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ORIGINAL RESEARCH

Global extended Krylov subspace methods for large-scale differential Sylvester matrix equations

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Abstract

In this paper, we present a new numerical methods for solving large-scale differential Sylvester matrix equations with low rank right hand sides. These differential matrix equations appear in many applications such as robust control problems, model reduction problems and others. We present two approaches based on extended global Arnoldi process. The first one is based on approximating exponential matrix in the exact solution using the global extended Krylov method. The second one is based on a low-rank approximation of the solution of the corresponding Sylvester equation using the extended global Arnoldi algorithm. We give some theoretical results and report some numerical experiments to show the effectiveness of the proposed methods compared with the extended block Krylov method given in Hached and Jbilou (Numer Linear Algebra Appl 255:e2187, 2018).

Keywords Extended global Krylov subspaces · Matrix exponential · Low-rank approximation · Differential Sylvester equations

Mathematics Subject Classification 65F · 15A

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1 Introduction

In this paper, we consider the differential Sylvester matrix equation (DSE in short) of the form

$$\begin{cases} \dot{X}(t) = AX(t) + X(t)B + EF^T, & t \in [t_0, T_f] \\ X(t_0) = X_0 \end{cases} \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{p\times p}$, and $E \in \mathbb{R}^{n\times s}$, $F \in \mathbb{R}^{p\times s}$, are full rank matrices, with $s \ll n, p$. The initial condition is given in a factored form as $X_0 = Z_0\tilde{Z}_0^T$ and the matrices A and B are assumed to be large and sparse.

Differential Sylvester matrix equations play a fundamental role in many problems in control, filter design theory, model reduction problems, differential equations and robust control problems; see, e.g., [1,13,19] and the references therein.

The exact solution of the differential Sylvester matrix Eq. (1) is given by the following result.

Theorem 1 [1] *The unique solution of the differential Sylvester equations* (1) *is defined by*

$$X(t) = e^{(t-t_0)A}X_0e^{(t-t_0)B} + \int_{t_0}^{t} e^{(t-\tau)A}EF^Te^{(t-\tau)B}d\tau. \tag{2}$$

There are several methods for solving small or medium-sized differential Sylvester matrix equations. One can see, for example backward differentiation formula (BDF) and Rosenbrock method [12,31].

During the last years, there is a large variety of methods to compute the solution of large scale matrix differential equations such as differential Lyapunov matrix equation, differential Sylvester equation and Riccati differential equation. For more details see [4,5,7,18,19,26–28,35]. For large-scale problems, the effective methods are based on Krylov subspaces. Some methods have been proposed for solving large matrix equation, see, e.g., [2,8,9,18–20,34]. The main idea employed in these methods is to use a extended Krylov subspace and then apply the Galerkin-type orthogonality condition. In [19], Jbilou and Hached are presented two approaches to solving the large differential Sylvester matrix equation, by using the block extended Krylov subspaces. The main idea in this work is using the extended global Krylov subspaces to solve (1).

The rest of the paper is organized as follows. In the next section, we give expression based on the global Arnoldi process of the unique solution $X(t)$ of the differential Sylvester matrix Eq. (1). In Sect. 3, we recall the extended global Arnoldi algorithm with some of its properties. In Sect. 4, we defined EGA-exp method based on extended global Krylov subspaces and quadrature method to approximate matrix exponential and computing numerical solution of Eq. (1). In Sect. 5, we present a low-rank approximation of solution of differential Eq. (1) using projection onto extended global Krylov subspaces $\mathcal{K}_m^g(A, E)$ and $\mathcal{K}_m^g(B^T, F)$. Finally, Sect. 6 is devoted to numerical experiments showing effectiveness of proposed methods.

Throughout the paper, we use the following notations. The Frobenius inner product of the matrices X and Y is defined by $\langle X, Y\rangle_F = tr(X^TY)$, where $tr(Z)$ denotes the trace of a square matrix Z. The associated norm is the Frobenius norm denoted by $\|\cdot\|_F$. The Kronecker product $A \otimes B = [a_{i,j}B]$ where $A = [a_{i,j}]$. This product satisfies the properties: $(A \otimes B)(C \otimes D) = (AC \otimes BD)$ and $(A \otimes B)^T = A^T \otimes B^T$.

We also use the matrix product $\diamond$ defined in [11]. The following proposition gives some properties satisfied by this product.

Proposition 1 *Let $A,\ B,\ C \in \mathbb{R}^{n\times ps}$, $D \in \mathbb{R}^{n\times n}$, $L \in \mathbb{R}^{p\times p}$ and $\alpha \in \mathbb{R}$. Then we have,*

1. $(A+B)^T \diamond C = A^T \diamond C + B^T \diamond C$.
2. $A^T \diamond (B+C) = A^T \diamond B + A^T \diamond C$.
3. $(\alpha A)^T \diamond C = \alpha(A^T \diamond C)$.
4. $(A^T \diamond B)^T = B^T \diamond A$.
5. $A^T \diamond (B(L \otimes I_s)) = (A^T \diamond B)L$.

A block matrix $\mathcal{V}_m = [V_1, V_2, \ldots, V_m]$ is F-orthonormal if $\mathcal{V}_m^T \diamond \mathcal{V}_m = I_m$. We have the following result.

Lemma 1 [24] *Let $\mathcal{V}_m = [V_1, V_2, \ldots, V_m]$ be an $n \times ms$ F-orthonormal block matrix, $Z \in \mathbb{R}^{m\times s}$ and $Y \in \mathbb{R}^{ms\times q}$. Then we have*

$$\|\mathcal{V}_m\,(Z \otimes I_s)\,\|_F = \|Z\|_F \quad \text{and} \quad \|\mathcal{V}_m Y\|_F \leq \|Y\|_F.$$

2 Expression of the exact solution of the differential Sylvester equation

In this section we will give the expression of the unique solution $X(t)$ of the differential Sylvester matrix Eq. (1). This expression is based on the global Arnoldi process. The modified global Arnoldi process constructs an F-orthonormal basis $V_1,\ V_2, \ldots, V_m$ of the matrix Krylov subspace

$$\mathcal{K}_m(A, V) = span\left\{V, AV, A^2V, \ldots, A^{m-1}V\right\}.$$

Algorithm 1 The modified global Arnoldi process(MGA)

Inputs: $A \in \mathbb{R}^{n\times n}$, $V \in \mathbb{R}^{n\times s}$ and m an integer.

1. Set $V_1 = \frac{V}{\|V\|_F}$;
2. **For** $j = 1, \ldots, m$
3. $\quad \widetilde{V} = AV_j$;
4. $\quad$ **For** $i = 1, \ldots, j$
5. $\qquad h_{i,j} = \langle V_i, \widetilde{V}\rangle_F$;
6. $\qquad \widetilde{V} = \widetilde{V} - h_{i,j}V_i$;
7. $\quad$ **end for(i)**.
8. $\quad$ Compute $h_{j+1,j} = \|\widetilde{V}\|_F$;
9. $\quad V_{j+1} = \widetilde{V}/h_{j+1,j}$;
10. **end for(j)**.

Let $\mathcal{V}_m = [V_1, V_2, \ldots, V_m]$ and $\widetilde{H}_m^A$ is the $(m+1)\times m$ upper Hessenberg matrix whose entries $h_{i,j}$ are defined by Algorithm 1 and H_m^A is the $m \times m$ matrix obtained from $\widetilde{H}_m^A$ by deleting its last row. We have the following relation

$$A\mathcal{V}_m = \mathcal{V}_m\left(H_m^A \otimes I_s\right) + h_{m+1,m}V_{m+1}\left(e_m^T \otimes I_s\right),$$

where $e_m^T = [0, 0, \ldots, 0, 1]$.

