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ORIGINAL RESEARCH

Singularly perturbed delay differential equations of convection–diffusion type with integral boundary condition

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Abstract In this paper we consider a class of singularly perturbed delay differential equations of convection diffusion type with integral boundary condition. A finite difference scheme with an appropriate piecewise Shishkin type mesh is suggested to solve the problem. We prove that the method is of almost first order convergent. An error estimate is derived in the discrete norm. Numerical experiments support our theoretical results.

Keywords Singularly perturbed problems · Delay differential equation · Finite difference scheme · Shishkin mesh · Integral boundary condition

Mathematics Subject Classification 65L11 · 65L12 · 65L20

1 Introduction

Differential Equations with integral boundary conditions have plenty of applications. A Parabolic equation with nonlocal boundary conditions arising from electro chemistry is well studied by Choi and Chan [8]. In [10], Day have discussed Parabolic equations and thermodynamics. Cannon [6] have worked for the solution of the heat equation subject to the specification of energy. etc. The authors of [4,12,19] have proved that the problem of differential equations with integral boundary conditions is well posed. The authors of [1,5,7,17,26] have developed various numerical schemes on uniform

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meshes for singularly perturbed first and second order differential equations with integral boundary conditions.

A differential equation is said to be singularly perturbed delay differential equation, if it includes at least one delay term, involving unknown functions occuring with various different arguments and also the highest derivative term is multiplied by a small parameter. Such type of delay differential equations play a very important role in the mathematical models of science and engineering, such as, the human pupil-light reflex with mixed delay type [20], variational problems in control theory with small state problem [13], models of HIV infection [9] and signal transition [11], etc.

The standard numerical methods used for solving singularly perturbed differential equation are some time ill posed and fail to give analytical solution when the perturbation parameter ε is small. Therefore, it is necessary to improve suitable numerical methods which are uniformly convergent to solve this type of differential equations. Many authors have worked on singularly perturbed differential equations with small and large delay using uniformly convergent numerical methods. In [18], Lange and Miura have discussed singularly perturbed linear second order differential-difference equations with small delay. In [14–16,21,23–25] finite difference and finite element methods are proposed to solve this kind of equations with large and small shifts.

In the present paper, motivated by the works of [1–3], we analyze a fitted finite difference scheme on a piecewise uniform mesh for the numerical solution of second order singularly perturbed convection diffusion equations with negative shift and integral boundary condition.

The present paper is arranged as follows. Statement of the problem is given in Sect. 2. In Sect. 3, maximum principle, stability result and appropriate bounds for the derivatives of the solution of the problem are presented. Section 4 describes the numerical method. Error analysis for approximate solution is given Sect. 5. Numerical results are given in Sect. 6. Conclusion are given in Sect. 7.

Throughout our analysis C and C_1 are generic positive constants that are independent of the parameter ε and number of mesh points $2N$. We assume that $\varepsilon \leq CN^{-1}$, $\overline{\Omega} = [0, 2]$, $\Omega = (0, 2)$, $\Omega_1 = (0, 1)$ and $\Omega_2 = (1, 2)$. Further, $\Omega^* = \Omega_1 \cup \Omega_2$, $\overline{\Omega}^{2N}$ is denoted by $\{0, 1, 2, \ldots, 2N\}$, Ω_1^{2N} is denoted by $\{1, 2, \ldots, N-1\}$, Ω_2^{2N} is denoted by $\{N+1, N+2, \ldots, 2N-1\}$. The supremum norm used for studying the convergence of the numerical solution to the exact solution of a singular perturbation problem is

$$\|u\|_{\Omega} = \sup_{x \in \Omega} |u|.$$

2 Statement of the problem

We consider the following singularly perturbed delay differential equation with integral boundary condition:

$$\mathcal{L}u(x) = -\varepsilon u''(x) + a(x)u'(x) + b(x)u(x) + c(x)u(x-1) = f(x), \quad x \in \Omega, \tag{1}$$

$$u(x) = \phi(x), \quad x \in [-1, 0], \tag{2}$$

$$\mathcal{K}u(2) = u(2) - \varepsilon \int_0^2 g(x)u(x)dx = l, \tag{3}$$

where $\phi(x)$ is sufficiently smooth on $[-1, 0]$. For all $x \in \overline{\Omega}$, it is assumed that the sufficient smooth functions $a(x), b(x)$ and $c(x)$ satisfy $a(x) > \alpha_1 > \alpha > 0, b(x) \geq \beta \geq 0, c(x) \leq \gamma \leq 0, \alpha + \beta + \gamma > 0$ and $\beta + \gamma \geq 0$.

Furthermore, $g(x)$ is non negative and monotonic with $\int_0^2 g(x)dx < 1$.

The above assumptions ensure that $u \in X = C^0(\overline{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega_1 \cup \Omega_2)$.

The problem (1)–(3) is equivalent to

$$\mathcal{L}u(x) = F(x),$$

where

$$\mathcal{L}u(x) = \begin{cases} \mathcal{L}_1 u(x) = -\varepsilon u''(x) + a(x)u'(x) + b(x)u(x), & x \in \Omega_1, \\ \mathcal{L}_2 u(x) = -\varepsilon u''(x) & \\ \qquad + a(x)u'(x) + b(x)u(x) + c(x)u(x-1), & x \in \Omega_2 \end{cases} \tag{4}$$

$$F(x) = \begin{cases} f(x) - c(x)\phi(x-1), & x \in \Omega_1, \\ f(x), & x \in \Omega_2, \end{cases} \tag{5}$$

with boundary conditions

$$u(x) = \phi(x), \quad x \in [-1, 0], \tag{6}$$

$$u(1^-) = u(1^+), \quad u'(1^-) = u'(1^+), \tag{7}$$

$$\mathcal{K}u(2) = u(2) - \varepsilon \int_0^2 g(x)u(x)dx = l. \tag{8}$$

3 Stability result

Lemma 1 (Maximum Principle) *Let $\psi(x)$ be any function in X such that $\psi(0) \geq 0, \mathcal{K}\psi(2) \geq 0, \mathcal{L}_1\psi(x) \geq 0, \forall x \in \Omega_1, \mathcal{L}_2\psi(x) \geq 0, \forall x \in \Omega_2$, and $[\psi'](1) \leq 0$ then $\psi(x) \geq 0, \forall x \in \overline{\Omega}$.*

Proof Define a test function

$$s(x) = \begin{cases} \frac{1}{8} + \frac{x}{2}, & x \in [0, 1] \\ \frac{3}{8} + \frac{x}{4}, & x \in [1, 2] \end{cases} \tag{9}$$

Note that $s(x) > 0, \forall x \in \overline{\Omega}$, $\mathcal{L}s(x) > 0, \forall x \in \Omega_1 \cup \Omega_2$, $s(0) > 0$, $\mathcal{K}s(2) > 0$ and $[s'](1) < 0$. Let

$$\mu = \max\left\{\frac{-\psi(x)}{s(x)} : x \in \overline{\Omega}\right\}.$$

Then there exists $x_0 \in \overline{\Omega}$ such that $\psi(x_0)+\mu s(x_0) = 0$ and $\psi(x)+\mu s(x) \geq 0, \forall x \in \overline{\Omega}$. Therefore, the function $(\psi + \mu s)$ attains its minimum at $x = x_0$. Suppose the theorem does not hold true, then $\mu > 0$.

Case (i) $x_0 = 0$

$$0 < (\psi + \mu s)(0) = \psi(0) + \mu s(0) = 0$$

It is a contradiction.

Case (ii) $x_0 \in \Omega_1$

$$\begin{aligned} 0 < \mathcal{L}(\psi + \mu s)(x_0) &= -\varepsilon(\psi + \mu s)''(x_0) + a(x_0)(\psi + \mu s)'(x_0) \\ &\quad + b(x_0)(\psi + \mu s)(x_0) \leq 0 \end{aligned}$$

It is a contradiction.

Case (iii) $x_0 = 1$

$$0 \leq [(\psi + \mu s)'](1) = [\psi'](1) + \mu[s'](1) < 0$$

It is a contradiction.

