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ORIGINAL RESEARCH

Modified viscosity implicit rules for proximal split feasibility and fixed point problems

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Abstract

The purpose of this paper is to present a modified implicit rules for finding a common element of the set of solutions of proximal split feasibility problem and the set of fixed point problems for ϑ-strictly pseudo-contractive mappings in Hilbert spaces. First, we prove strong convergence results for finding a point which minimizes a convex function such that its image under a bounded linear operator minimizes another convex function which is also a solution to fixed point of ϑ-strictly pseudo-contractive mapping. Our second algorithm generates a strong convergent sequence to approximate common solution of non-convex minimization feasibility problem and fixed point problem. In all our results in this work, our iterative scheme is proposed by a way of selecting the step size such that their implementation does not need any prior information about the operator norm because the calculation or at least an estimate of an operator norm is not an easy task. Finally, we gave numerical example to study the efficiency and implementation of our schemes.

Keywords Generalized implicit rule · Proximal split feasibility problems · Pseudo-contractive mapping · Fixed point · Hilbert space

Mathematics Subject Classification 47H09 · 47H10 · 49J20 · 49J40

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1 Introduction

Let H_1 and H_2 be two real Hilbert spaces. Suppose that $f : H_1 \to \mathbb{R} \cup \{+\infty\}$, $g : H_2 \to \mathbb{R} \cup \{+\infty\}$ are two proper, convex and lower semicontinuous functions and $A_1 : H_1 \to H_2$ is a bounded linear operator. In this paper, we consider the following problem: find a solution $\bar{x} \in H_1$ such that

$$\min_{x \in H_1} \{f(x) + g_\lambda(Ax)\}, \tag{1.1}$$

where $g_\lambda(y) = \min_{u \in H_2}\{g(u) + \frac{1}{2\lambda}\|u - y\|^2\}$ stands for the Moreau-Yosida approximate of the function g of parameter λ.

Based on an idea introduced in Lopez et al. [14], Moudafi and Thakur [18] proved weak convergence results for solving (1.1) in the case when $\arg\min f \cap A^{-1}(\arg\min) \neq \emptyset$, or in other words: in finding a minimizer $\bar{x}$ of f such that $A\bar{x}$ minimizes g, namely

$$\bar{x} \in \arg\min f \quad \text{such that} \quad A\bar{x} \in \arg\min g, \tag{1.2}$$

f, g being two proper, lower semicontinuous convex functions, $\arg\min f := \{\bar{x} \in H_1 : f(\bar{x}) \leq f(x),\ \forall x \in H_1\}$ and $\arg\min g := \{\bar{y} \in H_2 : g(\bar{y}) \leq g(y),\ \ \forall y \in H_2\}$. We shall denote the solution set of (1.2) by Γ. Concerning problem (1.2), moudafi and Thakur [18] introduced a new way of selecting the step size: Set $\theta(x) := \sqrt{\|\nabla h(x)\|^2 + \|\nabla l(x)\|^2}$ with $h(x) = \frac{1}{2}\|(I - prox_{\lambda g})Ax\|^2$, $l(x) = \frac{1}{2}\|(I - prox_{\lambda\mu_n f})x\|^2$ and introduced the following algorithm:
Split Proximal Algorithm Given an initial point $x_1 \in H_1$. Assume that $\{x_n\}$ has been constructed and $\theta(x_n) \neq 0$, then compute x_{n+1} via the rule

$$x_{n+1} = prox_{\lambda\mu_n f}(x_n - \mu_n A^*(I - prox_{\lambda g})Ax_n), \ \ n \geq 1, \tag{1.3}$$

where $prox_{\mu\lambda f}(y) = \arg\min_{x \in H_1}\{f(u) + \frac{1}{2\mu\lambda}\|u - y\|^2\}$, stepsize $\mu_n := \rho_n \frac{h(x_n)+l(x_n)}{\theta^2(x_n)}$ with $0 \leq \rho_n < 4$. If $\theta(x_n) = 0$, then $x_{n+1} = x_n$ is a solution of (1.1) and the iterative process stops, otherwise, we set $n = n + 1$ and go to (1.3). Using the split proximal algorithm (1.3), Moudafi and Thakur [18] proved a weak convergence theorem for approximating a solution of (1.2).

In 2015, Shehu and Ogbuisi [20] studied the following algorithm:
Algorithm 1 Given an initial point $x_1 \in H_1$, compute u_n using $u_n = (1 - \alpha_n)x_n$ and $\theta(u_n) \neq 0$, then compute x_{n+1} via the rule

$$\begin{cases} u_n = (1 - \alpha_n)x_n, \\ y_n = prox_{\lambda\mu_n f}(u_n - \mu_n A^*(I - prox_{\lambda g})Au_n), \\ x_{n+1} = (1 - \beta_n)y_n + \beta_n T y_n, \ n \geq 1. \end{cases} \tag{1.4}$$

where stepsize $\mu_n := \rho_n \frac{h(u_n)+l(u_n)}{\theta^2(u_n)}$ with $0 < \rho_n < 4$. If $\theta(u_n) = 0$, then $x_{n+1} = x_n$ is a solution of (1.1) which is also a fixed point of a k-strictly pseudo contractive mapping T and the iterative process stops, otherwise, we set $n := n + 1$ and go to (1.4).

Using (1.4), they prove the following strong convergence theorem for approximation of solution of (1.1) which is also a fixed point of a k strictly pseudocontractive mapping of H_1 into itself.

Theorem 1.1 *Assume that f and g are two proper convex lower-semicontinuous functions and that* (1.1) *is consistent (i.e., $\Gamma \neq 0$). Let T be a k-strictly pseudocontractive mapping of H_1 into itself such that $\Gamma \cap F(T) \neq \emptyset$. Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be real sequences in* $(0, 1)$ *satisfying the following conditions*

(A1) $\lim_{n\to\infty} \alpha_n = 0$;
(A2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(A3) $\epsilon \leq \rho_n \leq \frac{4h(u_n)}{h(u_n)+l(u_n)} - \epsilon$ *for some* $\epsilon > 0$;
(A4) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1 - k$,

then the sequence $\{x_n\}$ generated by (1.4) *converges strongly to $\bar{x} \in \Gamma \cap F(T)$.*

It should be noted that all the above-mentioned results on proximal split feasibility problems, the iterative schemes are proposed with a way of selecting the step sizes such that their implementation does not need any information about the norm of the bounded linear operator A because the calculation or at least an estimate of the operator norm $\|A\|$ is not an easy task. Please see [1,20–23] for more recent results on split feasibility problems.

Also note that by taking $f = \delta_C$ [defined as $\delta_C(x) = 0$ if $x \in C$ and $+\infty$ otherwise], $g = \delta_Q$, the indicator functions of two nonempty, closed and convex sets C, Q of H_1 and H_2 respectively, Problem (1.1) reduces to

$$\min_{x\in H_1} \{\delta_C(x) + (\delta_Q)(Ax)\} \Leftrightarrow \min_{x\in C} \left\{ \frac{1}{2\lambda} \|(I - P_Q)(Ax)\|^2 \right\} \tag{1.5}$$

which when $C \cap A^{-1}(Q) \neq \emptyset$, is equivalent to the Split Feasibility Problem (SFP): find a point

$$x \in C \quad \text{such that} \quad Ax \in Q. \tag{1.6}$$

where $A : H_1 \to H_2$ is a bounded linear operator. The SFP in finite-dimensional Hilbert space was first introduced by Censor and Elfving [7] for modeling inverse problems which arises from phase retrievals and in medical image reconstruction [4]. The SFP attracts the attention of many authors due to its application in signal processing. Various algorithms have been invented to solve it (see, e.g. [4,5,16,19,29, 31–33] and references therein). For more current and update survey on SFP please see [2,12,13,24,25,34,35].

Very recently, the implicit midpoint rules for solving fixed point problems of nonexpansive mappings have been studied by many authors, since it is a powerful numerical method for solving ordinary differential equation; see [3,9,11,26,30] and references therein. For example, Xu et al. [30] studied the following viscosity implicit midpoint rule:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T\left(\frac{x_n + x_{n+1}}{2}\right), \ n \geq 0. \tag{1.7}$$

Precisely, they obtained the following strong convergence theorems.

Theorem 1.2 (Xu et al. [30]) *Let H be a Hilbert space, C a closed convex subset of H, $T : C \to C$ a nonexpansive mapping with $S := F(T) \neq \emptyset$, and $f : C \to C$ a contraction with $\alpha \in [0, 1)$. Let $\{x_n\}$ be generated by the viscosity implicit midpoint rule (1.7), where $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying*

(C1) $\lim\limits_{n\to\infty} \alpha_n = 0$,
(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(C3) *either* $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *or* $\lim\limits_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1$.

Then $\{x_n\}$ converges in norm to a fixed point q of T, which is also the unique solution of the variational inequality

$$\langle (I - f)q, x - q\rangle \geq 0, \quad \forall\, x \in S.$$

Ke and Ma [11] studied the following generalized viscosity implicit rules:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T(s_n x_n + (1-s_n)x_{n+1}), \quad \forall\, n \geq 0. \tag{1.8}$$

To be more exact, they proved the followings main results.

Theorem 1.3 (Ke and Ma [11]) *Let C be a nonempty closed convex subset of real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and $f : C \to C$ be a contraction with coefficient $\theta \in [0, 1)$. Pick any $x_0 \in C$, let $\{x_n\}$ be a sequence generated by (1.8), where $\{\alpha_n\}, \{s_n\} \subset (0, 1)$ satisfying the following conditions:*

(1) $\lim_{n\to\infty} \alpha_n = 0$,
(2) $\sum_{n=0}^{\infty} \alpha_n = \infty$,
(3) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$,
(4) $0 < \epsilon \leq s_n \leq s_{n+1} < 1$ *for all* $n \geq 0$.

