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ORIGINAL RESEARCH

Study on flow shop scheduling with sum-of-logarithm-processing-times-based learning effects

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Abstract
This paper addresses flow shop scheduling problems with sum-of-logarithm-processing-times-based learning effects. The objective is to minimize the total completion time, the makespan, the total weighted completion time, and the sum of the quadratic job completion times, respectively. Heuristic algorithms based on the optimal schedules for the corresponding flow shop scheduling problems are presented and their worst-case error bounds are also analyzed.

Keywords Scheduling · Heuristic algorithm · Flow shop · Learning effect

Mathematics Subject Classification 90B35 · 68M20

1 Introduction

Scheduling problems have received considerable attention for many years (see Gonzalez and Sahni [1], Smutnicki [2], Hoogeveen and Kawaguchi [3], Koulamas and Kyparisis [4], Easwaran et al. [5], Pinedo [6]). Most research of deterministic scheduling assumes that processing time of a job is independent of its position in the process schedule. However, additional constraints such as learning effects of machines (workers) and flow shop machines setting are of interest to increase the efficiency of manufacture system. In the "learning effect", the production facility (a machine, a worker) improves continuously over time and the production time of a given product is shorter if it is scheduled later (Biskup [7,8], Wang et al. [9], Cheng et al. [10], Eren [11], Lee and Wu [12], Yin et al. [13], Yin et al. [14], Wu et al. [15], Yin et al. [16], and Zhao and Tang [17]).

Cheng et al. [10] considered single machine scheduling problems with sum-of-logarithm-processing-times-based learning effects. However, in the manufacturing

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enterprise, the flow shop scheduling problems are important and usual. Hence, in this paper we consider the same model as that of Cheng et al. [10], but with flow shop scheduling setting. The objective functions are to minimize the total completion time, the makespan, the total weighted completion time, and the sum of the quadratic job completion times, respectively. We give a heuristic for each criteria, and analysis its worst-case error bound. For some studies about flow shop scheduling with learning effects, we refer the readers to Wang and Xia [18], Xu et al. [19], Wang and Wang [20], Li et al. [21], Sun et al. [22], Wang and Wang [23], Sun et al. [24], Wang et al. [25], Wang and Zhang [26], He [27], Bai et al. [28], and Wang et al. [29]. Wang and Xia [18] and Xu et al. [19] considered flow shop scheduling with learning effect, i.e., the actual processing time p_{ijr} of job J_j on machine M_i is $p_{ijr} = p_{ij}(\alpha - \beta r)$ and $p_{ijr} = p_{ij}r^{\delta}, i = 1, 2, \ldots, m; r, j = 1, 2, \ldots, n$, where p_{ij} is the normal processing time of job J_j on machine M_i, $\alpha > 0, \beta \geq 0, \alpha - \beta(n+1) > 0$, and $\delta \leq 0$ is the learning rate. Using the three-field notation scheme (Graham et al. [30]), Wang and Xia [18] proposed heuristic algorithms with worst-case error bounds for the problems $Fm\left|p_{ijr} = p_{ij}(\alpha - \beta r)\right|\theta$ and $Fm\left|p_{ijr} = p_{ij}r^{\delta}\right|\theta$, where $\theta \in \{C_{\max}, \sum_{j=1}^{n} C_j\}$, Xu et al. [19] proposed heuristic algorithms with worst-case error bounds for the problems $Fm\left|p_{ijr} = p_{ij}(\alpha - \beta r)\right|\theta$ and $Fm\left|p_{ijr} = p_{ij}r^{\delta}\right|\theta$, where $\theta \in \{\sum_{j=1}^{n} w_j C_j, \sum_{j=1}^{n} C_j^2, \sum_{j=1}^{n} w_j(1-e^{-\gamma C_j})\}$, where C_j (w_j) is the completion time of job J_j, $w_j > 0$ is a weight associated with job J_j, $C_{\max}$ is the maximum completion time, $0 < \gamma < 1$. Wang and Wang [20] considered flow shop scheduling with learning effect in which $p_{ijr} = p_{ij}b^{r-1}$, where b denotes the learning ratio with $0 < b \leq 1$. Wang and Wang [20] proposed heuristic algorithms with worst-case error bounds for $Fm\left|p_{ijr} = p_{ij}b^{r-1}\right|\theta$, where $\theta \in \{\sum_{j=1}^{n} C_j, \sum_{j=1}^{n} w_j C_j, \sum_{j=1}^{n} C_j^2, \sum_{j=1}^{n} w_j(1 - e^{-\gamma C_j})\}$. Li et al. [21] considered flow shop scheduling with learning effect in which $p_{ijr} = p_{ij}\left(1 + \sum_{l=1}^{r-1} p_{i[l]}\right)^a$, where $a \leq 0$ denotes the learning ratio. They proposed heuristic algorithms with worst-case error bounds for $Fm\left|p_{ijr} = p_{ij}\left(1 + \sum_{l=1}^{r-1} p_{i[l]}\right)^a\right|\theta$, where $\theta \in \{C_{\max}, \sum_{j=1}^{n} C_j, \sum_{j=1}^{n} w_j C_j, \sum_{j=1}^{n} C_j^2, \sum_{j=1}^{n} w_j(1 - e^{-\gamma C_j})\}$. Sun et al. [22] proposed new heuristic algorithms with worst-case error bounds for $Fm\left|A\right|\sum_{j=1}^{n} w_j C_j$, where $A \in \{p_{ijr} = p_{ij}(\alpha - \beta r), p_{ijr} = p_{ij}r^{\delta}, p_{ijr} = p_{ij}b^{r-1}\}$. Wang and Wang [23] and Sun et al. [24] considered flow shop scheduling with learning effect in which $p_{ijr} = p_{ij}g(r)$, where $g(r)$ is a non-increasing function on r. Wang and Wang [23] and Sun et al. [24] proposed heuristic algorithms with worst-case error bounds for $Fm\left|p_{ijr} = p_{ij}g(r)\right|\theta$, where $\theta \in \{C_{\max}, \sum_{j=1}^{n} C_j, \sum_{j=1}^{n} C_j^2, \sum_{j=1}^{n} w_j C_j, \sum_{j=1}^{n} w_j(1 - e^{-\gamma C_j})\}$. Wang et al. [25] considered flow shop scheduling with learning effect in which $p_{ijr} = p_{ij}\max\{r^{\delta}, \zeta\}$, where ζ is a truncation parameter with $0 \leq \zeta \leq 1$. They proposed heuristic algorithms with worst-case error bounds for $Fm\left|p_{ijr} = p_{ij}\max\{r^{\delta}, \zeta\}\right|\theta$, where $\theta \in \{C_{\max}, \sum_{j=1}^{n} C_j, \sum_{j=1}^{n} C_j^2, \sum_{j=1}^{n} w_j C_j, \sum_{j=1}^{n} w_j(1 - e^{-\gamma C_j})\}$. Wang and Zhang [26] considered the problem $Fm\left|p_{ijr} = p_{ij}\left(1 + \sum_{l=1}^{r-1} \beta_l \ln p_{i[l]}\right)^a r^c\right|\lambda C_{\max} + (1-\lambda)\sum_{j=1}^{n} C_j$, where $0 < \beta_1 \leq \beta_2 \ldots \leq \beta_n$, $c \leq 0$, and $0 \leq \lambda \leq 1$. He [27] considered the maximum lateness minimization flow shop scheduling with a general exponential learning effect, he proposed a branch-and-bound algorithm, several heuristics, and a nested-partition-based solution approach to solve this problem. Bai et al. [28] considered the flow shop scheduling problems $Fm\left|p_{ijr} = p_{ij}g\left(r\right), r_j\right|\theta$, where $\theta \in \{C_{\max}, \sum_{j=1}^{n} C_j, \sum_{j=1}^{n} C_j^2\}$, r_j is the release dates of job J_j. Wang et al. [29] considered flow shop scheduling problems with truncated exponential sum of logarithm processing times based and position-based learning effects. For the makespan and total weighted completion time minimizations, they proposed several heuristics and a branch-and-bound algorithm.