Case (iv) $x_0 \in \Omega_2$

$$\begin{aligned} 0 < \mathcal{L}(\psi + \mu s)(x_0) &= -\varepsilon(\psi + \mu s)''(x_0) + a(x_0)(\psi + \mu s)'(x_0) \\ &\quad + b(x_0)(\psi + \mu s)(x_0) + c(x_0)(\psi + \mu s)(x_0 - 1) \leq 0 \end{aligned}$$

It is a contradiction.

Case (v) $x_0 = 2$

$$0 < \mathcal{K}(\psi + \mu s)(2) = (\psi + \mu s)(2) - \varepsilon \int_0^2 g(x)(\psi + \mu s)(x)dx \leq 0.$$

It is a contradiction. Hence the proof of the theorem. □

Lemma 2 (Stability Result) *The solution $u(x)$ of the problem* (1)–(3)*, satisfies the bound*

$$|u(x)| \leq C \max\left\{|u(0)|, |\mathcal{K}u(2)|, \sup_{x\in\Omega^*} |\mathcal{L}u(x)|\right\}, \quad x \in \overline{\Omega}$$

Proof This theorem can be proved by using Lemma 1 and the barrier functions $\theta^{\pm}(x) = CMs(x) \pm u(x),\ x \in \overline{\Omega}$, where $M = \max\left\{|u(0)|, |\mathcal{K}u(2)|, \sup_{x\in\Omega^- \cup \Omega^+} |\mathcal{L}u(x)|\right\}$ and $s(x)$ is the test function as in Lemma 1. □

Lemma 3 *Let $u(x)$ be the solution of (1)–(3). Then we have the following bounds:*

$$\|u^{(k)}(x)\|_{\Omega^*} \leq C\varepsilon^{-k}, \quad for \quad k = 1, 2, 3$$

Proof To bound $u'(x)$ on the interval Ω_1, we consider,

$$\mathcal{L}_1 u(x) = -\varepsilon u''(x) + a(x)u'(x) + b(x)u(x) = f(x).$$

Integrating the above equation on both sides,we have

$$-\varepsilon(u'(x) - u'(0)) = -[a(x)u(x) - a(0)u(0)] + \int_0^x a'(t)u(t)dt - \int_0^x b(t)u(t)dt + \int_0^x [f(t) - c(t)\phi(t-1)]dt$$

Therefore,

$$\varepsilon u'(0) = \varepsilon u'(x) - [a(x)u(x) - a(0)u(0)] + \int_0^x a'(t)u(t)dt - \int_0^x b(t)u(t)dt + \int_0^x [f(t) - c(t)\phi(t-1)]dt$$

Then by the Mean value theorem, there exits $z \in (0, \varepsilon)$ such that $|\varepsilon u'(z)| \leq C(\|u(x)\|, \|f(x)\|, \|\phi\|_{[-1,0]})$ and $|\varepsilon u'(0)| \leq C(\|u(x)\| + \|f(x)\| + \|\phi(x)\|)$.

Hence,

$$|\varepsilon u'(x)| \leq C \max(\|u(x)\|, \|f(x)\|, \|\phi\|).$$

By a similar argument we can bound $u'(x)$ on Ω_2, as $|\varepsilon u'(x)| \leq C$. From (4) and (5) we have $\|u^{(k)}(x)\|_{\Omega^*} \leq C\varepsilon^{-k}$, $k = 2, 3$. Hence the proof. □

3.1 Decomposition of the solution

The Shishkin decomposition of the solution $u(x)$ of (1)–(3) is $u(x) = v(x) + w(x)$, where $v(x)$ and $w(x)$ are regular and singular components respectively. Also $v(x) =$

$v_0(x)+\varepsilon v_1(x)+\varepsilon^2 v_2(x)$ where $v_0(x)$, $v_1(x)$ and $v_2(x)$ are solutions of the following problems respectively:

$$\begin{aligned} &a(x)\frac{dv_0(x)}{dx}+b(x)v_0(x)+c(x)v_0(x-1)=f(x), x\in\Omega\cup\{2\}\\ &v_0(x)=\phi(x),\quad x\in[-1,0] \end{aligned} \tag{10}$$

Find $v_1(x)\in C^0(\overline{\Omega})\cap C^1(\{0\}\cup\Omega^*)$ such that

$$\begin{aligned} &a(x)\frac{dv_1(x)}{dx}+b(x)v_1(x)+c(x)v_1(x-1)=v_0''(x),\quad x\in\Omega^*\cup\{2\}\\ &v_0(x)=0,\quad x\in[-1,0] \end{aligned} \tag{11}$$

Find $v_2(x)\in X$ such that

$$\begin{aligned} &-\varepsilon\frac{d^2v_2(x)}{dx^2}+a(x)\frac{dv_2(x)}{dx}+b(x)v_2(x)+c(x)v_2(x-1)=v_1''(x),\ x\in\Omega^*\\ &v_2(x)=0,\quad x\in[-1,0],\quad \mathcal{K}v_2(2)=0, \end{aligned} \tag{12}$$

The smooth component $v(x)$ satisfies the following problem:
Find $v(x)\in C^0(\overline{\Omega})\cap C^2(\Omega^*)$, such that

$$\begin{aligned} &\mathcal{L}v(x)=-\varepsilon\frac{d^2v(x)}{dx^2}+a(x)\frac{dv(x)}{dx}+b(x)v(x)+c(x)v(x-1)=f(x),\ x\in\Omega^*\\ &v(x)=\phi(x), x\in[-1,0],\quad v(1)=v_0(1)+\varepsilon v_1(1)+\varepsilon^2 v_2(1),\\ &\mathcal{K}v(2)=\mathcal{K}v_0(2)+\varepsilon\mathcal{K}v_1(2), \end{aligned} \tag{13}$$

Further $w(x)$ satisfies the following problem:
Find $w(x)\in C^0(\overline{\Omega})\cap C^2(\Omega^*)$ such that

$$\begin{aligned} &\mathcal{L}w(x)=-\varepsilon\frac{d^2w(x)}{dx^2}+a(x)\frac{dw(x)}{dx}+b(x)w(x)+c(x)w(x-1)=0,\ x\in\Omega^*\\ &w(x)=0,\quad x\in[-1,0],\quad [w'](1)=-[v'](1),\quad \mathcal{K}w(2)=\mathcal{K}u(2)-\mathcal{K}v(2). \end{aligned} \tag{14}$$

We further decompose $w(x)$ as $w(x)=w_B(x)+w_I(x)$,where the function $w_B(x)$ is boundary layer component and $w_I(x)$ is interior layer component, which are the solution of the following problems respectively:
Find $w_B(x)\in X$ such that

$$\begin{aligned} &\mathcal{L}w_B(x)=-\varepsilon\frac{d^2w_B(x)}{dx^2}+a(x)\frac{dw_B(x)}{dx}+b(x)w_B(x)+c(x)w_B(x-1)=0,\\ &w_B(x)=0,\quad x\in[-1,0],\quad \mathcal{K}w_B(2)=\mathcal{K}u(2)-\mathcal{K}v(2). \end{aligned} \tag{15}$$

Find $w_I(x) \in C^0(\overline{\Omega}) \cap C^2(\Omega^*)$ such that

$$\mathcal{L}w_I(x) = -\varepsilon \frac{d^2 w_I(x)}{dx^2} + a(x)\frac{dw_I(x)}{dx} + b(x)w_I(x) + c(x)w_I(x-1) = 0,$$
$$w_I(x) = 0, \quad x \in [-1,0], \quad [w_I'](1) = -[v'](1), \quad \mathcal{K}w_I(2) = 0. \tag{16}$$

Theorem 1 *Let $u(x)$ be the solution of the problem (1)–(3) and $v_0(x)$ be the reduced problem solution defined by (10). Then,*

$$|u(x) - v_0(x)| \le C\left(\varepsilon + exp\left(\frac{-\alpha(2-x)}{\varepsilon}\right)\right), \quad x \in \overline{\Omega}$$

Proof Consider the barrier functions

$$\theta^{\pm}(x) = C\left(\varepsilon s(x) + exp\left(\frac{-\alpha(2-x)}{\varepsilon}\right)\right) \pm (u(x) - v_0(x)), \quad x \in \overline{\Omega}.$$

Clearly $\theta^{\pm}(x) \in C^0(\overline{\Omega}) \cap C^2(\Omega_1 \cup \Omega_2)$. Note that, $\theta^{\pm}(0) \ge 0$,

$$\begin{aligned}\mathcal{K}\theta^{\pm}(2) &= \theta^{\pm}(2) - \varepsilon\int_0^2 g(x)\theta^{\pm}(x)dx \\ &= C\varepsilon\left[s(2) - \varepsilon\int_0^2 g(x)s(x)dx\right] \\ &\quad + C\left[1 - \varepsilon\int_0^2 g(x)exp\left(\frac{-\alpha(2-x)}{\varepsilon}\right)dx\right] \pm \mathcal{K}u(2) \\ &\ge C\varepsilon\mathcal{K}s(2) + C\left[1 - \varepsilon\int_0^2 g(x)dx\right] \pm \mathcal{K}u(2) \\ &\ge C\varepsilon\mathcal{K}s(2) \pm \mathcal{K}u(2) \ge 0.\end{aligned}$$

for a suitable choice of $C > 0$.