Then $\{x_n\}$ converges strongly to a fixed point q of the nonexpansive T, which is also the unique solution of the variational inequality

$$\langle (I - f)q, x - q\rangle \geq 0, \quad \forall\, x \in F(T).$$

In other words, q is the unique fixed point of the contraction $P_{F(T)}f$, that is $P_{F(T)}f(q) = q$.

Remark 1.4 We like to emphasize that, approximating a common solution of SFPs and fixed point problems have been used for many applications in various fields of science and technology, such as in signal processing and image reconstruction, and especially applied in medical fields such as intensity-modulated radiation therapy (IMRT). Example of such problems can be seen in ([6,10]).

Motivated by the result of Shehu and Ogbuisi [20] and other above mentioned related results on proximal split feasibility problems, we investigate the new viscosity implicit rules for finding a common solution of proximal split feasibility problems for the case of convex and nonconvex function [i.e (1.2) and (4.1)] which is also a fixed point of a ϑ-strictly pseudocontractive mapping and prove the strong convergence of the sequence generated by our scheme in Hilbert spaces. We mentioned here that our iterative scheme is proposed with a way of selecting the stepsize such that their implementation does not need any prior information about the bounded operator norm. Finally we gave numerical comparisons of our results with other established results to verify the efficiency and implementation of our new method.

2 Preliminaries

We state the following well-known lemmas which will be used in the sequel.
Let T be nonlinear mapping from C into itself. We use $F(T)$ to denote the set of fixed points of T. Now we recall the following basic concepts.
A mapping $V : C \to C$ is called to be a strict contraction, if there exists a fixed constant $\alpha \in (0, 1)$ such that

$$\|V(x) - V(y)\| \leq \alpha\|x - y\|, \quad \forall x, y \in C.$$

A mapping $T : C \to H$ is called to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

A mapping $T : C \to H$ is said to be ϑ-pseudocontractive if for $0 \leq \vartheta < 1$

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \vartheta\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C. \tag{2.1}$$

It is easy to show that (2.1) is equivalent to

$$\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2 - \frac{1 - \vartheta}{2}\|(I - T)x - (I - T)y\|^2. \tag{2.2}$$

Observe that if T is ϑ-strictly pseudocontractive, then for $z \in F(T)$, we can easily obtain that

$$(1 - \vartheta)\|x - Tx\|^2 \leq 2\langle x - z, x - Tx\rangle. \tag{2.3}$$

A mapping $A : C \to H$ is called monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall\ x, y \in C.$$

A mapping $A : C \to H$ called α -inverse strongly monotone if there exists a positive real number α satisfying

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \ \ \forall x, y \in C.$$

A mapping $A : C \to H$ is called η-strongly monotone if there exists a positive constant η such that

$$\langle Ax - Ay, x - y\rangle \geq \eta\|x - y\|^2, \quad \forall x, y \in C.$$

A mapping $A : C \to H$ is called k-Lipschitzian if there exists a positive constant k such that

$$\|Ax - Ay\| \leq k\|x - y\|, \quad \forall x, y \in C.$$

Lemma 2.1 *Let H be a real Hilbert space. Then there holds the following well-known results:*

(i) $\|x + y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2, \ \ \forall x, y \in H.$
(ii) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \ \ \forall x, y \in H.$

Lemma 2.2 *Let C be a nonempty, closed and convex subset of a real Hibert space H. Let $T : C \to C$ be nonexpansive mapping. Then $I - T$ is semiclosed at 0, i.e., if $x_n \rightharpoonup x \in C$ and $x_n - Tx_n \to 0$, then $x = Tx$.*

Lemma 2.3 [28] *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\sigma_n + \gamma_n, \quad n \geq 1,$$

where

(i) $\{a_n\} \subset [0, 1], \ \ \sum \alpha_n = \infty;$
(ii) $\limsup \sigma_n \leq 0;$
(iii) $\gamma_n \geq 0; \ \ (n \geq 1), \ \ \sum \gamma_n < \infty.$

Then, $a_n \to 0$ as $n \to \infty$.

Lemma 2.4 [27] *Let λ be a number in $(0, 1]$ and $T : C \to H$ be a nonexpansive mapping, we define the mapping $T^\lambda : C \to H$ by*

$$T^\lambda x = Tx - \lambda\mu F(Tx), \quad \forall x \in C,$$

where $F : H \to H$ is k-Lipschitzian and η-strongly monotone. Then T^λ is a contraction provided $0 < \mu < \frac{2\eta}{k^2}$; that is,

$$\|T^\lambda x - T^\lambda y\| \leq (1 - \lambda\tau)\|x - y\|, \ \ \forall x, y \in C,$$

where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu k^2)} \in (0, 1]$.

3 Main results

Let T be a ϑ-strictly pseudocontractive mapping of H_1 into itself, $F : H_1 \to H_1$ be a k-Lipschitz and η-strongly monotone mapping with $k > 0$ and $\eta > 0$, and $V : H_1 \to H_1$ be a δ-Lipschitz mapping with $\delta > 0$. Let $0 < \rho < 2\eta/k^2$ and $0 < \gamma\delta < \tau$, where $\tau = 1 - \sqrt{1 - \rho(2\eta - \rho k^2)}$. Set $\theta(x) := \sqrt{\|\nabla h(x)\|^2 + \|\nabla l(x)\|^2}$ with $h(x) = \frac{1}{2}\|(I - prox_{\lambda g})Ax\|^2$, $l(x) = \frac{1}{2}\|(I - prox_{\lambda\mu_n f})x\|^2$ and introduce the following algorithm:

Algorithm 3.1 *Given an initial point $x_1 \in H_1$, compute u_n using $u_n = s_n x_n + (1 - s_n)y_n$ and $\theta(u_n) \neq 0$, then compute x_{n+1} via initial rule*

$$\begin{cases} u_n = s_n x_n + (1 - s_n)y_n, \\ y_n = prox_{\lambda\mu_n f}(u_n - \mu_n A^*(I - prox_{\lambda g})Au_n), \\ x_{n+1} = \alpha_n \gamma V(x_n) + (1 - \alpha_n \rho F)[\beta_n T y_n + (1 - \beta_n)y_n], \ n \geq 1, \end{cases} \tag{3.1}$$

where the stepsize $\mu_n := \rho_n \frac{h(u_n)+l(u_n)}{\theta^2(u_n)}$ with $0 < \rho_n < 4$ and $A^* : H_2 \to H_1$ is the dual of the bounded linear operator A. If $\theta(u_n) = 0$, then $x_{n+1} = x_n$ is a solution of (1.2) which is also a fixed point of a nonexpansive mapping T and the iterative process stops, otherwise, we set $n := n + 1$ and go to (3.1).

Theorem 3.2 *Assume that f and g are two proper convex lower-semicontnuous functions and that (1.2) is consistent (i.e., $\Gamma \neq \emptyset$). Let T be a nonexpansive mapping of H_1 into itself such that $\Omega = \Gamma \cap F(T) \neq \emptyset$. Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be a real sequence in $(0, 1)$ satisfying the following conditions*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $\epsilon \leq \rho_n \leq \frac{4h(u_n)}{h(u_n)+l(u_n)} - \epsilon$ *for some* $\epsilon > 0$;
(iii) $0 < \epsilon \leq s_n \leq 1$.

then the sequence $\{x_n\}$ generated by (3.1) strongly converges to $z \in \Omega$ which is also the unique solution of the variational inequality (VI)

$$z \in \Omega, \ \ \langle(\rho F - \gamma V)z, x - z\rangle \geq 0, \ \ \forall x \in \Omega. \tag{3.2}$$

We prove Theorem 3.2 using Mainge's technique [17].

Proof Since $F : H_1 \to H_1$ is a κ-Lipschitz and η-strongly monotone mapping and $V : H_1 \to H_1$ is a ρ-Lipschitz mapping, we have for all $x, y \in H_1$ that

$$\begin{aligned} \|(I - \rho F)x - (I - \rho F)y\|^2 &= \|x - y\|^2 - 2\langle x - y, Fx - Fy\rangle + \rho^2\|Fx - Fy\|^2 \\ &\leq (1 - 2\rho\eta + \rho^2\kappa^2)\|x - y\|^2 \\ &= (1 - \tau)^2\|x - y\|^2, \end{aligned} \tag{3.3}$$

where $\tau = 1 - \sqrt{1-\rho(2\eta - \rho\kappa^2)}$. Furthermore

$$\begin{aligned}
&\|P_\Omega(I-\rho F+\gamma V)x - P_\Omega(I-\rho F+\gamma V)y\| \\
&\quad \le \|(I-\rho F+\gamma V)x - (I-\rho F+\gamma V)y\| \\
&\quad \le \|(I-\rho F)x - (I-\rho F)y\| + \gamma\|V(x)-V(y)\| \\
&\quad \le (1-\tau)\|x-y\| + \gamma\delta\|x-y\| \\
&\quad = (1-(\tau-\gamma\delta))\|x-y\|.
\end{aligned}$$

This implies that $P_\Omega(I-\rho F+\gamma V)$ is a contraction of H_1 into itself which implies that there exists a unique element $z \in H_1$ such that $z = P_\Omega(I-\rho F+\gamma V)z$.

