \varphi^{[n]}(s)\right) ds \\ & - \frac{1}{2}\int_0^T (T-s)^2 f\left(\varphi^{[0]}(s), \varphi^{[1]}(s), \varphi^{[2]}(s), \ldots, \varphi^{[n]}(s)\right) ds.\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\frac{\alpha\eta^2}{2T-\alpha\eta^2}\int_0^T (T-s)^2 f\left(\varphi^{[0]}(s),\varphi^{[1]}(s),\varphi^{[2]}(s),\ldots,\varphi^{[n]}(s)\right)ds\\
&\quad -\frac{\alpha T}{3\left(2T-\alpha\eta^2\right)}\int_0^\eta (\eta-s)^3 f\left(\varphi^{[0]}(s),\varphi^{[1]}(s),\varphi^{[2]}(s),\ldots,\varphi^{[n]}(s)\right)ds,
\end{aligned}
$$

and

$$
\begin{aligned}
\alpha\int_0^\eta (\mathcal{A}\varphi)(t)dt &= \alpha\int_0^\eta \frac{t}{2T-\alpha\eta^2}\left(\int_0^T (T-s)^2 f\left(\varphi^{[0]}(s),\varphi^{[1]}(s),\ldots,\varphi^{[n]}(s)\right)ds\right)dt\\
&\quad -\alpha\int_0^\eta \frac{\alpha t}{3\left(2T-\alpha\eta^2\right)}\left(\int_0^\eta (\eta-s)^3 f\left(\varphi^{[0]}(s),\varphi^{[1]}(s),\ldots,\varphi^{[n]}(s)\right)ds\right)dt\\
&\quad -\frac{1}{2}\alpha\int_0^\eta\left(\int_0^t (t-s)^2 f\left(\varphi^{[0]}(s),\varphi^{[1]}(s),\ldots,\varphi^{[n]}(s)\right)ds\right)dt\\
&= \frac{1}{2}\frac{\alpha\eta^2}{2T-\alpha\eta^2}\int_0^T (T-s)^2 f\left(\varphi^{[0]}(s),\varphi^{[1]}(s),\ldots,\varphi^{[n]}(s)\right)ds\\
&\quad -\frac{1}{6}\frac{\alpha^2\eta^2}{2T-\alpha\eta^2}\int_0^\eta (\eta-s)^3 f\left(\varphi^{[0]}(s),\varphi^{[1]}(s),\ldots,\varphi^{[n]}(s)\right)ds\\
&\quad -\frac{1}{6}\alpha\int_0^\eta (\eta-s)^3 f\left(\varphi^{[0]}(s),\varphi^{[1]}(s),\ldots,\varphi^{[n]}(s)\right)ds.
\end{aligned}
$$

This gives

$$
\begin{aligned}
\alpha\int_0^\eta (\mathcal{A}\varphi)(t)dt &= \frac{1}{2}\frac{\alpha\eta^2}{2T-\alpha\eta^2}\int_0^T (T-s)^2 f\left(\varphi^{[0]}(s),\varphi^{[1]}(s),\ldots,\varphi^{[n]}(s)\right)ds\\
&\quad +\frac{1}{3}\frac{T\alpha}{\alpha\eta^2-2T}\int_0^\eta (\eta-s)^3 f\left(\varphi^{[0]}(s),\varphi^{[1]}(s),\ldots,\varphi^{[n]}(s)\right)ds.
\end{aligned}
$$

Hence $\alpha\int_0^\eta (\mathcal{A}\varphi)(t)dt = (\mathcal{A}\varphi)(T)$. Consequently $\mathcal{A}$ is well defined. □

Lemma 5 *Suppose that condition (2.1) holds. Then the operator $\mathcal{A} : \mathcal{CB}_{Int}(L, M) \longrightarrow \mathcal{CB}_{Int}$ given by (3.1) is continuous.*