Let $x \in \Omega_1$. Then

$$\begin{aligned}\mathcal{L}_1\theta^{\pm}(x) = C\Bigg[&\left(\frac{\alpha}{\varepsilon}(a(x) - \alpha) + b(x)\right)exp\left(\frac{-\alpha(2-x)}{\varepsilon}\right) \\ &+ \varepsilon(a(x) + b(x)s(x))\Bigg] \pm \varepsilon v_0''(x), \\ \ge C&\left[\left(\frac{\alpha}{\varepsilon}(\alpha_1 - \alpha) + \beta\right)exp\left(\frac{-\alpha(2-x)}{\varepsilon}\right) + \varepsilon\left(\alpha + \frac{\beta}{8}\right)\right] \pm C\varepsilon, \\ \ge 0&.\end{aligned}$$

for a suitable choice of $C > 0$.

Let $x \in \Omega_2$. Then

$$\begin{aligned}\mathcal{L}_2\theta^{\pm}(x) &= C\left[\left(\frac{\alpha}{\varepsilon}(a(x)-\alpha)+b(x)+c(x)exp\left(-\frac{\alpha}{\varepsilon}\right)\right)exp\left(\frac{-\alpha(2-x)}{\varepsilon}\right)\right.\\ &\quad +\varepsilon(a(x)+b(x)s(x)+c(x)s(x-1)]\pm\varepsilon v_0''(x),\\ &\geq C\left[\left(\frac{\alpha}{\varepsilon}(\alpha_1-\alpha)+\beta+\gamma exp\left(-\frac{\alpha}{\varepsilon}\right)\right)exp\left(\frac{-\alpha(2-x)}{\varepsilon}\right)\right.\\ &\quad \left.+\varepsilon\left(\alpha+\frac{\beta}{8}+\frac{\gamma}{8}\right)\right]\pm C\varepsilon,\\ &\geq 0.\end{aligned}$$

for a suitable choice of $C>0$.

Then by the Lemma 1,we have $\theta^{\pm}(x)\geq 0$, $x\in\overline{\Omega}$. Hence the proof of the theorem. □

Lemma 4 *The regular component $v(x)$ and the singular component $w(x)$ of the solution $u(x)$ satisfy the following bounds.*

$$\|v^k(x)\|_{\Omega^*}\leq C(1+\varepsilon^{2-k}),\quad for\ k=0,1,2,3 \tag{17}$$

$$|w_B^k(x)|\leq C\varepsilon^{-k}exp\left(\frac{-\alpha(2-x)}{\varepsilon}\right),\quad x\in\Omega^*,\quad k=0,1,2,3 \tag{18}$$

$$|w_I^k(x)|\leq C\begin{cases}\varepsilon^{1-k}exp(\frac{-\alpha(1-x)}{\varepsilon}), & x\in\Omega_1,\\ \varepsilon^{1-k}, & x\in\Omega_2,\end{cases}\quad k=0,1,2,3 \tag{19}$$

Proof Integrating (10) and (12) and using the stability result, the inequality (17) can be proved easily.

To prove the inequalities (18), consider the barrier functions

$$\Phi^{\pm}(x)=C_1\left(exp\left(\frac{-\alpha(2-x)}{\varepsilon}\right)\right)\pm w_B(x),\quad x\in\overline{\Omega}.$$

It easy to see that $\Phi^{\pm}(0)\geq 0$.

Further,

$$\begin{aligned}\mathcal{K}\Phi^{\pm}(2) &= \Phi^{\pm}(2)-\varepsilon\int_0^2 g(x)\Phi^{\pm}(x)dx\\ &= C\left[1-\varepsilon\int_0^2 g(x)exp\left(\frac{-\alpha(2-x)}{\varepsilon}\right)dx\right]\pm\mathcal{K}w_B(2)\\ &\geq 0\end{aligned}$$

and also

$$\begin{aligned}\mathcal{L}\Phi^{\pm}(x) &= C_1\left[\frac{\alpha}{\varepsilon}(a(x)-\alpha)+b(x)+c(x)exp\left(-\frac{\alpha}{\varepsilon}\right)\right]\\ &\quad\times exp\left(\frac{-\alpha(2-x)}{\varepsilon}\right)\pm\ \mathcal{L}w_B(x)\\ &\geq C_1\left[\frac{\alpha}{\varepsilon}(\alpha_1-\alpha)+\beta+\gamma exp\left(-\frac{\alpha}{\varepsilon}\right)\right]exp\left(\frac{-\alpha(2-x)}{\varepsilon}\right)\pm\ 0\\ &\geq 0.\end{aligned}$$

By the Lemma 1,

$$|w_B(x)|\leq Cexp\left(\frac{-\alpha(2-x)}{\varepsilon}\right).$$

Integration of (15) yields the estimates of $|w'_B(x)|$. From the differential equations (14), one can derive the rest of the derivative estimates (18).

To prove the inequalities (19), consider the barrier functions

$$\Phi^{\pm}(x)=C_1\varepsilon\left(\exp\left(\frac{-\alpha(1-x)}{\varepsilon}\right)\right)\pm w_I(x),\quad x\in[0,1].$$

Clearly, $\Phi^{\pm}(0)\geq 0$ and $\Phi^{\pm}(1)\geq 0$ and also $\mathcal{L}_1\Phi(x_i)\geq 0$, easily proves first inequality.

Similarly, consider the following barrier function

$$\Phi^{\pm}(x)=C_1x\varepsilon\pm w_I(x),\quad x\in[1,2].$$

Note that $\Phi^{\pm}(1)\geq 0$, and

$$\begin{aligned}\mathcal{K}\Phi^{\pm}(2) &= \Phi^{\pm}(2)-\varepsilon\int_0^2 g(x)\Phi^{\pm}(x)dx\\ &= C\varepsilon\left[2-\varepsilon\int_0^2 xg(x)dx\right]\pm\mathcal{K}w_I(2)\\ &= 2C\varepsilon\left[1-\varepsilon\int_0^2 g(x)dx\right]\pm\ 0\\ &\geq 0\\ \mathcal{L}_2\Phi^{\pm}(x) &= -\varepsilon(\Phi^{\pm})''(x)+a(x)(\Phi^{\pm})'+b(x)\Phi^{\pm}(x)+c(x)\Phi^{\pm}(x-1)\geq 0\end{aligned}$$

Hence the proof. □

Note: From the above theorem, it is not difficult to prove

$$|u(x) - v(x)| \le C \begin{cases} \varepsilon\, exp(\frac{-\alpha(1-x)}{\varepsilon}) + exp(\frac{-\alpha(2-x)}{\varepsilon}), & x \in \Omega_1 \\ \varepsilon + exp(\frac{-\alpha(2-x)}{\varepsilon}), & x \in \Omega_2. \end{cases} \tag{20}$$

4 The discrete problem

4.1 Mesh selection procedure

The BVP (1)–(3) exhibits strong boundary layer at $x = 2$ and interior layer at $x = 1$.