The remaining part of this paper is organized as follows. In Sect. 2 we give the formulation of the model. In Sect. 3, we propose a heuristic with a worst-case error bound for several regular objective functions, respectively. In Sect. 4, computational results are given. The last section is the summary and future research.

2 Problem formulation

There is a set of n jobs $J = \{J_1, J_2, \ldots, J_n\}$ to be processed on m machines $M_1, M_2, \ldots, M_m$. Each job J_j must first be prossed on M_1, and then executed on M_2, and so on. As in Cheng et al. [10], in this paper, we consider flow shop scheduling with sum-of-logarithm-processing-times-based learning effects, i.e., the actual processing time p_{ijr} of job J_j on machine M_i is

$$p_{ijr} = p_{ij}\left(1 + \sum_{l=1}^{r-1} \ln p_{i[l]}\right)^a, i = 1, 2, \ldots, m; r, j = 1, 2, \ldots, n, \qquad (1)$$

where p_{ij} is the normal processing time (i.e., the processing time without any learning effects) of job J_j on machine M_i, $a \leq 0$ is the learning index, $\ln p_{ij} \geq 1$ and $\sum_{l=1}^{0} \ln p_{i[l]} = 0$.

Considering a schedule π, $C_{ij} = C_{ij}(\pi)$ denotes the completion time of job J_j on machine M_i, and $C_j = C_{mj}$ represents the completion time of job J_j, $\sum_{j=1}^{n} C_j$ is the total completion time, $C_{\max} = \max\{C_j | j = 1, 2, \ldots, n\}$ is the makespan of all jobs, $\sum_{j=1}^{n} w_j C_j$ denotes the total weighted completion time (where $w_j > 0$ is a weight associated with job J_j), $\sum_{j=1}^{n} C_j^2$ is the sum of the quadratic job completion times (Townsend [31]).

3 Main results

Lemma 1 (Cheng et al. [10]) *For the problem* $1|p_{jr} = p_j\left(1 + \sum_{l=1}^{r-1} \ln p_{[l]}\right)^a|\sum C_j$, *an optimal schedule can be obtained by sequencing the jobs in non-decreasing order of* p_j *(i.e., the SPT rule).*

Lemma 2 (Cheng et al. [10]) *For the problem* $1|p_{jr} = p_j\left(1 + \sum_{l=1}^{r-1} \ln p_{[l]}\right)^a|C_{\max}$, *an optimal schedule can be obtained by sequencing the jobs in non-decreasing order of* p_j *(i.e., the SPT rule).*

Lemma 3 (Smith [32]) *For the problem* $1||\sum w_jC_j$, *an optimal schedule can be obtained by sequencing the jobs in non-decreasing order of* $\frac{p_j}{w_j}$ *(i.e., the weighted shortest processing time first (WSPT) rule).*

Lemma 4 (Townsend [31]) *"For the problem* $1||\sum C_j^2$, *an optimal schedule can be obtained by sequencing the jobs in non-decreasing order of* p_j *(i.e., the SPT rule)."*

Obviously, the problems $Fm|prmu, p_{ijr} = p_{ij}\left(1+\sum_{l=1}^{r-1}\ln p_{i[l]}\right)^a|C_{\max}$ $(m \geq 3)$, $Fm|prmu, p_{ijr} = p_{ij}\left(1+\sum_{l=1}^{r-1}\ln p_{i[l]}\right)^a|\sum C_j$ $(m \geq 2)$, $Fm|prmu, p_{ijr} = p_{ij}\left(1+\sum_{l=1}^{r-1}\ln p_{i[l]}\right)^a|\sum w_jC_j$ $(m \geq 2)$, and $Fm|prmu, p_{ijr} = p_{ij}\left(1+\sum_{l=1}^{r-1}\ln p_{i[l]}\right)^a|\sum C_j^2$ $(m \geq 2)$ are NP-complete, respectively.

Let $V_j = \sum_{i=1}^{m} p_{ij}$, from Lemma 1, the SPT (in order of non-decreasing V_j) rule can be used as an approximate algorithm to solve $Fm|prmu, p_{ijr} = p_{ij}\left(1+\sum_{l=1}^{r-1}\ln p_{i[l]}\right)^a|\sum C_j$.

Theorem 1 *Let* S^* *be an optimal schedule and* S *be an SPT schedule for the problem* $Fm|prmu, p_{ijr} = p_{ij}\left(1+\sum_{l=1}^{r-1}\ln p_{i[l]}\right)^a|\sum C_j$. *Then* $\frac{\sum C_j(S)}{\sum C_j(S^*)} \leq \frac{m}{(1+\ln P_{\max})^a}$, *and this bound is tight, where* $\ln P_{\max} = \max\{\sum_{l=1}^{n}\ln p_{il} - \ln p_{i\min}|i = 1, 2, \ldots, m\}$ *and* $p_{i\min} = \min\{p_{ij}|j = 1, 2, \ldots, n\}$.