Let P_A be the minimal polynomial of A associated to E and P_{B^T} be the minimal polynomial of B^T associated to F, q the degree of P_A and q' the degree of P_{B^T}. The following result shows that the solution X of (1) can be expressed in terms of the global Arnoldi basis.

Theorem 2 *Let $\mathcal{V}_q = \left[V_1, V_2, \ldots, V_q\right]$ and $\mathcal{W}_{q'} = \left[W_1, W_2, \ldots, W_{q'}\right]$ be the F-orthonormal block matrices obtained by applying simultaneously q and q' steps of the global Arnoldi algorithm to the pairs (A, E) and (B^T, F) respectively. Then the unique solution X of* (1) *can be expressed as:*

$$X(t) = \mathcal{V}_q(Y_{qq'}(t) \otimes I_s)\mathcal{W}_{q'}^T, \tag{3}$$

where $Y_{qq'}(t)$ is the solution of the low-order differential Sylvester equation

$$\dot{Y}_m(t) - H_q^A Y_m(t) - Y_m(t)\left(H_{q'}^B\right)^T - \widetilde{E}_q\widetilde{F}_{q'}^T = 0,$$

with $\widetilde{E}_q = \|E\|_F e_1^{(q)}$ and $\widetilde{F}_{q'} = \|F\|_F e_1^{(q')}$.

Proof Let $Z(t)$ be the matrix defined by $\mathcal{V}_q(Y_{qq'}(t) \otimes I_s)\mathcal{W}_{q'}^T$. Then we have,

$$\begin{aligned}
&\dot{Z}(t) - AZ(t) - Z(t)B - EF^T \\
&\quad = \mathcal{V}_q(\dot{Y}_{qq'}(t) \otimes I_s)\mathcal{W}_{q'}^T - A\mathcal{V}_q(Y_{qq'}(t) \otimes I_s)\mathcal{W}_{q'}^T - \mathcal{V}_q(Y_{qq'}(t) \otimes I_s)\mathcal{W}_{q'}^T B - EF^T. \\
&\quad = \mathcal{V}_q\left[\left(\dot{Y}_{qq'}(t) - H_q^A Y_{qq'}(t) - Y_{qq'}(t)\left(H_{q'}^B\right)^T - \widetilde{E}_q\widetilde{F}_{q'}^T\right) \otimes I_s\right]\mathcal{W}_{q'}^T.
\end{aligned}$$

Since $Y_{qq'}$ is the solution of the lower order differential Sylvester equation then $\dot{Y}_{qq'}(t) - \mathbb{T}_{q,A}Y_{qq'}(t) - Y_{qq'}(t)\left(H_{q'}^B\right)^T - \widetilde{E}_q\widetilde{F}_{q'}^T = 0$. We get

$$\dot{Z}(t) - AZ(t) - Z(t)B - EF^T = 0.$$

Therefore, using the fact that the solution of (1) is unique, it follows that $X(t) = \mathcal{V}_q(Y_{qq'}(t) \otimes I_s)\mathcal{W}_{q'}^T$. □

3 The extended global Arnoldi process

In this section, we recall the extended global Krylov subspace and extended global Arnoldi process. Let V be a matrix of dimension $n \times s$. Then the extended global Krylov subspace associated to (A, V) is given by

$$\mathcal{K}_m^g(A, V) = span\left\{V, A^{-1}V, AV, A^{-2}V, A^2V, \ldots, A^{m-1}V, A^{-m}V\right\} \tag{4}$$

The extended global Arnoldi process constructs an F-orthonormal basis $\{V_1,\ V_2, \ldots, V_m\}$ of the extended global Krylov subspace $\mathcal{K}_m^g(A, V)$ [20]. The algorithm is summarized as follows

Algorithm 2 The extended global Arnoldi process(EGA)

Inputs: $A \in \mathbb{R}^{n\times n}$, $V \in \mathbb{R}^{n\times s}$ and m an integer.

1. Compute the global QR $[V, A^{-1}V]$, i.e., $[V, A^{-1}V] = V_1(\Omega \otimes I_s)$;
2. Set $\mathbb{V}_0 = [\ ]$;
3. **For** $j = 1, \ldots, m$
4. Set $V_j^{(1)} = V_j(:, 1 : s)$; $V_j^{(2)} = V_j(:, s+1 : 2s)$;
5. $\mathbb{V}_j = [\mathbb{V}_{j-1}, V_j]$; $U = [AV_j^{(1)}, A^{-1}V_j^{(2)}]$;
6. **For** $i = 1, \ldots, j$
7. $H_{i,j} = V_i^T \diamond U$;
8. $U = U - V_i(H_{i,j} \otimes I_s)$;
9. **end for(i)**.
10. Compute the global QR decomposition of U, i.e., $U = V_{j+1}(H_{j+1,j} \otimes I_s)$;
11. **end for(j)**.

Let $\mathbb{V}_m = [V_1, V_2, \ldots, V_m]$ with $V_i \in \mathbb{R}^{n\times 2s}$ and $2m \times 2m$ upper block Hessenberg matrix

$$\mathbb{T}_{m,A} = \mathbb{V}_m^T \diamond (A\mathbb{V}_m).$$

We have the following relation

$$\begin{aligned} A\mathbb{V}_m &= \mathbb{V}_{m+1}(\overline{\mathbb{T}}_{m,A} \otimes I_s) \\ &= \mathbb{V}_m(\mathbb{T}_{m,A} \otimes I_s) + V_{m+1}(T^A_{m+1,m} E_m^T \otimes I_s), \end{aligned}$$

where $\overline{\mathbb{T}}_{m,A} = \mathbb{V}_{m+1}^T \diamond (A\mathbb{V}_m)$, and $E_m^T = [0_{2\times 2(m-1)}, I_2]$ is the matrix formed with the last 2 columns of the $2m \times 2m$ identity matrix I_{2m}.

4 Matrix exponential approximation and Gauss quadrature method

In this section, we compute a approximation of the solution of the large differential Sylvester matrix Eq. (1). Our approach is based on two steps. First approximation of matrix exponential is given using extended global Krylov subspace and then the Gaussian quadrature method is applied.

The exact solution of (1) is given by

$$X(t) = e^{(t-t_0)A} X_0 e^{(t-t_0)B} + \int_{t_0}^{t} e^{(t-\tau)A} E F^T e^{(t-\tau)B} d\tau. \tag{5}$$

We use extended global Krylov subspace method to approximate $e^{(t-\tau)A}E$ and $e^{(t-\tau)B^T}F$.

By Algorithm 2 we compute $\mathbb{V}_m = [V_1, \ldots, V_m]$ and $\mathbb{W}_m = [W_1, \ldots, W_m]$ the F-orthonormal matrices whose columns form an basis of the subspace $\mathcal{K}_m^g(A, E)$ and $\mathcal{K}_m^g(B^T, F)$, respectively.

The approximation $Z_{m,A}$ of $Z_A = e^{(t-\tau)A}E$ is obtained by

$$Z_{m,A} = \mathbb{V}_m \left(e^{(t-\tau)\mathbb{T}_{m,A}} \mathbb{V}_m \diamond E \otimes I_s \right),$$

where $\mathbb{T}_{m,A} = \mathbb{V}_m^T \diamond (A\mathbb{V}_m)$ (see [32,33,36]).

In the same way, an approximation of $Z_B = e^{(t-\tau)B^T}F$ is given by

$$Z_{m,B} = \mathbb{W}_m (e^{(t-\tau)\mathbb{T}_{m,B}} \mathbb{W}_m \diamond F \otimes I_s),$$

where $\mathbb{T}_{m,B} = \mathbb{W}_m^T \diamond (B^T\mathbb{W}_m)$.