The interval [0, 1] is partitioned into $[0, 1-\sigma]$ and $[1-\sigma, 1]$ and the interval [1, 2] is partitioned as $[1, 2-\sigma]$ and $[2-\sigma, 2]$, where σ is transition parameter for this mesh defined by

$$\sigma = \min\left\{\frac{1}{2}, 2\frac{\varepsilon}{\alpha}\ln N\right\}.$$

The mesh $\bar{\Omega}^{2N} = \{x_0, x_1, \ldots, x_{2N}\}$ is defined by

$$\begin{aligned} &x_0 = 0, \\ &x_i = x_0 + iH, i = 1to\tfrac{N}{2} \\ &x_{i+\frac{N}{2}} = x_{\frac{N}{2}} + ih, i = 1to\tfrac{N}{2} \\ &x_{i+N} = x_N + iH, i = 1to\tfrac{N}{2} \\ &x_{i+\frac{3N}{2}} = x_{\frac{3N}{2}} + ih, i = 1to\tfrac{N}{2} \end{aligned}$$

where $h = \frac{2\sigma}{N}$, $H = \frac{2(1-\sigma)}{N}$.

4.2 Discrete problem

The discrete scheme corresponding to the original problem (1)–(3) is as follows:
For i = 1, 2, …, $N - 1$,

$$\mathcal{L}_1^N U_i = f_i - b_i\phi_{i-N}, \tag{21}$$

For i = $N + 1, \ldots, 2N - 1$,

$$\mathcal{L}_2^N U_i = f_i, \tag{22}$$

subject to the boundary conditions:

$$U_i = \phi_i,\ i = -N, -N+1, \ldots, 0 \tag{23}$$

$$\mathcal{K}^N U_{2N} = U_{2N} - \sum_{i=1}^{2N} \frac{g_{i-1}Ui - 1 + g_i U_i}{2} h_i \tag{24}$$

and

$$D^{-}U_N = D^{+}U_N \tag{25}$$

where

$$\mathcal{L}_1^N U_i = -\varepsilon\delta^2 U(x_i) + a(x_i)D^{-}U(x_i) + b(x_i)U(x_i)$$
$$\mathcal{L}_2^N U_i = -\varepsilon\delta^2 U(x_i) + a(x_i)D^{-}U(x_i) + b(x_i)U(x_i) + c(x_i)U(x_{i-N})$$

Lemma 5 (Discrete Maximum Principle) *Assume that*

$$\sum_{i=1}^{2N} \frac{g_{i-1}+g_i}{2} h_i = \rho < 1$$

and mesh function $\Psi(x_i)$ *satisfies* $\Psi(x_0) \geq 0$, *and* $\mathcal{K}^N\Psi(x_{2N}) \geq 0$, *Then* $\mathcal{L}_1^N\Psi(x_i) \geq 0$, $\forall\, x_i \in \Omega_1^{2N}$, $\mathcal{L}_2^N\Psi(x_i) \geq 0$, $\forall\, x_i \in \Omega_2^{2N}$ *and* $D^{+}(\Psi(x_N)) - D^{-}(\Psi(x_N)) \leq 0$ *imply that* $\Psi(x_i) \geq 0$, $\forall\, x_i \in \overline{\Omega}^{2N}$.

Proof

$$\text{Define } s(x_i) = \begin{cases} \frac{1}{8} + \frac{x_i}{2}, & x_i \in [0,1] \cap \overline{\Omega}^{2N}, \\ \frac{3}{8} + \frac{x_i}{4}, & x_i \in [1,2] \cap \overline{\Omega}^{2N}, \end{cases}$$

Note that $s(x_i) > 0, \forall x_i \in \overline{\Omega}^{2N}$, $\mathcal{L}s(x_i) > 0, \forall x_i \in \Omega_1^{2N} \cup \Omega_2^{2N}$, $s(0) > 0$, $\mathcal{K}s(x_{2N}) > 0$ and $[s'](x_N) < 0$. Let

$$\mu = \max\left\{ \frac{-\psi(x_i)}{s(x_i)} : x_i \in \overline{\Omega}^{2N} \right\}.$$

Then there exists $x_k \in \overline{\Omega}^{2N}$ such that $\psi(x_k) + \mu s(x_k) = 0$ and $\psi(x_i) + \mu s(x_i) \geq 0, \forall x_i \in \overline{\Omega}^{2N}$. Therefore, the function $(\psi + \mu s)$ attains its minimum at $x = x_k$. Suppose the theorem does not hold true, then $\mu > 0$.
Case (i): $x_k = x_0$

$$0 < (\psi + \mu s)(x_0) = 0$$

It is a contradiction.
Case (ii): $x_k \in \Omega_1^{2N}$

$$0 < \mathcal{L}_1^N(\psi + \mu s)(x_k) \leq 0$$

It is a contradiction.

Case (iii): $x_k = x_N$

$$0 \leq [D(\psi + \mu s)](x_N) < 0$$

It is a contradiction.
Case (iv): $x_k \in \Omega_2^{2N}$

$$0 < \mathcal{L}_2^N(\psi + \mu s)(x_k) \leq 0$$

It is a contradiction.
Case (v): $x_k = x_{2N}$

$$\begin{aligned} 0 < \mathcal{K}^N(\psi + \mu s)x_{2N} &= (\psi + \mu s)x_{2N} \\ &- \sum_{i=1}^{2N} \frac{g_{i-1}(\psi + \mu s)x_{i-1} + g_i(\psi + \mu s)x_i}{2} h_i \leq 0 \end{aligned}$$

It is a contradiction. Hence the proof of the theorem. □

Lemma 6 *Let $\Psi(x_i)$ be any mesh function then for $0 \leq i \leq 2N$,*

$$|\Psi(x_i)| \leq C \max\left\{|\Psi(x_0)|, |\mathcal{K}^N\Psi(x_{2N})|, \max_{i \in \Omega_1^{2N} \cup \Omega_2^{2N}} |\mathcal{L}^N\Psi(x_i)|\right\}.$$

Proof Consider the barrier functions

$$\theta^{\pm}(x_i) = CM \pm \Psi(x_i), \quad \forall\, x_i \in \overline{\Omega}^{2N}. \tag{26}$$

where

$$M = \max\{|\Psi(x_0)|, |\mathcal{K}^N\Psi(x_{2N})|, \max_{i \in \Omega_1^{2N} \cup \Omega_2^{2N}} |\mathcal{L}^N\Psi(x_i)|\}.$$

From Eq. (26) it is clear that $\theta^{\pm}(x_0) \geq 0$ and $\mathcal{K}^N\theta^{\pm}(x_{2N}) \geq 0$,

$$\begin{aligned} &\mathcal{L}_1^N\theta^{\pm}(x_i) \geq 0, \quad \forall\, x_i \in \Omega_1^{2N}, \\ &\mathcal{L}_2^N\theta^{\pm}(x_i) \geq 0, \quad \forall\, x_i \in \Omega_2^{2N}, \\ &D^+\theta^{\pm}(x_N) - D^-\theta^{\pm}(x_N) \leq 0. \end{aligned}$$

Using Lemma 5, $\theta^{\pm}(x_i) \geq 0, \quad \forall\, x_i \in \overline{\Omega}^{2N}$. □

5 Error estimate

To calculate the error estimate for the numerical solution, we decompose the discrete solution $U(x_i)$ into $V(x_i)$ and $W(x_i)$ as $U(x_i) = V(x_i) + W(x_i)$, where $V(x_i)$ and $W(x_i)$ satisfy the following differential equations respectively.

$$\begin{aligned}\mathcal{L}^N V(x_i) &= -\varepsilon\delta^2 V(x_i) + a(x_i)D^- V(x_i) + b(x_i)V(x_i)\\ &\quad + c(x_i)V(x_i - x_N) = f_i, \quad i \in I_{2N\setminus\{0,N,2N\}}\\ V(x_0) &= v(0),\ [D]V(x_N) = [v'](1),\ \mathcal{K}V(x_{2N}) = \mathcal{K}v(2). \qquad (27)\\ \mathcal{L}^N W(x_i) &= -\varepsilon\delta^2 W(x_i) + a(x_i)D^- W(x_i) + b(x_i)W(x_i)\\ &\quad + c(x_i)W(x_i - x_N) = f_i, \quad i \in I_{N\setminus\{0,N,2N\}}\\ W(x_0) &= w(0),\ [D]W(x_N) = -[D]V(x_N),\ \mathcal{K}W(x_{2N}) = \mathcal{K}w(2). \qquad (28)\end{aligned}$$

Theorem 2 *Let $U(x_i)$ be a numerical solution of* (1)–(3) *defined by* (21)–(25) *and $V(x_i)$ be a numerical solution of* (13) *defined by* (27). *Then*

$$|U(x_i) - V(x_i)| \le C \begin{cases} N^{-1}, & i = 0, 1, \dots, \frac{3N}{2} \\ N^{-1} + |l - V(x_{2N})|, & i = \frac{3N}{2} + 1, \dots, 2N. \end{cases}$$

Proof Consider the barrier functions

$$\theta^{\pm}(x_i) = C_1 N^{-1} s(x_i) + C_1 x_i \Psi(x_i) \pm (U(x_i) - V(x_i)), \forall x_i \in \overline{\Omega}^{2N},$$
$$\text{where} \quad \Psi(x_i) = \begin{cases} 0, & i = 0, 1, \dots, \frac{3N}{2}, \\ |l - V(x_{2N})|, & i = \frac{3N}{2} + 1, \dots, 2N. \end{cases}$$

It is clear that $\theta^{\pm}(x_0) \ge 0$ and $\mathcal{K}\theta^{\pm}(x_{2N}) \ge 0$.

