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ORIGINAL RESEARCH

Dynamics of a predator–prey model with harvesting and reserve area for prey in the presence of competition and toxicity

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Abstract The aim of this paper is to present and study a three-dimensional continuous time dynamical system modeling a predator–prey with harvesting and reserve zone for the prey in the presence of competition and toxicity. We first prove that our model is ecologically and mathematically well-posed. In addition, the stability analysis is investigated by direct and indirect Lyapunov methods. By using the Pontryagin's maximum principle, an optimal harvesting policy is established. Furthermore, numerical simulations are given in order to illustrate our theoretical results.

Keywords Predator–prey model · Competition · Toxicity · Stability · Optimal harvesting policy

Mathematics Subject Classification 34K18 · 34K20 · 92B20

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1 Introduction

In recent years, various species are extinct due to several ecological and economic factors such as the over predation and harvesting, the pollution, and the mismanagement of commercial exploitation of the biological resource like fisheries and forestry. For these reasons, many mathematical models have been proposed and developed to better describe the relationship between predator and prey populations by taking into account the harvesting [6,8], the optimal harvesting policy [9,11,13], the toxicity [2,3] and the harvesting and reserve area for the prey in the presence of toxicity [12].

In reality, there is always a competition between species that is an important topic in ecology. This phenomenon is very common in a habitat with finite common resources and it is considered by many authors (see, for example [5,10]). The purpose of this paper is to investigate the effects of competition and toxicity on dynamics of the following predator–prey model:

$$\begin{aligned} \frac{dx}{dt} &= r_1 x\left(1-\frac{x}{K}\right) - \sigma_1 x + \sigma_2 y - ux^2 - \frac{axz}{b+x} - q_1 Ex - n_1 xy, \\ \frac{dy}{dt} &= r_2 y + \sigma_1 x - \sigma_2 y - vy^2 - n_2 xy, \\ \frac{dz}{dt} &= \frac{\beta axz}{b+x} - dz - wz - q_2 Ez, \end{aligned} \tag{1}$$

where $x(t)$, $y(t)$ and $z(t)$ denote the biomass densities of the prey species inside the unreserved area, reserved area, and the predator species at time t, respectively. The intrinsic growth rates of prey species inside the unreserved and reserved areas are r_1 and r_2, respectively. The carrying capacity of fish species in the unreserved area is K. The catchability coefficients for the predator species in the unreserved area are q_1 and q_2, respectively. The effort applied for harvesting the fish population in the unreserved area and the predator populations in the unreserved area is E. Migration rate from unreserved area to reserved area and reserved area to unreserved are σ_1 and σ_2, respectively. The terms ux^2 and vy^2 represent the infection of the prey species by an external toxic substance and the term wz for the predator species. The term $\frac{axz}{b+x}$ denotes the Holling type II functional response. The parameter d is the death rate of predator species. Finally, n_1 and n_2 are the competition coefficients. It very important to note that when $n_1 = n_2 = 0$, we get the model presented by Yang and Jia in [12].

From [4], we know that if there is no migration of fish population from reserve area to free fishing zone (i.e. $\sigma_2 = 0$) and $r_1\sigma_1 - q_1 E < 0$, then $\frac{dx(t)}{dt} < 0$. Similarly, if there is no migration of fish population from free fishing zone to reserve area (i.e. $\sigma_1 = 0$) and $r_2 - \sigma_2 < 0$, then $\frac{dy(t)}{dt} < 0$. The first three terms of the third equation of (1) is always negative if $\beta a - d - w < 0$. Hence throughout our analysis, we assume that

$$r_1 - \sigma_1 - q_1 E > 0, \quad r_2 - \sigma_2 > 0, \quad \beta a - d - w > 0. \tag{2}$$

The rest of the paper is organized as follows. The next section deals with basic results and equilibria. The stability analysis is investigated in Sect. 3. The optimal harvesting policy is discussed in Sect. 4. Some numerical simulations are presented in Sect. 5 in order to illustrate the theoretical results. The paper ends with discussion and conclusion in Sect. 6.

2 Basic results and equilibria

In this section, we first prove the uniform boundedness of the solutions of system (1).

Lemma 2.1 *All solutions of system* (1) *with positive initial value* (x_0, y_0, z_0) *are positively invariant within* Ω*, where*

$$\Omega = \left\{(x, y, z) \in \mathbf{R}^3 : x + y + \frac{1}{\beta} z \leq \frac{\mu}{d + w + q_2 E}\right\},$$

with

$$\mu = \frac{(r_1 - q_1 E + d + w + q_2 E)^2}{4\left(\frac{r_1}{K} + u\right)} + \frac{(r_2 + d + w + q_2 E)^2}{4v}.$$

Proof Let $X(t) = x(t) + y(t) + \frac{1}{\beta} z(t)$. Then

$$\begin{aligned} \frac{dX(t)}{dt} + (d + w + q_2 E)X(t) &= (r_1 - q_1 E + d + w + q_2 E)x(t) - \left(\frac{r_1}{K} + u\right) x^2(t) \\ &\quad - (n_1 + n_2)x(t)y(t) + (r_2 + d + w + q_2 E)y(t) \\ &\quad - vy^2(t) \\ &\leq \frac{(r_1 - q_1 E + d + w + q_2 E)^2}{4\left(\frac{r_1}{K} + u\right)} + \frac{(r_2 + d + w + q_2 E)^2}{4v} \\ &= \mu. \end{aligned}$$

Applying the theory of differential inequality [1,7], we get

$$\begin{aligned} X(t) \leq & \frac{\mu}{d + w + q_2 E} \\ & - \left(\frac{\mu}{d + w + q_2 E} - (x(0) + y(0) + \frac{1}{\beta} z(0))\right) e^{-(d+w+q_2 E)t} \end{aligned}$$

and for $t \to +\infty$, we have $X(t) \leq \frac{\mu}{d+w+q_2 E}$. This completes the proof. □