Let $z \in \Gamma \cap F(T)$. Observe that $\nabla h(x) = A^*(I - prox_{\mu g})Ax$, $\nabla l(x) = (I - prox_{\mu\lambda f})x$. Using the fact that $prox_{\mu\lambda f}$ is nonexpansive, q verifies (1.11) (since minimizers of any function are exactly fixed points of its proximal mapping) and having in hand

$$\begin{aligned}
\langle \nabla h(x_n), x_n - z\rangle &= \langle (I - prox_{\lambda g})Ax_n, Ax_n - Az\rangle \\
&\ge \|(I - prox_{\lambda g}Ax_n)\|^2 \\
&= 2h(x_n).
\end{aligned}$$

Using the fact that $I - prox_{\lambda g}$ is firmly nonexpansive, we can write

$$\begin{aligned}
\|y_n - z\|^2 &= \|prox_{\lambda\mu_n f}(u_n - \mu_n A^*(I - prox_{\lambda g})Au_n) - z\|^2 \\
&\le \|u_n - \mu_n A^*(I - prox_{\lambda g})Au_n - z\|^2 \\
&= \|u_n - z\|^2 + \mu^2\|\nabla h(u_n)\|^2 - 2\mu_n\langle \nabla h(u_n), u_n - z\rangle \\
&\le \|u_n - z\|^2 + \mu_n^2\|\nabla h(u_n)\|^2 - 4\mu_n h(u_n) \\
&= \|u_n - z\|^2 + \rho_n^2\frac{(h(u_n)+l(u_n))^2}{(\theta^2(u_n))^2}\|\nabla h(u_n)\|^2 - 4\rho_n\frac{h(u_n)+l(u_n)}{\theta^2(u_n)}h(u_n) \\
&\le \|u_n - z\|^2 + \rho_n^2\frac{(h(u_n)+l(u_n))^2}{(\theta^2(u_n))^2} - 4\rho_n\frac{(h(u_n)+l(u_n))^2}{\theta^2(u_n)}\frac{h(u_n)}{h(u_n)+l(u_n)} \\
&= \|u_n - z\|^2 - \rho_n\left(\frac{4h(u_n)}{h(u_n)+l(u_n)} - \rho_n\right)\frac{(h(u_n)+l(u_n))^2}{\theta^2(u_n)}.
\end{aligned} \tag{3.4}$$

We also, obtain that

$$\begin{aligned}
\|u_n - z\| &= \|s_n x_n + (1-s_n)y_n - z\| \\
&= \|s_n(x_n - z) + (1-s_n)(y_n - z)\| \\
&\le s_n\|x_n - z\| + (1-s_n)\|y_n - z\| \\
&= s_n\|x_n - z\| + (1-s_n)\|u_n - z\|,
\end{aligned}$$

which implies that

$$\|u_n - z\| \le \|x_n - z\| \tag{3.5}$$

Let $w_n = \beta_n T y_n + (1-\beta_n) y_n$, we have that

$$\begin{aligned}
\|w_n - z\| &\leq \|\beta_n T y_n + (1-\beta_n) y_n - z\| \\
&= \|(1-\beta_n)(y_n - z) + \beta_n (T y_n - z)\|^2 \\
&= (1-\beta_n)^2 \|y_n - z\|^2 + \beta_n^2 \|T y_n - z\|^2 + 2\beta_n (1-\beta_n) \langle y_n - z, T y_n - z\rangle \\
&\leq (1-\beta_n)^2 \|y_n - z\|^2 + \beta_n \left[\|y_n - z\|^2 + \vartheta \|y_n - T y_n\|^2\right] \\
&\quad + 2\beta_n (1-\beta_n) \left[\|y_n - z\|^2 - \frac{1-\vartheta}{2} \|y_n - T y_n\|^2\right] \\
&= (1 - 2\beta_n + \beta_n^2) \|y_n - z\|^2 + \beta_n^2 \left[\|y_n - z\|^2 + \vartheta \|y_n - T y_n\|^2\right] \\
&\quad + 2\beta_n \|y_n - z\|^2 - 2\beta_n^2 \|y_n - z\|^2 - \beta_n (1-\beta_n)(1-\vartheta) \|y_n - T y_n\|^2 \\
&= \|y_n - z\|^2 + \left[\beta_n^2 \vartheta - \beta_n (1-\beta_n)(1-\vartheta)\right] \|y_n - T y_n\|^2 \\
&= \|y_n - z\|^2 + \beta_n [\vartheta + \beta_n - 1] \|y_n - T y_n\|^2 \\
&\leq \|y_n - z\|^2.
\end{aligned} \tag{3.6}$$

It follows from (3.1) that

$$\begin{aligned}
\|x_{n+1} - z\| &= \|\alpha_n \gamma V(x_n) + (I - \alpha_n \rho F) w_n - z\| \\
&\leq \alpha_n \|\gamma V(x_n) - \rho F(z)\| + \|(I - \alpha_n \rho F) w_n - (I - \alpha_n \rho F) z\| \\
&\leq \alpha_n \|\gamma V(x_n) - \rho F(z)\| + \|(I - \alpha_n \rho F) y_n - (I - \alpha_n \rho F) z\| \\
&\leq (1 - \alpha_n \tau) \|w_n - z\| + \alpha_n \|\gamma V(x_n) - \rho F(z)\| \\
&\leq (1 - \alpha_n \tau) \|x_n - z\| + \alpha_n \gamma \|V(x_n) - V(z)\| + \alpha_n \|V(z) - \rho F(z)\| \\
&\leq (1 - \alpha_n \tau) \|x_n - z\| + \alpha_n \gamma \delta \|x_n - z\| + \alpha_n \|\gamma V(z) - \rho F(z)\| \\
&= (1 - \alpha_n (\tau - \gamma\delta)) \|x_n - z\| + \alpha_n (\tau - \gamma\delta) \frac{\|\gamma V(z) - \rho F(z)\|}{\tau - \gamma\delta} \\
&\leq \max\left\{\|x_n - z\|, \frac{\|\gamma V(z) - \rho F(z)\|}{\tau - \gamma\delta}\right\} \\
&\vdots \\
&\leq \max\left\{\|x_1 - z\|, \frac{\|\gamma V(z) - \rho F(z)\|}{\tau - \gamma\delta}\right\}
\end{aligned} \tag{3.7}$$

This implies that the sequence $\{x_n\}$ is bounded. Consequently $\{y_n\}$ $\{T y_n\}$ and $\{u_n\}$ are bounded.
It follows from (3.1) that

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \|\alpha_n \gamma V(x_n) + (I - \alpha_n \rho F) w_n - z\|^2 \\
&= \|\alpha_n (\gamma V(x_n) - \rho F(z)) + (I - \alpha_n \rho F) w_n - (I - \alpha_n \rho F) z\|^2 \\
&\leq \|(I - \alpha_n \rho F) w_n - (I - \alpha_n \rho F) z\|^2 + 2\alpha_n \langle \gamma V(x_n) - \rho F(z), x_{n+1} - z\rangle \\
&\leq (1 - \alpha_n \tau)^2 \|w_n - z\|^2 + 2\alpha_n \langle \gamma V(x_n) - \rho F(z), x_{n+1} - z\rangle
\end{aligned}$$

$$
\begin{aligned}
&\le (1-\alpha_n\tau)^2\|y_n - z\|^2 + 2\alpha_n\langle \gamma V(x_n) - \rho F(z), x_{n+1} - z\rangle \\
&\le (1-\alpha_n\tau)^2\|x_n - z\|^2 + 2\alpha_n\langle \gamma V(x_n) - \rho F(z), x_{n+1} - z\rangle \\
&= (1-\alpha_n\tau)^2\|x_n - z\|^2 + 2\alpha_n\gamma\langle V(x_n) - V(z), x_{n+1} - z\rangle \\
&\quad + 2\alpha_n\langle \gamma V(z) - \rho F(z), x_{n+1} - z\rangle \\
&\le (1-\alpha_n\tau)^2\|x_n - z\|^2 + 2\alpha_n\gamma\delta\|x_n - z\|\|x_{n+1} - z\| \\
&\quad + 2\alpha_n\langle \gamma V(z) - \rho F(z), x_{n+1} - z\rangle \\
&\le (1-\alpha_n\tau)^2\|x_n - z\|^2 + \alpha_n\gamma\delta(\|x_n - z\|^2 + \|x_{n+1} - z\|^2) \\
&\quad + 2\alpha_n\langle \gamma V(z) - \rho F(z), x_{n+1} - z\rangle \\
&= (1-\alpha_n\tau)^2\|x_n - z\|^2 + \alpha_n\gamma\delta\|x_n - z\|^2 + \alpha_n\gamma\delta\|x_{n+1} - z\|^2 \\
&\quad + 2\alpha_n\langle \gamma V(z) - \rho F(z), x_{n+1} - z\rangle.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\|x_{n+1} - z\|^2 &\le \frac{(1-\alpha_n\tau)^2 + \alpha_n\gamma\delta}{1-\alpha_n\gamma\delta}\|x_n - z\|^2 + \frac{2\alpha_n}{1-\alpha_n\gamma\delta}\langle \gamma V(z) - \rho F(z), x_{n+1} - z\rangle \\
&= \left(1 - \frac{2(\tau-\gamma\delta)\alpha_n}{1-\alpha_n\gamma\delta}\right)\|x_n - z\|^2 + \frac{\alpha_n^2\tau^2}{1-\alpha_n\gamma\delta}\|x_n - z\|^2 \\
&\quad + \frac{2\alpha_n}{1-\alpha_n\gamma\delta}\langle \gamma V(z) - \rho F(z), x_{n+1} - z\rangle \\
&\le \left(1 - \frac{2(\tau-\gamma\delta)\alpha_n}{1-\alpha_n\gamma\delta}\right)\|x_n - z\|^2 + \frac{2(\tau-\gamma\delta)\alpha_n}{1-\alpha_n}\bigg\{\frac{\alpha_n\rho^2 M_3}{2(\tau-\gamma\delta)} \\
&\quad + \frac{1}{\tau-\gamma\delta}\left[\langle \gamma V(z) - \rho F(z), x_{n+1} - z\rangle\right]\bigg\} \\
&= (1-\delta_n)\|x_n - z\|^2 + \delta_n\sigma_n,
\end{aligned}
\tag{3.8}
$$

where $\|x_n - z\|^2 \le M_3$, $\delta_n = \frac{2(\tau-\gamma\delta)\alpha_n}{1-\alpha_n}$, $\sigma_n = \frac{\alpha_n\rho^2 M_3}{2(\tau-\gamma\delta)} + \frac{1}{\tau-\gamma\delta}\left[\langle \gamma V(z) - \rho F(z), x_{n+1} - z\rangle\right]$

The rest of the proof will be divided into two parts.