If $\forall\ x_i \in \Omega_1^{2N}$

$$\mathcal{L}_1^N \theta^{\pm}(x_i) \ge 0$$

If $\forall\ x_i \in \Omega_2^{2N}$

$$\mathcal{L}_2^N \theta^{\pm}(x_i) \ge 0, \text{ and}$$

$[D]^+\theta^{\pm}(x_N) = -C_1 \frac{N^{-1}}{4} \pm [v'](1) < 0$ for a suitable choice of $C_1 > 0$.

By Lemma 5, this thorem gets proved. □

Theorem 3 *Let $V(x_i)$ be a numerical solution of* (13) *defined by* (27). *Then*

$$|v(x_i) - V(x_i)| \le C N^{-1}, \quad x_i \in \overline{\Omega}^{2N}.$$

Proof If $x_i \in \Omega_1^{2N}$ and $x_i \in \Omega_2^{2N}$ then by [22], we have

$$|L^N(v(x_i) - V(x_i))| \le CN^{-1}, i \in \Omega_1^{2N} \cup \Omega_2^{2N}.$$

By the Lemma 6, we have

$$|v(x_i) - V(x_i)| \le CN^{-1}, \;\; i \in \Omega_1^{2N} \cup \Omega_2^{2N}.$$

At the point $x_i = x_{2N}$,

$$\begin{aligned}
\mathcal{K}^N(V - v)(x_{2N}) &= \mathcal{K}^N V(x_{2N}) - \mathcal{K}^N v(x_{2N}) \\
&= l - \mathcal{K}^N v(x_{2N}) \\
&= \mathcal{K}v(x_{2N}) - \mathcal{K}^N v(x_{2N}) \\
&= v(x_{2N}) - \int_{x_0}^{x_{2N}} g(x)v(x)dx - v(x_{2N}) + \sum_{i=1}^{2N} \frac{g_{i-1}v_{i-1} + g_i v_i}{2} h_i \\
|\mathcal{K}^N(V - v)(x_{2N})| &\le C\varepsilon((h_1^3 v''(\chi_1) + \cdots + h_{2N}^3 v''(\chi_{2N})) \\
&\le C\varepsilon(h_1^3 + \cdots + h_{2N}^3) \\
&\le CN^{-2} \\
&\le CN^{-1}, \quad \text{where } \; x_{i-1} \le \chi_i \le x_i, 1 \le i \le 2N.
\end{aligned}$$

Applying Lemma 6 we have $|(V - v)(x_{2N})| \le CN^{-1}$.

Hence $|v(x_i) - V(x_i)| \le CN^{-1}, \;\; i \in \overline{\Omega}^{2N}$. □

Theorem 4 *Let $W(x_i)$ be a numerical solution of* (14) *defined by* (28). *Then*

$$|w(x_i) - W(x_i)| \le CN^{-1} ln^2 N, \;\; x_i \in \overline{\Omega}^{2N}$$

Proof Note that

$$|w(x_i) - W(x_i)| \le |u(x_i) - U(x_i)| + |v(x_i) - V(x_i)|$$

Then by (20), Theorems 1 and 3, we have

$$|u(x_i) - U(x_i)| \le |U(x_i) - V(x_i)| + |v(x_i) - V(x_i)| + |u(x_i) - v(x_i)|.$$

Now,

$$\begin{aligned}
|w(x_i) - W(x_i)| &\le |U(x_i) - V(x_i)| + 2|v(x_i) - V(x_i)| + |u(x_i) - v(x_i)|, \\
&\le C_1 N^{-1} + C_1 \exp(\frac{-\alpha(2 - x)}{\varepsilon}) + \varepsilon \\
&\le C_1 \exp(\frac{-\alpha\sigma}{\varepsilon}) + C_1 N^{-1} \le CN^{-1}, i = 0 \text{ to } \frac{3N}{2} \qquad (29)
\end{aligned}$$

Consider mesh functions

$$\phi^{\pm}(x_i) = C_1 N^{-1} s(x_i) + C_1 N^{-1} \frac{\sigma}{\varepsilon^2}(x_i - (2-\sigma)) \\ \pm (w(x_i) - W(x_i)) \;\; x_i \in [2-\sigma, 2] \cap \overline{\Omega}^{2N}.$$

From (29), it is easy to check $\phi^{\pm}(x_{\frac{3N}{2}}) \geq 0$ and $\mathcal{K}\phi^{\pm}(x_{2N}) \geq 0$. For a suitable choice of $C_1 > 0$.

$$\mathcal{L}^N \phi^{\pm}(x_i) \geq 0$$

Then by the Lemma 5, we have $\phi^{\pm}(x_i) \geq 0, \quad x_i \in \overline{\Omega}^{2N}$. Therefore

$$|w(x_i) - W(x_i)| \leq CN^{-1} ln^2 N, \quad x_i \in \overline{\Omega}^{2N}$$

Hence the proof. □

Theorem 5 *Let $U(x_i)$ be the numerical solution of* (1)–(3) *defined by* (21)–(25). *Then*

$$|u(x_i) - U(x_i)| \leq CN^{-1} ln^2 N, \quad x_i \in \overline{\Omega}^{2N}$$

Proof The descried estimate follows from the fact that $u_k = v_k + w_k$, $U_k = V_k + W_k$ and from the above Theorems 3 and 4. □

6 Numerical experiments

In this section, four examples are given to illustrate the numerical method discussed above. The exact solution of the test problems are not known. Therefore, we use the double mesh principle to estimate the error and compute the experiment rate of convergence to the computed solution. For this we put

$$D_\varepsilon^N = \max_{0 \leq i \leq 2N} |U_i^N - U_{2i}^{2N}|$$

where U_i^N and U_{2i}^{2N} are the ith components of the numerical solutions on meshs of N and $2N$, respectively. We compute the uniform error and the rate of convergence as

$$D^N = \max_{\varepsilon} D_\varepsilon^N \quad \text{and} \quad p^N = log_2\left(\frac{D^N}{D^{2N}}\right)$$

The numerical results are presented for the values of the perturbation parameter $\varepsilon \in \{2^{-2}, 2^{-3}, \ldots, 2^{-20}\}$

Example 6.1

$$-\varepsilon u''(x) + 3u'(x) + u(x) - u(x-1) = 1, \quad \text{for} \;\; x \in (0,1) \cup (1,2),$$

$$u(x) = 1 \quad \text{for} \quad x \in [-1, 0], \ \mathcal{K}u(2) = u(2) - \varepsilon \int_0^2 \frac{x}{3} u(x) dx = 2.$$

Example 6.2

$$-\varepsilon u''(x) + 5u'(x) + (x+1)u(x) - u(x-1) = x^2, \quad \text{for} \quad x \in (0,1) \cup (1,2),$$

$$u(x) = 1 \quad \text{for} \quad x \in [-1, 0], \mathcal{K}u(2) = u(2) - \varepsilon \int_0^2 \frac{x}{3} u(x) dx = 2.$$

Example 6.3

$$-\varepsilon u''(x) + 3u'(x) - u(x-1) = 0, \quad \text{for} \quad x \in (0,1) \cup (1,2),$$

$$u(x) = 1 \quad \text{for} \quad x \in [-1, 0], \ \mathcal{K}u(2) = u(2) - \varepsilon \int_0^2 \frac{x}{3} u(x) dx = 2.$$

Example 6.4

$$-\varepsilon u''(x) + (x+10)u'(x) - u(x-1) = x^2, \quad \text{for} \quad x \in (0,1) \cup (1,2),$$

$$u(x) = 1 \quad \text{for} \quad x \in [-1, 0], \mathcal{K}u(2) = u(2) - \varepsilon \int_0^2 \frac{x}{3} u(x) dx = 2.$$

7 Conclusion

We have solved singularly perturbed delay differential equations of convection diffusion type with integral boundary condition (1)–(3), using finite difference method on Shishkin mesh. The method is shown to be of order $O(N^{-1} \ln^2 N)$, that is, the method has almost first order convergence with respect to ε. Four examples are given to illustrate the numerical method. Our numerical results reflect the theoretical estimates. Maximum pointwise errors and order of convergence of the Examples 6.1, 6.2, 6.3 and 6.4 are given in Tables 1, 2, 3 and 4 respectively. The maximum errors of Examples 6.1, 6.2, 6.3 and 6.4 are shown in Figs. 1, 2, 3, and 4 respectively.

