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ORIGINAL RESEARCH

Local exponential stabilization for a class of uncertain nonlinear impulsive periodic switched systems with norm-bounded input

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Abstract This paper investigates stabilization for a class of uncertain nonlinear impulsive periodic switched systems under a norm-bounded control input. The proposed approach studies stabilization criteria locally where the nonlinear dynamics satisfy the Lipschitz condition only on a subspace containing the origin, not on $\mathbb{R}^n$. This makes the proposed approach applicable in most practical cases where the region of validity is limited due to physical issues. In presence of different resources of non-vanishing uncertainties, the main objective is to find a stabilizing control signal such that not only trajectories exponentially converge to a sufficient small ultimate bound, but also have the largest region of attraction. To this, for a more general model, we first propose several sufficient conditions using the common Lyapunov function approach. The proposed strategy allows the Lyapunov function to increase in some intervals, which is suitable when some of the subsystems are unstable and uncontrollable. We then apply these conditions to the targeted system, and the sufficient criteria are extracted in the forms of linear and bilinear matrix inequalities. To achieve the main goal, an optimization problem is also formulated which is solvable using augmented Lagrangian methods. Finally, some illustrative examples are presented to demonstrate the proposed approach.

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1 Introduction

Impulsive switched systems, as a particularly challenging class of hybrid systems, consist of a finite number of subsystems and a switching law where the state variables may jump at switching instants [1]. The impulsive switched systems have numerous applications in modeling and control of real-world processes, such as in drugs distribution in human body [2,3], in population ecology [4,5], in mechanical systems [6,7], and in chaotic systems [8–10]. For more practical applications, reader may refer to [1].

The high applicability of impulsive switched systems raises attention to the fundamental issues in control theory. Most of the developed approaches use various types of Lyapunov function candidate as a powerful tool. For example, they use discretized Lyapunov function when they are faced with jumps in state variables or when they are planning to have less conservatism [11–13]. For more details, we may refer to [14,15] and references therein.

Since stability and stabilization under any arbitrary switching signal is not always possible [16], the Lyapunov-based efforts usually study constrained switching laws, especially time-dependent ones. In these cases, researchers consider either minimum and/or maximum duration between two consecutive switching instants (called minimum/maximum dwell-time) [17–21] or using average dwell-time [11,13,22–27]. When stability criteria are defined on a subspace, use of average-based methods is lacking in efficiency, because there is no guarantee that trajectories remain in this subspace. Therefore, most studies that use the average dwell-time techniques develop the stability criteria on the entire state space. As discussed later, this may be conservative. Nevertheless, by knowing minimum and/or maximum dwell-time, we can ensure that trajectories never leave this subspace.

Furthermore, to guarantee the stability, finding a common Lyapunov function (CLF) or Multiple Lyapunov functions (MLF) is essential. Although CLF approach leads to fewer parameters as compared with MLF, it makes a lot of conservatism. MLF techniques have better results than CLF ones, especially when more details of the switching signal are available. The main idea of the MLF techniques is based on this fact that the Lyapunov functions do not necessarily always have non-positive time-derivative. To guarantee convergence of Lyapunov functions over the time, the Lyapunov functions are upper-bounded, and relations between them is also considered at switching instants [10–13,21,23–26,28–33]. Obviously, for an arbitrary switching signal, number of the relations grows exponentially with the number of subsystems. Hence, in order to decrease the number of parameters and required conditions, this paper take into account CLF approach, but the Lyapunov function is allowed to increase in some intervals.

The above-mentioned studies have two other limitations. First, they often do not consider actuator saturation. Second, proposed stability criteria are usually global which is conservative when control input is norm-bounded or when region of validity for state variables is limited due to practical/physical issues.

In the field of switched systems and impulsive systems, to the best of authors' knowledge, there are only a few works that address actuator saturation [19,34–37]. However, in practice, several important aspects of switched systems should be considered simultaneously, such as impulsive jumps at switching instants, system nonlinearity, non-vanishing uncertainties and norm-boundedness of control signal to avoid actuator saturation. Heretofore, these aspects have not been addressed at the same time.

On the other hand, in most approaches where nonlinear dynamics are considered, they take into account the Lipschitz condition on $\mathbb{R}^n$ and then, they establish stability criteria. Unfortunately, in many practical systems, the Lipschitz condition is not met on $\mathbb{R}^n$, or may leads to very large Lipschitz constants and infeasibility. Note, as reported in [19,37], in the presence of input constraints, obtaining a global stabilizing controller is difficult. Besides, the region of validity for state variables may be limited by physical issues [38]. Therefore, considering global Lipschitz condition and obtaining the global stability criteria are conservative.

The above-mentioned limitations along with the wide practical use of impulsive switched systems for modelling and control of real-world processes motivate us to study stabilization for a class of nonlinear impulsive periodic switched systems that concurrently embraces most important challenging aspects of real-world processes. These aspects include various types of non-vanishing uncertainties, norm-boundedness of control input to avoid actuator saturation, region of validity for state variables, and realistic Lipschitz condition for nonlinear dynamics. The proposed method develops stabilization criteria in terms of a common Lyapunov function candidate. Unlike the traditional CLF approaches, we allow the Lyapunov function to have positive derivative in some intervals. Hence, our method not only has fewer parameters than MLF approaches, but also is applicable when there are unstable and uncontrollable subsystems. In addition, unlike other methods that consider norm-bounded inputs, and then propose the stability criteria globally (e.g. see [37]), our approach develops stabilization conditions locally (on a subspace containing the equilibrium point). This simplifies necessary assumption such as Lipschitz condition for known nonlinear dynamics, and makes it applicable when the region of validity is limited. However, to ensure that trajectories never leave the mentioned subspace, the stability conditions are extracted with respect to the minimum dwell-time and the maximum period of switching cycle. The established criteria are then reformulated as matrix inequalities using S-Lemma and Schur complement [37,39].