Case 1 Suppose there exists $n_0 \in \mathbb{N}$ such that $\{\|x_n - z\|^2\}_{n=n_0}^{\infty}$ is nonincreasing. Then $\{\|x_n - z\|^2\}_{n=1}^{\infty}$ converges and

$$
\|x_n - z\|^2 - \|x_{n+1} - z\|^2 \to 0,\ n \to \infty. \tag{3.9}
$$

From (3.7), we have for some $M^* > 0$ that

$$
\begin{aligned}
\|x_{n+1} - z\|^2 &\le (\|y_n - z\| + \alpha_n\|\gamma V(x_n) - \rho F(z)\|)^2 \\
&= \|y_n - z\|^2 + 2\alpha_n M^*
\end{aligned}
$$

By (3.4), we have

$$\begin{aligned}&\rho_n\left(\frac{4h(u_n)}{h(u_n)+l(u_n)}-\rho_n\right)\left(\frac{(h(u_n)+l(u_n))^2}{\theta^2(u_n)}\right)\\&\leq \|u_n-z\|^2-\|y_n-z\|^2\\&\leq \|x_n-z\|^2-\|x_{n+1}-z\|^2+2\alpha_n M^*\end{aligned}\tag{3.10}$$

Since $\alpha_n \to 0$ as $n\to\infty$, by (3.10), we obtain that

$$\rho_n(1-\beta_n)\left(\frac{4h(u_n)}{h(u_n)+l(u_n)}-\rho_n\right)\left(\frac{(h(u_n)+l(u_n))^2}{\theta^2(u_n)}\right)\to 0,\ n\to\infty.$$

Hence, we obtain

$$\frac{(h(u_n)+l(u_n))^2}{\theta^2(u_n)}\to 0,\ n\to\infty.\tag{3.11}$$

Consequently, we have

$$\lim_{n\to\infty}(h(u_n)+l(u_n))=0 \Leftrightarrow \lim_{n\to\infty}h(u_n)=0 \text{ and } \lim_{n\to\infty}l(u_n)=0,$$

because $\theta^2(u_n)=\|\nabla h(u_n)\|^2+\|\nabla l(u_n)\|^2$ is bounded. This follows from the fact that ∇h is Lipschitz continuous with constant $\|A\|^2$, ∇l is nonexpansive and $\{u_n\}$ is bounded. More precisely, for any z which solves (1.9), we have

$$\begin{aligned}\|\nabla h(u_n)\|&=\|\nabla h(u_n)-\nabla z\|\leq \|A\|^2\|u_n-z\| \text{ and}\\ \|\nabla l(u_n)\|&=\|\nabla l(u_n)-\nabla z\|\leq \|u_n-z\|.\end{aligned}$$

Now, let $\bar{x}$ be a weak cluster point of $\{u_n\}$, there exists a subsequence $\{u_{n_j}\}$ which converges weakly to $\bar{x}$. The lower-semicontinuity of h then implies that

$$0\leq h(\bar{x})\leq \liminf_{j\to\infty}h(u_{n_j})=\lim_{n\to\infty}h(u_n)=0.$$

That is, $h(\bar{x})=\frac{1}{2}\|(1-prox_{\lambda g}A\bar{x})\|=0$, i.e., $A\bar{x}$ is a fixed point of the proximal mapping of g or equivalently, $0\in\partial g(A\bar{x})$. In other words, $A\bar{x}$ is a minimizer of g.

Likewise, the lower-semicontinuity of l implies that

$$0\leq l(\bar{x})\leq \liminf_{j\to\infty}l(u_{n_j})=\lim_{n\to\infty}l(u_n)=0.$$

That is $l(\bar{x})=\frac{1}{2}\|(I-prox_{\mu_n\lambda f})\bar{x}\|=0$, i.e., $\bar{x}$ is a fixed point of the proximal mapping of f or equivalently, $0\in\partial f(\bar{x})$. In other words, $\bar{x}$ is a minimizer of f. Hence $\bar{x}\in\Gamma$.

Next, we show that $\bar{x} \in F(T)$. $z = prox_{\lambda\mu_n f}(z - \mu_n A^*(I - prox_{\lambda g})Az)$ and $A^*(I - prox_{\lambda g})A$ is Lipschitz with constant $\|A\|^2$, we have from Algorithm 3.1 that

$$\begin{aligned}
\|y_n - z\|^2 &= \|prox_{\lambda\mu_n f}(u_n - \mu_n A^*(I - prox_{\lambda g})Au_n) \\
&\quad - prox_{\lambda\mu_n f}(z - \mu_n A^*(I - prox_{\lambda g})Az)\|^2 \\
&\leq \langle (u_n - \mu_n A^*(I - prox_{\lambda g})Au_n) \\
&\quad -(z - \mu_n A^*(I - prox_{\lambda g})Az), y_n - z\rangle \\
&= \frac{1}{2}\Big[\|(u_n - \mu_n A^*(I - prox_{\lambda g})Au_n) \\
&\quad -(z - \mu_n A^*(I - prox_{\lambda g})Az)\|^2 + \|y_n - z\|^2 \\
&\quad -\|(u_n - \mu_n A^*(I - prox_{\lambda g})Au_n) \\
&\quad -(z - \mu_n A^*(I - prox_{\lambda g})Az) - (y_n - z)\|^2\Big] \\
&\leq \frac{1}{2}\Big[(1 + \mu_n\|A\|^2)^2\|u_n - z\|^2 + \|y_n - z\|^2 \\
&\quad - \|u_n - y_n - \mu_n(A^*(I - prox_{\lambda g})Au_n - A^*(I - prox_{\lambda g})Az)\|^2\Big] \\
&\leq \frac{1}{2}\Big[(1 + \mu_n\|A\|^2)^2\|u_n - z\|^2 + \|y_n - z\|^2 - \|u_n - y_n\|^2 \\
&\quad +2\mu_n\langle u_n - y_n, A^*(I - prox_{\lambda g})Au_n - A^*(I - prox_{\lambda g})Az\rangle \\
&\quad - \mu_n^2\|A^*(I - prox_{\lambda g})Au_n - A^*(I - prox_{\lambda g})Az\|^2\Big].
\end{aligned}$$

Thus,

$$\begin{aligned}
\|y_n - z\|^2 \leq{}& (1 + \mu_n\|A\|^2)^2\|x_n - z\|^2 - \|u_n - y_n\|^2 \\
&+ 2\mu_n\langle u_n - y_n, A^*(I - prox_{\lambda g})Au_n - A^*(I - prox_{\lambda g})Az\rangle \\
&- \mu^2\|A^*(I - prox_{\lambda g})Au_n - A^*(I - prox_{\lambda g})Az\|^2. \qquad (3.12)
\end{aligned}$$

We observe that

$$0 < \mu_n < 4\frac{h(u_n) + l(u_n)}{\theta^(u_n)} \to 0,\ n \to \infty$$

implies that $\mu_n \to 0$, $n \to \infty$. Furthermore, we obtain from Algorithm 3.1 and 3.12 that

$$\begin{aligned}
\|u_n - y_n\|^2 &\leq (1 + \mu_n\|A\|^2)^2\|u_n - z\|^2 - \|y_n - z\|^2 \\
&\quad +2\mu_n\langle u_n - y_n, A^*(I - prox_{\lambda g})Au_n - A^*(I - prox_{\lambda g})Az\rangle \\
&= \|u_n - z\|^2 + \mu_n\|A\|^2(2 + \mu_n\|A\|^2)\|u_n - z\|^2 - \|y_n - z\|^2 \\
&\quad +2\mu_n\langle u_n - y_n, A^*(I - prox_{\lambda g})Au_n - A^*(I - prox_{\lambda g})Az\rangle \\
&\leq \|x_n - z\|^2 + \mu_n\|A\|^2(2 + \mu_n\|A\|^2)\|u_n - z\|^2 - \|x_{n+1} - z\|^2 + 2\alpha_n M^* \\
&\quad +2\mu_n\langle u_n - y_n, A^*(I - prox_{\lambda g})Au_n - A^*(I - prox_{\lambda g})Az\rangle
\end{aligned}$$

$$= \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + 2\alpha_n M^* + \mu_n \|A\|^2 (2 + \mu_n \|A\|^2) \|u_n - z\|^2$$
$$+ 2\mu_n \langle u_n - y_n, A^*(I - prox_{\lambda g}) A u_n - A^*(I - prox_{\lambda g}) A z \rangle. \quad (3.13)$$