Table 1 Computed maximum pointwise errors D_ε^N, computed ε-uniform errors D^N and the ε-uniform orders of convergence p^N for Example 6.1

Number of mesh points 2N						
ε	32	64	128	256	512	1024
2^{-2}	4.4340e−04	2.2662e−04	1.1448e−04	5.7525e−05	2.8833e−05	1.4434e−05
2^{-3}	5.2125e−04	2.6472e−04	1.3345e−04	6.7003e−05	3.3571e−05	1.6803e−05
2^{-4}	5.6085e−04	2.8481e−04	1.4353e−04	7.2064e−05	3.6107e−05	1.8072e−05
2^{-5}	5.8182e−04	2.9536e−04	1.4881e−04	7.4709e−05	3.7431e−05	1.8735e−05
2^{-6}	5.9334e−04	3.0073e−04	1.5154e−04	7.6060e−05	3.8108e−05	1.9073e−05
2^{-7}	2.8182e−03	9.8118e−04	1.5145e−04	7.6754e−05	3.8450e−05	1.9244e−05
2^{-8}	3.8249e−03	1.6274e−03	6.7972e−04	2.8307e−04	1.2223e−04	1.9330e−05
2^{-9}	4.5189e−03	2.0698e−03	9.4517e−04	4.3487e−04	2.0544e−04	1.0288e−04
2^{-10}	5.0010e−03	2.3755e−03	1.1275e−03	5.3851e−04	2.6147e−04	1.3121e−04
2^{-11}	5.3378e−03	2.5881e−03	1.2538e−03	6.0996e−04	2.9986e−04	1.5046e−04
2^{-12}	5.5738e−03	2.7367e−03	1.3418e−03	6.5956e−04	3.2641e−04	1.6369e−04
2^{-13}	5.7397e−03	2.8409e−03	1.4034e−03	6.9419e−04	3.4487e−04	1.7284e−04
2^{-14}	5.8565e−03	2.9141e−03	1.4466e−03	7.1844e−04	3.5778e−04	1.7922e−04
2^{-15}	5.9389e−03	2.9657e−03	1.4770e−03	7.3548e−04	3.6683e−04	1.8368e−04
2^{-16}	5.9970e−03	3.0021e−03	1.4985e−03	7.4747e−04	3.7319e−04	1.8681e−04
2^{-17}	6.0380e−03	3.0278e−03	1.5136e−03	7.5592e−04	3.7767e−04	1.8902e−04
2^{-18}	6.0670e−03	3.0459e−03	1.5242e−03	7.6189e−04	3.8083e−04	1.9057e−04
2^{-19}	6.0875e−03	3.0587e−03	1.5318e−03	7.6610e−04	3.8306e−04	1.9166e−04
2^{-20}	6.1020e−03	3.0678e−03	1.5371e−03	7.6907e−04	3.8464e−04	1.9243e−04
D^N	6.1020e−03	3.0678e−03	1.5371e−03	7.6907e−04	3.8464e−04	1.9243e−04
P^N	9.9207e−01	9.9697e−01	9.9904e−01	9.9960e−01	9.9913e−01	

Table 2 Computed maximum pointwise errors D_ε^N, computed ε-uniform errors D^N and the ε-uniform orders of convergence p^N for Example 6.2

Number of mesh points 2N						
ε	32	64	128	256	512	1024
2^{-2}	2.5742e−03	1.2344e−03	6.0382e−04	2.9852e−04	1.4841e−04	7.3992e−05
2^{-3}	2.4238e−03	1.1581e−03	5.6534e−04	2.7920e−04	1.3873e−04	6.9148e−05
2^{-4}	2.3476e−03	1.1194e−03	5.4587e−04	2.6944e−04	1.3384e−04	6.6700e−05
2^{-5}	2.3093e−03	1.1000e−03	5.3611e−04	2.6454e−04	1.3138e−04	6.5473e−05
2^{-6}	1.6439e−03	9.7352e−04	5.3122e−04	2.6209e−04	1.3016e−04	6.4858e−05
2^{-7}	1.2200e−03	6.1816e−04	3.5791e−04	2.1289e−04	1.2761e−04	6.4551e−05
2^{-8}	1.3550e−03	7.0083e−04	3.3820e−04	1.6023e−04	8.0417e−05	4.8983e−05
2^{-9}	1.4460e−03	7.7823e−04	3.8998e−04	1.8922e−04	9.0350e−05	4.2882e−05
2^{-10}	1.5023e−03	8.2944e−04	4.2515e−04	2.1093e−04	1.0303e−04	4.9926e−05

Table 2 continued

Number of mesh points 2N						
ε	32	64	128	256	512	1024
2^{-11}	1.5379e−03	8.6376e−04	4.4918e−04	2.2591e−04	1.1186e−04	5.4957e−05
2^{-12}	1.5610e−03	8.8702e−04	4.6570e−04	2.3629e−04	1.1800e−04	5.8471e−05
2^{-13}	1.5763e−03	9.0297e−04	4.7715e−04	2.4351e−04	1.2228e−04	6.0926e−05
2^{-14}	1.5865e−03	9.1398e−04	4.8511e−04	2.4855e−04	1.2528e−04	6.2646e−05
2^{-15}	1.5935e−03	9.2163e−04	4.9067e−04	2.5209e−04	1.2738e−04	6.3852e−05
2^{-16}	1.5983e−03	9.2698e−04	4.9458e−04	2.5457e−04	1.2885e−04	6.4701e−05
2^{-17}	1.6016e−03	9.3072e−04	4.9732e−04	2.5631e−04	1.2989e−04	6.5298e−05
2^{-18}	1.6039e−03	9.3336e−04	4.9925e−04	2.5754e−04	1.3063e−04	6.5720e−05
2^{-19}	1.6055e−03	9.3521e−04	5.0061e−04	2.5841e−04	1.3114e−04	6.6017e−05
2^{-20}	1.6067e−03	9.3652e−04	5.0157e−04	2.5902e−04	1.3151e−04	6.6227e−05
D^N	2.5742e−03	1.2344e−03	6.0382e−04	2.9852e−04	1.4841e−04	7.3992e−05
P^N	1.0602e+00	1.0317e+00	1.0162e+00	1.0082e+00	1.0041e+00	

Table 3 Computed maximum pointwise errors D_ε^N, computed ε-uniform errors D^N and the ε-uniform orders of convergence p^N for Example 6.3