Besides, due to non-vanishing uncertainties, the proposed approach only guarantees convergence of trajectories to an ultimate bound containing the origin, not to the origin. To achieve the smallest ultimate bound along with the largest region of attraction, this paper also propose an optimization problem with linear and bilinear constraints. Here, all bilinear terms contain scalar variables. Therefore, in addition to augmented Lagrangian methods (such as PENBMI [40]), some other reliable LMI-based algorithms can be used to solve this optimization problem, for example see [37].

We can summarize the main contributions of this paper as:

- Considering norm-bounded control input for a model that includes impulse effect, nonlinear dynamics, and different types of uncertainties. The model of real-world systems includes linear and nonlinear dynamics along with non-vanishing uncertainties. Control signal is also norm-bounded to avoid actuator saturation. Most of the previous works (especially in the field of the impulsive switched systems) do not consider these challenging aspects concurrently.
- Providing stabilization conditions locally over a subspace. This enables us to use them in switched systems with locally Lipschitz nonlinearities, and to use them in practical systems where the region of validity is limited. Note, as seen in the literature, researchers often develop condition on the entire state space and consider global Lipschitz assumption for nonlinear dynamics, which may have two problems. Firstly, most of real-world systems may not satisfy this assumption globally. Secondly, if a system meets it globally, it may lead to large Lipschitz coefficients, and as a result, may lead to larger upper limits for the time-derivative of Lyapunov function. Consequently, established stability conditions for such systems may be infeasible. Besides, due to practical/physical issues, the system model may only be valid on a subspace of state space (called region of validity). In these cases, developing of the stability conditions locally ensures the validity of results.
- Considering positive derivatives for Lyapunov function candidate and bounding them by an exponential function. They enable us to use our approach when there are unstable and uncontrollable subsystems, and enable us to prove exponential convergence.
- Developing an optimization problem to find the largest region of attraction along with the smallest ultimate bound. This also takes into account many practical control issues. Clearly, increasing gain of controller can reduce the size of ultimate bound. However, this will increase the norm of control signal and thus reduce region of attraction. Therefore, we develop the objective function such that compromise between the size of region of attraction and the size of ultimate bound, while the norm-boundedness of control signal and the region of validity are considered as constraints. According to the authors' knowledge, in the field of impulsive switched systems, previous works have not taken into account these aspects simultaneously.

The rest of this paper is organized as follows. Section 2 introduces the problem formulation, including some useful lemmas. In Sect. 3, firstly, for a more general model of impulsive switched systems, sufficient stability conditions are proposed in terms of a CLF, and then, for the targeted system, they are reformulated as linear and bilinear matrix inequalities using Schur complement and S-Lemma. Simulation of the resulting control law for some illustrative examples are provided in Sect. 4. Finally, conclusions are drawn in Sect. 5.

Notations The notations that are used throughout this paper are standard. The sets of positive integers, real non-negative scalars, real n-dimensional vectors and $n \times m$ real matrices are presented by $\mathbb{N}^+$, $\mathbb{R}^+$, $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$, respectively. The symmetric positive (or semi-positive) definite matrix A is indicated by $A > 0$ (or $A \geq 0$).

In addition, $\|x\| := \left(x^T x\right)^{1/2}$ specifies the Euclidean norm of $x \in \mathbb{R}^n$ where the superscript "T" represents the transpose. The identity and zero matrices are denoted by I and 0, respectively. Moreover, the ellipsoid $\mathcal{E}(P, r)$, which is associated with the matrix $P > 0$ and the scalar $r > 0$, is given by $\left\{x \in \mathbb{R}^n | x^T P x \leq r\right\}$. Besides, for simplicity, we show the Lyapunov function $V(x(t))$ as $V(t)$. Furthermore, the symbol "$*$" in matrix inequalities denotes the matrix's symmetric part. Finally, if dimensions of some matrices are not explicitly stated, it is assumed that they have appropriate dimensions for algebraic operations

2 Problem formulation

This paper studies local stabilization for the following impulsive switched uncertain nonlinear system in which the control inputs vector $u(t) \in \mathbb{R}^m$ is norm-bounded,

$$\begin{cases} \dot{x}(t) = (A_i + \Delta A_i)\, x + f_{ci}(x) + (B_i + \Delta B_i)\, u(t) + \phi_i(t), & t \neq t_k \\ x(t^+) = C_i x(t), & t = t_k \end{cases} \tag{1}$$

where $\{t_k\}_{k=1}^{\infty} := \{t_1, t_2, \ldots, t_k, \ldots\}$ is a strictly increasing sequence of impulse instants composed by the switching signal $\sigma(t) : \mathbb{R}^+ \to \{1, 2, \ldots, m\}$ and m is the number of subsystems. Also, $i \in \{1, 2, \ldots, m\}$ represents the index of active subsystem determined by the switching signal $\sigma(t)$. At impulse instants, when a subsystem switches to another one, the state vector $x(t) \in \mathbb{R}^n$ suddenly changes according to the known jump matrix $C_i \in \mathbb{R}^{n \times n}$. It is assumed that the state vector $x(t)$ is left continuous at impulse instants such that,

$$x(t_k) = x\left(t_k^-\right) = lim_{\zeta \to 0^+} x(t_k - \zeta).$$

In addition, the right limit of the state vector $x(t)$ at the impulse instant t_k is defined as,

$$x\left(t_k^+\right) = lim_{\zeta \to 0^+} x(t_k + \zeta).$$

The matrices $A_i \in \mathbb{R}^{n \times n}$ and $B_i \in \mathbb{R}^{n \times m}$ are known and constant. Also, the vector function $f_{ci}(x) : \mathbb{R}^n \to \mathbb{R}^n$ is known that represents nonlinear dynamics of the subsystem i with $f_{ci}(0) = 0$. Besides, ΔA_i, ΔB_i and $\phi_i(t)$ represent different types of uncertainties and unknown perturbations for the subsystem i. In this paper, we consider a class of system (1) with the following periodic switching scheme,

$$\sigma(t') = \sigma(t''), \quad \forall t' \in (t_{k-1}, t_k] \quad and\ \forall t'' \in (t_{k+m-1}, t_{k+m}],$$

where $t_0 \geq 0$ is the initial time. In addition, without loss of generality, we assume,

$$\sigma(t) = k, \quad \forall t \in (t_{k-1}, t_k] \quad and\ 1 \leq k \leq m,$$

which is equivalent to the following switching sequences,

$$\mathcal{T} := \{(t_0, 1), (t_1, 2), \ldots, (t_{m-1}, m), (t_m, 1), (t_{m+1}, 2), \ldots, (t_{2m-1}, m), \ldots\}.$$