Since $\mu_n \to 0,\ n \to \infty$ and $\alpha_n \to 0,\ n \to \infty$, we obtain that

$$\lim_{n \to \infty} \|u_n - y_n\| = 0. \quad (3.14)$$

We observe that

$$\|u_n - y_n\| = \|s_n x_n + (1 - s_n) y_n - y_n\| = s_n \|y_n - x_n\|.$$

It follows that

$$\begin{aligned} \|x_n - y_n\| &\leq \frac{\|u_n - y_n\|}{s_n} \\ &\leq \frac{\|u_n - y_n\|}{\epsilon} \to 0,\ n \to \infty. \end{aligned} \quad (3.15)$$

We also obtain from (3.14) and (3.15) that

$$\|u_n - x_n\| = \|u_n - y_n\| + \|y_n - x_n\| \to 0.$$

By (3.1) and (2.3), we have

$$\begin{aligned} \|x_{n+1} - z\|^2 &= \|\alpha_n \gamma V(x_n) + (I - \alpha_n \rho F) w_n - z\|^2 \\ &= \|\alpha_n (\gamma V(x_n) - \rho F(z)) + (I - \alpha_n \rho F) w_n - (I - \alpha_n \rho F) z\|^2 \\ &\leq (1 - \alpha_n \tau)^2 \|w_n - z\|^2 + 2\alpha_n \langle \gamma V(x_n) - \rho F(z), x_{n+1} - z \rangle \\ &\leq (1 - \alpha_n \tau)^2 \left[\|(y - z) + \beta_n (T y_n - y_n)\|^2 \right] \\ &\quad + 2\alpha_n \langle \gamma V(x_n) - \rho F(z), x_{n+1} - z \rangle \\ &\leq (1 - \alpha_n \tau)^2 \left[\|y_n - z\|^2 + \beta_n^2 \|y_n - T y_n\|^2 - 2\beta_n \langle y_n - z, y_n - T y_n \rangle, \right] \\ &\quad + 2\alpha_n \langle \gamma V(x_n) - \rho F(z), x_{n+1} - z \rangle \\ &\leq (1 - \alpha_n \tau)^2 \left[\|y_n - z\|^2 + \beta_n (\beta_n - (1 - \vartheta)) \|y_n - T y_n\|^2 \right] \\ &\quad + 2\alpha_n \langle \gamma V(x_n) - \rho F(z), x_{n+1} - z \rangle \\ &\leq (1 - \alpha_n \tau)^2 \|x_n - z\|^2 + (1 - \alpha_n \tau)^2 \beta_n (\beta_n - (1 - \vartheta)) \|y_n - T y_n\|^2 \\ &\quad + 2\alpha_n \langle \gamma V(x_n) - \rho F(z), x_{n+1} - z \rangle \\ &\leq \|x_n - z\|^2 + \alpha_n^2 \tau^2 \|x_n - z\|^2 + (1 - \alpha_n \tau)^2 \beta_n (\beta_n - (1 - \vartheta)) \\ &\qquad \|y_n - T y_n\|^2 + \alpha_n \gamma \delta \|x_n - z\|^2 \\ &\quad + \alpha_n \gamma \|x_{n+1} - z\|^2 + 2\alpha_n \langle \gamma V(z) - \rho F(z), x_{n+1} - z \rangle \end{aligned}$$

This implies that

$$
\begin{aligned}
&(1-\alpha_n\tau)^2\beta_n((1-\vartheta)-\beta_n)\|y_n-Ty_n\|^2\\
&\quad\le \|x_n-z\|^2-\|x_{n+1}-z\|^2+\alpha_n[2\langle\gamma V(z)-\rho F(z),x_{n+1}-z\rangle\\
&\qquad+\gamma\|x_{n+1}-z\|^2+\alpha_n\tau^2\|x_n-z\|^2+\gamma\delta\|x_n-z\|^2]
\end{aligned}
$$

From the last inequality, (3.9) and condition (i) we obtain

$$\lim_{n\to\infty}\|Ty_n-y_n\|=0 \tag{3.16}$$

Using the fact that $u_{n_j}\rightharpoonup\bar{x}\in H_1$ and $\|u_n-y_n\|\to 0, n\to\infty$, we have that $y_{n_j}\rightharpoonup\bar{x}\in H_1$. Similarly $x_{n_j}\rightharpoonup\bar{x}\in H_1$ since $\|u_n-x_n\|\to 0, n\to\infty$. Using Lemma 2.2 and (3.16), we have that $\bar{x}\in F(T)$. Therefore, $\bar{x}\in\Gamma\cap F(T)$.

Next, we show that $\limsup_{n\to\infty}\langle(\rho F-\gamma V)z, z-x_n\rangle\le 0$, where $z=P_{\Gamma\cap F(T)}(I-\rho F+\gamma V)z$ is a unique solution of the variational inequality:

$$\langle(\rho F-\gamma V)z, x^*-z\rangle\ge 0,\quad \forall x^*\in\Gamma\cap F(T).$$

Then, we obtain that

$$
\begin{aligned}
\limsup_{n\to\infty}\langle(\rho F-\gamma V)z, z-x_n\rangle &= \lim_{j\to\infty}\langle\rho F-\gamma V)z, z-x_{n_j}\rangle\\
&=\langle(\rho F-\gamma V)z, z-\bar{x}\rangle\\
&\le 0.
\end{aligned}
$$

Next we prove that $\{x_n\}$ converges strongly to z, where z is the unique solution of the VI (3.2). It is easy to see in (3.8) that $\delta_n\to 0, n\to\infty$, $\sum_{n=1}^{\infty}\delta_n=\infty$ and $\limsup_{n\to\infty}\sigma_n\le 0$. Using Lemma 2.3 in (3.8), we obtain

$$\lim_{n\to\infty}\|x_n-z\|=0.$$

Thus, $x_n\to z, n\to\infty$.

Case 2 Assume that $\{\|x_n-z\|\}_{n=1}^{\infty}$ is not monotonically decreasing sequence. Set $\Gamma_n=\|x_n-z\|$, $\forall n\ge 1$ and let $\tau:\mathbb{N}\to\mathbb{N}$ be a mapping for all $n\ge n_0$ (for some n_0 large enough) by

$$\tau(n):=\max\{k\in\mathbb{N}:k\le n,\Gamma_k\le\Gamma_{k+1}\}.$$

Clearly, τ is non decreasing sequence such that $\tau(n)\to\infty$ as $n\to\infty$ and

$$0\le\Gamma_{\tau(n)}\le\Gamma_{\tau(n)+1},\ \forall n\ge n_0.$$

After a similar conclusion from (3.16), it is easy to see that

$$\lim_{n\to\infty}\|Ty_{\tau(n)}-y_{\tau(n)}\|=0.$$

Furthermore, we can show that

$$\lim_{n\to\infty} h(x_{\tau(n)}) = 0 \text{ and } \lim_{n\to\infty} l(x_{\tau(n)}) = 0.$$

Since $\{x_{\tau(n)}\}$ is bounded, there exists a subsequence of $\{x_{\tau(n)}\}$, still denoted by $\{x_{\tau(n)}\}$, which converges weakly to $\bar{x} \in \Gamma \cap F(T)$. By similar argument as above in Case 1, we conclude immediately that $\lim_{n\to\infty} \delta_{\tau(n)} = 0$ and $\limsup_{n\to\infty} \sigma_{\tau(n)} \leq 0$. From (3.8) we have that

$$\|x_{\tau(n)+1} - z\|^2 \leq (1 - \delta_{\tau(n)})\|x_{\tau(n)} - z\|^2 + \delta_{\tau(n)}\sigma_{\tau(n)} \tag{3.17}$$

which implies that (noting that $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ and $\alpha_{\tau(n)} > 0$)

$$\|x_{\tau(n)} - z\| \leq \sigma_{\tau(n)}.$$

This implies that

$$\limsup_{n\to\infty} \|x_{\tau(n)} - z\|^2 \leq 0.$$

Thus,

$$\lim_{n\to\infty} \|x_{\tau(n)} - z\| = 0. \tag{3.18}$$

Again from (3.17), we obtain

$$\limsup_{n\to\infty} \|x_{\tau(n)+1} - z\|^2 \leq \limsup_{n\to\infty} \|x_{\tau(n)} - z\|^2.$$

Therefore,

$$\lim_{n\to\infty} \|x_{\tau(n)+1} - z\| = 0.$$

Furthermore, for $n \geq n_0$, it is easy to see that $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ if $n \neq \tau(n)$ (that is, $\tau(n) < n$), because $\Gamma_j \leq \Gamma_{j+1}$ for $\tau(n) + 1 \leq j \leq n$. As a consequences, we obtain for all $n \geq n_0$,

$$0 \leq \Gamma_n \leq \max\{\Gamma_{\tau(n)}, \Gamma_{\tau(n)+1}\} = \Gamma_{\tau(n)+1}.$$

Hence $\lim \Gamma_n = 0$, that is $\{x_n\}$ converges strongly to z. □

We can obtain the following result easily.