Number of mesh points 2N						
ε	32	64	128	256	512	1024
2^{-2}	4.0354e−04	1.7378e−04	8.0500e−05	3.8715e−05	1.8980e−05	9.3969e−06
2^{-3}	2.2608e−04	9.2684e−05	4.1255e−05	1.9356e−05	9.3601e−06	4.6752e−06
2^{-4}	1.5371e−04	5.6514e−05	2.3170e−05	1.0313e−05	4.8390e−06	2.3400e−06
2^{-5}	1.1754e−04	3.8429e−05	1.4128e−05	5.7926e−06	2.5784e−06	1.2097e−06
2^{-6}	9.9464e−05	2.9387e−05	9.6073e−06	3.5321e−06	1.4481e−06	6.4461e−07
2^{-7}	5.0968e−03	1.8528e−03	7.4292e−05	2.4018e−06	8.8303e−07	3.6204e−07
2^{-8}	6.7448e−03	2.8576e−03	1.2074e−03	5.1368e−04	2.2650e−04	2.2075e−07
2^{-9}	7.9129e−03	3.5690e−03	1.6222e−03	7.4603e−04	3.5172e−04	1.7433e−04
2^{-10}	8.7402e−03	4.0726e−03	1.9156e−03	9.1036e−04	4.3976e−04	2.1871e−04
2^{-11}	9.3259e−03	4.4288e−03	2.1232e−03	1.0265e−03	5.0201e−04	2.5009e−04
2^{-12}	9.7404e−03	4.6808e−03	2.2700e−03	1.1087e−03	5.4603e−04	2.7227e−04
2^{-13}	1.0033e−02	4.8591e−03	2.3738e−03	1.1668e−03	5.7716e−04	2.8796e−04
2^{-14}	1.0241e−02	4.9852e−03	2.4472e−03	1.2079e−03	5.9917e−04	2.9905e−04
2^{-15}	1.0387e−02	5.0744e−03	2.4991e−03	1.2370e−03	6.1474e−04	3.0690e−04
2^{-16}	1.0491e−02	5.1374e−03	2.5358e−03	1.2575e−03	6.2574e−04	3.1244e−04
2^{-17}	1.0564e−02	5.1820e−03	2.5618e−03	1.2720e−03	6.3352e−04	3.1637e−04
2^{-18}	1.0616e−02	5.2135e−03	2.5801e−03	1.2823e−03	6.3902e−04	3.1914e−04
2^{-19}	1.0653e−02	5.2358e−03	2.5931e−03	1.2896e−03	6.4292e−04	3.2110e−04

Table 3 continued

Number of mesh points 2N						
ε	32	64	128	256	512	1024
2^{-20}	1.0679e−02	5.2516e−03	2.6023e−03	1.2947e−03	6.4567e−04	3.2249e−04
D^N	1.0679e−02	5.2516e−03	2.6023e−03	1.2947e−03	6.4567e−04	3.2249e−04
P^N	1.0240e+00	1.0129e+00	1.0071e+00	1.0038e+00	1.0015e+00	

Table 4 Computed maximum pointwise errors D_ε^N, computed ε-uniform errors D^N and the ε-uniform orders of convergence p^N for Example 6.4

Number of mesh points 2N						
ε	32	64	128	256	512	1024
2^{-2}	1.6831e−03	7.9159e−04	3.8327e−04	1.8850e−04	9.3466e−05	4.6537e−05
2^{-3}	1.6141e−03	7.5745e−04	3.6630e−04	1.8004e−04	8.9243e−05	4.4427e−05
2^{-4}	1.5801e−03	7.4066e−04	3.5795e−04	1.7588e−04	8.7165e−05	4.3389e−05
2^{-5}	1.5632e−03	7.3233e−04	3.5381e−04	1.7381e−04	8.6135e−05	4.2874e−05
2^{-6}	3.4094e−03	1.0758e−03	3.5175e−04	1.7279e−04	8.5622e−05	4.2618e−05
2^{-7}	4.2525e−03	1.7858e−03	7.6415e−04	3.3000e−04	9.6236e−05	4.2490e−05
2^{-8}	4.8569e−03	2.1306e−03	9.6068e−04	4.4038e−04	2.0505e−04	9.7821e−05
2^{-9}	5.2885e−03	2.3753e−03	1.0995e−03	5.1813e−04	2.4756e−04	1.2028e−04
2^{-10}	5.5958e−03	2.5488e−03	1.1977e−03	5.7296e−04	2.7750e−04	1.3608e−04
2^{-11}	5.8142e−03	2.6717e−03	1.2671e−03	6.1165e−04	2.9860e−04	1.4720e−04
2^{-12}	5.9692e−03	2.7588e−03	1.3161e−03	6.3898e−04	3.1349e−04	1.5504e−04
2^{-13}	6.0790e−03	2.8204e−03	1.3508e−03	6.5829e−04	3.2400e−04	1.6057e−04
2^{-14}	6.1568e−03	2.8640e−03	1.3753e−03	6.7194e−04	3.3142e−04	1.6447e−04
2^{-15}	6.2119e−03	2.8948e−03	1.3927e−03	6.8158e−04	3.3667e−04	1.6723e−04
2^{-16}	6.2509e−03	2.9166e−03	1.4049e−03	6.8840e−04	3.4038e−04	1.6918e−04
2^{-17}	6.2785e−03	2.9321e−03	1.4136e−03	6.9322e−04	3.4300e−04	1.7056e−04
2^{-18}	6.2980e−03	2.9430e−03	1.4197e−03	6.9662e−04	3.4485e−04	1.7154e−04
2^{-19}	6.3118e−03	2.9507e−03	1.4241e−03	6.9903e−04	3.4616e−04	1.7222e−04
2^{-20}	6.3216e−03	2.9562e−03	1.4271e−03	7.0074e−04	3.4709e−04	1.7271e−04
D^N	6.3216e−03	2.9562e−03	1.4271e−03	7.0074e−04	3.4709e−04	1.7271e−04
P^N	1.0965e+00	1.0505e+00	1.0262e+00	1.0135e+00	1.0069e+00	