In the following, we discuss the existence of equilibria of the system (1).
Clearly, the vanishing equilibrium point $P_0(0, 0, 0)$ always exists.
The predator free-equilibrium point $\bar{P}(\bar{x}, \bar{y}, 0)$, where $(\bar{x}, \bar{y})$ is the positive solution of the following equations:

$$\begin{aligned} r_1 x \left(1 - \frac{x}{K}\right) - \sigma_1 x + \sigma_2 y - u x^2 - q_1 E x - n_1 x y &= 0, \\ r_2 y + \sigma_1 x - \sigma_2 y - v y^2 - n_2 x y &= 0. \end{aligned} \tag{3}$$

Consequently, x is satisfied by the following cubic equation

$$A_3x^3 + A_2x^2 + A_1x + A_0 = 0,$$

where

$$\begin{aligned}
A_3 &= \left(u + \frac{r_1}{K}\right)\left[-v\left(u + \frac{r_1}{K}\right) + n_1n_2\right],\\
A_2 &= 2v\left(u + \frac{r_1}{K}\right)(r_1 - \sigma_1 - q_1E) - n_2\sigma_2\left(u + \frac{r_1}{K}\right)\\
&\quad - n_1(r_2 - \sigma_2)\left(u + \frac{r_1}{K}\right) - n_1n_2(r_1 - \sigma_1 - q_1E) + n_1^2\sigma_1,\\
A_1 &= -v(r_1 - \sigma_1 - q_1E)^2 + (n_2\sigma_2 + n_1(r_2 - \sigma_2))(r_1 - \sigma_1 - q_1E)\\
&\quad + \left(u + \frac{r_1}{K}\right)(r_2 - \sigma_2)\sigma_2 - 2\sigma_1\sigma_2n_1,\\
A_0 &= -\sigma_2(r_2 - \sigma_2)(r_1 - \sigma_1 - q_1E) + \sigma_1\sigma_2^2.
\end{aligned}$$

$A_1 > 0$ if $E > \frac{1}{q_1}\left(r_1 - \sigma_1 - \frac{\sqrt{\Delta_1 + n_2\sigma_2 + n_1(r_2-\sigma_2)}}{2v}\right)$,
where $\Delta_1 = (n_2\sigma_2 + n_1(r_2 - \sigma_2))^2 + 4v(u + \frac{r_1}{K})(r_2 - \sigma_2)\sigma_2$,
$A_0 > 0$ if $E > \frac{1}{q_1}\left(r_1 - \sigma_1 - \frac{\sigma_1\sigma_2}{r_2-\sigma_2}\right)$. Then,

$$E > \max\left(\frac{1}{q_1}\left(r_1 - \sigma_1 - \frac{\sigma_1\sigma_2}{r_2 - \sigma_2}\right), \frac{1}{q_1}\left(r_1 - \sigma_1 - \frac{\sqrt{\Delta_1 + n_2\sigma_2 + n_1(r_2 - \sigma_2)}}{2v}\right)\right) \tag{4}$$

$$A_3 < 0 \text{ if } n_1n_2 < v\left(u + \frac{r_1}{K}\right) \tag{5}$$

$$A_2 > 0 \text{ if } v(r_1 - \sigma_1 - q_1E) > n_2\sigma_2 + n_1(r_2 - \sigma_2). \tag{6}$$

The first equation of (3), we have

$$\bar{y} = \left(\frac{\left(u + \frac{r_1}{K}\right)\bar{x} - (r_1 - \sigma_1 - q_1E)}{\sigma_2 - n_1\bar{x}}\right)\bar{x} > 0, \tag{7}$$

where

$$\frac{r_1 - \sigma_1 - q_1E}{u + \frac{r_1}{K}} < \bar{x} < \frac{\sigma_2}{n_1} \quad \text{or} \quad \frac{\sigma_2}{n_1} < \bar{x} < \frac{r_1 - \sigma_1 - q_1E}{u + \frac{r_1}{K}}. \tag{8}$$

For the equilibrium point $P^*(x^*, y^*, z^*)$, (x^*, y^*, z^*) is the positive solution of the equations:

$$\begin{aligned}
r_1x\left(1 - \frac{x}{K}\right) - \sigma_1x + \sigma_2y - ux^2 - \frac{axz}{b + x} - q_1Ex - n_1xy &= 0,\\
r_2y + \sigma_1x - \sigma_2y - vy^2 - n_2xy &= 0,\\
\frac{\beta axz}{b + x} - dz - wz - q_2Ez &= 0.
\end{aligned} \tag{9}$$

Then, the third equation

$$x^* = \frac{b(d + w + q_2 E)}{\beta a - d - w - q_2 E} > 0. \tag{10}$$

if

$$E < \frac{1}{q_2}(\beta a - d - w). \tag{11}$$

Substituting x^* to the second equation of (9), this equation has unique positive equation

$$y^* = \frac{r_2 - \sigma_2 - n_2 x^* + \sqrt{(r_2 - \sigma_2 - n_2 x^*)^2 + 4v\sigma_1 x^*}}{2v}. \tag{12}$$

Substituting x^*, y^* to the first equation of (9)

$$z^* = \frac{(b + x^*)\left[-\left(u + \frac{r_1}{K}\right)x^{*2} + (r_1 - \sigma_1 - q_1 E - n_1 y^*)x^* + \sigma_2 y^*\right]}{ax^*}. \tag{13}$$

Then, $z^* > 0$ if

$$x^* < \frac{(r_1 - \sigma_1 - q_1 E - n_1 y^*) + \sqrt{(r_1 - \sigma_1 - q_1 E - n_1 y^*)^2 + 4\left(u + \frac{r_1}{K}\right)\sigma_2 y^*}}{2\left(u + \frac{r_1}{K}\right)}. \tag{14}$$

Taking (14) into account, we can get the following theorem.

Theorem 2.2 *The trivial equilibrium point* $P_0(0, 0, 0)$ *exists. If* (4)–(8) *hold, the predator equilibrium* $\bar{P}(\bar{x}, \bar{y}, 0)$ *exists. If* (11)–(14) *holds, the interior equilibrium* $P^*(x^*, y^*, z^*)$ *exists.*

3 Stability analysis

In this section, we establish the local stability of equilibria.