Proof Let $V_1 \leq V_2 \leq \cdots \leq V_n$, we have

$$C_j(S) \leq V_1 + V_2\left(1+p_{\min\{i1\}}\right)^a + V_3\left(1+p_{\min\{i1+i2\}}\right)^a + \cdots + V_j\left(1+p_{\min\{i1+i2+\cdots+i,j-1\}}\right)^a, \tag{2}$$

where $p_{\min\{i1+i2+\cdots+i,j-1\}} = \min\{\ln p_{i1} + \ln p_{i2} + \cdots + \ln p_{i,j-1}|i = 1, 2, \ldots, m\}$ and so

$$\sum_{j=1}^{n} C_j(S) \leq \sum_{j=1}^{n}\sum_{l=1}^{j} V_l. \tag{3}$$

Let $\bar{S} = (J_{[1]}, J_{[2]}, \ldots, J_{[n]})$ be any schedule, where $[j]$ denotes the job that occupies the jth position in S^*, we have

$$\begin{aligned}
C_{1[j]} &= p_{1[1]} + p_{1[2]}\left(1+\ln p_{1[1]}\right)^a + \cdots + p_{1[j]} \\
&\quad (1+\ln p_{1[1]} + \ln p_{1[2]} + \cdots + \ln p_{1[j-1]})^a \\
C_{2[j]} &\geq p_{2[1]} + p_{2[2]}\left(1+\ln p_{2[1]}\right)^a + \cdots + p_{2[j]} \\
&\quad (1+\ln p_{2[1]} + \ln p_{2[2]} + \cdots + \ln p_{2[j-1]})^a \\
&\cdots \\
C_{m[j]} &\geq p_{m[1]} + p_{m[2]}\left(1+\ln p_{m[1]}\right)^a + \cdots + \\
&\quad p_{m[j]}\left(1+\ln p_{m[1]} + \ln p_{m[2]} + \cdots + \ln p_{m[j-1]}\right)^a,
\end{aligned}$$

hence

$$C_{[j]}(\bar{S}) \geq \frac{1}{m}\sum_{l=1}^{j} V_{[j]}\left(1+\ln P_{\max}\right)^{a}, \tag{4}$$

where $\ln P_{\max} = \max\{\sum_{l=1}^{n} \ln p_{il} - \ln p_{i\min} | i = 1, 2, \ldots, m\}$ and $p_{i\min} = \min\{p_{ij} | j = 1, 2, \ldots, n\}$, and for the optimal schedule S^*, we have

$$\sum_{j=1}^{n} C_j(S^*) \geq \frac{1}{m}\left(1+\ln P_{\max}\right)^{a} \sum_{j=1}^{n}\sum_{j=1}^{i} V_{[j]} \geq \frac{1}{m}\left(1+\ln P_{\max}\right)^{a} \sum_{i=1}^{n}\sum_{j=1}^{i} V_j, \tag{5}$$

as the term $\sum_{i=1}^{n}\sum_{j=1}^{i} L_{[j]}$ is minimized by the increasing order of L_j (Lemma 1). Consequently, form (3) and (5), we have that $\frac{\sum C_j(S)}{\sum C_j(S^*)} \leq \frac{m}{(1+\ln P_{\max})^a}$.

We show that the bound $\frac{m}{(1+\ln P_{\max})^a}$ is tight. Consider the following instance. Learning takes place by the 100%-learning curve (a learning rate of 100% means that no learning is taking place), thus $a = 0$, i.e., the bound $\frac{m}{(1+\ln P_{\max})^a} = m$. The bound m of the SPT rule for $Fm|prmu|\sum C_j$ is tight as shown in Gonzalez and Sahni [1] and therefore the bound for the problem $Fm|prmu, p_{ijr} = p_{ij}\left(1+\sum_{l=1}^{r-1}\ln p_{i[l]}\right)^{a}|\sum C_j$ is also tight. □

Gonzalez and Sahni [1] proposed the ARB (any busy schedule) rule to solve $Fm|prmu|\sum C_j$, hence, we can also use the ARB rule as an approximate algorithm to solve $Fm|prmu, p_{ijr} = p_{ij}\left(1+\sum_{l=1}^{r-1}\ln p_{i[l]}\right)^{a}|\sum C_j$.

Theorem 2 *Let S^* (S) be an optimal (ARB rule) schedule for the problem $Fm|prmu$, $p_{ijr} = p_{ij}\left(1+\sum_{l=1}^{r-1}\ln p_{i[l]}\right)^{a}|\sum C_j$. Then $\frac{\sum_{j=1}^{n} C_j(S)}{\sum_{j=1}^{n} C_j(S^*)} \leq \frac{n}{(1+\ln P_{\max})^a}$, and this bound is tight, where $\ln P_{\max} = \max\{\sum_{l=1}^{n}\ln p_{il} - \ln p_{i\min} | i = 1, 2, \ldots, m\}$ and $p_{i\min} = \min\{p_{ij} | j = 1, 2, \ldots, n\}$.*

Proof Let $\sum_{j=1}^{n} V_j = T$, we assume that $V_1 \leq V_2 \leq \cdots \leq V_n$. Let $C_j(S)$ be the completion time of job J_j using SPT schedule S. Then from Theorem 1, we have $C_j(S) \leq \sum_{l=1}^{j} V_l$ and so

$$\sum_{j=1}^{n} C_j(S) \leq \sum_{j=1}^{n}\sum_{l=1}^{j} V_l \leq nT.$$

For the optimal schedule S^*, from Theorem 1, we have $C_{[j]}(S^*) \geq V_{[j]}\left(1+\ln P_{\max}\right)^{a}$ and so

$$\sum_{j=1}^{n} C_j(S^*) \geq \sum_{j=1}^{n} V_{[j]}(1+\ln P_{\max})^{a} \geq \sum_{i=1}^{n} V_{[i]}(1+\ln P_{\max})^{a} = T(1+\ln P_{\max})^{a}.$$

Consequently, we have

$$\frac{\sum_{j=1}^{n} C_j(S)}{\sum_{j=1}^{n} C_j(S^*)} \leq \frac{n}{(1+\ln P_{\max})^a}.$$

We show that the bound $\frac{n}{(1+\ln P_{\max})^a}$ is tight. Consider the following instance. Learning takes place by the 100%-learning curve, i.e., $a = 0$ and the bound $\frac{n}{(1+\ln P_{\max})^a} = n$. The bound n of the ARB rule for $Fm|prmu|\sum C_j$ is tight as shown in Gonzalez and Sahni [1] and therefore the bound for the problem $Fm|prmu, p_{ijr} = p_{ij}\left(1+\sum_{l=1}^{r-1}\ln p_{i[l]}\right)^a|\sum C_j$ is also tight. □

Gonzalez and Sahni [1] proposed the ARB rule (any busy schedule) as an approximate algorithm to solve $Fm|prmu|C_{\max}$, hence, we can also use the ARB rule as an approximate algorithm to solve $Fm|prmu, p_{ijr} = p_{ij}\left(1+\sum_{l=1}^{r-1}\ln p_{i[l]}\right)^a|C_{\max}$.