That leads us to the following approximation

$$e^{(t-\tau)A} E F^T e^{(t-\tau)B} \approx Z_{m,A}(t) Z_{m,B}^T(t). \tag{6}$$

Assuming that $X(t_0) = 0$, then the approximate solution of the differential Sylvester Eq. (1) is obtained by

$$X_m(t) = \mathbb{V}_m (\mathbb{X}_m(t) \otimes I_s) \mathbb{W}_m^T, \tag{7}$$

where

$$\mathbb{X}_m(t) = \int_{t_0}^{t} \mathbb{X}_{m,A}(\tau) \mathbb{X}_{m,B}^T(\tau) d\tau, \tag{8}$$

with

$$\begin{cases} \mathbb{X}_{m,A}(\tau) = e^{(t-\tau)\mathbb{T}_{m,A}} \mathbb{V}_m^T \diamond E \\ \mathbb{X}_{m,B}(\tau) = e^{(t-\tau)\mathbb{T}_{m,B}} \mathbb{W}_m^T \diamond F. \end{cases}$$

Since m is generally very small ($m << n$), the factors $\mathbb{X}_{m,A}$ and $\mathbb{X}_{m,B}$ can be computed using the **expm** function of Matlab, and we calculate the approximation of the integral of (8) by Gauss quadrature formulae.

The next result shows that the $2m \times 2m$ matrix function $\mathbb{X}_m(t)$ is solution of a low-order differential Sylvester matrix equation.

Theorem 3 *The matrix function $\mathbb{X}_m(t)$ defined by (8) satisfies the following low-order differential Sylvester matrix equation*

$$\dot{\mathbb{X}}_m(t) = \mathbb{T}_{m,A} \mathbb{X}_m(t) + \mathbb{X}_m(t) \mathbb{T}_{m,B}^T + \widetilde{E}_m \widetilde{F}_m^T, \quad t \in [t_0, T_f], \tag{9}$$

where $\widetilde{E}_m = \mathbb{V}_m^T \diamond E$ and $\widetilde{F}_m = \mathbb{W}_m^T \diamond F$.

Proof The proof can be easily derived from the expression (8) and the result of Theorem 1. □

Let $R_m(t) = \dot{X}_m(t) - AX_m(t) - X_m(t)B - EF^T$ be the residual associated to the approximation $X_m(t)$.

Theorem 4 *If $X_m(t) = \mathbb{V}_m(\mathbb{X}_m(t) \otimes I_s)\mathbb{W}_m^T$ is the approximation obtained at step m by the extended global Arnoldi algorithm. Then the residual $R_m(t)$ satisfies the inequality*

$$\|R_m(t)\|_F^2 \leq \|T^A_{m+1,m}\overline{\mathbb{X}}_m(t)\|_F^2 + \|T^B_{m+1,m}\overline{\mathbb{X}}_m(t)\|_F^2, \tag{10}$$

for the 2-norm, we have

$$\|R_m(t)\|_2 \leq \max\{\|T^A_{m+1,m}\overline{\mathbb{X}}_m(t)\|_2, \|T^B_{m+1,m}\overline{\mathbb{X}}_m(t)\|_2\},$$

where $\overline{\mathbb{X}}_m(t)$ is the $2 \times 2m$ matrix corresponding to the last 2 rows of $\mathbb{X}_m(t)$.

Proof We have

$$R_m(t) = \dot{X}_m(t) - AX_m(t) - X_m(t)B - EF^T,$$
$$\textit{where} \quad X_m(t) = \mathbb{V}_m(\mathbb{X}_m(t) \otimes I_s)\mathbb{W}_m^T.$$

Therefore

$$R_m(t) = \mathbb{V}_m\left(\dot{\mathbb{X}}_m(t) \otimes I_s\right)\mathbb{W}_m^T - A\mathbb{V}_m\left(\mathbb{X}_m(t) \otimes I_s\right)\mathbb{W}_m^T - \mathbb{V}_m(\mathbb{X}_m(t) \otimes I_s)\mathbb{W}_m^T B - EF^T.$$

We use the following properties

$$\begin{cases} A\mathbb{V}_m = \mathbb{V}_{m+1}\left(\widehat{\mathbb{T}}_{m,A} \otimes I_s\right), \\ \mathbb{W}_m^T B = \left(\widehat{\mathbb{T}}^T_{m,B} \otimes I_s\right)\mathbb{W}^T_{m+1}, \\ \mathbb{V}_m = \mathbb{V}_{m+1}\begin{bmatrix} I_{2sm} \\ 0_{2s,2s} \end{bmatrix}, \\ \mathbb{W}_m^T = \begin{bmatrix} I_{2sm} & 0_{2s,2s} \end{bmatrix}\mathbb{W}^T_{m+1}, \end{cases}$$

and

$$\begin{cases} \widehat{\mathbb{T}}_{m,A} = \begin{bmatrix} \mathbb{T}_{m,A} \\ T^A_{m+1,m}E_m^T \end{bmatrix}, \\ \widehat{\mathbb{T}}^T_{m,B} = \begin{bmatrix} \mathbb{T}^T_{m,B} & E_m(T^B_{m+1,m})^T \end{bmatrix}, \end{cases}$$

we obtained

$$\begin{aligned} R_m(t) &= \mathbb{V}_{m+1}\begin{bmatrix} \dot{\mathbb{X}}_m(t) \otimes I_s & 0 \\ 0 & 0 \end{bmatrix}\mathbb{W}^T_{m+1} - \mathbb{V}_{m+1}\begin{bmatrix} \mathbb{T}_{m,A}\mathbb{X}_m(t) \otimes I_s & 0 \\ T^A_{m+1,m}E_m^T\mathbb{X}_m(t) \otimes I_s & 0 \end{bmatrix}\mathbb{W}^T_{m+1} \\ &- \mathbb{V}_{m+1}\begin{bmatrix} \mathbb{X}_m(t)\mathbb{T}^T_{m,B} \otimes I_s & \mathbb{X}_m(t)E_m(T^B_{m+1,m})^T \otimes I_s \\ 0 & 0 \end{bmatrix}\mathbb{W}^T_{m+1} \\ &- \mathbb{V}_{m+1}\begin{bmatrix} \mathbb{V}_m^T \diamond EF^T \diamond \mathbb{W}_m & 0 \\ 0 & 0 \end{bmatrix}\mathbb{W}^T_{m+1} \end{aligned}$$

$\mathbb{V}_{m+1}\left(\left[\mathcal{S}_m\left(\mathbb{X}_m(t)\right) - \mathbb{X}_m(t)E_m(T^B_{m+1,m})^T - T^A_{m+1,m}E_m^T\mathbb{X}_m(t)0\right] \otimes I_s\right)\mathbb{W}^T_{m+1}$, where

$\mathcal{S}_m\left(\mathbb{X}_m(t)\right) = \dot{\mathbb{X}}_m(t) - \mathbb{T}_{m,A}\mathbb{X}_m(t) - \mathbb{X}_m(t)\mathbb{T}^T_{m,B} - \widetilde{E}_m\widetilde{F}_m^T.$

Since $\mathbb{X}_m(t)$ is the exact solution of the equation

$$\dot{\mathbb{X}}_m(t) = \mathbb{T}_{m,A}\mathbb{X}_m(t) + \mathbb{X}_m(t)\mathbb{T}_{m,B}^T + \widetilde{E}_m\widetilde{F}_m^T.$$

Then

$$R_m(t) = \mathbb{V}_{m+1}\left(\begin{bmatrix} 0 & -\mathbb{X}_m(t)E_m(T_{m+1,m}^B)^T \\ -T_{m+1,m}^A E_m^T\mathbb{X}_m(t) & 0 \end{bmatrix} \otimes I_s\right)\mathbb{W}_{m+1}^T.$$

If $\overline{\mathbb{X}}_m(t) = E_m^T\mathbb{X}_m(t)$, then

$$\|R_m(t)\|_F^2 \leq \|T_{m+1,m}^A\overline{\mathbb{X}}_m(t)\|_F^2 + \|T_{m+1,m}^B\overline{\mathbb{X}}_m(t)\|_F^2.$$

The same way for the 2-norm, we have
$\|R_m(t)\|_2 \leq \max\left\{\|T_{m+1,m}^A\overline{\mathbb{X}}_m(t)\|_2, \|T_{m+1,m}^B\overline{\mathbb{X}}_m(t)\|_2\right\}.$ □

The following result shows that the approximation $X_m(t)$ is an exact solution of a perturbed differential Sylvester equation.