Theorem 3.1 *Suppose that* (2) *holds. The trivial equilibrium* $P_0(0, 0, 0)$ *is always unstable.*

Proof The Jacobian matrix

$$V(0, 0, 0) = \begin{pmatrix} r_1 - \sigma_1 - q_1 E & \sigma_2 & 0 \\ \sigma_1 & r_2 - \sigma_2 & 0 \\ 0 & 0 & -(d + w + q_2 E) \end{pmatrix}.$$

Then the characteristic equation of system at P_0 is

$$(\lambda + d + w + q_2 E)\left((\lambda - (r_1 - \sigma_1 - q_1 E))(\lambda - (r_2 - \sigma_2)) - \sigma_1\sigma_2\right) = 0$$

Firstly, It is clear that $\lambda_1 = -(d + w + q_2 E) < 0$. If λ_2 and λ_3 are the two other roots, then $\lambda_2 + \lambda_3 > 0$ with (2) holds. Therefore λ_2 and λ_3 have one positive value. Hence, P_0 is unstable. □

Theorem 3.2 *Suppose that* (2) *holds, if* $\bar{x} < \frac{b(d+w+q_2E)}{\beta a - d - w - q_2 E} = x^*$, *the predator equilibrium point* $\bar{P}(\bar{x}, \bar{y}, 0)$ *is locally asymptotically stable.*

Proof The Jacobian matrix of the point $(\bar{x}, \bar{y}, 0)$ is

$$V(\bar{x}, \bar{y}, 0) = \begin{pmatrix} V_1 & \sigma_2 - n_1\bar{x} & \frac{-a\bar{x}}{b+\bar{x}} \\ \sigma_1 - n_2\bar{y} & V_2 & 0 \\ 0 & 0 & V_3 \end{pmatrix},$$

where

$$\begin{aligned} V_1 &= -2\left(u + \frac{r_1}{K}\right)\bar{x} + (r_1 - \sigma_1 - q_1 E) - n_1\bar{y}, \\ V_2 &= r_2 - \sigma_2 - 2v\bar{y} - n_2\bar{x}, \\ V_3 &= \frac{\beta a\bar{x}}{b+\bar{x}} - (d + w + q_2 E). \end{aligned}$$

Then, the characteristic equation of system at $\bar{P}$ is

$$\begin{aligned} &\left(\lambda - \frac{\beta a\bar{x}}{b+\bar{x}} + (d + w + q_2 E)\right)\left(\left(\lambda + 2\left(u + \frac{r_1}{K}\right)\bar{x}\right.\right. \\ &\left. -(r_1 - \sigma_1 - q_1 E) + n_1\bar{y}\right)(\lambda - r_2 + \sigma_2 + 2v\bar{y} + n_2\bar{x}) \\ &\left. -(\sigma_2 - n_1\bar{x})(\sigma_1 - n_2\bar{y})\right) = 0. \end{aligned}$$

Obviously, $\lambda_1 = \frac{\beta a\bar{x}}{b+\bar{x}} - (d + w + q_2 E)$. This root is $\lambda_1 < 0$ if $\bar{x} < \frac{b(d+w+q_2E)}{\beta a - d - w - q_2 E} = x^*$. Let λ_2 and λ_3 be the two other eigenvalues. These are the roots of the equation:

$$\lambda^2 + s_1\lambda + s_2 = 0,$$

where,

$$\begin{aligned} s_1 &= 2\left(u + \frac{r_1}{K}\right)\bar{x} - (r_1 - \sigma_1 - q_1 E) + n_1\bar{y} - r_2 + \sigma_2 + 2v\bar{y} + n_2\bar{x}, \\ s_2 &= \left(2\left(u + \frac{r_1}{K}\right)\bar{x} - (r_1 - \sigma_1 - q_1 E) + n_1\bar{y}\right)(-r_2 + \sigma_2 + 2v\bar{y} + n_2\bar{x}) \\ &\quad - (\sigma_2 - n_1\bar{x})(\sigma_1 - n_2\bar{y}). \end{aligned}$$

Using to $\bar{y} = \frac{(u+\frac{r_1}{K})\bar{x} - (r_1 - \sigma_1 - q_1 E)}{\sigma_2 - n_1\bar{x}}\bar{x}$ and $\bar{x} = \frac{v\bar{y}^2 + (-r_2+\sigma_2)\bar{y}}{\sigma_1 - n_2\bar{y}}$, we get

$$\begin{aligned} s_1 &= \frac{\bar{y}(\sigma_2 - n_1\bar{x})}{\bar{x}} + \left(u + \frac{r_1}{K}\right)\bar{x} + \frac{\bar{x}(\sigma_1 - n_2\bar{y})}{\bar{y}} + v\bar{y} + n_2\bar{x} + n_1\bar{y} > 0, \\ s_2 &= \frac{\bar{y}(\sigma_2 - n_1\bar{x})}{\bar{x}}(v\bar{y} + n_2\bar{x}) \end{aligned}$$

$$+\left(\left(u+\frac{r_1}{K}\right)\bar{x}+n_1\bar{y}\right)\left(\frac{\bar{x}(\sigma_1-n_2\bar{y})}{\bar{y}}+v\bar{y}+n_2\bar{x}\right)>0.$$

Then, it is to verify that $\lambda_2+\lambda_3=-s_1<0$ and $\lambda_2\lambda_3=s_2>0$.
So $\lambda_1,\ \lambda_2,\ \lambda_3<0$. Thus the predator equilibrium point $\bar{P}(\bar{x},\bar{y},0)$ of the system (1) is locally asymptotically stable. □

Theorem 3.3 *Under assumptions* (11)–(14)*, the predator equilibrium* $\bar{P}(\bar{x},\bar{y},0)$ *exists. Then it is globally asymptoticlly stable if* $n_1+\frac{\sigma_2\bar{y}n_2}{\sigma_1\bar{x}}<2\min\left(\left(\frac{r_1}{K}+u\right),\frac{\sigma_2\bar{y}v}{\sigma_1\bar{x}}\right)$.