Theorem 3 *Let S^* (S) be an optimal (ARB rule) schedule for the problem $Fm|prmu, p_{ijr} = p_{ij}\left(1+\sum_{l=1}^{r-1}\ln p_{i[l]}\right)^a|C_{\max}$. Then $C_{\max}(S)/C_{\max}(S^*) \leq \frac{m}{(1+\ln P_{\max})^a}$, and this bound is tight, where $\ln P_{\max} = \max\{\sum_{l=1}^{n}\ln p_{il} - \ln p_{i\min}|i = 1, 2, \ldots, m\}$ and $p_{i\min} = \min\{p_{ij}|j = 1, 2, \ldots, n\}$.*

Proof Similar to the proof of Theorem 2 except that: $C_{\max}(S) \leq \sum_{j=1}^{n} V_j = T$ and

$$C_{\max}(S^*) \geq \frac{1}{m}\sum_{l=1}^{n} V_{[j]}(1+\ln P_{\max})^a = \frac{1}{m}(1+\ln P_{\max})^a T,$$

hence, we have $C_{\max}(S)/C_{\max}(S^*) \leq \frac{m}{(1+\ln P_{\max})^a}$.

We show that the bound $\frac{m}{(1+\ln P_{\max})^a}$ is tight. Consider the following instance. Learning takes place by the 100%-learning curve, i.e., $a = 0$ and $\frac{m}{(1+\ln P_{\max})^a} = m$. The bound m of the ARB rule for $Fm|prmu|C_{\max}$ is tight as shown in Gonzalez and Sahni [1] and therefore the bound for the problem $Fm|prmu, p_{ijr} = p_{ij}\left(1+\sum_{l=1}^{r-1}\ln p_{i[l]}\right)^a|C_{\max}$ is also tight. □

Since for the problem $1|p_{jr} = p_j\left(1+\sum_{l=1}^{r-1}\ln p_{[l]}\right)^a|C_{\max}$, SPT rule generates an optimal solution (Lemma 2), so we use the SPT rule as an approximate algorithm to solve $Fm|prmu, p_{ijr} = p_{ij}\left(1+\sum_{l=1}^{r-1}\ln p_{i[l]}\right)^a|C_{\max}$.

Corollary 1 *Let S^* (S) be an optimal (SPT) schedule for $Fm|prmu, p_{ijr} = p_{ij}\left(1+\sum_{l=1}^{r-1}\ln p_{i[l]}\right)^a|C_{\max}$. Then $C_{\max}(S)/C_{\max}(S^*) \leq \frac{m}{(1+\ln P_{\max})^a}$, and this bound is tight, where $\ln P_{\max} = \max\{\sum_{l=1}^{n}\ln p_{il} - \ln p_{i\min}|i = 1, 2, \ldots, m\}$ and $p_{i\min} = \min\{p_{ij}|j = 1, 2, \ldots, n\}$.*

From Lemma 3, we can use the WSPT (in order of non-decreasing $\frac{V_j}{w_j}$) rule as an approximate algorithm for problem $Fm|prmu, p_{ijr} = p_{ij}\left(1+\sum_{l=1}^{r-1}\ln p_{i[l]}\right)^a|$ $\sum w_jC_j$.

Theorem 4 *Let S^* (S) be an optimal (a WSPT) schedule for the problem $Fm|prmu,$ $p_{ijr} = p_{ij}\left(1+\sum_{l=1}^{r-1}\ln p_{i[l]}\right)^a|\sum w_jC_j$. Then $\frac{\sum_{j=1}^n w_jC_j(S)}{\sum_{j=1}^n w_jC_j(S^*)} \le \frac{m}{(1+\ln P_{\max})^a}$, and this bound is tight, where $\ln P_{\max} = \max\{\sum_{l=1}^n \ln p_{il} - \ln p_{i\min}|i = 1, 2, \ldots, m\}$ and $p_{i\min} = \min\{p_{ij}|j = 1, 2, \ldots, n\}$.*

Proof Without loss of generality we assume that $\frac{V_1}{w_1} \le \frac{V_2}{w_2} \le \cdots \le \frac{V_n}{w_n}$. Let $C_j(S)$ be the completion time of job J_j using WSPT schedule S. Then from Theorem 1, we have $C_j(S) \le \sum_{l=1}^j V_l$ and so

$$\sum_{j=1}^n w_jC_j(S) \le \sum_{j=1}^n w_j\sum_{l=1}^j V_l.$$

Let $(J_{[1]}, J_{[2]}, \ldots, J_{[n]})$ be the order in which jobs complete in the optimal schedule S^*. Then from Theorem 1, we have $C_{[j]}(S^*) \ge \frac{1}{m}\sum_{l=1}^j V_{[j]}\,(1+\ln P_{\max})^a$ and so

$$\sum_{j=1}^n w_jC_j(S^*) \ge \frac{(1+\ln P_{\max})^a}{m}\sum_{j=1}^n w_{[j]}\sum_{l=1}^j V_{[l]} \ge \frac{(1+\ln P_{\max})^a}{m}\sum_{i=1}^n w_i\sum_{j=1}^i V_j,$$

as the term $\sum_{j=1}^n w_{[j]}\sum_{l=1}^j V_{[l]}$ is minimized by the increasing order of $\frac{V_j}{w_j}$ (Lemma 3). Consequently, we have

$$\frac{\sum_{j=1}^n w_jC_j(S)}{\sum_{j=1}^n w_jC_j(S^*)} \le \frac{m}{(1+\ln P_{\max})^a}.$$