Furthermore, we suppose that minimum dwell-time τ_{min} and maximum period of switching cycle τ_{pmax} are known as follow,

$$\tau_{min} := inf_{k\in\mathbb{N}^+} (t_k - t_{k-1}), \quad (2)$$
$$\tau_{pmax} := sup_{k\in\mathbb{N}^+} (t_{k+m} - t_k). \quad (3)$$

We also assume the following assumptions,
(A1) Uncertainties and perturbations satisfy the following form,

$$\left[\, \Delta A_i (t) \;\; \Delta B_i (t) \;\; \phi_i (t) \,\right] = D_i F_i (t) \left[\, E_{ai} \;\; E_{bi} \;\; E_{\phi i} \,\right],$$

where D_i, E_{ai}, E_{bi} and $E_{\phi i}$ are known constant matrices and $F_i (t)$ is an unknown time varying matrix with Lebesgue measurable elements such that $F_i^T (t) F_i (t) \leq I$ for all $t \in \mathbb{R}^+$.

Remark There are two main classes of uncertainties: time-varying uncertainties and polytopic type uncertainties. This paper considers the time-varying uncertainties, and we shall to investigate other kinds of uncertainties in future researches.

(A2) The vector function $f_{ci} (x)$ satisfies the following Lipschitz condition locally, on a region of validity $\mathbb{D} \subset \mathbb{R}^n$,

$$f_{ci} (x)^T f_{ci} (x) = \| f_{ci} (x) \|^2 \leq \|M_i x\|^2 = x^T M_i^T M_i x, \quad \forall x \in \mathbb{D}.$$

where M_i is a constant matrix with appropriate dimension.

Remark The assumption (A2) is common in the field of impulsive nonlinear systems. To guarantee validation of proposed stability/stabilization criteria, researchers usually consider (A2) globally, especially when they use the average dwell-time methods. Here, we replace it with a more relaxed one, i.e. locally Lipschitz condition.

Before developing main results, we introduce several lemmas used to prove the subsequent theorems and can increase the readability of the paper.

Lemma 1 (Λ-inequality in [37]) *Given real matrices $S_1, S_2 \in \mathbb{R}^{n\times m}$. If Λ be a symmetric real positive definite matrix with appropriate dimensions, the following inequality holds,*

$$S_1^T S_2 + S_2^T S_1 \leq S_1^T \Lambda S_1 + S_2^T \Lambda^{-1} S_2.$$

Lemma 2 [41] *Given real matrices D, E and F with appropriate dimensions. If $F^T F \leq I$, then for any scalar $\gamma > 0$, the following inequality holds,*

$$DFE + E^T F^T D^T \leq \gamma DD^T + \gamma^{-1} E^T E.$$

Lemma 3 (Schur complement in [39]) *Suppose A, B, C, D are respectivley $n \times n$, $n \times p$, $p \times n$ and $p \times p$ matrices, and D is invertible. Let*

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

Then the Schur complement of the block D of the matrix M is the $n \times n$ matrix $A - BD^{-1}C$. Let D be positive definite. Then M is positive semi-definite if and only if the Schur complement of D in M is positive semi-definite.

Lemma 4 (S-Lemma in [37]) *Consider the real scalar α_0, α_1, ..., α_k and the quadratic forms,*

$$f_i(x) = x^T A_i x, \quad i = 0, 1, \ldots, k,$$

where $x \in \mathbb{R}^n$ and $A_i \in \mathbb{R}^{n \times n}$ is symmetric. If there exist real non-negative scalars $\tau_1, \tau_2, \ldots, \tau_m$ such that,

$$\alpha_0 - \tau_1\alpha_1 - \tau_2\alpha_2 - \ldots - \tau_m\alpha_m \geq 0,$$
$$\tau_1 A_1 + \tau_2 A_2 + \ldots + \tau_m A_m - A_0 \geq 0,$$

then the system of inequalities,

$$f_i(x) \leq \alpha_i, \quad i = 1, 2, \ldots, k,$$

implies the specific inequality,

$$f_0(x) \leq \alpha_0.$$

3 Main results

This section develops an approach to design a stabilizing control signal for the impulsive switched system (1). Here, we try to find a Lyapunov-like function $V(x)$ that has certain conditions for all $x \in \Omega_c \backslash \Omega_f$ where Ω_c is a region of attraction, and Ω_f is an inner subspace. Under these conditions, we show that $V(x)$ exponentially decreases over the time, and the trajectories ultimately converge to an ultimate bound Ω_u, and remain in it for future times (see Fig. 1). In fact, this is an extension of the Attractive Ellipsoid (AE) approach [37] to the impulsive switched systems. It is also similar to [42,43] where they study the non-impulsive switched systems. In this paper, we not only consider the impulse effect, but also provide conditions locally.

At first, we propose some sufficient stability conditions for the following general nonlinear impulsive periodic switched system,

$$\begin{cases} \dot{x}(t) = f_i(x(t)), & t \neq t_k, k \in \mathbb{N}^+ \\ x(t^+) = g_i(x(t)), & t = t_k, k \in \mathbb{N}^+ \end{cases}, \tag{4}$$