Proof The Lyapunov function is given by

$$V(x,y,0)=\left(x-\bar{x}-\bar{x}\ln\left(\frac{x}{\bar{x}}\right)\right)+l\left(y-\bar{y}-\bar{y}\ln\left(\frac{y}{\bar{y}}\right)\right),$$

where l for is positive constant to be determined in the subsequent steps.

$$\begin{aligned}\frac{dV}{dt}&=(x-\bar{x})\left(-\left(\frac{r_1}{K}+u\right)(x-\bar{x})+\sigma_2\left(\frac{y}{x}-\frac{\bar{y}}{\bar{x}}\right)-n_1(y-\bar{y})\right)\\&\quad+l(y-\bar{y})\left(-v(y-\bar{y})+\sigma_1\left(\frac{x}{y}-\frac{\bar{x}}{\bar{y}}\right)-n_2(x-\bar{x})\right)\end{aligned}$$

Let us choose $l=\frac{\sigma_2\bar{y}}{\sigma_1\bar{x}}$, we have,

$$\begin{aligned}\frac{dV}{dt}&=-\left(\frac{r_1}{K}+u\right)(x-\bar{x})^2-\frac{\sigma_2\bar{y}v}{\sigma_1\bar{x}}(y-\bar{y})^2-\frac{\sigma_2}{x\bar{x}y}(y\bar{x}-x\bar{y})^2\\&\quad-\left(n_1+\frac{\sigma_2\bar{y}n_2}{\sigma_1\bar{x}}\right)(x-\bar{x})(y-\bar{y}),\\&\le\left(-\left(\frac{r_1}{K}+u\right)+\frac{1}{2}\left(n_1+\frac{\sigma_2\bar{y}n_2}{\sigma_1\bar{x}}\right)\right)(x-\bar{x})^2\\&\quad+\left(\frac{1}{2}\left(n_1+\frac{\sigma_2\bar{y}n_2}{\sigma_1\bar{x}}\right)-\frac{\sigma_2\bar{y}v}{\sigma_1\bar{x}}\right)(y-\bar{y})^2-\frac{\sigma_2}{x\bar{x}y}(y\bar{x}-x\bar{y})^2.\end{aligned}$$

Then, $\frac{dV}{dt}\le 0$ if $n_1+\frac{\sigma_2\bar{y}n_2}{\sigma_1\bar{x}}<2\min\left(\left(\frac{r_1}{K}+u\right),\frac{\sigma_2\bar{y}v}{\sigma_1\bar{x}}\right)$. □

Theorem 3.4 *Under assumptions* (11)–(14)*, the interior equilibrium point* $P^*(x^*,y^*,z^*)$ *exists. Then* P^* *is locally asymptotically stable if* $m_0>0$*,* $m_2>0$ *and* $m_1m_2-m_0>0$*, where* m_0*,* m_1 *and* m_2 *are given in the proof.*

Proof The Jacobian matrix of the point (x^*,y^*,z^*) is

$$V(x^*,y^*,z^*)=\begin{pmatrix}V_4 & \sigma_2-n_1x^* & \frac{-ax^*}{b+x^*}\\ \sigma_1-n_2y^* & V_5 & 0\\ \frac{\beta abz^*}{(b+x^*)^2} & 0 & V_6\end{pmatrix},$$

where

$$V_4 = -2\left(u + \frac{r_1}{K}\right)x^* + (r_1 - \sigma_1 - q_1 E) - n_1 y^* - \frac{abz^*}{(b + x^*)^2},$$
$$V_5 = r_2 - \sigma_2 - 2vy^* - n_2 x^*,$$
$$V_6 = \frac{\beta a x^*}{b + x^*} - (d + w + q_2 E).$$

The characteristic equation of system at P^* is

$$\lambda^3 + m_2\lambda^2 + m_1\lambda + m_0 = 0,$$

where

$$m_2 = -Tr(V(x^*, y^*, z^*)) = -(V_4 + V_5 + V_6),$$
$$m_1 = V_4V_5 + V_5V_6 + V_6V_4 - (\sigma_2 - n_1\bar{x})(\sigma_1 - n_2\bar{y}) + \frac{\beta a^2 b x^* z^*}{(b + x^*)^3},$$
$$m_0 = -V_4V_5V_6 - V_5\frac{\beta a^2 b x^* z^*}{(b + x^*)^3} + V_6(\sigma_2 - n_1\bar{x})(\sigma_1 - n_2\bar{y}).$$

Then, the interior equilibrium point P^* of the system is locally asymptotically stable if $m_2 > 0$ and $m_1m_2 - m_0 > 0$. □

Theorem 3.5 *Suppose the equilibrium point* $P^*(x^*, y^*, z^*)$ *exists, if* $n_1 + \frac{\sigma_2 y^* n_2}{\sigma_1 x^*} < 2\min\left(\left(\frac{r_1}{K} + u\right) - \frac{az^*}{b(b+x^*)}, \frac{\sigma_2 y^* v}{\sigma_1 x^*}\right)$, *then* P^* *is globally asymptotically stable.*

Proof The Lyapunov function is given by

$$V(x, y, z) = \left(x - x^* - x^*\ln\left(\frac{x}{x^*}\right)\right) + l_1\left(y - y^* - y^*\ln\left(\frac{y}{y^*}\right)\right)$$
$$+ l_2\left(z - z^* - z^*\ln\left(\frac{z}{z^*}\right)\right).$$

where l_i for $i = 1, 2$ are positive constants to be determined in the subsequent steps.

$$\frac{dV}{dt} = (x - x^*)\left(-\left(\frac{r_1}{K} + u\right)(x - x^*) - a\left(\frac{z}{b + x} - \frac{z^*}{b + x^*}\right)\right.$$
$$\left. + \sigma_2\left(\frac{y}{x} - \frac{y^*}{x^*}\right) - n_1(y - y^*)\right)$$
$$+ l_1(y - y^*)\left(-v(y - y^*) + \sigma_1\left(\frac{x}{y} - \frac{x^*}{y^*}\right) - n_2(x - x^*)\right)$$
$$+ l_2(z - z^*)\beta a\left(\frac{x}{b + x} - \frac{x^*}{b + x^*}\right).$$