Let T be a ϑ-pseudocontractive mapping of H_1 into itself, $F : H_1 \to H_1$ be a k-Lipschitz and η-strongly monotone mapping with $k > 0$ and $\eta > 0$, and $V : H_1 \to H_1$ be a δ-Lipschitz mapping with $\delta > 0$. Let $0 < \rho < 2\eta/k^2$ and $0 < \gamma\delta < \tau$, where $\tau = 1 - \sqrt{1 - \rho(2\eta - \rho k^2)}$. Set $\theta(x) := \sqrt{\|\nabla h(x)\|^2 + \|\nabla l(x)\|^2}$ with $h(x) = \frac{1}{2}\|(I - prox_{\lambda g})Ax\|^2$, $l(x) = \frac{1}{2}\|(-prox_{\lambda\mu_n f})\|^2$ and introduce the following algorithm:

Algorithm 3.3 *Given an initial point* $x_1 \in H_1$, *the compute* u_n *using* $u_n = \frac{x_n+y_n}{2}$ *and* $\theta(u_n) \neq 0$, *then compute* x_{n+1} *via initial rule*

$$\begin{cases} u_n = \frac{x_n+y_n}{2}, \\ y_n = prox_{\lambda\mu_n f}(u_n - \mu_n A^*(I - prox_{\lambda g})Au_n), \\ x_{n+1} = \alpha_n \gamma V(x_n) + (1 - \alpha_n \rho F)[\beta_n T y_n + (1 - \beta_n)y_n], \ n \geq 1, \end{cases} \tag{3.19}$$

where stepsize $\mu_n := \rho_n \frac{h(u_n)+l(u_n)}{\theta^2(u_n)}$ with $0 < \rho_n < 4$ and $A^* : H_2 \to H_1$ is the dual of the bounded linear operator A. If $\theta(u_n) = 0$, then $x_{n+1} = x_n$ is a solution of (1.2) which is also a fixed point of a nonexpansive mapping T and the iterative process stops, otherwise, we set $n := n + 1$ and go to (3.3).

Corollary 3.4 *Assume that* f *and* g *are two proper convex lower-semicontnuous function and that* (1.2) *is consistent (i.e.,* $\Gamma \neq \emptyset$*). Let* T *be a nonexpansive mapping of* H_1 *into itself such that* $\Omega = \Gamma \cap F(T) \neq \emptyset$. *Let* $\{\alpha_n\}_{n=1}^{\infty}$ *and* $\{\beta_n\}_{n=1}^{\infty}$ *be a real sequence in* $(0, 1)$ *satisfying the following conditions*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $\epsilon \leq \rho_n \leq \frac{4h(u_n)}{h(u_n)+l(u_n)} - \epsilon$ *for some* $\epsilon > 0$;

then sequence $\{x_n\}$ *generated by* (3.3) *strongly converges to* $z \in \Omega$ *which is also the unique solution of the variational inequality (VI)*

$$z \in \Omega, \ \ \langle(\rho F - \gamma V)z, x - z\rangle \geq 0, \ \ x \in \Omega. \tag{3.20}$$

4 Strong convergence for nonconvex minimization feasibility problem

Throughout this section g is assumed to be prox-regular. The following problem:

$$0 \in \partial f(\bar{x}) \ \text{ such that } \ 0 \in \partial_{pg}(A\bar{x}), \tag{4.1}$$

is very general in the sense that it includes, as special cases, g is convex and g is a lower-C^2 function which is of great importance in optimization and can be locally expressed as a difference $g - \frac{r}{2}\|\cdot\|^2$, where g is finite convex function, hence a large core of problems of interest in variational analysis and optimization. It should be noticed that examples abound of practitioners needing algorithms for solving nonconvex problems, for instance in crystallography, astronomy, and, more recently in inverse scattering; see for example, [15]. In what follows, we shall represent the set of solution of (4.1) by Υ.

Definition 4.1 Let $g : H_2 \to \mathbb{R}$ be a function and let $\bar{x} \in domg$, i.e., $g(\bar{x}) < +\infty$. A vector v is in proximal subdifferential $\partial_{pg}(\bar{x})$ if there exists some $r > 0$ and $\epsilon > 0$ such that for all $x \in B(\bar{x}, \epsilon)$,

$$\langle v, x - \bar{x}\rangle \leq g(x) - g(\bar{x}) + \frac{r}{2}\|x - \bar{x}\|^2.$$

when $g(\bar{x}) = +\infty$, one puts $\partial_{pg}(\bar{x}) = \emptyset$.

Before starting the definition of prox-regularity of g and properties of its proximal mapping, we recall that g is locally lower semicontinuous at $\bar{x}$ if its epigraph is closed relative to a neighborhood of $(\bar{x}, g(\bar{x})$, prox-bounded if g is minorized by quadratic function, and recall that for $\epsilon > 0$, the g-attentive ϵ-localization of $\partial_{pg}(\bar{x})$ around $(\bar{x}, \bar{v})$, is the mapping $T_\epsilon : H_2 \to 2^{H_2}$ defined by

$$\begin{cases} \{v \in \partial_{pg},\ \|v - \bar{v}\| < \epsilon\} & \text{if } \|x - \bar{x}\| < \epsilon \text{ and } |g(x) - g(\bar{x})| < \epsilon, \\ \emptyset & \text{otherwise} \end{cases} \tag{4.2}$$

Definition 4.2 A function g is said to be prox-regular at $\bar{x}$ for $\bar{v} \in \partial_{pg}(\bar{x})$ if there exists some $r > 0$ and $\epsilon > 0$ such that for all $x, x' \in B(\bar{x}, \epsilon)$ with $|g(x) - g(x')| < \epsilon$ and all $v \in B(\bar{v}, \epsilon)$ with $v \in \partial_{pg}(\bar{x})$ one has

$$g(x') \geq g(x) + \langle v, x' - x \rangle - \frac{r}{2}\|x' - x\|^2.$$

It the property holds for all vectors $\bar{v} \in \partial_{pg}(\bar{x})$, the function is said to be prox-regular at $\bar{x}$. Fundamental insights into the properties of a function g come from the study of its Moreau-Yosida regularization g_λ and the associated proximal mapping $prox_{\lambda g}$ defined for $\lambda > 0$, respectively, by

$$g_\lambda(x) = \inf_{u \in H_2}\left\{g(u) + \frac{1}{2\lambda}\|u - x\|^2\right\} \text{ and } prox_{\lambda g} := \arg\min_{u \in H_2}\left\{g(u) + \frac{1}{2\lambda}\|u - x\|^2\right\}.$$

The Latter is a fundamental tool in optimization and it was shown that a fixed point iteration on the proximal mapping could be used to develop a simple optimization algorithm, namely, the proximal point algorithm.

Note also, see, for example, Section 1 in [8], that local minima are zeros of the proximal subdifferential and that the proximal subdifferential and the convex one coincide in the convex case.

Now, let us state the following key property of the proximal mapping complement, which was proved in Remark 3.2 of Moudafi and Thakur [18].

Lemma 4.3 (Moudafi and Thakur [18]) *Suppose that g is locally lower-semicontinuous at $\bar{x}$ and prox-regular at for $\bar{v} = 0$ with respect to r and ϵ. Let T_ϵ be the g-attentive ϵ-localization of ∂_{pg} around $(\bar{x}, \bar{v})$. Then for each $\lambda \in (0, \frac{1}{r})$ and x_1, x_2 in a neigborhood U_λ of $\bar{x}$, one has*

$$\begin{aligned} &\langle (I - prox_{\lambda g})(x_1) - (I - prox_{\lambda g})(x_2), x_1 - x_2 \rangle \\ &\geq \|(I - prox_{\lambda g})(x_1) - (I - prox_{\lambda g})(x_2)\|^2 \\ &\quad - \frac{\lambda r}{(1 - \lambda r)^2}\|x_1 - x_2\|^2. \end{aligned}$$

Observe that when$r = 0$, which amounts to saying that g is convex, we recover the fact that the mapping $I - prox_{\lambda g}$ is firmly nonexpansive.

Now, the regularization parameter λ are allowed to vary in the algorithm (3.1), namely considering possibly variable parameter $\lambda \in (0, \frac{1}{r} - \epsilon)$ (for some $\epsilon > 0$ small enough) and $\mu_n > 0$, our interest is in studying the following the convergence properties of the following:

Algorithm 4.4 *Let T be a ϑ-strictly pseudocontractive mapping of H_1 into itself, $F : H_1 \to H_1$ be a k-Lipschitz and η-strongly monotone mapping with $k > 0$ and $\eta > 0$, and $V : H_1 \to H_1$ be a δ-Lipschitz mapping with $\delta > 0$. Let $0 < \rho < 2\eta/k^2$ and $0 < \gamma\delta < \tau$, where $\tau = 1 - \sqrt{1 - \rho(2\eta - \rho k^2)}$. Given an initial point $x_1 \in H_1$, the compute u_n using $u_n = s_n x_n + (1 - s_n)y_n$ and $\theta(u_n) \neq 0$, then compute x_{n+1} via initial rule*

$$\begin{cases} u_n = s_n x_n + (1 - s_n)y_n, \\ y_n = prox_{\lambda_n \mu_n f}(u_n - \mu_n A^*(I - prox_{\lambda_n g})Au_n), \\ x_{n+1} = \alpha_n \gamma V(x_n) + (1 - \alpha_n \rho F)[\beta_n T y_n + (1 - \beta_n)y_n], \ n \geq 1, \end{cases} \tag{4.3}$$

where stepsize $\mu_n := \rho_n \frac{h(u_n)+l(u_n)}{\theta^2(u_n)}$ with $0 < \rho_n < 4$ and and $A^* : H_2 \to H_1$ is the dual of the bounded linear operator A. If $\theta(u_n) = 0$, then $x_{n+1} = x_n$ is a solution of (1.2) which is also a fixed point of a nonexpansive mapping T and the iterative process stops, otherwise, we set $n := n + 1$ and go to (4.4).