Theorem 5 *Let $X_m(t)$ be the approximate solution given by (7). Then we have*

$$\dot{X}_m(t) = (A - F_{m,A})X_m(t) + X_m(t)(B - F_{m,B}) + EF^T, \tag{11}$$

where

$$\begin{cases} F_{m,A} = V_{m+1}\left(T_{m+1,m}^A E_m^T \otimes I_s\right)\left(\mathbb{V}_m^T\mathbb{V}_m\right)^{-1}\mathbb{V}_m^T, \\ F_{m,B} = \mathbb{W}_m^T\left(\mathbb{W}_m\mathbb{W}_m^T\right)^{-1}\left[E_m\left(T_{m+1,m}^B\right)^T \otimes I_s\right]W_{m+1}^T. \end{cases}$$

Proof By multiplying (9) on the left by $\mathbb{V}_m$ and on the right by $\mathbb{W}_m^T$, we obtained

$$\begin{aligned} \mathbb{V}_m(\dot{\mathbb{X}}_m(t) \otimes I_s)\mathbb{W}_m^T &= \mathbb{V}_m(\mathbb{T}_{m,A}\mathbb{X}_m(t) \otimes I_s)\mathbb{W}_m^T \\ &\quad + \mathbb{V}_m(\mathbb{X}_m(t)\mathbb{T}_{m,B}^T \otimes I_s)\mathbb{W}_m^T + \mathbb{V}_m(\widetilde{E}_m\widetilde{F}_m^T \otimes I_s)\mathbb{W}_m^T \\ &= \mathbb{V}_m(\mathbb{T}_{m,A} \otimes I_s)(\mathbb{X}_m(t) \otimes I_s)\mathbb{W}_m^T \\ &\quad + \mathbb{V}_m(\mathbb{X}_m(t) \otimes I_s)(\mathbb{T}_{m,B}^T \otimes I_s)\mathbb{W}_m^T \\ &\quad + \mathbb{V}_m(\widetilde{E}_m \otimes I_s)(\widetilde{F}_m^T \otimes I_s)\mathbb{W}_m^T. \end{aligned}$$

Since

$$\begin{cases} A\mathbb{V}_m = \mathbb{V}_m(\mathbb{T}_{m,A} \otimes I_s) + V_{m+1}(T_{m+1,m}^A E_m^T \otimes I_s), \\ \mathbb{W}_m^T B = (\mathbb{T}_{m,B}^T \otimes I_s)\mathbb{W}_m^T + (E_m(T_{m+1,m}^B)^T \otimes I_s)W_{m+1}^T, \end{cases}$$

then

$$\begin{aligned}\dot{X}_m(t) &= [A\mathbb{V}_m - V_{m+1}(T^A_{m+1,m}E^T_m \otimes I_s)](\mathbb{X}_m(t)\otimes I_s)\mathbb{W}^T_m \\ &\quad +\mathbb{V}_m(\mathbb{X}_m(t)\otimes I_s)[B\mathbb{W}^T_m - (E_m(T^B_{m+1,m})^T \otimes I_s)W^T_{m+1}] + EF^T \\ &= [A - V_{m+1}(T^A_{m+1,m}E^T_m \otimes I_s)(\mathbb{V}^T_m\mathbb{V}_m)^{-1}\mathbb{V}^T_m]X_m(t) \\ &\quad +X_m(t)[B - \mathbb{W}^T_m(\mathbb{W}_m\mathbb{W}^T_m)^{-1}(E_m(T^B_{m+1,m})^T \otimes I_s)W^T_{m+1}] + EF^T.\end{aligned}$$

Finally

$$\dot{X}_m(t) = (A - F_{m,A})X_m(t) + X_m(t)(B - F_{m,B}) + EF^T.$$

□

The next result states that the error $\mathcal{E}_m(t) = X(t) - X_m(t)$ satisfies also a differential Sylvester matrix equation.

Theorem 6 *Let $X(t)$ be the exact solution of (1) and let $X_m(t)$ be the approximate solution obtained at step m of Algorithm 2. The error $\mathcal{E}_m(t) = X(t) - X_m(t)$ satisfies the following equation*

$$\dot{\mathcal{E}}_m(t) = A\mathcal{E}_m(t) + \mathcal{E}_m(t)B - R_m(t), \tag{12}$$

Proof We have

$$\begin{cases} \dot{X}(t) = AX(t) + X(t)B + EF^T, \\ R_m(t) = \dot{X}_m(t) - AX_m(t) - X_m(t)B - EF^T, \end{cases}$$

then

$$\begin{aligned}\dot{\mathcal{E}}_m(t) &= \dot{X}(t) - \dot{X}_m(t) \\ &= AX(t) + X(t)B + EF^T - AX_m(t) - X_m(t)B - EF^T - R_m(t) \\ &= A(X(t) - X_m(t)) + (X(t) - X_m(t))B - R_m(t) \\ &= A\mathcal{E}_m(t) + \mathcal{E}_m(t)B - R_m(t).\end{aligned}$$

□

Notice that from Theorem 6, the error $\mathcal{E}_m(t)$ can be expressed in the integral form as follows

$$\mathcal{E}_m(t) = e^{(t-t_0)A}\mathcal{E}_{m,0}e^{(t-t_0)B} + \int_{t_0}^{t} e^{(t-\tau)A}R_m(\tau)e^{(t-\tau)B}d\tau, \quad t \in [t_0, T_f]. \tag{13}$$

where $\mathcal{E}_{m,0} = \mathcal{E}_m(t_0)$.

Next, we give an upper bound for the norm of the error by using the 2-logarithmic norm defined by $\mu_2(A) = \frac{1}{2}\lambda_{max}(A + A^T)$.

Theorem 7 *Assume that the matrices A and B are such that $\mu_2(A) + \mu_2(B) \neq 0$. Then at step m of the extended global Arnoldi process, we have the following upper bound for the norm of the error $\mathcal{E}_m(t)$,*

$$\|\mathcal{E}_m(t)\|_2 \leq \|\mathcal{E}_{m,0}\|_2 e^{(t-t_0)(\mu_2(A)+\mu_2(B))} + \alpha_m \frac{e^{(t-t_0)(\mu_2(A)+\mu_2(B))} - 1}{(\mu_2(A) + \mu_2(B))}, \tag{14}$$

where

$$\alpha_m = \max_{\xi \in [t_0,t]} (\max\{\|T^A_{m+1,m}\overline{\mathbb{X}}_m(\xi)\|_2, \|T^B_{m+1,m}\overline{\mathbb{X}}_m(\xi)\|_2\}).$$

The matrix $\overline{\mathbb{X}}_m$ is the $2 \times 2m$ matrix corresponding to the last 2 rows of $\mathbb{X}_m(t)$.