We show that the bound $\frac{m}{(1+\ln P_{\max})^a}$ is tight. Consider the following instance. Learning takes place by the 100%—learning curve, i.e., $a = 0$ and $\frac{m}{(1+\ln P_{\max})^a} = m$. The bound m of the WSPT rule for $Fm|prmu|\sum w_jC_j$ is tight as shown in Smutnicki [2] and therefore the bound for the problem $Fm|prmu, p_{ijr} = p_{ij}\left(1+\sum_{l=1}^{r-1}\ln p_{i[l]}\right)^a|\sum w_jC_j$ is also tight. □

Theorem 5 *Let S^* (S) be an optimal (ARB) schedule for the $Fm|prmu, p_{ijr} = p_{ij}\left(1+\sum_{l=1}^{r-1}\ln p_{i[l]}\right)^a|\sum w_jC_j$ problem. Then $\frac{\sum_{j=1}^n w_jC_j(S)}{\sum_{j=1}^n w_jC_j(S^*)} \le \frac{1+(n-1)(\overline{w}/\underline{w})}{(1+\ln P_{\max})^a}$, and this bound is tight, where $\underline{w} = \min_{j\in J} w_j$ and $\overline{w} = \max_{j\in J} w_j$.*

Proof Let $\sum_{j=1}^{n} V_j = T$ and $S = (J_1, J_2, \ldots, J_n)$ be any busy schedule, we have $C_i(S) \le \sum_{j=1}^{i} V_j$ and so

$$\sum_{i=1}^{n} w_i C_i(S) \le \sum_{i=1}^{n} w_i \sum_{j=1}^{i} V_j \le \sum_{i=1}^{n} w_i \sum_{j=1}^{i} L_j \le \sum_{i=1}^{n} w_i T \le T(\underline{w} + (n-1)\overline{w}).$$

Let $S^* = (J_{[1]}, J_{[2]}, \ldots, J_{[n]})$ be an optimal schedule. For S^* we have $C_{[i]}(S^*) \ge V_{[i]}\,(1 + \ln P_{\max})^a$ and so

$$\sum_{i=1}^{n} w_i C_i(S^*) \ge \sum_{i=1}^{n} w_{[i]} V_{[i]}\,(1 + \ln P_{\max})^a \ge \underline{w} \sum_{i=1}^{n} V_{[i]}\,(1 + \ln P_{\max})^a$$
$$= \underline{w} T\,(1 + \ln P_{\max})^a .$$

Consequently, we have

$$\frac{\sum_{j=1}^{n} w_j C_j(S)}{\sum_{j=1}^{n} w_j C_j(S^*)} \le \frac{1 + (n-1)(\overline{w}/\underline{w})}{(1 + \ln P_{\max})^a}.$$

We show that the bound $\frac{1+(n-1)(\overline{w}/\underline{w})}{(1+\ln P_{\max})^a}$ is tight. Consider the following instance. Learning takes place by the 100%-learning curve, i.e., $a = 0$ and $\frac{1+(n-1)(\overline{w}/\underline{w})}{(1+\ln P_{\max})^a} = 1 + (n-1)(\overline{w}/\underline{w})$. The bound $1 + (n-1)(\overline{w}/\underline{w})$ of any busy schedule for $Fm|prmu|\sum w_j C_j$ is tight as shown in Smutnicki [2] and therefore the bound for the problem $Fm|prmu, p_{ijr} = p_{ij}\left(1 + \sum_{l=1}^{r-1} \ln p_{i[l]}\right)^a |\sum w_j C_j$ is also tight. □

Theorem 6 *Let S^* (S) be an optimal (SPT) schedule for the $Fm|prmu$, $p_{ijr} = p_{ij}\left(1 + \sum_{l=1}^{r-1} \ln p_{i[l]}\right)^a |\sum C_j^2$ problem. Then $\frac{\sum_{j=1}^{n} C_j^2(S)}{\sum_{j=1}^{n} C_j^2(S^*)} \le \left(\frac{m}{(1+\ln P_{\max})^a}\right)^2$, and this bound is tight, where $\ln P_{\max} = \max\{\sum_{l=1}^{n} \ln p_{il} - \ln p_{i\min} | i = 1, 2, \ldots, m\}$ and $p_{i\min} = \min\{p_{ij} | j = 1, 2, \ldots, n\}$.*

Proof Similar to the proof of Theorem 1, we assume that $V_1 \le V_2 \le \cdots \le V_n$, we have $C_j(S) \le \sum_{l=1}^{j} V_l$ and so

$$\sum_{j=1}^{n} C_j^2(S) \le \sum_{j=1}^{n} \left(\sum_{l=1}^{j} V_l\right)^2 .$$

Let $S^* = (J_{[1]}, J_{[2]}, \ldots, J_{[n]})$, we have $C_{[j]}(S^*) \ge \frac{(1+\ln P_{\max})^a}{m} \sum_{l=1}^{j} V_{[j]}$ and so

$$\sum_{j=1}^{n} C_j^2(S^*) \ge \sum_{j=1}^{n} \left(\frac{(1 + \ln P_{\max})^a}{m} \sum_{l=1}^{j} V_{[l]}\right)^2$$

Table 1 Results for $C_{\max}$ and $m = 3$

n	a	$\frac{C_{\max}(SPT)}{C_{\max}(S^*)}$		$\frac{m}{(1+\ln P_{\max})^a}$		$\frac{C_{\max}(ARB)}{C_{\max}(S^*)}$		$\frac{m}{(1+\ln P_{\max})^a}$	
		Mean	Max.	Mean	Max.	Mean	Max.	Mean	Max.
	−0.15	1.063	1.221	5.021	5.135	1.562	1.991	5.021	5.135
	−0.25	1.087	1.231	7.075	7.348	1.331	1.846	7.075	7.348
8	−0.35	1.066	1.208	9.942	10.515	1.521	1.727	9.942	10.515
	−0.45	1.058	1.175	14.068	15.047	1.169	1.626	14.068	15.047
	−0.15	1.051	1.156	5.113	5.217	1.416	1.982	5.113	5.217
	−0.25	1.016	1.131	7.296	7.544	1.358	1.839	7.296	7.544
9	−0.35	1.043	1.164	10.412	10.911	1.354	1.857	10.412	10.911
	−0.45	1.073	1.206	14.812	15.777	1.550	1.935	14.812	15.777
	−0.15	1.071	1.217	5.197	5.292	1.391	1.914	5.197	5.292
	−0.25	1.061	1.171	7.496	7.726	1.553	1.963	7.496	7.726
10	−0.35	1.062	1.156	10.814	11.281	1.211	1.654	10.814	11.281
	−0.45	1.071	1.236	15.591	16.469	1.493	1.969	15.591	16.469
	−0.15	1.081	1.151	5.274	5.361	1.378	1.733	5.274	5.361
	−0.25	1.056	1.137	7.682	7.896	1.061	1.887	7.682	7.896
11	−0.35	1.023	1.271	11.190	11.629	1.246	1.484	11.190	11.629
	−0.45	1.049	1.088	16.299	17.127	1.206	1.916	16.299	17.127