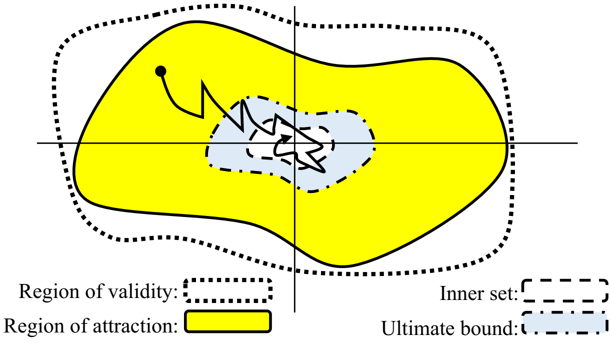

Fig. 1 All trajectories starting from Ω_c ultimately converge Ω_u and remain in it

where f_i and g_i are nonlinear continuous- and discrete-time dynamic regarding to the active subsystem i. The active subsystem i is determined by the periodic switching signal $\sigma(t)$. We show that all trajectories starting in Ω_c ultimately converge to Ω_u where

$$\Omega_c := \nu(V, r) = \left\{x \in \mathbb{R}^n | V(x) \leq r\right\},$$
$$\Omega_u := \nu\left(V, \delta''\right) = \left\{x \in \mathbb{R}^n | V(x) \leq \delta''\right\}$$

V is a suitable Lyapunov function candidate, and $r > \delta'' > 0$ are real scalars. To this, Theorem 1 simply supposes that the subsystem $i = 1$ satisfies stable/controllable assumption, and then, Corollary 1 considers another case where the subsystem $i \neq 1$ is stable (or controllable). After that, Theorem 2 takes into account an impulsive periodic switched system that has more than one stable (or controllable) subsystem.

Theorem 1 *Given real scalars $0 < \delta < r$. Suppose there exist a suitable Lyapunov function candidate V, real positive scalars ρ_{on}, ρ_{off}, $\mu \geq 1$, and $\delta' > \delta$ such that,*

$$\begin{cases} \dot{V} \leq -\rho_{on} V, & \sigma(t) = 1 \\ \dot{V} \leq \rho_{off} V, & \sigma(t) \neq 1 \end{cases}, \quad \forall x \in \upsilon(V, r) \backslash \upsilon(V, \delta), \forall t \in [t_k, t_{k+1}), \tag{5}$$

$$V\left(t_k^+\right) \leq \mu V(t_k), \quad \forall x \in \upsilon(V, r) \backslash \upsilon(V, \delta), \tag{6}$$

$$V\left(t_k^+\right) \leq \delta', \quad \forall x \in \upsilon(V, \delta), \tag{7}$$

$$\alpha := \left(\rho_{on} + \rho_{off}\right) \frac{\tau_{min}}{\tau_{pmax}} - \frac{ln\, \mu}{\tau_{min}} - \rho_{off} > 0, \tag{8}$$

$$\mu^{m-1} \delta' exp\left(\rho_{off}\left(\tau_{pmax} - \tau_{min}\right)\right) < r, \tag{9}$$

where τ_{min} and τ_{pmax} are defined in (2) and (3), respectively, and,

$$\nu(V, r) := \left\{x \in \mathbb{R}^n | V(x) \leq r\right\}, \quad \nu(V, \delta) := \left\{x \in \mathbb{R}^n | V(x) \leq \delta\right\}.$$

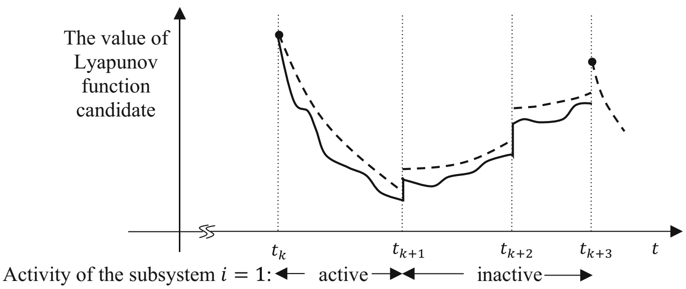

Fig. 2 Changes in the value of Lyapunov function candidate during a switching cycle (solid line). They are bounded by some exponential functions (dashed lines). The value of the Lyapunov function candidate can also increase immediately after an impulse instant

Then, for all initial conditions $x_0 \in \upsilon(V, r)$, system (4) is ultimate bounded stable with the ultimate bound,

$$\upsilon\left(V, \delta''\right) := \left\{x \in \mathbb{R}^n | V(x) \leq \delta''\right\},$$

where δ'' is a real positive scalar such that,

$$\mu^{m-1}\delta' exp\left(\rho_{off}\left(\tau_{pmax} - \tau_{min}\right)\right) \leq \delta'' \leq r.$$

In addition, all trajectories exponentially converge to $\upsilon\left(V, \delta''\right)$.

Before proving Theorem 1, it is recommended to pay attention to the following remarks.

Remark Before converging to $\nu(V, \delta)$, (5) ensures that none of the subsystems has finite time escape and $V(t)$ falls under a decreasing or incremental function. Also, at switching instants, (6) states that the Lyapunov function candidate can be discrete. Fig. 2 illustrates (5) and (6) for a typical impulsive switched system with three subsystems.

Remark When a trajectory reaches to $\nu(V, \delta)$, the impulse must be such that trajectory does not leave the outer set $\nu(V, r)$. This is guaranteed by considering (7) and (9). Also, the condition (8) guarantees that the value of Lyapunov function candidate generally falls under a descending exponential function, before the trajectory converges to $\nu\left(V, \delta''\right)$.

Proof We first show that all trajectories starting in $\upsilon(V, r) \setminus \upsilon(V, \delta)$ exponentially converge to $\upsilon(V, \delta)$. Assume that the convergence occurs during $(k+1)^{th}$ period of switching scheme (i.e. at $t_\delta \in \left[t_{k\times m}, t_{(k+1)\times m}\right)$ where m is the number of subsystems). After that, we show that if a trajectory reaches to $\upsilon(V, \delta)$, it remains in the invariant set $\upsilon\left(V, \delta''\right)$.