Theorem 4.5 *Assume that f is a proper convex lower-semicontnuous function, g is locally lower semicontinuous at Az, prox-bounded and prox-regular at Az for $\bar{v} = 0$. Let T be a nonexpansive mapping of H_1 into itself such that $\Upsilon \cap F(T) \neq \emptyset$ and A a bounded linear operator which is surjective with a dense. Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{s_n\}_{n=1}^{\infty}$.be a real sequence in $(0, 1]$ satisfying the following conditions*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $\epsilon \leq \rho_n \leq \frac{4h(u_n)}{h(u_n)+l(u_n)} - \epsilon$ *for some* $\epsilon > 0$;
(iii) $\sum_{n=1}^{\infty} \lambda_n < \infty$;
(iv) $0 < \epsilon \leq s_n \leq 1$,

and if $\|x_1 - z\|$ is small enough, then sequence $\{x_n\}$ generated by (4.4) strongly converges to $z \in \Upsilon \cap F(T)$ which is also the unique solution of the variational inequality (VI)

$$z \in \Upsilon \cap F(T), \ \ \langle(\rho F - \gamma V)z, x - z\rangle \geq 0, \ \ x \in \Upsilon \cap F(T). \tag{4.4}$$

Proof Using the fact that $prox_{\lambda_n \mu_n f}$ is nonexpansive, z verifies (4.1) (critical points of any function are exactly fixed-point of its proximal mapping) and having in mind Lemma 4.3, we can write

$$\begin{aligned} \|y_n - z\|^2 &= \|u_n - z\|^2 + \mu_n^2 \|\nabla h(u_n)\|^2 - 2\mu_n \langle \nabla h(u_n), u_n - z\rangle \\ &\leq \|u_n - z\|^2 + \mu_n^2 \|\nabla h(u_n)\|^2 - 2\mu_n \left(2h(u_n) - \frac{\lambda_n r \|A\|^2}{(1 - \lambda_n r)^2}\|u_n - z\|^2\right) \\ &= \|u_n - z\|^2 + 2\mu_n \frac{\lambda_n r \|A\|^2}{(1 - \lambda_n r)^2}\|u_n - z\|^2 - 4\mu_n h(u_n) + \mu_n^2 \|\nabla h(u_n)\|^2 \end{aligned}$$

$$
\begin{aligned}
&\le \|u_n - z\|^2 + 2\rho_n \frac{h(u_n)+l(u_n)}{\|\nabla h(u_n)\|^2+\|\nabla l(u_n)\|^2}\frac{\lambda_n r\|A\|^2}{(1-\lambda_n r)^2}\|u_n-z\|^2\\
&\quad -\rho_n\left(\frac{4h(u_n)}{h(u_n)+l(u_n)}-\rho_n\right)\frac{(h(u_n)+l(u_n))^2}{\theta^2(u_n)}\\
&\le \left(1+\lambda_n\rho_n\left(\frac{2h(u_n)}{\|\nabla h(u_n)\|^2}+\frac{2l(u_n)}{\|\nabla l(u_n)\|^2}\right)\frac{r\|A\|^2}{(1-\lambda_n r)^2}\|u_n-z\|^2\right)\\
&\quad -\rho_n\left(\frac{4h(u_n)}{h(u_n)+l(u_n)}-\rho_n\right)\frac{(h(u_n)+l(u_n))^2}{\theta^2(u_n)}\\
&= \left(1+\lambda_n\rho_n\left(1+\frac{2h(u_n)}{\nabla h(u_n)\|^2}\right)\frac{r\|A\|^2}{(1-\lambda_n r)^2}\|u_n-z\|^2\right)\\
&\quad -\rho_n\left(\frac{4h(u_n)}{h(u_n)+l(u_n)}-\rho_n\right)\frac{(h(u_n)+l(u_n))^2}{\theta^2(u_n)}. \qquad (4.5)
\end{aligned}
$$

Recall that A is surjective with a dense domain $\Leftrightarrow \exists\zeta > 0$ such that $\|A^*x\| \ge \zeta\|x\|$,

$$
\frac{2h(u_n)}{\|\nabla h(u_n)\|^2} = \frac{\|(I-prox_{\lambda_n g})(Au_n)\|^2}{\|A^*(I-prox_{\lambda_n g})(Au_n)\|^2} \le \frac{1}{\zeta^2}.
$$

Conditions on the parameters λ_n and ρ_n assure the existence of a positive constant M such that

$$
\|y_n - z\|^2 \le (1+M\lambda_n)\|u_n-z\|^2 - \rho_n\left(\frac{4h(u_n)}{h(u_n)+l(u_n)}-\rho_n\right)\frac{(h(u_n)+l(u_n))^2}{\theta^2(u_n)}. \qquad (4.6)
$$

By (3.5), (4.6) and (3.4) (taking into account that $1+x \le e^x,\ x \ge 0$), we obtain that

$$
\begin{aligned}
\|x_{n+1}-z\| &= \|\alpha_n\gamma V(x_n) + (I-\alpha_n\rho F)w_n - z\|\\
&\le \alpha_n\|\gamma V(x_n)-\rho F(z)\| + \|(I-\alpha_n\rho F)w_n-(I-\alpha_n\rho F)z\|\\
&\le \alpha_n\|\gamma V(x_n)-\rho F(z)\| + \|(I-\alpha_n\rho F)y_n-(I-\alpha_n\rho F)z\|\\
&\le (1-\alpha_n\tau)\|w_n-z\| + \alpha_n\|\gamma V(x_n)-\rho F(z)\|\\
&\le (1-\alpha_n\tau)\|y_n-z\| + \alpha_n\gamma\|V(x_n)-V(z)\| + \alpha_n\|V(z)-\rho F(z)\|\\
&\le (1-\alpha_n\tau)(1+M\lambda_n)^{1/2}\|x_n-z\| + \alpha_n\gamma\delta\|x_n-z\|\\
&\quad +\alpha_n\|\gamma V(z)-\rho F(z)\|\\
&= (e^{M\lambda_n})^{\frac{1}{2}}(1-\alpha_n(\tau-\gamma\delta))\|x_n-z\| + \alpha_n(\tau-\gamma\delta)\frac{\|\gamma V(z)-\rho F(z)\|}{\tau-\gamma\delta}\\
&\le e^{\frac{M}{2}\lambda_n}\left(\max\left\{\|x_n-z\|, \frac{\|\gamma V(z)-\rho F(z)\|}{\tau-\gamma\delta}\right\}\right)\\
&\ \ \vdots\\
&\le e^{\frac{M}{2}\sum_{n=1}^{\infty}\lambda_n}\left(\max\left\{\|x_1-z\|, \frac{\|\gamma V(z)-\rho F(z)\|}{\tau-\gamma\delta}\right\}\right) \qquad (4.7)
\end{aligned}
$$

This implies that the sequence $\{x_n\}$ is bounded. Consequently $\{y_n\}$ $\{Ty_n\}$ and $\{u_n\}$ are bounded.

Following the method of proof of Theorem 3.2, we can show that

$$\lim_{n\to\infty}(h(u_n)+l(u_n))=0 \Leftrightarrow \lim_{n\to\infty} h(u_n)=0 \text{ and } \lim_{n\to\infty} l(u_n)=0.$$

If $\bar{x}$ is a weak cluster point of $\{u_n\}$, then there exists a subsequence $\{u_{n_j}\}$ which weakly converges to $\bar{x}$ From the proof of Theorem 3.2, we can show that

(i) $0 \in \partial f(\bar{x})$ such that $0 \in \partial_{pg}(A\bar{x})$,
(ii) $\|Ty_n - y_n\| \to 0, n \to \infty$,
(iii) $\lim_{n\to\infty}\|u_n - y_n\| = 0 = \lim_{n\to\infty}\|x_n - y_n\|$ and
(iv) $\bar{x} \in F(T)$. Therefore, $z \in \Upsilon \cap F(T)$.

Finally, from Algorithm 4.4, we have

$$\begin{aligned}
\|x_{n+1}-z\|^2 &= \|\alpha_n\gamma V(x_n) + (I-\alpha_n\rho F)w_n - z\|^2\\
&= \|\alpha_n(\gamma V(x_n) - \rho F(z)) + (I-\alpha_n\rho F)w_n - (I-\alpha_n\rho F)z\|^2\\
&\le \|(I-\alpha_n\rho F)w_n - (I-\alpha_n\rho F)z\|^2 + 2\alpha_n\langle\gamma V(x_n) - \rho F(z), x_{n+1}-z\rangle\\
&\le (1-\alpha_n\tau)^2\|w_n - z\|^2 + 2\alpha_n\langle\gamma V(x_n)-\rho F(z), x_{n+1}-z\rangle\\
&\le (1-\alpha_n\tau)^2\|y_n - z\|^2 + 2\alpha_n\langle\gamma V(x_n)-\rho F(z), x_{n+1}-z\rangle\\
&\le (1-\alpha_n\tau)^2(1+M\lambda_n)\|x_n - z\|^2 + 2\alpha_n\langle\gamma V(x_n)-\rho F(z), x_{n+1}-z\rangle\\
&= (1-\alpha_n\tau)^2\|x_n - z\|^2 + M\lambda_n(1-\alpha_n\tau)^2\|x_n\\
&\quad - z\|^2 + 2\alpha_n\gamma\langle V(x_n) - V(z), x_{n+1}-z\rangle\\
&\quad + 2\alpha_n\langle\gamma V(z) - \rho F(z), x_{n+1}-z\rangle\\
&\le (1-\alpha_n\tau)^2\|x_n - z\|^2 + \alpha_n\gamma\delta\|x_n - z\|^2 + \alpha_n\gamma\delta\|x_{n+1}\\
&\quad - z\|^2 + M\lambda_n(1-\alpha_n\tau)^2\|x_n - z\|^2\\
&\quad + 2\alpha_n\langle\gamma V(z) - \rho F(z), x_{n+1}-z\rangle.
\end{aligned}$$