Proof For $\varphi, \psi \in \mathcal{CB}_{Int}(L, M)$, we have

$$
\begin{aligned}
\left|(\mathcal{A}\varphi)(t)-(\mathcal{A}\psi)(t)\right| &\le \frac{t}{\left|2T-\alpha\eta^2\right|}\int_0^T (T-s)^2\left|f\left(\varphi^{[0]}(s),\varphi^{[1]}(s),\varphi^{[2]}(s),\ldots,\varphi^{[n]}(s)\right)\right.\\
&\quad \left.-f\left(\psi^{[0]}(s),\psi^{[1]}(s),\psi^{[2]}(s),\ldots,\psi^{[n]}(s)\right)\right|ds\\
&\quad +\frac{1}{3}\frac{|\alpha|\,t}{\left|\left(2T-\alpha\eta^2\right)\right|}\int_0^\eta \left|(\eta-s)^3\right|\left|f\left(\varphi^{[0]}(s),\varphi^{[1]}(s),\varphi^{[2]}(s),\ldots,\varphi^{[n]}(s)\right)\right.\\
&\quad \left.-f\left(\psi^{[0]}(s),\psi^{[1]}(s),\psi^{[2]}(s),\ldots,\psi^{[n]}(s)\right)\right|ds\\
&\quad +\frac{1}{2}\int_0^t (t-s)^2\left|f\left(\varphi^{[0]}(s),\varphi^{[1]}(s),\varphi^{[2]}(s),\ldots,\varphi^{[n]}(s)\right)\right.\\
&\quad \left.-f\left(\psi^{[0]}(s),\psi^{[1]}(s),\psi^{[2]}(s),\ldots,\psi^{[n]}(s)\right)\right|ds.
\end{aligned}
$$

By (2.1), we obtain

$$
\begin{aligned}
|(\mathcal{A}\varphi)(t)-(\mathcal{A}\psi)(t)| \leq\ & \frac{t}{|2T-\alpha\eta^2|}\left(\int_0^T (T-s)^2 ds\right)\left(\sum_{i=1}^n c_i \left\|\varphi^{[i]}-\psi^{[i]}\right\|\right) \\
&+\frac{1}{3}\frac{|\alpha|\,t}{|(2T-\alpha\eta^2)|}\left(\int_0^\eta \left|(\eta-s)^3\right| ds\right)\left(\sum_{i=1}^n c_i \left\|\varphi^{[i]}-\psi^{[i]}\right\|\right) \\
&+\frac{1}{2}\left(\int_0^t (t-s)^2 ds\right)\left(\sum_{i=1}^n c_i \left\|\varphi^{[i]}-\psi^{[i]}\right\|\right) \\
=\ & \frac{1}{3}\frac{tT^3}{|2T-\alpha\eta^2|}\left(\sum_{i=1}^n c_i \left\|\varphi^{[i]}-\psi^{[i]}\right\|\right) \\
&+\frac{1}{12}\frac{|\alpha|\,\eta^4 t}{|(2T-\alpha\eta^2)|}\left(\sum_{i=1}^n c_i \left\|\varphi^{[i]}-\psi^{[i]}\right\|\right) \\
&+\frac{1}{6}t^3\left(\sum_{i=1}^n c_i \left\|\varphi^{[i]}-\psi^{[i]}\right\|\right) \\
\leq\ & \frac{1}{3}\frac{T^4}{|2T-\alpha\eta^2|}\left(\sum_{i=1}^n c_i \left\|\varphi^{[i]}-\psi^{[i]}\right\|\right) \\
&+\frac{1}{12}\frac{|\alpha|\,\eta^4 T}{|(2T-\alpha\eta^2)|}\left(\sum_{i=1}^n c_i \left\|\varphi^{[i]}-\psi^{[i]}\right\|\right) \\
&+\frac{1}{6}T^3\left(\sum_{i=1}^n c_i \left\|\varphi^{[i]}-\psi^{[i]}\right\|\right),
\end{aligned}
$$

which implies that

$$
\begin{aligned}
|(\mathcal{A}\varphi)(t)-(\mathcal{A}\psi)(t)| &\leq \left(\frac{1}{3}\frac{tT^4}{|2T-\alpha\eta^2|}+\frac{1}{12}\frac{|\alpha|\,\eta^4 T}{|(2T-\alpha\eta^2)|}+\frac{1}{6}T^3\right)\left(\sum_{i=1}^n c_i \left\|\varphi^{[i]}-\psi^{[i]}\right\|\right) \\
&= \left(\frac{1}{12}\frac{T\left(4T^3+|\alpha|\,\eta^4\right)}{|2T-\alpha\eta^2|}+\frac{1}{6}T^3\right)\left(\sum_{i=1}^n c_i \left\|\varphi^{[i]}-\psi^{[i]}\right\|\right).
\end{aligned}
$$