Proof We first point out that $\|e^{tA}\|_2 \leq e^{\mu_2(A)t}$. Using the expression (13) of $\mathcal{E}_m(t)$, we obtain the following relation

$$\|\mathcal{E}_m(t)\|_2 = \|e^{(t-t_0)A}\mathcal{E}_{m,0}e^{(t-t_0)B}\|_2 + \int_{t_0}^{t} \|e^{(t-\tau)A} R_m(\tau) e^{(t-\tau)B}\|_2 d\tau.$$

Therefore, using (13) and the fact that $\|e^{(t-\tau)A}\|_2 \leq e^{(t-\tau)\mu_2(A)}$, we get

$$\begin{aligned}
\|\mathcal{E}_m(t)\|_2 &\leq \|\mathcal{E}_{m,0}\|_2 e^{(t-t_0)(\mu_2(A)+\mu_2(B))} \\
&\quad + \max_{\xi \in [t_0,t]} \|R_m(\xi)\|_2 \int_{t_0}^{t} e^{(t-\tau)\mu_2(A)} e^{(t-\tau)\mu_2(B)} d\tau \\
&\leq \|\mathcal{E}_{m,0}\|_2 e^{(t-t_0)(\mu_2(A)+\mu_2(B))} \\
&\quad + \max_{\xi \in [t_0,t]} \|R_m(\xi)\|_2 e^{t(\mu_2(A)+\mu_2(B))} \int_{t_0}^{t} e^{\tau(\mu_2(A)+\mu_2(B))} d\tau.
\end{aligned}$$

Using the result of Theorem 4, we obtain $\max_{\xi \in [t_0,t]} \|R_m(\xi)\|_2 \leq \alpha_m$ and then

$$\|\mathcal{E}_m(t)\|_2 \leq \|\mathcal{E}_{m,0}\|_2 e^{(t-t_0)(\mu_2(A)+\mu_2(B))} + \alpha_m \frac{e^{(t-t_0)(\mu_2(A)+\mu_2(B))} - 1}{(\mu_2(A) + \mu_2(B))}.$$

Next, we give another upper bound for the norm of the error $\mathcal{E}_m(t)$. □

Theorem 8 *The error $\mathcal{E}_m(t)$ satisfies the following inequality*

$$\|\mathcal{E}_m(t)\|_2 \leq \|F\|_2 e^{t\mu_2(B)} \Gamma_{1,m}(t) + \|\widetilde{E}_m\|_2 e^{t\mu_2(A)} \Gamma_{2,m}(t), \tag{15}$$

where

$$\begin{cases} \Gamma_{1,m}(t) = \int_{t_0}^{t} e^{-\tau\mu_2(B)} \|Z_A(\tau) - Z_{m,A}(\tau)\|_2 d\tau, \\ \Gamma_{2,m}(t) = \int_{t_0}^{t} e^{-\tau\mu_2(A)} \|Z_B(\tau) - Z_{m,B}(\tau)\|_2 d\tau. \end{cases}$$

Proof From the expressions of $X(t)$ and $X_m(t)$, we have

$$\begin{aligned}
\|\mathcal{E}_m(t)\|_2 &= \|X(t) - X_m(t)\|_2 \\
&= \|\int_{t_0}^{t} \left(Z_A(\tau)Z_B^T(\tau) - Z_{m,A}(\tau)Z_{m,B}(\tau)^T \right) d\tau \|_2 \\
&= \|\int_{t_0}^{t} \left(Z_A(\tau)Z_B^T(\tau) - Z_{m,A}(\tau)Z_B^T(\tau) + Z_{m,A}(\tau)Z_B^T(\tau) - Z_{m,A}(\tau)Z_{m,B}^T(\tau) \right) d\tau \|_2 \\
&= \|\int_{t_0}^{t} [(Z_A(\tau) - Z_{m,A}(\tau))Z_B^T(\tau) + Z_{m,A}(\tau)(Z_B(\tau) - Z_{m,B}(\tau))^T] d\tau \|_2 \\
&\leq \int_{t_0}^{t} \|Z_B(\tau)\|_2 \|(Z_A(\tau) - Z_{m,A}(\tau))\|_2 + \|Z_{m,A}(\tau)\|_2 \|(Z_B(\tau) - Z_{m,B}(\tau))\|_2 d\tau.
\end{aligned}$$

Now as

$$\begin{aligned}
\mu_2(\mathbb{T}_{m,A}) &= \frac{1}{2}\lambda_{\max}\left(\mathbb{T}_{m,A} + \mathbb{T}_{m,A}^T\right) \\
&\leq \frac{1}{2}\lambda_{\max}(A + A^T) \\
&= \mu_2(A),
\end{aligned}$$

and since

$$\begin{cases} \|Z_B(\tau)\|_2 \leq e^{(t-\tau)\mu_2(B)} \|F\|_2, \\ \|Z_{m,A}(\tau)\|_2 \leq e^{(t-\tau)\mu_2(\mathbb{T}_{m,A})} \|\widetilde{E}_m\|_2 \leq e^{(t-\tau)\mu_2(A)} \|\widetilde{E}_m\|_2, \end{cases}$$

we also have

$$\begin{aligned}
\|\mathcal{E}_m(t)\|_2 \leq \|F\|_2 e^{t\mu_2(B)} &\int_{t_0}^{t} e^{-\tau\mu_2(B)} \|Z_A(\tau) - Z_{m,A}(\tau)\|_2 d\tau \\
&+ \|\widetilde{E}_m\|_2 e^{t\mu_2(A)} \int_{t_0}^{t} e^{-\tau\mu_2(A)} \|Z_B(\tau) - Z_{m,B}(\tau)\|_2 d\tau.
\end{aligned}$$

We get

$$\|\mathcal{E}_m(t)\|_2 \leq \|F\|_2 e^{t\mu_2(B)} \Gamma_{1,m}(t) + \|\widetilde{E}_m\|_2 e^{t\mu_2(A)} \Gamma_{2,m}(t).$$

□

One can use some known results [22,33] to derive upper bounds for $\|Z_A(\tau) - Z_{m,A}(\tau)\|_2$, and $\|Z_B(\tau) - Z_{m,B}(\tau)\|_2$, when using the extended global Krylov subspaces.

Lemma 2

$$\begin{aligned}
&\|e_{m,A}(\tau)\|_2 := \|Z_A(\tau) - Z_{m,A}(\tau)\|_2 \\
&\leq \|V_{m+1}(T_{m+1,m}^A \otimes I_s)\|_2 \int_0^{\tau} e^{(u-\tau)\nu_2(A)} \|L_{m,A}(u)\|_2 du, \qquad (16)
\end{aligned}$$

where

$$\begin{cases} L_{m,A}(u) = E_m^T e^{(t-u)\mathbb{T}_{m,A}} \widetilde{E}_m \otimes I_s, \\ \nu_2(A) = \lambda_{\min}\left(\frac{A+A^T}{2}\right). \end{cases}$$

Proof We have

$$\begin{cases} Z_A(\tau) = e^{(t-\tau)A} E, \\ Z_{m,A}(\tau) = \mathbb{V}_m \left(e^{(t-\tau)\mathbb{T}_{m,A}} \widetilde{E}_m \otimes I_s\right), \end{cases}$$

then

$$Z'_A(\tau) = -Ae^{(t-\tau)A} E = -AZ_A(\tau),$$

and

$$\begin{aligned} Z'_{m,A}(\tau) &= -\mathbb{V}_m(\mathbb{T}_{m,A} \otimes I_s)\left(e^{(t-\tau)\mathbb{T}_{m,A}} \widetilde{E}_m \otimes I_s\right) \\ &= -\left[A\mathbb{V}_m - V_{m+1}(T^A_{m+1,m} E_m^T \otimes I_s)\right]\left(e^{(t-\tau)\mathbb{T}_{m,A}} \widetilde{E}_m \otimes I_s\right) \\ &= -AZ_{m,A}(\tau) + V_{m+1}(T^A_{m+1,m} \otimes I_s) L_{m,A}(\tau). \end{aligned}$$