Let us choose $l_1 = \frac{\sigma_2 y^*}{\sigma_1 x^*}$ and $l_2 = \frac{b+x^*}{b\beta}$, then we have,

$$\begin{aligned}\frac{dV}{dt} &= -\left(\frac{r_1}{K}+u\right)(x-x^*)^2 - \frac{\sigma_2 y^* v}{\sigma_1 x^*}(y-y^*)^2 - \frac{\sigma_2}{xx^*y}(yx^*-xy^*)^2 \\ &\quad + \frac{az^*}{(b+x)(b+x^*)}(x-x^*)^2 - \left(n_1 + \frac{\sigma_2 y^* n_2}{\sigma_1 x^*}\right)(x-x^*)(y-y^*), \\ &\le \left(\frac{az^*}{b(b+x^*)} - \left(\frac{r_1}{K}+u\right) + \frac{1}{2}\left(n_1 + \frac{\sigma_2 y^* n_2}{\sigma_1 x^*}\right)\right)(x-x^*)^2 \\ &\quad - \left(\frac{\sigma_2 y^* v}{\sigma_1 x^*} - \frac{1}{2}\left(n_1 + \frac{\sigma_2 y^* n_2}{\sigma_1 x^*}\right)\right)(y-y^*)^2 - \frac{\sigma_2}{xx^*y}(yx^*-xy^*)^2.\end{aligned}$$

Then, $\frac{dV}{dt} \le 0$ if $n_1 + \frac{\sigma_2 y^* n_2}{\sigma_1 x^*} < 2\min\left(\left(\frac{r_1}{K}+u\right) - \frac{az^*}{b(b+x^*)}, \frac{\sigma_2 y^* v}{\sigma_1 x^*}\right)$. □

4 Optimal harvesting policy

In this section, the Pontryagins Principle is used to obtain a path of optimal harvesting policy. Let D be the constant harvesting cost per unit effort, p_1 is the constant price per unit biomass of the prey in the unreserved zone, p_2 is the constant price per unit biomass of the predator.

The net economic revenue at any time t is given by

$$\pi(x, y, z, E) = p_1 q_1 E(t) x(t) + p_2 q_2 E(t) z(t) - D E(t).$$

In what follows, our goal is to solve the problem of maximization

$$\begin{aligned} I &= \int_0^T \pi(x, y, z, E)\, e^{-\delta t} dt, \\ &= \int_0^T \left(p_1 q_1 x(t) + p_2 q_2 z(t) - D \right) E(t)\, e^{-\delta t} dt,\end{aligned}$$

where δ is the instantaneous discount rate, subject to the state equations (9) and the control contraints $0 \le E \le E_{\max}$.

Thus, to solve the problem of maximization, we use the Pontryagin's maximum principle. The Hamiltonian function H is given by

$$\begin{aligned} H = e^{-\delta t}&\left(p_1 q_1 x(t) + p_2 q_2 z(t) - D \right) E(t) + \lambda_1(t)\left(r_1 x(t)\left(1 - \frac{x(t)}{K}\right)\right. \\ &\left. -\sigma_1 x(t) + \sigma_2 y(t) - u x(t)^2 - \frac{a x(t) z(t)}{b + x(t)} - q_1 E x(t) - n_1 x(t) y(t)\right) \\ &+\lambda_2(t)\left(r_2 y(t) + \sigma_1 x(t) - \sigma_2 y(t) - v y(t)^2 - n_2 x(t) y(t)\right)\end{aligned}$$

$$+\lambda_3(t)\left(\frac{\beta a x(t) z(t)}{b+x(t)} - dz(t) - wz(t) - q_2 E(t) z(t)\right),$$

where λ_i for $i = 1, 2, 3$ are the adjoint variables.

H will be maximized under the control set $0 \le E \le E_{\max}$ if the switching function given by

$$\psi(t) := \frac{\partial H}{\partial E} = \big(p_1 q_1 x(t) \ + \ p_2 q_2 z(t) \ - \ D\big) e^{-\delta t} \\ - q_1 \lambda_1(t) x(t) \ - \ q_2 \lambda_3(t) z(t) = 0.$$

This is a necessary condition for singular control to be optimal. Using the Maximum Principle, we get the adjoint equations

$$\begin{aligned} \frac{d\lambda_1}{dt} &= -\frac{\partial H}{\partial x} \\ &= -p_1 q_1 e^{-\delta t} E(t) - \lambda_1(t)\bigg(r_1 - 2\frac{r_1}{K} x(t) - \sigma_1 - 2ux(t) - \frac{abz(t)}{(b+x(t))^2} \\ &\qquad -q_1 E(t) - n_1 y(t)\bigg) - \lambda_2(t)\Big(\sigma_1 - n_2 y(t)\Big) - \lambda_3(t)\frac{\beta a b z(t)}{(b+x(t))^2}, \\ \frac{d\lambda_2}{dt} &= -\frac{\partial H}{\partial y} \\ &= -\lambda_1(t)\Big(\sigma_2 - n_1 x(t)\Big) - \lambda_2(t)\Big(r_2 - \sigma_2 - 2vy(t) - n_2 x(t)\Big), \\ \frac{d\lambda_3}{dt} &= -\frac{\partial H}{\partial z} \\ &= -p_2 q_2 e^{-\delta t} E(t) - \lambda_1(t)\frac{a x(t)}{b+x(t)} - \lambda_3(t)\bigg(\frac{\beta a x(t)}{b+x(t)} - (d + w + q_2 E(t))\bigg). \end{aligned}$$

Let denote that the optimal harvesting policy is

$$\begin{cases} E = E_{max} & \text{when } \psi(t) > 0, \\ E = E^* & \text{when } \psi(t) = 0, \\ E = 0 & \text{when } \psi(t) < 0, \end{cases} \tag{15}$$

where E^* the optimal control.

If $\psi(t) = 0$, then

$$q_1 \lambda_1(t) e^{\delta t} x \ + \ q_2 \lambda_3(t) e^{\delta t} z \ = \ p_1 q_1 x \ + \ p_2 q_2 z \ - \ D \ = \ \frac{\partial \pi}{\partial E}.$$

To find the optimal equilibrium solution for this system, we consider x, y, z and E as constants.