It follows from Lemma 3 that

$$
|(\mathcal{A}\varphi)(t)-(\mathcal{A}\psi)(t)| \leq \left(\frac{T}{6}\left(\frac{1}{2}\frac{\left(4T^3+|\alpha|\,\eta^4\right)}{|2T-\alpha\eta^2|}+T^2\right)\sum_{i=1}^n c_i \sum_{j=0}^{j=i-1} M^j\right)\|\varphi-\psi\|,
$$

which shows that the operator $\mathcal{A}$ is continuous. □

Lemma 6 *Let* $0 < \alpha \leq \dfrac{2T}{\eta^2}$. *Suppose that condition* (2.1) *holds. If*

$$\frac{1}{6}\zeta T\left(\frac{1}{2}\frac{\left(4T^3+|\alpha|\,\eta^4\right)}{\left|2T-\alpha\eta^2\right|}+T^2\right)\leq L, \tag{3.2}$$

and

$$\frac{\zeta}{6}\left(\frac{1}{2}\frac{4T^3+|\alpha|\,\eta^4}{\left|2T-\alpha\eta^2\right|}+4T^2\right)\leq M, \tag{3.3}$$

then $\mathcal{A}\left(\mathcal{CB}_{Int}\left(L,M\right)\right)\subset\mathcal{CB}_{Int}\left(L,M\right)$.

Proof For φ in $\mathcal{CB}_{Int}\left(L,M\right)$ we have

$$\begin{aligned}
|(\mathcal{A}\varphi)(t)| \leq{}& \frac{t}{\left|2T-\alpha\eta^2\right|}\int_0^T (T-s)^2\left|f\left(\varphi^{[0]}(s),\varphi^{[1]}(s),\varphi^{[2]}(s),\ldots,\varphi^{[n]}(s)\right)\right|ds\\
&+\frac{|\alpha|\,t}{3\left|2T-\alpha\eta^2\right|}\int_0^\eta\left|(\eta-s)^3\right|\left|f\left(\varphi^{[0]}(s),\varphi^{[1]}(s),\varphi^{[2]}(s),\ldots,\varphi^{[n]}(s)\right)\right|ds\\
&+\frac{1}{2}\int_0^t (t-s)^2\left|f\left(\varphi^{[0]}(s),\varphi^{[1]}(s),\varphi^{[2]}(s),\ldots,\varphi^{[n]}(s)\right)\right|ds.
\end{aligned}$$

But

$$\begin{aligned}
\left|f\left(\varphi^{[0]}(s),\varphi^{[1]}(s),\ldots,\varphi^{[n]}(s)\right)\right| ={}& \left|f\left(s,\varphi(s),\ldots,\varphi^{[n]}(s)\right)\right.\\
&\left.-f\left(s,0,\ldots,0\right)+f\left(s,0,\ldots,0\right)\right|\\
\leq{}& \left|f\left(s,\varphi(s),\varphi^{[2]}(s),\ldots,\varphi^{[n]}(s)\right)\right.\\
&\left.-f\left(s,0,0,\ldots,0\right)\right|+\left|f\left(s,0,0,\ldots,0\right)\right|\\
\leq{}& \rho+\sum_{i=1}^{n}c_i\sum_{j=0}^{j=i-1}M^j\,\|\varphi\|\\
\leq{}& \rho+L\sum_{i=1}^{n}c_i\sum_{j=0}^{j=i-1}M^j=\zeta,
\end{aligned}$$

so

$$\begin{aligned}
|(\mathcal{A}\varphi)(t)| \leq{}& \frac{t\zeta}{\left|2T-\alpha\eta^2\right|}\int_0^T (T-s)^2\,ds\\
&+\frac{|\alpha|\,t\zeta}{3\left|2T-\alpha\eta^2\right|}\int_0^\eta\left|(\eta-s)^3\right|ds\\
&+\frac{1}{2}\zeta\int_0^t (t-s)^2 ds
\end{aligned}$$

$$= \frac{1}{3}T^3 \frac{t}{\left|2T - \alpha\eta^2\right|}\zeta + \frac{1}{12}t\frac{\eta^4}{\left|2T - \alpha\eta^2\right|}|\alpha|\,\zeta + \frac{1}{6}\zeta t^3$$
$$\leq \frac{1}{6}\zeta T\left(\frac{1}{2}\frac{\left(4T^3 + |\alpha|\,\eta^4\right)}{\left|2T - \alpha\eta^2\right|} + T^2\right) \leq L.$$