From (5), we have for all $t \in [t_0, t_1)$ where $\sigma(t) = 1$,

$$V(t) \leq V(t_0)\, exp\left(-\rho_{on}(t - t_0)\right).$$

This gives,

$$V(t_1) \leq V(t_0)\, exp\left(-\rho_{on}(t_1 - t_0)\right).$$

Based on (6), we obtain,

$$V\left(t_1^+\right) \leq \mu V(t_1) \leq \mu V(t_0)\, exp\left(-\rho_{on}(t_1 - t_0)\right). \tag{10}$$

However, during the time interval $(t_1, t_m]$, the first subsystem is inactive and V may increase according to (5). Therefore, using (5) and (10), we can conclude for all $t \in (t_1, t_2]$,

$$\begin{aligned} V(t) &\leq V\left(t_1^+\right) exp\left(\rho_{off}(t - t_1)\right) \\ &\leq \mu V(t_0)\, exp\left(-\rho_{on}(t_1 - t_0)\right) exp\left(\rho_{off}(t - t_1)\right), \end{aligned}$$

which yields,

$$V(t_2) \leq \mu V(t_0)\, exp\left(-\rho_{on}(t_1 - t_0)\right) exp\left(\rho_{off}(t_2 - t_1)\right),$$

and,

$$V\left(t_2^+\right) \leq \mu^2 V(t_0)\, exp\left(-\rho_{on}(t_1 - t_0)\right) exp\left(\rho_{off}(t_2 - t_1)\right).$$

By following the same procedure for other time intervals, we have,

$$\begin{aligned} V\left(t_m^+\right) &\leq \mu^m V(t_0)\, exp\left(-\rho_{on}(t_1 - t_0)\right) exp\left(\rho_{off}(t_m - t_1)\right) \\ &= \mu^m V(t_0)\, exp\left(-\left(\rho_{on} + \rho_{off}\right)(t_1 - t_0)\right) exp\left(\rho_{off}(t_m - t_0)\right) \\ &\leq \mu^m V(t_0)\, exp\left(-\left(\rho_{on} + \rho_{off}\right)\tau_{min}\right) exp\left(\rho_{off}(t_m - t_0)\right). \end{aligned}$$

where t_m^+ is the next switched-on instant for the subsystem $i = 1$. Similarly, we have for the second switching cycle,

$$\begin{aligned} V\left(t_{2m}^+\right) &\leq \mu^m V\left(t_m^+\right) exp\left(-\left(\rho_{on} + \rho_{off}\right)\tau_{min}\right) exp\left(\rho_{off}(t_{2m} - t_m)\right) \\ &\leq \mu^{2m} V(t_0)\, exp\left(-2\left(\rho_{on} + \rho_{off}\right)\tau_{min}\right) exp\left(\rho_{off}(t_{2m} - t_0)\right). \end{aligned}$$

Likewise, we can conclude,

$$\begin{aligned} V\left(t_{km}^+\right) &\leq \mu^{k\times m} V(t_0)\, exp\left(-k\left(\rho_{on} + \rho_{off}\right)\tau_{min}\right) exp\left(\rho_{off}(t_{km} - t_0)\right) \\ &\leq V(t_0)\, exp\left(km\, ln\mu - k\left(\rho_{on} + \rho_{off}\right)\tau_{min} + \rho_{off}(t_{km} - t_0)\right). \end{aligned}$$

where $t_{km} = t_{k\times m}$ is the last switching instant in switching cycle k. Since,

$$k \times m \leq \frac{t_{km} - t_0}{\tau_{min}}, \quad and, \quad \frac{t_{km} - t_0}{\tau_{pmax}} \leq k,$$

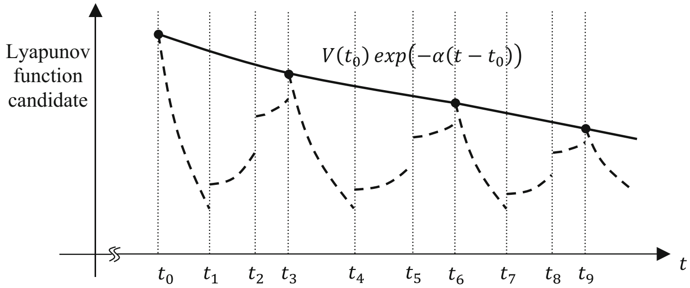

Fig. 3 The upper bound for the value of Lyapunov function candidate during the first three switching cycles (solid line) for a typical system with three subsystems. Dashed lines also indicate the upper bounds during the active and inactive intervals

we have,

$$V\left(t_{km}^{+}\right) \leq V\left(t_0\right) exp\left(\left(\frac{ln\,\mu}{\tau_{min}} - \frac{\tau_{min}}{\tau_{pmax}}\left(\rho_{on} + \rho_{off}\right) + \rho_{off}\right)\left(t_{km} - t_0\right)\right).$$

Therefore, we can conclude that the value of Lyapunov function candidate V falls under an exponential function, as follows,

$$V\left(t_{km}^{+}\right) \leq V\left(t_0\right) exp\left(-\alpha\left(t_{km} - t_0\right)\right), \quad \forall t \in \left(t_0, t_{km}\right], \tag{11}$$

where α is defined in (8). As stated in (8), if $\alpha > 0$, then the value of V decreases until the trajectory reaches $\upsilon\left(V, \delta\right)$. Fig. 3 illustrates it for a typical impulsive switched system.

Now, suppose that trajectory reaches the inner set $\upsilon\left(V, \delta\right)$ at t_δ. If $t_\delta \in \left[t_{km}, t_{km+1}\right)$, then the subsystem $i = 1$ is active and according to (5), the trajectory remains in $\upsilon\left(V, \delta\right)$ until the next switching instant t_{km+1}. Similar to what seen above, we have,

$$\begin{aligned}
V\left(t_{(k+1)m}^{+}\right) &\leq \mu^{m-1} V\left(t_{km+1}^{+}\right) exp\left(\rho_{off}\left(t_{(k+1)m} - t_{km+1}\right)\right)\\
&\leq \mu^{m-1} V\left(t_{km+1}^{+}\right) exp\left(\rho_{off}\left(\tau_{pmax} - \tau_{min}\right)\right)\\
&\leq \mu^{m-1} \delta' exp\left(\rho_{off}\left(\tau_{pmax} - \tau_{min}\right)\right) \leq \delta''.
\end{aligned}$$

Clearly, for other cases that $t_\delta \in \left[t_{km+1}, t_{(k+1)m}\right)$, the value of $V(t_{(k+1)m}^{+})$ is less than δ''. Therefore, if $\delta'' < r$, the trajectory remains in the outer set, and then converges again to $\upsilon\left(V, \delta\right)$. Hence, we can generally conclude that the system (4) is ultimate bounded stable with the ultimate bound $\upsilon\left(V, \delta''\right)$ and the trajectory exponentially converges to it. This completes the proof. □

Theorem 1 assumes that the subsystem $i = 1$ satisfies $\dot{V} \leq -\rho_{on} V$. The proposed approach in Theorem 1 is also applicable when the subsystem $i = 1$ is unstable and uncontrollable, but there is another subsystem satisfies this.