Table 2 Results for $C_{\max}$ and $m = 5$

n	a	$\frac{C_{\max}(SPT)}{C_{\max}(S^*)}$		$\frac{m}{(1+\ln P_{\max})^a}$		$\frac{C_{\max}(ARB)}{C_{\max}(S^*)}$		$\frac{m}{(1+\ln P_{\max})^a}$	
		Mean	Max.	Mean	Max.	Mean	Max.	Mean	Max.
	−0.15	1.042	1.146	8.327	8.557	1.566	1.709	8.327	8.557
	−0.25	1.041	1.131	11.782	12.247	1.635	1.918	11.782	12.247
8	−0.35	1.075	1.246	16.632	17.524	1.302	1.817	16.632	17.524
	−0.45	1.044	1.191	23.446	25.078	1.682	1.933	23.446	25.078
	−0.15	1.055	1.108	8.522	8.695	1.167	1.995	8.522	8.695
	−0.25	1.048	1.086	12.121	12.572	1.589	1.896	12.121	12.572
9	−0.35	1.084	1.206	17.351	18.185	1.335	1.814	17.351	18.185
	−0.45	1.047	1.201	24.762	26.293	1.141	1.443	24.762	26.293
	−0.15	1.055	1.109	8.662	8.820	1.616	1.855	8.662	8.820
	−0.25	1.079	1.273	12.361	12.877	1.365	1.626	12.361	12.877
10	−0.35	1.091	1.163	18.021	18.797	1.374	1.964	18.021	18.797
	−0.45	1.054	1.104	25.994	27.441	1.349	1.675	25.994	27.441
	−0.15	1.049	1.085	8.745	8.934	1.261	1.467	8.745	8.934
	−0.25	1.089	1.113	12.677	13.166	1.276	1.636	12.677	13.166
11	−0.35	1.095	1.281	18.556	19.381	1.329	1.592	18.556	19.381
	−0.45	1.083	1.162	27.166	28.545	1.458	1.804	27.166	28.545

Table 3 Results for $\sum C_j$ and $m = 3$

n	a	$\frac{\sum C_j(SPT)}{\sum C_j(S^*)}$		$\frac{m}{(1+\ln P_{\max})^a}$		$\frac{\sum C_j(ARB)}{\sum C_j(S^*)}$		$\frac{n}{(1+\ln P_{\max})^a}$	
		Mean	Max.	Mean	Max.	Mean	Max.	Mean	Max.
	−0.15	1.041	1.143	5.021	5.135	1.211	2.414	13.363	13.694
	−0.25	1.058	1.203	7.075	7.348	1.176	2.044	18.876	19.598
8	−0.35	1.060	1.216	9.942	10.515	1.254	2.852	26.611	28.041
	−0.45	1.069	1.242	14.068	15.047	1.778	2.665	37.514	40.128
	−0.15	1.049	1.121	5.113	5.217	1.466	2.118	15.332	15.651
	−0.25	1.097	1.225	7.296	7.544	1.107	2.905	21.386	22.633
9	−0.35	1.076	1.218	10.412	10.911	1.517	2.924	31.237	32.725
	−0.45	1.077	1.316	14.812	15.777	1.271	2.768	44.573	47.282
	−0.15	1.057	1.135	5.197	5.292	1.106	2.096	17.318	17.641
	−0.25	1.072	1.216	7.496	7.726	1.310	2.552	24.723	25.815
10	−0.35	1.071	1.181	10.814	11.281	1.516	2.615	36.059	37.639
	−0.45	1.091	1.186	15.591	16.469	1.173	2.471	51.668	54.239
	−0.15	1.085	1.219	5.274	5.361	1.225	2.111	19.119	19.625
	−0.25	1.065	1.097	7.682	7.896	1.104	2.167	28.121	28.618
11	−0.35	1.058	1.112	11.190	11.629	1.366	2.646	41.031	42.641
	−0.45	1.091	1.213	16.299	17.127	1.508	2.022	59.164	62.623

Table 4 Results for $\sum C_j$ and $m = 5$

n	a	$\frac{\sum C_j(SPT)}{\sum C_j(S^*)}$		$\frac{m}{(1+\ln P_{\max})^a}$		$\frac{\sum C_j(ARB)}{\sum C_j(S^*)}$		$\frac{n}{(1+\ln P_{\max})^a}$	
		Mean	Max.	Mean	Max.	Mean	Max.	Mean	Max.
	−0.15	1.056	1.212	8.327	8.557	1.508	2.002	13.363	13.694
	−0.25	1.043	1.178	11.782	12.247	1.473	2.124	18.876	19.598
8	−0.35	1.041	1.159	16.632	17.524	1.459	2.142	26.611	28.041
	−0.45	1.042	1.121	23.446	25.078	1.596	2.828	37.514	40.128
	−0.15	1.037	1.079	8.522	8.695	1.665	2.575	15.332	15.651
	−0.25	1.072	1.165	12.121	12.572	1.425	2.126	21.386	22.633
9	−0.35	1.071	1.162	17.351	18.185	1.438	2.019	31.237	32.725
	−0.45	1.057	1.153	24.762	26.293	1.657	2.505	44.573	47.282
	−0.15	1.065	1.236	8.662	8.820	1.477	2.180	17.318	17.641
	−0.25	1.065	1.159	12.361	12.877	1.492	2.481	24.723	25.815
10	−0.35	1.059	1.136	18.021	18.797	1.773	2.036	36.059	37.639
	−0.45	1.069	1.154	25.994	27.441	1.442	2.133	51.668	54.239
	−0.15	1.085	1.118	8.745	8.934	1.544	2.914	19.119	19.625
	−0.25	1.072	1.117	12.677	13.166	1.311	2.497	28.121	28.618
11	−0.35	1.085	1.175	18.556	19.381	1.359	2.250	41.031	42.641
	−0.45	1.042	1.078	27.166	28.545	1.444	2.895	59.164	62.623