From Lemma 2 and (3.2), we get

$$0 \leq (\mathcal{A}\varphi)(t) \leq |(\mathcal{A}\varphi)(t)| \leq L.$$

Let $t_1, t_2 \in [0, T]$ with $t_1 < t_2$, we have

$$\begin{aligned}|(\mathcal{A}\varphi)(t_2) - (\mathcal{A}\varphi)(t_1)| \leq{}& \frac{|t_2 - t_1|}{\left|2T - \alpha\eta^2\right|}\int_0^T (T-s)^2 \left| f\left(\varphi^{[0]}(s), \varphi^{[1]}(s), \ldots, \varphi^{[n]}(s)\right)\right| ds \\
&+ \frac{|\alpha|}{3}\frac{|t_2 - t_1|}{\left|2T - \alpha\eta^2\right|}\int_0^\eta \left|(\eta - s)^3\right| f\left(\varphi^{[0]}(s), \varphi^{[1]}(s), \ldots, \varphi^{[n]}(s)\right) ds \\
&+ \frac{1}{2}\left|\int_0^{t_1} (t_2 - s)^2 f\left(\varphi^{[0]}(s), \varphi^{[1]}(s), \ldots, \varphi^{[n]}(s)\right) ds\right. \\
&+ \frac{1}{2}\int_{t_1}^{t_2} (t_2 - s)^2 f\left(\varphi^{[0]}(s), \varphi^{[1]}(s), \ldots, \varphi^{[n]}(s)\right) ds \\
&\left. - \frac{1}{2}\int_0^{t_1} (t_1 - s)^2 f\left(\varphi^{[0]}(s), \varphi^{[1]}(s), \ldots, \varphi^{[n]}(s)\right) ds\right| \\
\leq{}& \frac{1}{3}\frac{\zeta T^3}{\left|2T - \alpha\eta^2\right|}|t_2 - t_1| + \frac{1}{12}\frac{\zeta\,|\alpha|\,\eta^4}{\left|2T - \alpha\eta^2\right|}|t_2 - t_1| \\
&+ \frac{1}{2}\zeta t_1 t_2\,|t_1 - t_2| + \frac{1}{6}\zeta\,(t_1 - t_2)^3 \\
\leq{}& \frac{\zeta}{3}\left(\frac{1}{6}\frac{4T^3 + |\alpha|\,\eta^4}{\left|2T - \alpha\eta^2\right|} + 2T^2\right)|t_1 - t_2|.\end{aligned}$$

Using (3.3), we obtain

$$|(\mathcal{A}\varphi)(t_2) - (\mathcal{A}\varphi)(t_1)| \leq M\,|t_1 - t_2|.$$

Since $\mathcal{A}$ is well defined, i.e. $(\mathcal{A}\varphi)(t) \in \mathcal{CB}_{Int}$ for all $\varphi \in \mathcal{CB}_{Int}(L, M)$, we conclude that $\mathcal{A}(\mathcal{CB}_{Int}(L, M)) \subset \mathcal{CB}_{Int}(L, M)$. □

Theorem 2 *Let $0 < \alpha \leq \dfrac{2T}{\eta^2}$. Suppose that conditions (2.1) and (3.2)–(3.3) hold. Then the problem (1.1)–(1.2) has at least one solution x in $\mathcal{CB}_{Int}(L, M)$.*

Proof From Lemma 1, the problem (1.1)–(1.2) has a solution x on $\mathcal{CB}_{Int}(L, M)$ if and only if the operator $\mathcal{A}$ defined by (3.1) has a fixed point.
From Lemmas 4, 5 and 6 all conditions of Schauder's fixed point theorem are satisfied. Consequently $\mathcal{A}$ has at least one fixed point on $\mathcal{CB}_{Int}(L, M)$ which is a solution of problem (1.1)–(1.2). □

3.2 Uniqueness

In this section, we establish the uniqueness of solutions of the problem (1.1)–(1.2).