Therefore, the error $e_{m,A}(\tau) = Z_A(\tau) - Z_{m,A}(\tau)$ is such that

$$e'_{m,A}(\tau) = -Ae_{m,A}(\tau) - V_{m+1}(T^A_{m+1,m} \otimes I_s) L_{m,A}(\tau),$$

which allows to give the following expression of $e_{m,A}$:

$$e_{m,A}(\tau) = -\int_0^\tau e^{(u-\tau)A} V_{m+1}(T^A_{m+1,m} \otimes I_s) L_{m,A}(u) du. \tag{17}$$

As $\tau - u > 0$, it follows that

$$\|e^{(u-\tau)A}\|_2 \leq e^{(\tau-u)\mu_2(-A)} = e^{(u-\tau)\nu_2(A)}.$$

Then, we get

$$\|e_{m,A}(\tau)\|_2 \leq \|T^A_{m+1,m}\|_2 \int_0^\tau e^{(u-\tau)\nu_2(A)} \|L_{m,A}(u)\|_2 du.$$

□

Notice that if $\nu_2(A)$ is not known but $\nu_2(A) \geq 0$ (which is the case for positive semidefinite matrices) then we get the upper bound

$$\|e_{m,A}(\tau)\|_2 \leq \|V_{m+1}(T^A_{m+1,m} \otimes I_s)\|_2 \int_0^\tau \|L_{m,A}(u)\|_2 du.$$

To define a new upper bound for the norm of the global error $\mathcal{E}_m(t)$, we can use the upper bounds for the errors $e_{m,A}$ and $e_{m,B}$ in the expression (15) stated in Theorem 8 to get

$$\|\mathcal{E}_m(t)\|_2 \leq \|F\|_2 e^{t\mu_2(B)} \int_{t_0}^{t} e^{-\tau\mu_2(B)} \|e_{m,A}(\tau)\|_2 d\tau + \|\widetilde{E}_m\|_2 e^{t\mu_2(A)} \int_{t_0}^{t} e^{-\tau\mu_2(A)} \|e_{m,B}(\tau)\|_2 d\tau,$$

and then we obtain

$$\|\mathcal{E}_m(t)\|_2 \leq \|F\|_2 e^{t\mu_2(B)} \|V_{m+1}(T^A_{m+1,m} \otimes I_s)\|_2 \int_{t_0}^{t} e^{-\tau\mu_2(B)} S_{m,A}(\tau) d\tau \tag{18}$$

$$+ \|\widetilde{E}_m\|_2 e^{t\mu_2(A)} \|W_{m+1}(T^B_{m+1,m} \otimes I_s)\|_2 \int_{t_0}^{t} e^{-\tau\mu_2(A)} S_{m,B}(\tau)\|_2 d\tau, \tag{19}$$

where

$$\begin{cases} S_{m,A}(\tau) = \int_0^\tau e^{(u-\tau)V(A)} \|L_{m,A}(u)\|_2 du, \\ S_{m,B}(\tau) = \int_0^\tau e^{(u-\tau)V(B)} \|L_{m,B}(u)\|_2 du. \end{cases}$$

The approximate solution $X_m(t)$ could be given as a product of two matrices of low rank. It is possible to decompose it as $X_m = Z_1 Z_2^T$ where the matrix Z_1 and Z_2 are of low rank (lower than $2m$). Consider the singular value decomposition of the $2m \times 2m$ matrix

$$\mathbb{X}_m(t) = \widetilde{G}_1 \Sigma \widetilde{G}_2^T,$$

where Σ is the diagonal matrix of the singular values of $\mathbb{X}_m$ sorted in decreasing order. Let $\mathbb{X}_{1,l}$ and $\mathbb{X}_{2,l}$ be the $2m \times l$ matrices of the first l columns of $\widetilde{G}_1$ and $\widetilde{G}_2$ respectively, corresponding to the l singular values of magnitude greater than some tolerance d_{tol}. We obtain the truncated SVD

$$\mathbb{X}_m(t) \approx \mathbb{X}_{1,l}\, \Sigma_l\, \mathbb{X}_{2,l}{}^T,$$

where $\Sigma_l = \text{diag}[\sigma_1, \ldots, \sigma_l]$. Setting $Z_{1,m} = \mathbb{V}_m \left(\mathbb{X}_{1,l}\, \Sigma_l^{1/2} \otimes I_s\right)$ and $Z_{2,m} = \mathbb{W}_m \left(\mathbb{X}_{2,l}\, \Sigma_l^{1/2} \otimes I_s\right)$ Leads to

$$X_m \approx Z_{1,m}\, Z_{2,m}^T. \tag{20}$$

This is very important for large problems when one doesn't need to compute and store the approximation X_m at each iteration, see [2,6,8].

We summarize the above method for solving large differential Sylvester matrix equations **(EGA-exp)** in following algorithm.

Algorithm 3 The extended global Arnoldi (EGA-exp) method for DSE's

1. Inputs $X_0 = X(t_0)$ an $n \times p$ matrix, A an $n \times n$ matrix, B an $p \times p$ matrix, E an $n \times s$ matrix and F an $p \times s$ matrix.
2. Choose a tolerance $tol > 0$ and an integer m_{max}.
3. For $m = 1 : m_{max}$
   (a) Compute $\mathbb{V}_m$ and $\mathbb{T}_{m,A}$ by Algorithm 2 applied to (A, E).
   (b) Compute $\mathbb{W}_m$ and $\mathbb{T}_{m,B}$ by Algorithm 2 applied to (B^T, F).
   (c) Set $\widetilde{E}_m = \|E\|_F e_1^{(2m)}$, $\widetilde{F}_m = \|F\|_F e_1^{(2m)}$ and compute $\mathbb{X}_{m,A}(\tau) = e^{(t-\tau)\mathbb{T}_{m,A}} \widetilde{E}_m$, $\mathbb{X}_{m,B}(\tau) = e^{(t-\tau)\mathbb{T}_{m,B}} \widetilde{F}_m$ by using the Matlab function $expm$.
   (d) Use a quadrature method to compute the integral (8) and get an approximation of $\mathbb{X}_m(t)$ for each $t \in [t_0, T_f]$.
   (e) If $\|R_m(t)\|_F < tol$ stop.
   (f) Using (20), the approximate solution $X_m(t)$ is given by $X_m \approx Z_{1,m} Z_{2,m}^T$.
4. End

5 Low-rank approximate solutions by extended global Arnoldi algorithm

We present in this section an approach that avoid exponential approximation and also avoid quadrature method that we use in previous sections. This approach is based on extended global Krylov projection of the differential Sylvester matrix Eq. (1). For more details on global Krylov projection method for solving large matrix equations see [2,8,9,24].