Corollary 1 *Given real scalars* $0 < \delta < r$. *Suppose there exist a suitable Lyapunov function candidate* V, *real positive scalars* ρ_{on}, ρ_{off}, $\mu \geq 1$, *and* $\delta' > \delta$ *such that* (6)–(9) *and the following condition hold,*

$$\begin{cases} \dot{V} \leq -\rho_{on} V, \ \sigma(t) = j \\ \dot{V} \leq \rho_{off} V, \ \ \sigma(t) \neq j \end{cases}, \quad \forall x \in \upsilon(V, r) \setminus \upsilon(V, \delta), \forall t \in [t_k, t_{k+1}),$$

where $j \in \{1, 2, \ldots, m\}$ *is the index of a desired subsystem. Then system* (4) *meets all the results mentioned in Theorem* 1 *for all initial conditions* $x_0 \in \upsilon\left(V, r'\right)$ *where,*

$$r' \leq \mu^{-(m-1)} r \, exp\left(-\rho_{off}\left(\tau_{pmax} - \tau_{min}\right)\right).$$

Proof Since none of the subsystems has finite time escape and the switching cycle time is upper bounded by τ_{pmax}, there is an instant during the first switching cycle when the subsystem j is activated, i.e. at t_{j-1}^{+}. From this moment onwards, Theorem 1 is in place, and we just have to show that the trajectories does not leave the outer set $\upsilon(V, r)$ during time interval $[t_0, t_{j-1}^{+}]$.

Since the initial condition x_0 belongs to $\upsilon\left(V, r'\right)$, it can be easily demonstrated that for all $t \in [t_0, t_{j-1}^{+}]$,

$$\begin{aligned} V(x(t)) &\leq \mu^{j-1} r' exp\left(\rho_{off}\left(t_{j-1} - t_0\right)\right) \\ &\leq \mu^{m-1} r' exp\left(\rho_{off}\left(\tau_{pmax} - \tau_{min}\right)\right) \leq r, \end{aligned}$$

and the trajectory remains in the outer set $\upsilon(V, r)$. This completes the proof. □

Although the Corollary 1 eliminates the stability assumption for the first subsystem, it is extremely conservative, and only considers decreasing constraint $\dot{V} \leq -\rho_{on} V$ for one of the subsystems while this may be available for a set of subsystems. In these cases, the invariant set $\upsilon\left(V, \delta''\right)$ may become smaller and the convergence rate may be faster.

Theorem 2 *Given real scalars* $0 < \delta < r$. *Suppose there exist a suitable Lyapunov function candidate* V, *real positive scalars* ρ_{on}, ρ_{off}, $\mu \geq 1$, $\delta' > \delta$, *and an* n-*element subset* $\mathcal{P}$ *of the subsystems* $\{1, 2, \ldots, m\}$ *where* $n \leq m$ *such that,*

$$\begin{cases} \dot{V} \leq -\rho_{on} V, \ \sigma(t) \in \mathcal{P} \\ \dot{V} \leq \rho_{off} V, \ \ \sigma(t) \notin \mathcal{P} \end{cases}, \quad \forall x \in \upsilon(V, r) \setminus \upsilon(V, \delta), \forall t \in [t_k, t_{k+1}), \tag{12}$$

$$V\left(t_k^{+}\right) \leq \mu V(t_k), \quad \forall x \in \upsilon(V, r) \setminus \upsilon(V, \delta), \tag{13}$$

$$V\left(t_k^{+}\right) \leq \delta', \quad \forall x \in \upsilon(V, \delta), \tag{14}$$

$$\alpha := \left(\rho_{on} + \rho_{off}\right) \frac{n\tau_{min}}{\tau_{pmax}} - \frac{ln \, \mu}{\tau_{min}} - \rho_{off} > 0, \tag{15}$$

$$\mu^{m-n} \delta' exp\left(\rho_{off}\left(\tau_{pmax} - n\tau_{min}\right)\right) < r, \tag{16}$$

where τ_{min} *and* τ_{pmax} *are defined in* (2) *and* (3)*, respectively. Then, system* (4) *is ultimate bounded stable with the ultimate bound* $\upsilon\left(V, \delta''\right)$ *for all initial conditions* $x_0 \in \upsilon\left(V, r''\right)$ *where,*

$$r' \leq \mu^{-(m-n)} r exp\left(-\rho_{off}\left(\tau_{pmax} - n\tau_{min}\right)\right), \tag{17}$$

and,

$$\mu^{m-n}\delta' exp\left(\rho_{off}\left(\tau_{pmax} - n\tau_{min}\right)\right) < \delta'' < r. \tag{18}$$

In addition, system (4) *exponentially converges to the invariant set* $\upsilon\left(V, \delta''\right)$*.*

Proof Similar to Theorem 1, assume that the subsystem $i = 1$ is stable, i.e. we have $\dot{V} \leq -\rho_{on} V$ for all $t \in [t_0, t_1)$. During a switching cycle, since n subsystems satisfy $\dot{V} \leq -\rho_{on} V$ and others fulfil $\dot{V} \leq \rho_{off} V$, the Lyapunov function candidate decreases for at least $n\tau_{min}$ and increases for up to $\left(\tau_{pmax} - n\tau_{min}\right)$. In addition, due to the impulse effect, there are m jumps in the changes of Lyapunov function candidate. So, we can conclude,

$$V\left(t_{km}^+\right) \leq V\left(t_0\right)\mu^{k\times m} exp\left(-k\left(\rho_{on} + \rho_{off}\right) n\tau_{min}\right) exp\left(\rho_{off}\left(t_{km} - t_0\right)\right),$$

which, similar to (11), yields,

$$\begin{aligned} V\left(t_{km}^+\right) &\leq V\left(t_0\right) exp\left(\left(\frac{ln\,\mu}{\tau_{min}} - \frac{n\tau_{min}}{\tau_{pmax}}\left(\rho_{on} + \rho_{off}\right) + \rho_{off}\right)\left(t_{km} - t_0\right)\right) \\ &= V\left(t_0\right) exp\ \left(-\alpha\left(t_{km} - t_0\right)\right). \end{aligned}$$

According to (15), the above inequality guarantees exponential convergence before reaching $\upsilon\left(V, \delta\right)$.