Theorem 3 *In addition to the assumptions of Theorem 2, if we suppose that*

$$\frac{T}{6}\left(\frac{1}{2}\frac{\left(4T^3+|\alpha|\,\eta^4\right)}{\left|2T-\alpha\eta^2\right|}+T^2\right)\sum_{i=1}^{n}c_i\sum_{j=0}^{j=i-1}M^j<1, \tag{3.4}$$

then the problem (1.1)–(1.2) *has a unique solution in* $\mathcal{CB}_{Int}\,(L,M)$.

Proof Let φ and ψ be two distinct fixed points of the operator $\mathcal{A}$. Similarly as in the proof of Lemma 5 we have

$$\begin{aligned}|\varphi(t)-\psi(t)| &= |(\mathcal{A}\varphi)(t)-(\mathcal{A}\psi)(t)|\\ &\le \frac{T}{6}\left(\frac{1}{2}\frac{\left(4T^3+|\alpha|\,\eta^4\right)}{\left|2T-\alpha\eta^2\right|}+T^2\right)\sum_{i=1}^{n}c_i\sum_{j=0}^{j=i-1}M^j\,\|\varphi-\psi\|.\end{aligned}$$

It follows from (3.4) that

$$\|\varphi-\psi\|<\|\varphi-\psi\|.$$

Therefore, we arrive at a contradiction. We conclude that $\mathcal{A}$ has a unique fixed point which is the unique solution of problem (1.1)–(1.2). □

3.3 Continuous dependence

In this section, we will show that the previously obtained solution depends continuously on the function f.

Structural stability

Theorem 4 *Suppose that the conditions of Theorem 3 hold. The unique solution of problem* (1.1)–(1.2) *depends continuously on the function* f.

Proof Let $f_1, f_2 : [0,T]\times\mathbb{R}^n \longrightarrow [0,+\infty)$ two continuous functions with respect to their arguments. From Theorem 3, it follows that there exist two unique corresponding functions x_1 and x_2 in $\mathcal{CB}_{Int}\,(L,M)$ such that

$$\begin{aligned}x_1(t) &= \frac{t}{2T-\alpha\eta^2}\int_0^T (T-s)^2 f_1\left(x_1^{[0]}(s),x_1^{[1]}(s),x_1^{[2]}(s),\ldots,x_1^{[n]}(s)\right)ds\\ &\quad-\frac{1}{3}\frac{\alpha t}{2T-\alpha\eta^2}\int_0^\eta(\eta-s)^2 f_1\left(x_1^{[0]}(s),x_1^{[1]}(s),x_1^{[2]}(s),\ldots,x_1^{[n]}(s)\right)ds\\ &\quad-\frac{1}{2}\int_0^t(t-s)^2 f_1\left(x_1^{[0]}(s),x_1^{[1]}(s),x_1^{[2]}(s),\ldots,x_1^{[n]}(s)\right)ds,\end{aligned}$$

and

$$x_2(t) = \frac{t}{2T - \alpha\eta^2}\int_0^T (T-s)^2 f_2\left(x_2^{[0]}(s), x_2^{[1]}(s), x_2^{[2]}(s), \ldots, x_2^{[n]}(s)\right) ds$$
$$- \frac{1}{3}\frac{\alpha t}{2T - \alpha\eta^2}\int_0^\eta (\eta-s)^3 f_2\left(x_2^{[0]}(s), x_2^{[1]}(s), x_2^{[2]}(s), \ldots, x_2^{[n]}(s)\right) ds$$
$$- \frac{1}{2}\int_0^t (t-s)^2 f_2\left(x_2^{[0]}(s), x_2^{[1]}(s), x_2^{[2]}(s), \ldots, x_2^{[n]}(s)\right) ds.$$

We have

$$|x_2(t) - x_1(t)| \leq \frac{t}{\left|2T - \alpha\eta^2\right|}\int_0^T (T-s)\left|f_2\left(x_2^{[0]}(s), x_2^{[1]}(s), \ldots, x_2^{[n]}(s)\right)\right.$$
$$\left. - f_1\left(x_1^{[0]}(s), x_1^{[1]}(s), \ldots, x_1^{[n]}(s)\right)\right| ds$$
$$+\frac{1}{3}\frac{|\alpha|\, t}{2T - \alpha\eta^2}\int_0^T \left|(\eta-s)^3\right|\left|f_2\left(x_2^{[0]}(s), x_2^{[1]}(s), \ldots, x_2^{[n]}(s)\right)\right.$$
$$\left. - f_1\left(x_1^{[0]}(s), x_1^{[1]}(s), \ldots, x_1^{[n]}(s)\right)\right| ds$$
$$+\frac{1}{2}\int_0^T (t-s)^2\left|f_2\left(x_2^{[0]}(s), x_2^{[1]}(s), \ldots, x_2^{[n]}(s)\right)\right.$$
$$\left. - f_1\left(x_1^{[0]}(s), x_1^{[1]}(s), \ldots, x_1^{[n]}(s)\right)\right| ds.$$