For the interior equilibrium P^* and under (10), (12), (13), we have

$$\frac{d\lambda_3}{dt} = M_1\lambda_3 - M_2 e^{-\delta t},$$

where $M_1 = \frac{-aq_2z^*}{q_1(b+x^*)}$, and $M_2 = p_2q_2E - \frac{\left(p_1q_1x^*+p_2q_2z^*-D\right)a}{q_1(b+x^*)}$. Then,

$$\lambda_3(t) = \frac{M_2}{M_1+\delta}\, e^{-\delta t}.$$

In the same way, we have

$$\frac{d\lambda_2}{dt} = N_1\lambda_2 - N_2 e^{-\delta t},$$

where $N_1 = -r_2 + \sigma_2 + 2vy^* + n_2x^*$ and $N_2 = \frac{\left(p_1q_1x^*+p_2q_2z^*-D\right)\left(\sigma_2-n_1x^*\right)}{q_1x^*} + \frac{M_2q_2z^*\left(\sigma_2-n_1x^*\right)}{\left(M_1+\delta\right)q_1x^*}$.
Hence

$$\lambda_2(t) = \frac{N_2}{N_1+\delta}\, e^{-\delta t}.$$

The expression of $\frac{d\lambda_1}{dt}$ can be written as

$$\frac{d\lambda_1(t)}{dt} = B_1\lambda_1(t) - B_2 e^{-\delta t},$$

where $B_1 = -\left(r_1 - \sigma_1 - q_1E\right) + \left(\frac{2r_1}{K} + 2u\right)x^* + \frac{abz^*}{(b+x^*)^2} + n_2y^*$ and
$B_2 = p_1q_1E - \frac{N_2\left(\sigma_1-n_2y^*\right)}{N_1+\delta} + \frac{\beta M_2abz^*}{\left(M_1+\delta\right)\left(b+x^*\right)^2}$.
By calculation, we get

$$\lambda_1(t) = \frac{B_2}{B_1+\delta}\, e^{-\delta t}.$$

Thus, the previous calculation, leads to

$$\frac{B_2}{B_1+\delta}q_1x^* + \frac{M_2}{M_1+\delta}q_2z^* = p_1q_1x^* + p_2q_2z^* - D. \tag{16}$$

So, in case of infinite discount rate, the net economic revenue to the society becomes zero and the fishery would remain closed.

We consider the function

$$
\begin{aligned}
F(x) = {} & \frac{K\left(\sigma_1 - n_2 y\right)\left(\sigma_2 - n_1 x\right)\left(p_1 q_1 x + p_2 q_2 z - D\right)\left(b + x\right)^2}{\Delta_2\left(\delta - r_2 + \sigma_2 - n_1 x + 2vy\right)} \\
& + \frac{K p_1 q_1^2 E\left(b + x\right)^2}{\Delta_2} - \frac{K\beta a^2 b q_1 x z\left(p_1 q_1 x + p_2 q_2 z - D\right)}{\Delta_2\left(\delta - r_2 + \sigma_2 - n_1 x + 2vy\right)} \\
& + \frac{K q_1 p_2 q_2^2 E z\left(\sigma_1 - n_2 y\right)\left(\sigma_2 - n_1 x\right)\left(b + x\right)^3}{\Delta_2\left(\delta - r_2 + \sigma_2 - n_1 x + 2vy\right)\left(\delta q_1 (b + x) - a q_2 z\right)} \\
& - \frac{K a q_2 z\left(\sigma_1 - n_2 y\right)\left(\sigma_2 - n_1 x\right)\left(p_1 q_1 x + p_2 q_2 z - D\right)\left(b + x\right)^2}{\Delta_2\left(\delta - r_2 + \sigma_2 - n_1 x + 2vy\right)\left(\delta q_1 (b + x) - a q_2 z\right)} \\
& + \frac{K\beta a b q_1^2 p_2 q_2 E x z\left(b + x\right)\left(\delta - r_2 + \sigma_2 - n_1 x + 2vy\right)}{\Delta_2\left(\delta - r_2 + \sigma_2 - n_1 x + 2vy\right)\left(\delta q_1 (b + x) - a q_2 z\right)} \\
& + \frac{q_1 p_2 q_2^2 E z\left(b + x\right) - a q_2 z\left(p_1 q_1 x + p_2 q_2 z - D\right)}{\delta q_1\left(b + x\right) - a q_2 z} \\
& - \left(p_1 q_1 x + p_2 q_2 z - D\right),
\end{aligned}
$$

where

$$
\begin{aligned}
\Delta_2 = {} & K\left(b + x\right)^2\left(\delta - r_1 + \sigma_1 + q_1 E\right) + 2\left(r_1 + uK\right)\left(b + x\right)^2 \\
& + n_2 y\left(b + x\right)^2 + abKx.
\end{aligned}
$$

Thus, the (16) can be written as $F(x^*) = 0$. On the one hand, the calculation gives $F(0) > 0$, and the other, $F'(x) < 0$ for $0 < x < K$. Then, there exists a unique positive solution $x^* = x_\delta$ of the equation $F(x) = 0$.
So, according to the above analysis, we propose the following theorem.

Theorem 4.1 *If* $\frac{\beta a x_\delta}{b+x_\delta} - d - w > 0$, *then the optimal harvesting control* $E_\delta = \frac{1}{q_2}\left(\frac{\beta a x_\delta}{b+x_\delta} - d - w\right)$ *and the corresponding solutions*

$$\begin{aligned} x_\delta &= x^*, \\ y_\delta &= \frac{r_2 - \sigma_2 - n_2 x_\delta + \sqrt{(r_2 - \sigma_2 - n_2 x_\delta)^2 + 4v\sigma_1 x_\delta}}{2v}, \\ z_\delta &= \frac{(b + x_\delta)\left[-\left(u + \frac{r_1}{K}\right) x_\delta^2 + (r_1 - \sigma_1 - q_1 E - n_1 y_\delta)x_\delta + \sigma_2 y_\delta\right]}{a x_\delta} \end{aligned}$$

exist that maximize I *over* $[\,0, E_{\max}]$.