Recall that when we Apply the extended global Arnoldi algorithm to the pairs (A, E) and (B^T, F), we get F-orthonormal bases $\{V_1, V_2, \ldots, V_m\}$ and $\{W_1, W_2, \ldots, W_m\}$ of extended global Krylov subspaces $\mathcal{K}_m^g(A, E)$ and $\mathcal{K}_m^g(B^T, F)$, respectively and we have

$$\mathbb{T}_{m,A} = \mathbb{V}_m^T \diamond (A\mathbb{V}_m) \quad \text{and} \quad \mathbb{T}_{m,B} = \mathbb{W}_m^T \diamond (B^T \mathbb{W}_m),$$

where

$$\mathbb{V}_m = [V_1, V_2, \ldots, V_m] \quad \text{and} \quad \mathbb{W}_m = [W_1, W_2, \ldots, W_m].$$

We then consider approximate solution of the large differential Sylvester matrix Eq. (1) that have the low-rank form

$$X_m(t) = \mathbb{V}_m (Y_m(t) \otimes I_s) \mathbb{W}_m^T, \tag{21}$$

where $Y_m(t)$ the solution of the reduced differential Sylvester matrix equation

$$\dot{Y}_m(t) - \mathbb{T}_{m,A} Y_m(t) - Y_m(t) \mathbb{T}_{m,B}^T - \widetilde{E}_m \widetilde{F}_m^T = 0. \tag{22}$$

with $\widetilde{E}_m = \|E\|_F e_1^{(2m)}$ and $\widetilde{F}_m = \|F\|_F e_1^{(2m)}$.

The following theorem gives a result that allows us the computation of the norm of the residual.

Theorem 9 *The Frobenius norm of the residual $R_m(t)$ associated to the approximation $X_m(t)$ satisfies the relation*

$$\|R_m(t)\|_F^2 \leq \|T^A_{m+1,m}\overline{Y}_m(t)\|_F^2 + \|T^B_{m+1,m}\overline{Y}_m(t)\|_F^2, \tag{23}$$

where $\overline{Y}_m(t)$ is the $2 \times 2m$ matrix corresponding to the last 2 rows of Y_m.

Proof See Theorem 4. □

To solve the reduced order differential Sylvester matrix Eq. (22) one can use Backward Differentiation Formula (BDF) or Rosenbrock method, see [12,19,31].

We summarize steps of this approach in the following algorithm

Algorithm 4 The extended global Arnoldi for DSE's (EGA)

1. Input X_0, a tolerance $tol > 0$, an integer m_{max}.
2. For $m = 1 : m_{max}$
   (a) Apply the extended global Arnoldi algorithm to (A, E) and (B^T, F) to get the $F-$orthonormal matrices $\mathbb{V}_m$ and $\mathbb{W}_m$ and the upper block Hessenberg matrices $\mathbb{T}_{m,A}$ and $\mathbb{T}_{m,B}$.
   (b) Use BDF or Rosenbrock method to solve the differential Sylvester Eq. (22).
   (c) If $\|R_m(t)\|_F < tol$ stop.
   (d) Compute the approximate solution $X_m(t)$ by using (20).
3. End

6 Numerical experiments

In this section, we present some numerical experiments of large and sparse differential Sylvester matrix equations. We compare the two approaches proposed in this work [Algorithm 3 (EGA-exp), Algorithm 4 using BDF (EGA–BDF) and Algorithm 4 using Rosenbrock (EGA–ROS)] whit the extended block Arnoldi (EBA-exp) given in [19]. All the experiments were performed on a laptop with an Intel Core i5 processor and 4 GB of RAM using Matlab2014. The n-by-s matrices E and F are given by random values uniformly distributed on [0, 1].

6.1 Example 1

In this first example, the matrices A and B are obtained from the centered finite difference discretization of the operators:

$$L_A(u) = \Delta u + f_1(x, y)\frac{\partial u}{\partial x} + f_2(x, y)\frac{\partial u}{\partial y} + f(x, y)u$$

$$L_B(u) = \Delta u + g_1(x, y)\frac{\partial u}{\partial x} + g_2(x, y)\frac{\partial u}{\partial y} + g(x, y)u$$

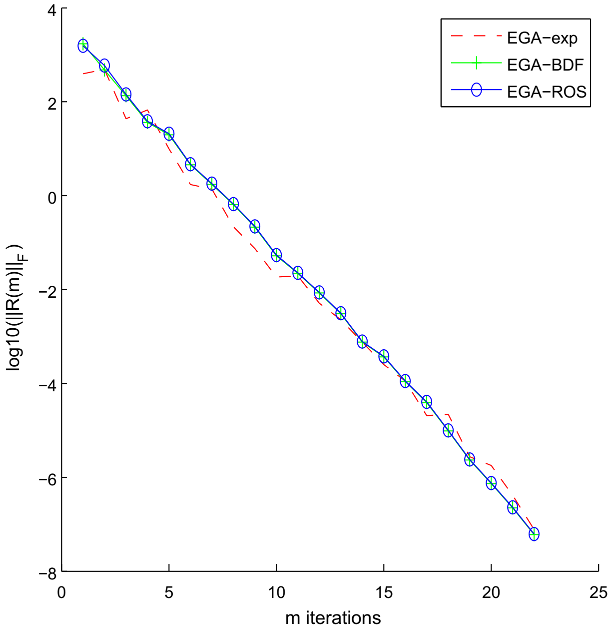

Fig. 1 Residual norm versus number m of extended global Arnoldi iterations

on the unit square $[0, 1] \times [0, 1]$ with homogeneous Dirichlet boundary conditions. The number of inner grid points in each direction was n_0 and p_0 for the operators L_A and L_B, respectively. The matrices A and B were obtained from the discretization of the operator L_A and L_B with the dimensions $n = n_0^2$ and $p = p_0^2$, respectively. The discretization of the operator $L_A(u)$ and $L_B(u)$ yields matrices extracted from the `Lyapack` package [30] using the command `fdm_2d_matrix` and denoted as $\mathrm{fdm}(n0,' f\_1(x, y)',' f\_2(x, y)',' f(x, y)')$. For this experiment, we consider $A = \mathrm{fdm}(n0, f_1(x, y), f_2(x, y), f(x, y))$ and $B = \mathrm{fdm}(p0, g_1(x, y), g_2(x, y), g(x, y))$ with $f_1(x, y) = -e^{xy}$, $f_2(x, y) = -\underline{\sin(xy)}$, $f(x, y) = y^2$, $g_1(x, y) = -100e^x$, $g_2(x, y) = -12xy$ and $g(x, y) = \sqrt{x^2 + y^2}$. we used $s = 2$. The time interval considered was $[0, 2]$ and the initial condition $X_0 = X(t_0)$ was $X_0 = Z_0 \widetilde{Z}_0^T$, where $Z_0 = 0_{n\times 2}$ and $\widetilde{Z}_0 = 0_{p\times 2}$. The tolerance was set to 10^{-7} for the stop test on the residual. For the EGA–BDF and Rosenbrock methods, we used a constant timestep $h = 0.1$.

In Fig. 1, we chose a size of 2500×2500, 2500×2500 for the matrices A and B, respectively, we plotted the Frobenius norms of the residuals $\|R_m(T_f)\|_F$ at final time T_f versus the number of extended global Arnoldi iterations for the EGA-exp, EGA–BDF and EGA–ROS methods.

In Fig. 2, the matrices A and B are obtained from the discretisation of the operator $L_A(u)$ and $L_B(u)$ with dimensions $n = 10{,}000$ and $p = 4900$, respectively. we plotted the Frobenius norms of the residuals $\|R_m(T_f)\|_F$ at final time T_f versus the number of extended global Arnoldi iterations for the EGA-exp, EGA–BDF and EGA–ROS methods.