To estimate the invariant set $\upsilon\left(V, \delta''\right)$, we also consider the worst case. Note that, when the active subsystem does not belong to $\mathcal{P}$, in the worst case, the value of Lyapunov function candidate is incremental. Therefore in the worst case, reaching to $\upsilon\left(V, \delta\right)$ will surely occur when the active subsystem belongs to $\mathcal{P}$. Absolutely, the trajectory remains in $\upsilon\left(V, \delta\right)$ until the next switching instant. Now suppose that during the next $(m - n)$ switches, none of the activate subsystems belong to $\mathcal{P}$. Hence, we can conclude,

$$\delta'' \geq \mu^{m-n}\delta' exp\left(\rho_{off}\left(\tau_{pmax} - n\tau_{min}\right)\right),$$

and in order to ensure that the trajectory remains in $\nu\left(V, r\right)$, δ'' should be lower than r.

Now, if the first subsystem does not belong to $\mathcal{P}$, it takes up to $\left(\tau_{pmax} - n\tau_{min}\right)$ to activate a subsystem from $\mathcal{P}$. In this case, during the time interval $\left[t_0,\ \tau_{pmax} - n\tau_{min}\right]$, the Lyapunov function may increase exponentially with the maximum rate ρ_{off}, and there are $(m - n)$ impulses. So,

$$V\left(t_{m-n}^+\right) \leq V\left(t_0\right)\mu^{m-n} exp\left(\rho_{off}\left(\tau_{pmax} - n\tau_{min}\right)\right).$$

Hence, in order to ensure that the trajectory remain in the outer set $\nu(V, r)$, the initial conditions x_0 must belong to $\nu\left(V, r'\right)$ where r' is defined in (17). This completes the proof. □

Remark In Theorem 2, if all subsystems satisfy the decreasing condition for the Lyapunov function candidate (i.e. $n = m$), then $\rho_{off} = 0$ and we can replace (15) with the following standard condition,

$$\alpha := \rho_{on} - \frac{ln\, \mu}{\tau_{min}} > 0.$$

Now, we are ready to reformulate the proposed conditions in Theorem 2 for the targeted impulsive uncertain nonlinear periodic switched system (1). Suppose the following control signal,

$$u(t) = u_\sigma(t) \tag{19}$$

where $u_i = K_i x,\ i \in \{1, 2, \ldots, m\}$ are some suitable control signals. According to the switching control law $\sigma(t)$, the control input u_i is active when the subsystem i is active. Also, consider the common quadratic Lyapunov function candidate $V = x^T P x$. During the time intervals that the subsystem i is active, using (A1), the time derivative of V is,

$$\dot{V} = \dot{x}^T P x + x^T P \dot{x} = x^T \left(\Upsilon_i + \Upsilon_i^T\right) x + \Pi_i + \Pi_i^T. \tag{20}$$

where,

$$\begin{aligned} \Upsilon_i &= A_i^T P + K_i^T B_i^T P + E_{ai}^T F_i^T D_i^T P + K_i^T E_{bi}^T F_i^T D_i^T P, \\ \Pi_i &= f_{ci}^T P x + E_{\phi i}^T F_i^T D_i^T P x. \end{aligned}$$

Using Λ-inequality, we have,

$$f_{ci}^T P x + x^T P f_{ci} \leq \zeta_i x^T P P x + \zeta_i^{-1} f_{ci}^T f_{ci} \leq x^T \left(\zeta_i P P + \zeta_i^{-1} M_i^T M_i\right) x, \tag{21}$$

where ζ_i is a real positive scalar, and M_i is the matrix of Lipschitz constants that are related to f_{ci}. In addition, using Lemma 2, we can obtain the following inequalities,

$$E_{\phi i}^T F_i^T D_i^T P x + x^T P D_i F_i E_{\phi i} \leq \gamma_{\phi i} x^T P D_i D_i^T P x + \gamma_{\phi i}^{-1} E_{\phi i}^T E_{\phi i}, \tag{22}$$

$$E_{ai}^T F_i^T D_i^T P + P D_i F_i E_{ai} \leq \gamma_{ai} P D_i D_i^T P + \gamma_{ai}^{-1} E_{ai}^T E_{ai}, \tag{23}$$

$$K_i^T E_{bi}^T F_i^T D_i^T P + P D_i F_i E_{bi} K_i \leq \gamma_{bi} P D_i D_i^T P + \gamma_{bi}^{-1} K_i^T E_{bi}^T E_{bi} K_i, \tag{24}$$

where γ_{ai}, γ_{bi} and $\gamma_{\phi i}$ are positive real scalars. Substituting (21)–(24) into (20) yields,

$$\dot{V} \leq x^T (\Psi_i + \Theta_i) x + \gamma_{\phi i}^{-1} E_{\phi i}^T E_{\phi i},$$

where,

$$\begin{aligned}\Psi_i &= A_i^T P + P A_i + K_i^T B_i^T P + P B_i K_i + \zeta_i P P + \left(\gamma_{\phi i} + \gamma_{ai} + \gamma_{bi}\right) P D_i D_i^T P,\\ \Theta_i &= \zeta_i^{-1} M_i^T M_i + \gamma_{ai}^{-1} E_{ai}^T E_{ai} + \gamma_{bi}^{-1} K_i^T E_{bi}^T E_{bi} K_i.\end{aligned}$$