Table 5 Results for $\sum w_j C_j$ and $m = 3$

n	a	$\frac{\sum w_j C_j(WSPT)}{\sum w_j C_j(S^*)}$		$\frac{m}{(1+\ln P_{\max})^a}$		$\frac{\sum w_j C_j(ARB)}{\sum w_j C_j(S^*)}$		$\frac{1+(n-1)(\overline{w}/\underline{w})}{(1+\ln P_{\max})^a}$	
		Mean	Max.	Mean	Max.	Mean	Max.	Mean	Max.
	−0.15	1.115	1.729	5.021	5.135	1.816	2.793	121.801	294.589
	−0.25	1.135	1.469	7.075	7.348	1.621	2.313	173.913	431.110
8	−0.35	1.137	1.407	9.942	10.515	1.513	2.054	248.865	616.905
	−0.45	1.173	1.454	14.068	15.047	1.430	2.193	356.118	882.772
	−0.15	1.242	1.715	5.113	5.217	1.899	3.194	138.068	349.546
	−0.25	1.251	1.714	7.296	7.544	1.662	2.516	197.253	505.488
9	−0.35	1.326	2.668	10.412	10.911	1.915	2.578	281.246	730.995
	−0.45	1.264	2.047	14.812	15.777	1.345	2.697	401.812	1057.117
	−0.15	1.148	1.629	5.197	5.292	1.715	2.361	157.652	398.681
	−0.25	1.183	1.749	7.496	7.726	2.054	3.133	227.408	582.065
10	−0.35	1.283	2.077	10.814	11.281	1.923	2.774	328.033	849.803
	−0.45	1.151	1.526	15.591	16.469	1.745	2.671	473.173	1240.687
	−0.15	1.197	1.441	5.274	5.361	1.926	2.421	177.558	448.607
	−0.25	1.194	1.368	7.682	7.896	2.074	2.958	258.635	660.662
11	−0.35	1.105	1.359	11.190	11.629	1.744	2.287	376.734	972.989
	−0.45	1.313	1.861	16.299	17.127	1.917	2.867	548.759	1432.954

Table 6 Results for $\sum w_j C_j$ and $m = 5$

n	a	$\frac{\sum w_j C_j(WSPT)}{\sum w_j C_j(S^*)}$		$\frac{m}{(1+\ln P_{\max})^a}$		$\frac{\sum w_j C_j(ARB)}{\sum w_j C_j(S^*)}$		$\frac{1+(n-1)(\overline{w}/\underline{w})}{(1+\ln P_{\max})^a}$	
		Mean	Max.	Mean	Max.	Mean	Max.	Mean	Max.
	−0.15	1.171	1.737	8.327	8.557	1.695	2.487	120.818	296.591
	−0.25	1.164	1.634	11.782	12.247	1.443	2.351	175.925	432.152
8	−0.35	1.167	1.544	16.632	17.524	1.694	3.048	251.851	617.925
	−0.45	1.336	2.172	23.446	25.078	1.526	3.372	345.182	886.721
	−0.15	1.241	2.218	8.522	8.695	2.014	3.234	137.081	348.561
	−0.25	1.204	1.577	12.121	12.572	1.485	2.045	199.231	514.498
9	−0.35	1.183	1.521	17.351	18.185	1.548	2.234	280.261	731.954
	−0.45	1.128	1.959	24.762	26.293	1.549	2.534	405.825	1076.174
	−0.15	1.270	1.785	8.662	8.820	1.577	2.229	155.625	399.618
	−0.25	1.230	1.476	12.361	12.877	1.533	2.456	229.481	583.055
10	−0.35	1.331	2.171	18.021	18.797	1.654	2.017	321.031	850.831
	−0.45	1.258	2.119	25.994	27.441	1.774	3.016	470.134	1241.677
	−0.15	1.135	1.808	8.745	8.934	1.781	2.117	176.585	449.671
	−0.25	1.221	1.546	12.677	13.166	1.541	2.272	260.653	661.664
11	−0.35	1.267	1.602	18.556	19.381	1.574	1.769	377.743	973.991
	−0.45	1.146	1.501	27.166	28.545	1.801	2.424	549.796	1434.944

Table 7 Results for $\sum C_j^2$ and $m = 3$

n	a	$\frac{\sum C_j^2(SPT)}{\sum C_j^2(S^*)}$		$\left(\frac{m}{(1+\ln P_{\max})^a}\right)^2$		$\frac{\sum C_j^2(ARB)}{\sum C_j^2(S^*)}$		$\left(\frac{n}{(1+\ln P_{\max})^a}\right)^2$	
		Mean	Max.	Mean	Max.	Mean	Max.	Mean	Max.
	– 0.15	1.131	1.412	25.210	26.368	3.449	6.651	178.570	187.526
	– 0.25	1.119	1.452	50.056	53.993	3.124	5.819	356.303	384.082
8	– 0.35	1.108	1.601	98.843	110.565	2.578	5.733	708.145	786.298
	– 0.45	1.156	1.482	197.9086	226.412	1.346	3.853	1407.300	1610.256
	– 0.15	1.102	1.348	26.143	27.217	2.613	5.663	235.070	244.954
	– 0.25	1.103	1.319	53.232	56.912	2.002	8.448	457.361	512.253
9	– 0.35	1.221	1.831	108.410	119.050	2.396	4.514	975.750	1070.926
	– 0.45	1.112	1.345	219.395	248.914	2.656	6.326	1986.752	2235.588
	– 0.15	1.115	1.263	27.009	28.005	2.767	5.062	299.913	311.205
	– 0.25	1.116	1.421	56.190	59.691	3.196	9.987	611.227	666.414
10	– 0.35	1.035	1.251	116.942	127.261	3.039	8.109	1300.251	1416.694
	– 0.45	1.177	1.553	243.079	271.228	3.136	5.577	2669.582	2941.869
	– 0.15	1.115	1.202	27.815	28.740	3.219	9.718	365.536	385.141
	– 0.25	1.239	1.488	59.013	62.347	2.168	4.908	790.791	818.990
11	– 0.35	1.024	1.391	125.216	135.234	2.872	4.995	1683.543	1818.255
	– 0.45	1.081	1.155	265.657	293.334	2.446	4.226	3500.379	3921.640

$$\geq \left(\frac{(1+\ln P_{\max})^a}{m}\right)^2 \sum_{j=1}^{n}\left(\sum_{l=1}^{j} V_{[l]}\right)^2$$
$$\geq \left(\frac{(1+\ln P_{\max})^a}{m}\right)^2 \sum_{j=1}^{n}\left(\sum_{l=1}^{j} V_{l}\right)^2,$$

as the term $\sum_{j=1}^{n}\left(\sum_{l=1}^{j} V_l\right)^2$ is minimized by the increasing order of V_j (Lemma 5). Consequently, we have

$$\frac{\sum_{j=1}^{n} C_j^2(S)}{\sum_{j=1}^{n} C_j^2(S^*)} \leq \left(\frac{m}{(1+\ln P_{\max})^a}\right)^2.$$