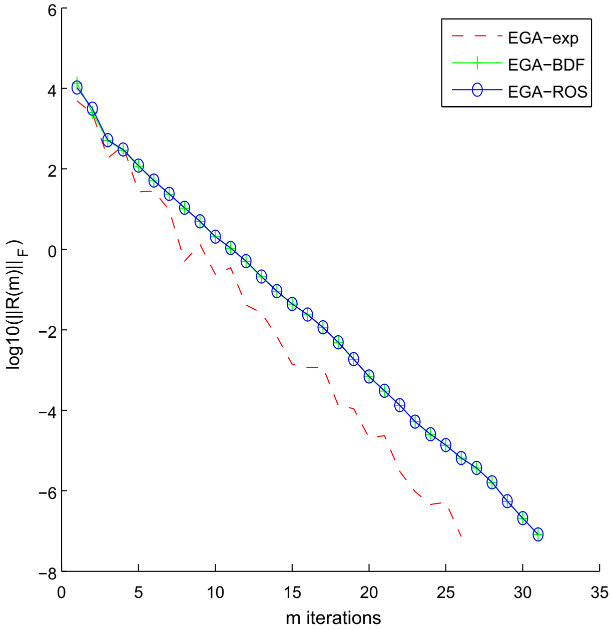

Fig. 2 Residual norm versus number m of extended global Arnoldi iterations

6.2 Example 2

For the second set of experiments, we use the matrices $add32$, $pde2961$, and $thermal$ from the University of Florida Sparse Matrix Collection [15] and from the Harwell Boeing Collection (http://math.nist.gov/MatrixMarket). The tolerance was set to 10^{-7} for the stop test on the residual. For the EGA–BDF and EGA–ROS methods, we used a constant timestep $h = 0.01$.

In Fig. 3, the matrices $A = thermal$ and $B = add32$ with dimensions $n = 3456$ and $p = 4960$, respectively, and $s = 3$. we plotted the Frobenius norms of the residuals $\|R_m(T_f)\|_F$ at final time T_f versus the number of extended global Arnoldi iterations for the EGA-exp, EGA–BDF and EGA–ROS methods.

In Fig. 4, we used the matrices $A = pde2961$ and $B = \text{fdm}(90, 100e^x, \sqrt{2x^2 + y^2}, y^2 - x^2)$ with dimensions $n = 2961$ and $p = 8100$, respectively, and $s = 4$. we plotted the Frobenius norms of the residuals $\|R_m(T_f)\|_F$ at final time T_f versus the number of extended global Arnoldi iterations for the EGA-exp, EGA–BDF and EGA–ROS methods.

6.3 Example 3

In this last example, we compare the performances of the extended global Arnoldi method associated to the different techniques for solving the reduced-order problem and the EBA-exp method given in [19].

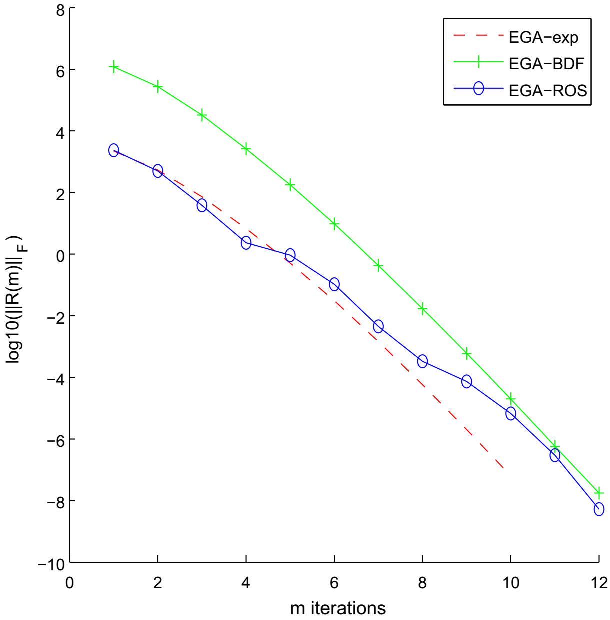

Fig. 3 Residual norm versus number m of extended global Arnoldi iterations

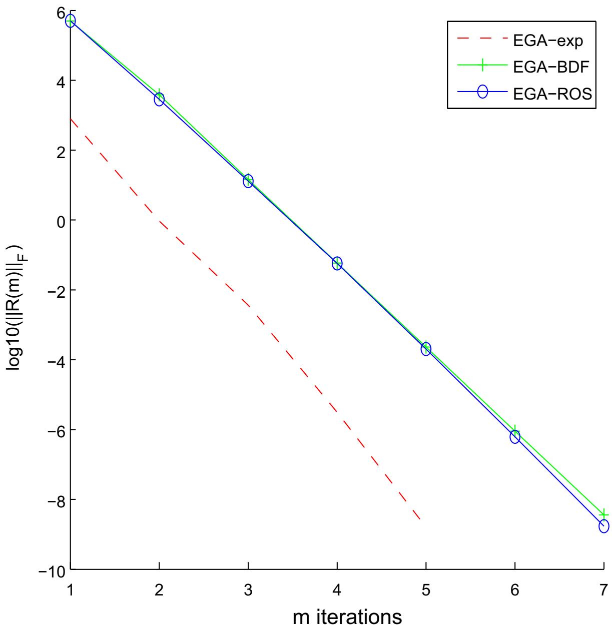

Fig. 4 Residual norm versus number m of extended global Arnoldi iterations

Table 1 Runtimes in seconds and the residual norms

Test problem	Method	CPU time (s)	Iterations	$\|R_m(T_f)\|_F$
$n = p = 2500$, $h = 0.1$ and $s = 2$	EGA-exp	**0.51**	15	1.47×10^{-10}
$A = fdm(-e^{xy}, \sin(xy), 1)$	EGA–BDF	18.31	20	2.69×10^{-10}
$B = fdm(xy, 12xy, \sqrt{x^2 + y^2})$	EGA–ROS	37.45	20	2.64×10^{-10}
	EBA-exp	5.09	19	1.43×10^{-10}
$n = 12{,}100$	EGA-exp	**2.36**	14	6.53×10^{-10}
$p = 8100$, $h = 0.1$ and $s = 3$	EGA–BDF	65.81	27	6.22×10^{-10}
$A = fdm(xy, y^2, 1)$	EGA–ROS	94.43	27	6.22×10^{-10}
$B = fdm(xy, \cos(xy), 10)$	EBA-exp	4.65	23	2.34×10^{-10}
$A = B = fdm(\cos(xy), e^{y^2 x}, 100)$	EGA-exp	**2.22**	11	9.10×10^{-10}
$n = p = 12{,}100$ and $s = 3$	EGA–BDF	48.06	18	5.40×10^{-10}
$h = 0.05$	EGA–ROS	59.03	19	5.44×10^{-10}
	EBA-exp	4.03	10	8.72×10^{-10}
$A = pde2961$, $n = 2961$,	EGA-exp	**3.21**	7	2.61×10^{-10}
$B=\texttt{fdm}(100e^x, \sqrt{2x^2 + y^2}, y^2 - x^2)$	EGA–BDF	3.73	8	4.78×10^{-10}
$p = 10{,}000$, $h = 0.1$ and $s = 4$	EGA–ROS	4.27	9	5.20×10^{-10}
	EBA-exp	5.13	8	5.64×10^{-10}

Bold values indicate the global extended exponential method EGA-exp is very effective compared to other methods

In Table 1, we list the Frobenius residual norms at final time $T_f = 2$ and the corresponding CPU time for each method. For this experiment, the algorithms are stopped when the residual norms are smaller than 10^{-9}.

The numerical results are promising, showing the effectiveness of the global extended exponential method EGA-exp compared with extended block exponential approach EBA-exp given in [19] in terms of precision and computation time.

7 Conclusion

We presented in this paper some iterative methods for computing numerical solutions for large scale differential Sylvester matrix equations with low rank right-hand sides. The first approach arises naturally from the exponential expression of the exact solution and the use of approximation approach of the exponential of a matrix times by extended global Krylov subspaces. The second approach is based on low-rank approximate solutions and extended global algorithm. The approximate solutions are given as products of two low rank matrices and allow for saving memory for large problems. The numerical experiments show that the proposed extended global Krylov based methods are effective for large and sparse problems.

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