But

$$\left|f_2\left(x_2^{[0]}(s), x_2^{[1]}(s), x_2^{[2]}(s), \ldots, x_2^{[n]}(s)\right)\right.$$
$$\left. -f_1\left(x_1^{[0]}(s), x_1^{[1]}(s), x_1^{[2]}(s), \ldots, x_1^{[n]}(s)\right)\right|$$
$$= \left|f_2\left(x_2^{[0]}(s), x_2^{[1]}(s), x_2^{[2]}(s), \ldots, x_2^{[n]}(s)\right)\right.$$
$$-f_2\left(x_1^{[0]}(s), x_1^{[1]}(s), x_1^{[2]}(s), \ldots, x_1^{[n]}(s)\right)$$
$$+ f_2\left(x_1^{[0]}(s), x_1^{[1]}(s), x_1^{[2]}(s), \ldots, x_1^{[n]}(s)\right)$$
$$\left. -f_1\left(x_1^{[0]}(s), x_1^{[1]}(s), x_1^{[2]}(s), \ldots, x_1^{[n]}(s)\right)\right|$$

Using (2.1) and Lemma 3, we arrive at

$$\left|f_2\left(x_2^{[0]}(s), x_2^{[1]}(s), x_2^{[2]}(s), \ldots, x_2^{[n]}(s)\right) - f_1\left(x_1^{[0]}(s), x_1^{[1]}(s), x_1^{[2]}(s), \ldots, x_1^{[n]}(s)\right)\right|$$
$$\leq \|f_2 - f_1\| + \sum_{i=1}^{n} c_i \sum_{j=0}^{j=i-1} M^j \|x_2 - x_1\|.$$

Hence

$$\|x_2 - x_1\| \leq \frac{T}{6}\left(\frac{1}{2}\frac{\left(4T^3 + |\alpha|\,\eta^4\right)}{\left|2T - \alpha\eta^2\right|} + T^2\right)\|f_2 - f_1\|$$
$$+\frac{T}{6}\left(\frac{1}{2}\frac{\left(4T^3 + |\alpha|\,\eta^4\right)}{\left|2T - \alpha\eta^2\right|} + T^2\right)\sum_{i=1}^{n} c_i \sum_{j=0}^{j=i-1} M^j\,\|x_2 - x_1\|\,.$$

Therefore

$$\|x_2 - x_1\| \leq \frac{\frac{T}{6}\left(\frac{1}{2}\frac{\left(4T^3 + |\alpha|\,\eta^4\right)}{\left|2T - \alpha\eta^2\right|} + T^2\right)}{1 - \frac{T}{6}\left(\frac{1}{2}\frac{\left(4T^3 + |\alpha|\,\eta^4\right)}{\left|2T - \alpha\eta^2\right|} + T^2\right)\sum_{i=1}^{n} c_i \sum_{j=0}^{j=i-1} M^j}\,\|f_2 - f_1\|\,.$$

This completes the proof of the theorem. □

4 Example

In this section we will give an example for illustrating our main results.

Consider the following boundary-value problem:

$$x'''(t) + \frac{1}{4} + \frac{1}{4}\cos t + \frac{1}{18}\left(\cos^2 t\right)x^{[1]}(t)$$
$$+\frac{1}{19}\left(\sin^2 t\right)x^{[2]}(t) + \frac{1}{20}\left(\sin^2 t\right)\left(\cos^2 t\right)x^{[3]}(t) = 0, \tag{4.1}$$
$$x(0) = x''(0) = 0,\ \alpha\int_0^{\eta} x(t)\,dt = x(T)\ \text{ with } \eta \in (0, T)\,, \tag{4.2}$$

where

$$f(t, x, y, z) = \frac{1}{4} + \frac{1}{4}\cos t + \frac{1}{18}x\left(\cos^2 t\right) + \frac{1}{19}y\left(\sin^2 t\right) + \frac{1}{20}z\left(\sin^2 t\right)\left(\cos^2 t\right).$$