5 Numerical simulations

For our simulation works, we take system parameters as:

$$\begin{aligned} &r_1 = 5\ , r_2 = 1,\ \ K = 4, \sigma_1 = 1,\ \ \sigma_2 = 1,\ \ a = 0.94,\ \ b = 0.7,\ \ q_1 = 0.1, \\ &q_2 = 0.2, E = 0.8, \beta = 0.998,\ \ u = 0.0001,\ \ v = 0.333,\ \ w = 0.00003, \\ &d = 0.03,\ \ n_1 = 0.5,\ \ n_2 = 0.3. \end{aligned} \tag{17}$$

For the set of value parameters mentioned above, we note that the positive equilibrium $P^*(x^*, y^*, z^*)$ exists and is given by

$$x^* = 0.1778, \quad y^* = 0.6550, \quad z^* = 6.5873.$$

We plot the dynamics of the system (1) for the set of values parameters (17). The behavior of x, y and z with respect to time t is plotted in Fig. 1. From this figure, we note that x, y and z increase for a short time and then they decrease and finally attain their equilibrium level.

As in Fig. 2, the Fig. 3 shows the behavior of x, y and z with different initial values. From this figure, we see that all trajectories starting with different initial points converge to $P^*(0.1778, 0.6550, 6.5873)$. Thus P^* is globally asymptotically stable.

We observe that n_1 and n_2 are also an important parameters which governs the dynamics of system (1). The behavior of x, y and z with respect to time t for different values of n_1 and n_2 are shown in Figs. 4 and 5. From Fig. 4, we note that x and y decrease in a short time and increase after as n_1 increases, but z decreases as n_1 increases. From Fig. 5, we note that x decreases in a short time and increases after as n_2 increases, but y and z decrease as n_2 increases.

The results found are compatible with the analysis and the study presented in the Sect. 3. From Figs. 4 and 5, x^* is invariant with respect to n_1 and n_2, z^* is variable with respect to n_1 and n_2 and y^* is invariant with respect to n_1 and is variable with respect to n_2. These results justify the formulas (10), (12) and (13) of which x^* does not depend on n_1 nor on n_2, y^* depends on n_2 but not on n_1. Finally z^* depends on n_1 and n_2.

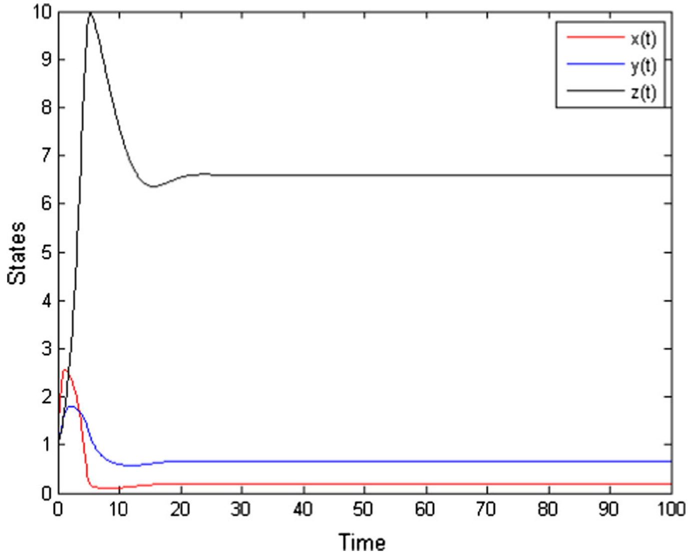

Fig. 1 Solution curves corresponding to the set values parameters (17), beginning with $x = 1$, $y = 1$, $z = 1$

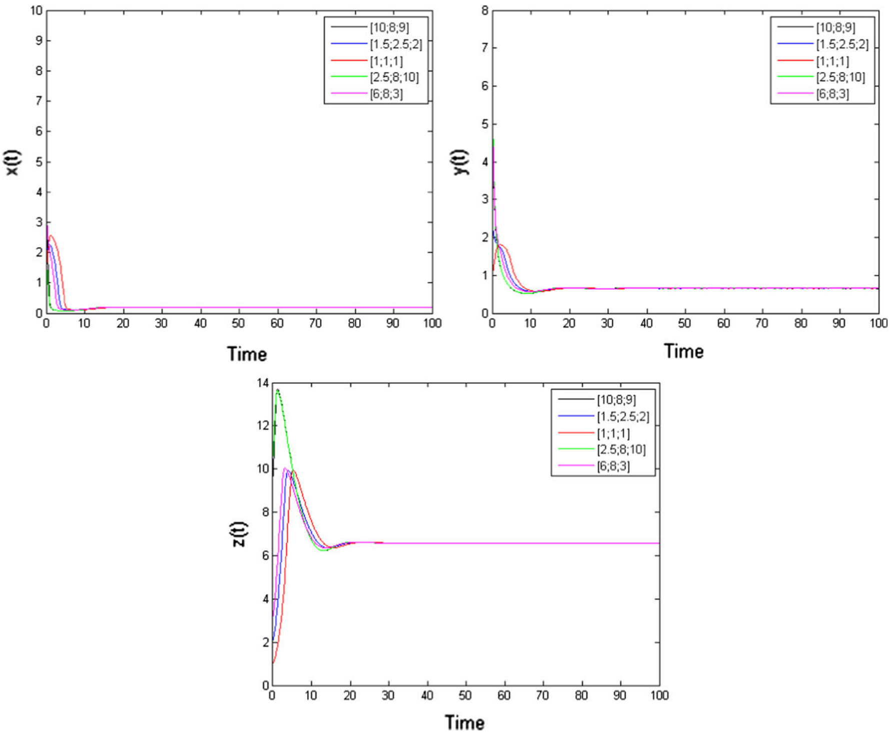

Fig. 2 The equilibrium point P^* of is globally asymptotically stable. x, y, z states for different initial points

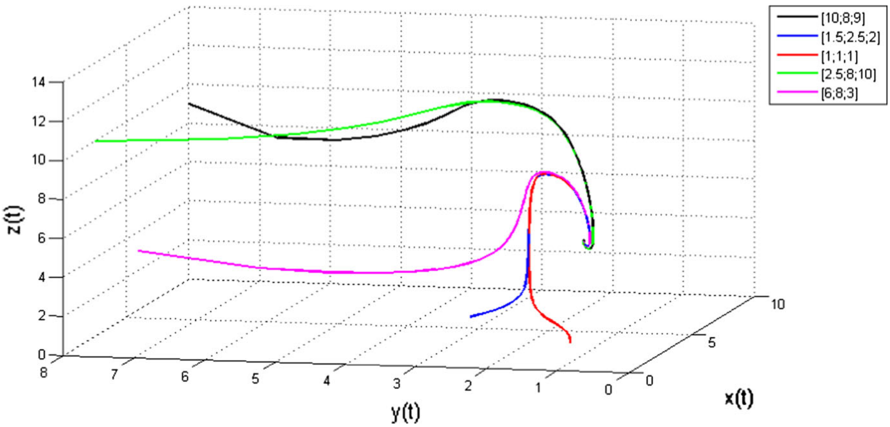

Fig. 3 The phase trajectory of the system for the set values parameters (17)