We show that the bound $\left(\frac{m}{(1+\ln P_{\max})^a}\right)^2$ is tight. Consider the following instance. Learning takes place by the 100%—learning curve (a learning rate of 100% means that no learning is taking place), thus $a = 0$, that is, the bound $\left(\frac{m}{(1+\ln P_{\max})^a}\right)^2 = m^2$. The bound m^2 of the SPT rule for $Fm|prmu|\sum C_j^2$ is tight as shown in Koula-

Table 8 Results for $\sum C_j^2$ and $m = 5$

n	a	$\frac{\sum C_j^2(SPT)}{\sum C_j^2(S^*)}$		$\left(\frac{m}{(\alpha a^{P_{\max}}+\beta)b^{n-1}}\right)^2$		$\frac{\sum C_j^2(ARB)}{\sum C_j^2(S^*)}$		$\left(\frac{n}{(1+\ln P_{\max})^a}\right)^2$	
		Mean	Max.	Mean	Max.	Mean	Max.	Mean	Max.
	−0.15	1.053	1.153	69.339	73.222	2.059	4.623	178.570	187.526
	−0.25	1.112	1.530	138.816	149.989	2.238	5.181	356.303	384.082
8	−0.35	1.072	1.278	276.623	307.091	2.277	5.208	708.145	786.298
	−0.45	1.107	1.827	549.715	628.906	2.276	4.331	1407.300	1610.256
	−0.15	1.051	1.157	72.624	75.603	3.341	9.863	235.070	244.954
	−0.25	1.117	1.318	146.919	158.055	2.025	4.321	457.361	512.253
9	−0.35	1.023	1.307	301.057	330.694	2.648	4.605	975.750	1070.926
	−0.45	1.091	1.303	613.157	691.322	2.215	5.576	1986.752	2235.588
	−0.15	1.128	1.427	75.030	77.792	1.683	2.233	299.913	311.205
	−0.25	1.018	1.302	152.794	165.817	1.613	3.844	611.227	666.414
10	−0.35	1.119	1.433	324.756	353.327	2.191	4.857	1300.251	1416.694
	−0.45	1.153	1.531	675.688	753.009	2.451	5.673	2669.582	2941.869
	−0.15	1.061	1.181	76.475	79.816	2.823	5.865	365.536	385.141
	−0.25	1.133	1.367	160.706	173.344	1.204	2.219	790.791	818.990
11	−0.35	1.073	1.216	344.325	375.623	2.072	3.451	1683.543	1818.255
	−0.45	1.108	1.244	737.992	814.817	2.315	4.316	3500.379	3921.640

mas and Kyparisis [4] and therefore the bound for the problem $Fm|prmu, p_{ijr} = p_{ij}\left(1+\sum_{l=1}^{r-1}\ln p_{i[l]}\right)^a|\sum C_j^2$ is also tight. □

Theorem 7 *Let S^* (S) be an optimal (ARB rule) schedule for the problem $Fm|prmu, p_{ijr} = p_{ij}\left(1+\sum_{l=1}^{r-1}\ln p_{i[l]}\right)^a|\sum C_j^2$. Then $\frac{\sum_{j=1}^n C_j^2(S)}{\sum_{j=1}^n C_j^2(S^*)} \leq \left(\frac{n}{(1+\ln P_{\max})^a}\right)^2$, and this bound is tight, where $\ln P_{\max} = \max\{\sum_{l=1}^n \ln p_{il} - \ln p_{i\min}|i = 1, 2, \ldots, m\}$ and $p_{i\min} = \min\{p_{ij}|j = 1, 2, \ldots, n\}$.*

Proof Similar to the proof of Theorems 2 and 6. □

4 Computational study

Computational experiments were employed here to evaluate the performance of the heuristic algorithms, and were coded in VC++ 6.0 and tested on a Pentium 4 with 2 GB RAM personal computer. The parameters of the test problems were generated as follows:

1. p_{ij} (w_j) were generated from a uniform distribution over [3, 100] ([1, 50]);
2. For small-sized instances n=8,9,10,11, $m = 3, 5$;
3. $a = -0.15, -0.25, -0.35, -0.45$;.

For small-sized instances of the studied problems, the percentage error of the solution produced by the heuristic algorithm is calculated as $\frac{F(Heur)}{F(Opt)}$, where $F \in \{C_{\max}, \sum_{j=1}^{n} C_j, \sum_{j=1}^{n} w_j C_j, \sum_{j=1}^{n} C_j^2\}$, Heur $\in$ {ARB, SPT, WSPT}, $F(Heur)$ is the objective value of the solution generated by the heuristic *Heur* and $F(Opt)$ is the objective value for the optimal solution (can be obtained by an enumerative algorithm). The experiments are run for each problem size, and 20 instances were randomly generated. The results are shown in Tables 1, 2, 3, 4, 5, 6, 7 and 8. From Tables 1, 2, 3, 4, 7 and 8, it can be seen that the performance of the SPT rule is more effective than the ARB rule for $C_{\max}$, $\sum C_j$ and $\sum C_j^2$. From Tables 5 and 6, it can be seen that the performance of the WSPT rule is more effective than the ARB rule for $\sum w_j C_j$.

5 Conclusion

In this paper, we studied flow shop scheduling with sum-of-logarithm-processing-times-based learning effects. We developed heuristic algorithms with tight worst-case bound for the flow shop scheduling with four regular objective functions, numerical experiments demonstrate the effectiveness of the heuristic algorithms. In addition, future research may focus on proposing more sophisticated and efficient heuristics, considering two-stage assembly flow shop (Wu et al. [33], and Wu et al. [34]), studying the other learning effect models (Wang et al. [35], Niu et al. [36], and Yin [37]), or addressing deterioration effects problems (Fan and Zhao [38], Li and Zhao [39], Wang et al. [40], Wang and Zhao [41], and Huang [42]).

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