We have

$$|f(t, y_1, y_2, y_3) - f(t, z_1, z_2, z_3)| \leq \frac{1}{18}|y_1 - z_1| + \frac{1}{19}|y_2 - z_2| + \frac{1}{20}|y_3 - z_3|\,,$$

then

$$|f(t, y_1, y_2, y_3) - f(t, z_1, z_2, z_3)| \leq \sum_{i=1}^{3} c_i\,\|y_i - z_i\|\,,$$

with $c_1 = \frac{1}{18}$, $c_2 = \frac{1}{19}$ and $c_2 = \frac{1}{20}$. Furthermore, if $T = 1$, $L = 3$ and $M = 4$ in the definition of $\mathcal{CB}_{Int}(L, M)$, then $f > 0$, $\rho = \sup_{s\in[0,1]} |f(s, 0, 0)| = \frac{1}{2}$ and $\zeta \simeq 4.606\,1$. For $\alpha = \frac{1}{4}$ and $\eta = \frac{1}{5}$, we have $2T = 2 \neq \alpha\eta^2 = \frac{1}{100}$, $\alpha = \frac{1}{4} \leq \frac{2}{\eta^2} = \frac{25}{2}$ and

$$\frac{1}{6}\zeta T\left(\frac{1}{2}\frac{\left(4T^3 + |\alpha|\,\eta^4\right)}{\left|2T - \alpha\eta^2\right|} + T^2\right) \simeq 1.5393 \leq L = 3,$$
$$\frac{\zeta}{6}\left(\frac{1}{2}\frac{4T^3 + |\alpha|\,\eta^4}{\left|2T - \alpha\eta^2\right|} + 4T^2\right) \simeq 3.8424 \leq M = 4,$$

and

$$\frac{T}{6}\left(\frac{1}{2}\frac{\left(4T^3 + |\alpha|\,\eta^4\right)}{\left|2T - \alpha\eta^2\right|} + T^2\right)\sum_{i=1}^{n} c_i \sum_{j=0}^{j=i-1} M^j \simeq 0.4574 < 1.$$

Then the problem (4.1)–(4.2) has a unique solution which depends continuously on the function f.

5 Conclusion and remarks

In the current paper, under some sufficient conditions on the nonlinearity, we established the existence, uniqueness and continuous dependence of a positive solution for a class of third order iterative differential equations with integral boundary conditions.

Our approach is based on converting the equation into an integral one and applying Schauder's fixed point theorem combined with Arzela–Ascoli theorem. In other words, the main idea in this work consists to convert problem (1.1)–(1.2) into an integral equation before transforming it into a fixed point problem by pursuing the following path:

First of all, we constructed an appropriate Banach space for making the iterative terms $x^{[2]}(t), \ldots, x^{[n]}(t)$ well-defined and applying the Schauder's fixed point theorem. The second step is the conversion of our equation into an equivalent integral one. Afterwards, from this last one and by using Arzela–Ascoli theorem, we succeeded in defining a continuous operator from a compact subset to it self. Finally, we concluded that all conditions of Schauder's fixed point theorem was satisfied. So, if x is a fixed point of the integral operator, then x is a solution of the integral equation, hence also of problem (1.1)–(1.2).

Our results are more general than those of [1,3–6,9] in four aspects. Firstly, for yielding more realistic models, the source term in problem (1.1)–(1.2) included iterative terms describing retarded interactions, while problems studied in [4–6] were without these delays.

Secondly, the approaches used in [1,3,4,9] relied on fixed point theory and the determination of an explicit representation of a Green's function and some of its useful properties. These last ones may be quite difficult to construct, if not impossible. Whereas we reached our desired targets by using the same theory but without needed to use these properties.

Thirdly, we relaxed conditions on the nonlinearity in (1.1) for example, we did not require its periodicity as in [3,9].

Fourthly, in order to make the model more realistic, we were interested in the positivity of the solution since this last one can represent for example thickness of a thin film of viscous fluid over a solid surface, a density, a number of individuals or a concentration which are positive quantities.

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