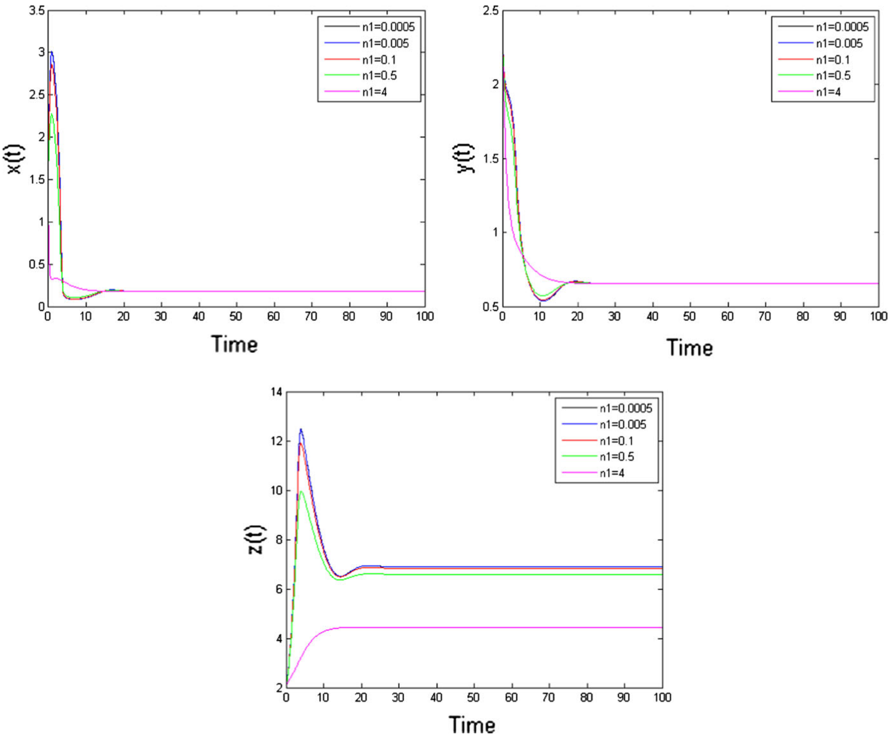

Fig. 4 Plot of x, y, z with respect to time t for different values of n_1, others values of parameters are same as given in (17)

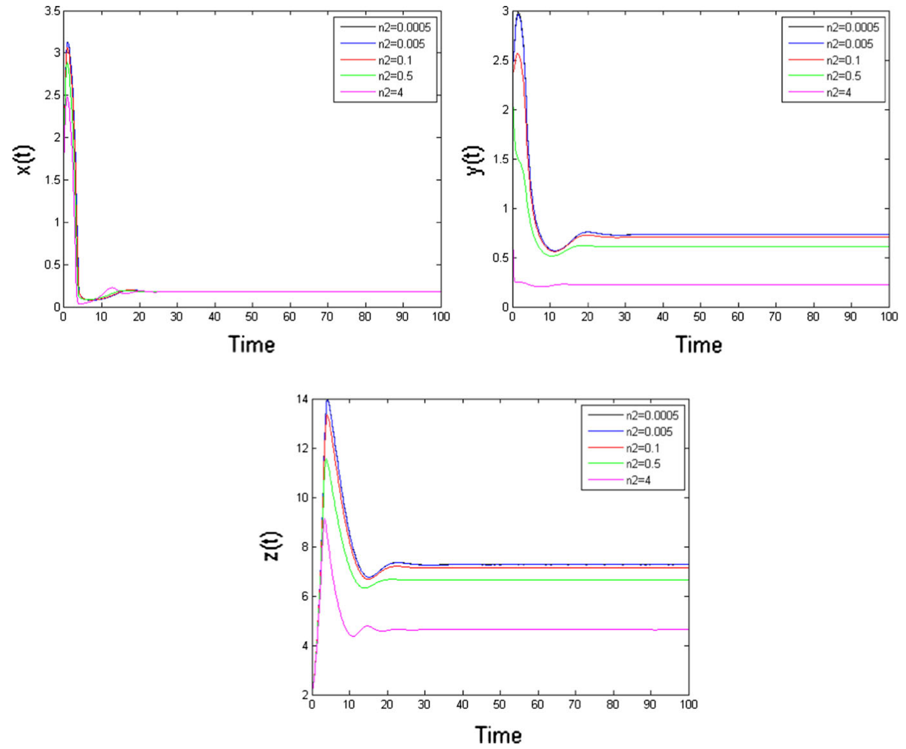

Fig. 5 Plot of x, y, z with respect to time t for different values of n_2 and $n_1 = 0.3$, others values of parameters are same as given in (17)

6 Conclusion

We proposed and analyzed a model of predator-predator Holling II functional response with harvest for reserve fishery resources and incorporate toxic substances released by external agents into natural systems taking into account competition and consider an ecosystem where a species Predator depends on simple prey species with harvest. And the habitat consists of an unrestricted zone, where the predator attacks its only food prey, and the reserved area, where the prey lived safely. Our model represents a development from other preliminary studies. We analyze the positivity and the limit of these solutions. We also study the criteria for the existence of all the possible equilibria of this system, as well as discuss the local stability of different equilibria of the system. We are also discussing the optimal harvest policy by the maximal Pontryagins principle. By simulation, we show the rich dynamic properties of the proposed system. First, from the conditions of Theorem, we see that the locally asymptotically stable free point of equilibrium of the predator $\bar{P}$ of the system (1). Secondly, the internal equilibrium point P^* of the system (1) is globally asymptotically stable, which is consistent with the theoretical analysis.

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