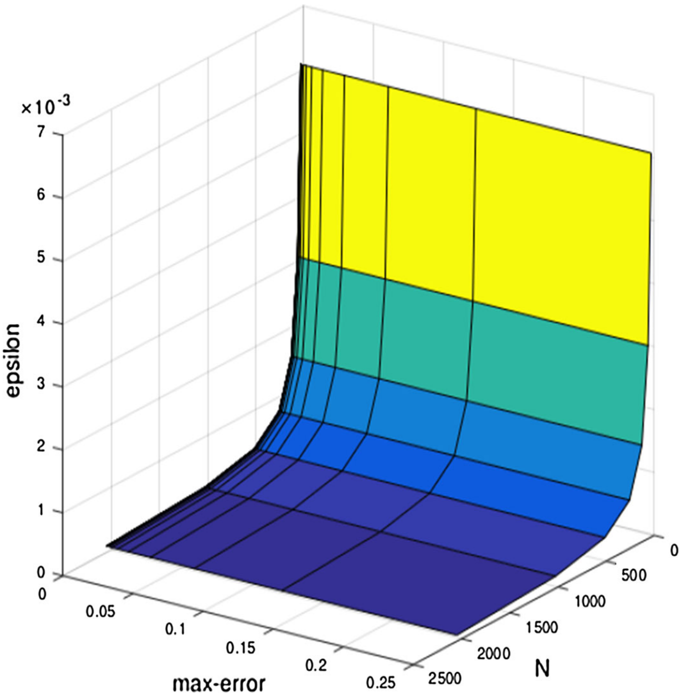

Fig. 1 Maximum pointwise error of the numerical solution of Example 6.1

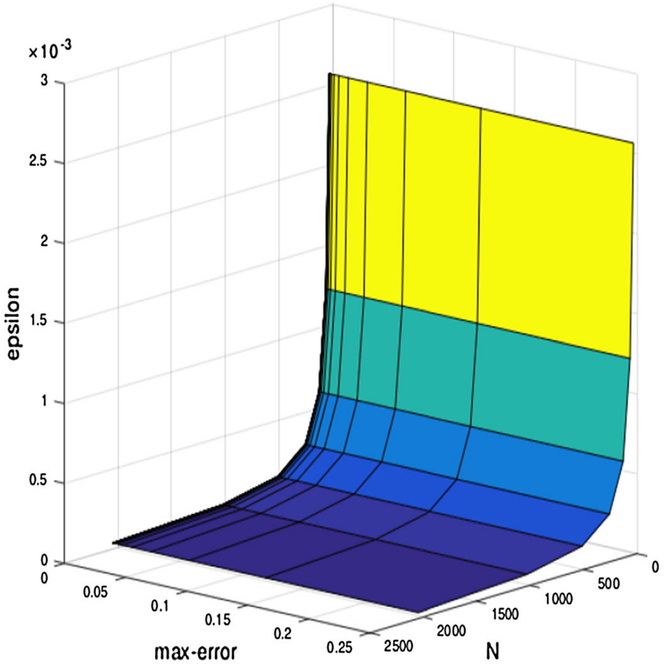

Fig. 2 Maximum pointwise error of the numerical solution of Example 6.2

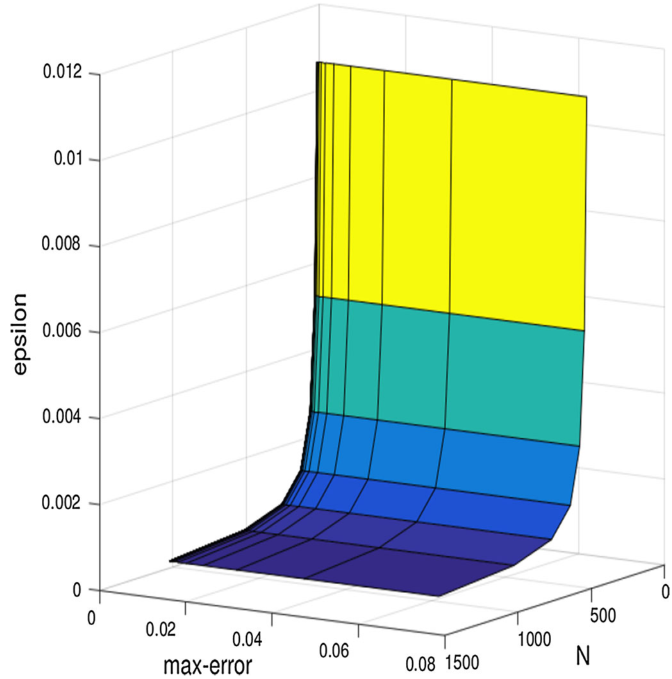

Fig. 3 Maximum pointwise error of the numerical solution of Example 6.3

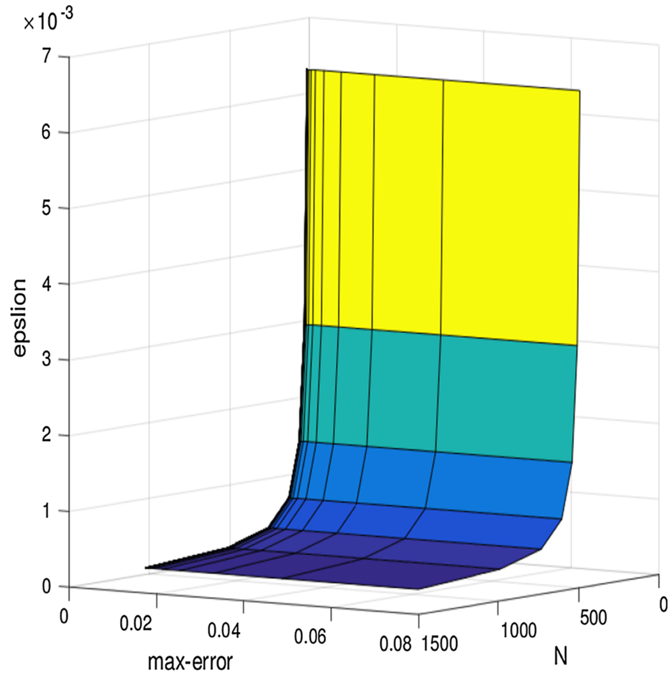

Fig. 4 Maximum pointwise error of the numerical solution of Example 6.4

References

1. Amiraliyev, G.M., Amiraliyev, I.G., Kudu, M.: A numerical treatment for singularly perturbed differential equations with integral boundary condition. Appl. Math. Comput. **185**, 574–582 (2007)
2. Bahuguna, D., Dabas, J.: Existence and uniqueness of a solution to a semilinear partial delay differential equation with an integral condition. Nonlinear Dyn. Syst. Theory **8**(1), 7–19 (2008)
3. Bahuguna, D., Abbas, S., Dabas, J.: Partial functional differential equation with an integral condition and applications to population dynamics. Nonlinear Anal. **69**, 2623–2635 (2008)
4. Boucherif, A.: Second order boundary value problems with integral boundary condition. Nonlinear Anal. **70**(1), 368–379 (2009)
5. Cakir, M., Amiraliyev, G.M.: A finite difference method for the singularly perturbed problem with nonlocal boundary condition. Appl. Math. Comput. **160**, 539–549 (2005)
6. Cannon, J.R.: The solution of the heat equation subject to the specification of energy. Q. Appl. Math. **21**, 155–160 (1963)
7. Cen, Z., Cai, X.: A Second Order Upwind Difference Scheme for a Singularly Perturbed Problem with Integral Boundary Condition in Netural Network, pp. 175–181. Springer, Berlin (2007)
8. Choi, Y.S., Chan, K.-Y.: A parabolic equation with nonlocal boundary conditions arising from electrochemistry. Nonlinear Anal. Theory Methods Appl. **18**(4), 317–331 (1992)
9. Culshaw, R.V., Ruan, S.: A delay differential equation model of HIV infection of $CD4^+$ T-cells. Math. Biosci. **165**, 27–39 (2000)
10. Day, W.A.: Parabolic equations and thermodynamics. Q. Appl. Math. **50**, 523–533 (1992)
11. Els'gol'ts, E.L.: Qualitative Methods in Mathematical Analysis in: Translations of Mathematical Monographs, vol. 12. American Mathematical Society, Providence (1964)
12. Feng, M., Ji, D., Weigao, G.: Positive solutions for a class of boundary value problem with integral boundary conditions in banach spaces. J. Comput. Appl. Math. **222**, 351–363 (2008)
13. Glizer, V.Y.: Asymptotic analysis and Solution of a finite-horizon H_∞ control problem for singularly perturbed linear systems with small state delay. J. Optim. Theory Appl. **117**, 295–325 (2003)
14. Kadalbajoo, M.K., Kumar, D.: Fitted mesh B-spline collocation method for singularly perturbed differential equations with small delay. Appl. Math. Comput. **204**, 90–98 (2008)
15. Kadalbajoo, M.K., Sharma, K.K.: Numerical treatment of boundary value problems for second order singularly perturbed delay differential equations. Comput. Appl. Math. **24**(2), 151–172 (2005)
16. Kadalbajoo, M.K., Sharma, K.K.: Parameter-uniform fitted mesh method for singularly perturbed delay differential equations with layer behavior. Electron. Trans. Numer. Anal. **23**, 180–201 (2006)
17. Kudu, M., Amiraliyev, G.: Finite difference method for a singularly perturbed differential equations with integral boundary condition. Int. J. Math. Comput. **26**, 72–79 (2015)
18. Lange, C.G., Miura, R.M.: Singularly perturbation analysis of boundary-value problems for differential-difference equations. SIAM J. Appl. Math. **42**(3), 502–530 (1982)
19. Li, H., Sun, F.: Existence of solutions for integral boundary value problems of second order ordinary differential equations. Bound. Value Probl. **1**, 147 (2012)
20. Longtin, A., Milton, J.: Complex oscillations in the human pupil light reflex with mixed and delayed feedback. Math. Biosci. **90**, 183–199 (1988)
21. Mahendran, R., Subburayan, V.: Fitted finite difference method for third order singularly perturbed delay differential equations of convection diffusion type. Int. J. Comput. Methods **15**(1), 1840007 (2018)
22. Miller, J.J.H., O'Riordan, E., Shishkin, G.I.: Fitted Numerical Methods for Singular Perturbation Problems. World Scientific Publishing Co., London (1996)
23. Nicaise, S., Xenophontos, C.: Robust approximation of singularly perturbed delay differential equations by the hp finite element method. Comput. Methods Appl. Math. **13**(1), 21–37 (2013)
24. Tang, Z.Q., Geng, F.Z.: Fitted reproducing kernel method for singularly perturbed delay initial value problems. Appl. Math. Comput. **284**, 169–174 (2016)
25. Zarin, H.: On discontinuous Galerkin finite element method for singularly perturbed delay differential equations. Appl. Math. Lett. **38**, 27–32 (2014)
26. Zhang, L., Xie, F.: Singularly perturbed first order differential equations with integral boundary condition. J. Shanghai Univ. (Eng) **13**, 20–22 (2009)