This implies that for some $M_1 > 0$ we have

$$\begin{aligned}
\|x_{n+1}-z\|^2 &\le \frac{(1-\alpha_n\tau)^2 + \alpha_n\gamma\delta}{1-\alpha_n\gamma\delta}\|x_n - z\|^2\\
&\quad + \frac{2\alpha_n}{1-\alpha_n\gamma\delta}\langle\gamma V(z) - \rho F(z), x_{n+1}-z\rangle + \lambda_n M_1\\
&= \left(1 - \frac{2(\tau-\gamma\delta)\alpha_n}{1-\alpha_n\gamma\delta}\right)\|x_n - z\|^2 + \frac{\alpha_n^2\tau^2}{1-\alpha_n\gamma\delta}\|x_n - z\|^2\\
&\quad + \frac{2\alpha_n}{1-\alpha_n\gamma\delta}\langle\gamma V(z) - \rho F(z), x_{n+1}-z\rangle + \lambda_n M_1\\
&\le \left(1 - \frac{2(\tau-\gamma\delta)\alpha_n}{1-\alpha_n\gamma\delta}\right)\|x_n - z\|^2 + \frac{2(\tau-\gamma\delta)\alpha_n}{1-\alpha_n}\left\{\frac{\alpha_n\rho^2 M_3}{2(\tau-\gamma\delta)}\right.
\end{aligned}$$

$$+\frac{1}{\tau-\gamma\delta}\left[\langle\gamma V(z)-\rho F(z), x_{n+1}-z\rangle\right]\Big\}+\lambda_n M_1$$
$$=(1-\delta_n)\|x_n-z\|^2+\delta_n\sigma_n+\lambda_n M_1, \tag{4.8}$$

where $\|x_n - z\|^2 \leq M_3$, $\delta_n = \frac{2(\tau-\gamma\delta)\alpha_n}{1-\alpha_n}$, $\sigma_n = \frac{\alpha_n\rho^2 M_3}{2(\tau-\gamma\delta)} + \frac{1}{\tau-\gamma\delta}$ $\left[\langle\gamma V(z)-\rho F(z), x_{n+1}-z\rangle\right]$

We conclude from condition (i),(iii) and Lemma 2.3 that $\{x_n\}$ converges strongly to $z \in \Upsilon \cap F(T)$.

5 Numerical example

In this section, we give a numerical example of Algorithm 3.1 in comparison with Algorithm 1.4 of Shehu and Ogbuisi in an infinite dimensional Hilbert space. Let $H_1 = H_2 = L_2([0, 1])$ be endowed with inner product

$$\langle x, y\rangle = \int_0^1 x(t)y(t)dt \ \ \forall\, x, y \in L_2([0, 1])$$

and norm

$$||x|| := \Big(\int_0^1 |x(t)|^2 dt\Big)^{\frac{1}{2}} \ \ \forall\, x, y \in L_2([0, 1]).$$

Let $C = \{x \in L_2([0, 1]) : \langle y, x\rangle \leq a\}$, where $y = 2t^2 + 1$ and $a = 2$. Then, we define $\mathrm{Prox}_{\lambda\mu f}$ as

$$\mathrm{Prox}_{\lambda\mu f}(x) = P_C(x) = \begin{cases} \frac{a-\langle y,x\rangle}{||y||^2_{L_2}} y + x, & \text{if } \langle y, x\rangle > a, \\ x, & \text{otherwise,} \end{cases}$$

where $f = \delta_C$ (the indicator function of C).
Now, let $Q = \{x \in L_2([0, 1]) \ : \ ||x - e||_{L_2} \leq b\}$, where $e = t + 2$ and $b = 1$, then we define $\mathrm{Prox}_{\lambda g}$ as

$$\mathrm{Prox}_{\lambda g}(x) = P_Q(x) = \begin{cases} x, & \text{if } x \in Q, \\ \frac{x-e}{||x-e||_{L_2}} b + e, & \text{otherwise,} \end{cases}$$

where $g = \delta_Q$.
Define the operator $A : L_2([0, 1]) \to L_2([0, 1])$ by

$$(Ax)(s) = \int_0^1 e^{-st} x(t)dt, \ \ x \in L_2([0, 1]).$$

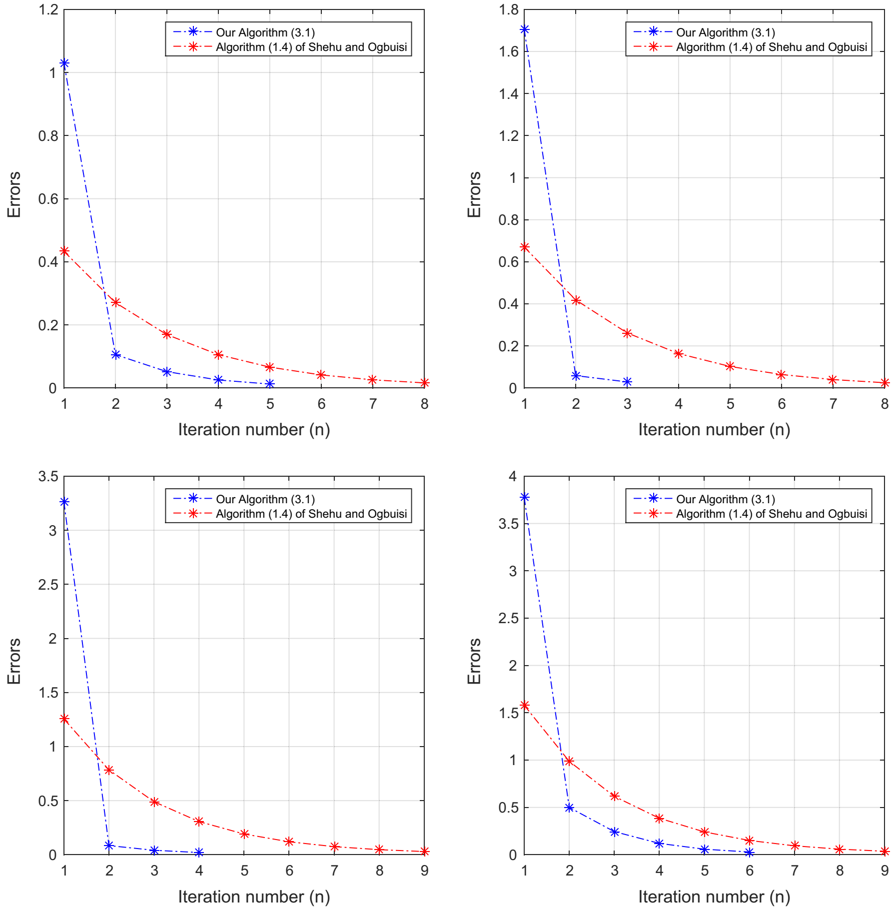

Fig. 1 Errors versus Iteration numbers(n): Case 1 (top left); Case 2 (top right); Case 3 (bottom left); Case 4 (bottom right)

Then A is a bounded linear mapping. Let $T, F, V : L_2([0,1]) \to L_2([0,1])$ be defined by $Tx(t) = -4x(t)$, $Fx(t) = x(t)$ and $Vx(t) = \frac{1}{2}x(t)$ for all $x(t) \in L_2([0,1])$. Now, take $\rho = 1 = \gamma$, $\alpha_n = \frac{1}{3n+2}$, $s_n = \frac{n}{2n+1}$ and $\beta_n = \frac{2n}{5n+3}$ for all $n \geq 1$, then the conditions in Theorem 3.2 are satisfied. We now consider the following four cases.

Case 1 Take $x_1(t) = 1 + e^{2t}$.

Case 2 Take $x_1(t) = e^{3t}$.

Case 3 Take $x_1(t) = t^2 + 3$.

Case 4 Take $x_1(t) = t^4 + e^t$.

By using these cases (Case 1–Case 4 above), we compared our Algorithm 3.1 with Algorithm 1.4 of Shehu and Ogbuisi [20] in Fig. 1. The graphs show that our algorithm converges faster than the Algorithm in [20]. This shows that our algorithm works well and have competitive advantages over the algorithm of Shehu and Ogbuisi [20].

6 Conclusion

Strong convergence of some new viscosity implicit rule methods for finding a common solution of proximal split feasibility problems (for convex and nonconvex functions) and fixed point problems for a ϑ-strictly pseudocontractive mapping is established in the framework of real Hilbert spaces. The strong convergent result is obtained under some relaxed assumptions, one of which is that our proposed methods uses a stepsize such that the implementation of our methods does not need any prior information about the bounded operator norm. Also, some numerical experiments of our method (Algorithm 3.1) in comparison with Algorithm 1.4 of Shehu and Ogbuisi [20] were carried out in infinite dimensional Hilbert spaces. In all our comparisons, the numerical results demonstrate that our method performs better and has competative advantage than the method in [20].

Moreover, the problem studied in this paper can be applied to real world problems such as Intensity-Modulated Radiation Therapy (IMRT) treatment planning, phase retrievals, signal processing, image restoration problems, data compression/compressed sensing, among others (see, for example [4–6,10,12,24]).

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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