According to (12), we should have,

$$\dot{V} + \lambda_i V \leq 0, \quad \forall x \in \Omega := \mathcal{E}(P, r) \backslash \mathcal{E}(P, \delta),$$

where,

$$\lambda_i = \begin{cases} \rho_{on}, & i = \sigma(t) \in \mathcal{P} \\ -\rho_{off}, & i = \sigma(t) \notin \mathcal{P} \end{cases}.$$

Hence, (12) is true if the following condition holds,

$$x^T (\Psi_i + \Theta_i + \lambda_i P) x \leq -\gamma_{\phi i}^{-1} E_{\phi i}^T E_{\phi i}, \quad \forall x \in \Omega := \mathcal{E}(P, r) \backslash \mathcal{E}(P, \delta).$$

Based on the S-Lemma, the above condition holds if there exist positive real scalars τ_{1i} and τ_{2i} such that,

$$-\gamma_{\phi i}^{-1} E_{\phi i}^T E_{\phi i} - \tau_{1i} r + \tau_{2i} \delta \geq 0, \tag{25}$$

and,

$$\tau_{1i} P - \tau_{2i} P - (\Psi_i + \Theta_i + \lambda_i P) \geq 0. \tag{26}$$

According to Schur complement, (25) is equivalent to,

$$\begin{bmatrix} -\tau_{1i} r + \tau_{2i} \delta & E_{\phi i}^T \\ * & \gamma_{\phi i} I \end{bmatrix} \geq 0, \tag{27}$$

Moreover, we can rewrite (26) as follows,

$$\begin{bmatrix} \tau_{1i} P - \tau_{2i} P - \Psi_i - \lambda_i P & M_i^T & E_{ai}^T & K_i^T E_{bi}^T \\ * & \zeta_i I & 0 & 0 \\ * & * & \gamma_{ai} I & 0 \\ * & * & * & \gamma_{bi} I \end{bmatrix} \geq 0.$$

Pre- and post-multiplying the above matrix inequality by $diag\left(P^{-1}, I, I, I\right)$ gives,

$$\begin{bmatrix} \tau_{1i} P^{-1} - \tau_{2i} P^{-1} - \psi_i & P^{-1} M_i^T & P^{-1} E_{ai}^T & P^{-1} K_i^T E_{bi}^T \\ * & \zeta_i I & 0 & 0 \\ * & * & \gamma_{ai} I & 0 \\ * & * & * & \gamma_{bi} I \end{bmatrix} \geq 0, \tag{28}$$

where,

$$\psi_i = P^{-1}A_i^T + A_i P^{-1} + P^{-1}K_i^T B_i^T + B_i K_i P^{-1} + \zeta_i I \\ + \left(\gamma_{\phi i} + \gamma_{ai} + \gamma_{bi}\right) D_i^T D_i + \lambda_i P^{-1}.$$

In summary, the criterion (12) holds if the matrix inequalities (27)–(28) are met for all $i \in \{1, 2, \ldots, m\}$.

Now, to obtain the property (13) in the form of matrix inequalities, substituting $V = x^T Px$ into (13) gives,

$$x^T \left(t_k^+\right) Px \left(t_k^+\right) \leq \mu x^T (t_k) Px (t_k), \quad \forall x \in \mathcal{E}(P, r) \backslash \mathcal{E}(P, \delta).$$

Using the jump functions of system (1), we can conclude that,

$$x^T (t_k) \left(C_i^T PC_i - \mu P\right) x (t_k) \leq 0, \quad \forall x \in \mathcal{E}(P, r) \backslash \mathcal{E}(P, \delta).$$

Hence, by applying S-Lemma, (13) is satisfied if there exist positive real scalars τ_{3i} and τ_{4i} such that,

$$-\tau_{3i} r + \tau_{4i}\delta \geq 0, \tag{29}$$

and

$$\tau_{3i} P - \tau_{4i} P - \left(C_i^T PC_i - \mu P\right) \geq 0. \tag{30}$$

Using Schur complement for the matrix inequality (30) and then pre- and post-multiplying it by $diag\left(P^{-1}, I\right)$, we have,

$$\begin{bmatrix} \tau_{3i} P^{-1} - \tau_{4i} P^{-1} + \mu P^{-1} & P^{-1} C_i^T \\ * & P^{-1} \end{bmatrix} \geq 0. \tag{31}$$

In this way, we reformulate (13) into the equivalent matrix inequalities (29) and (31). Similarly, by substituting $V\left(t_k^+\right) = x^T\left(t_k^+\right) Px\left(t_k^+\right) = x^T (t_k) C_i^T PC_i x (t_k)$ into (14) and then applying the S-Lemma, we can conclude that the condition (14) also holds if there exist real positive scalar τ_{5i} such that,

$$\delta' - \tau_{5i}\delta \geq 0, \tag{32}$$

and

$$\tau_{5i} P - C_i^T PC_i \geq 0. \tag{33}$$

Again, applying the Schur complement and pre- and post-multiplying by $diag\left(P^{-1}, I\right)$, we can rewrite (33) as follows,

$$\begin{bmatrix} \tau_{5i} P^{-1} & P^{-1} C_i^T \\ * & P^{-1} \end{bmatrix} \geq 0. \tag{34}$$

Therefore, considering (27), (28), (29), (31), (32), and (34), the conditions (12)–(14) are held. In addition, if (15) and (16) be also true then all trajectories of the system

(1) under the control input (19) remain in the outer set $\mathcal{E}(P, r)$ for all initial conditions $x_0 \in \mathcal{E}(P, r')$ and exponentially converge to the ultimate bound $\mathcal{E}(P, \delta'')$. However, in practice, the maximum admissible control magnitude is always restricted [37], for example with the following inequality,

$$\|u\|^2 = u^T u \leq u_{max}^2,$$

where u_{max} is a given real positive scalar. Hence, we also derive some other sufficient conditions that guarantee the norm-boundedness of the control signal (19) on the outer set $\mathcal{E}(P, r)$. To have a norm-bounded input, the following statement is necessary for all $i \in \{1, 2, \ldots, m\}$,

$$\|u_i\|^2 = u_i^T u_i = x^T K_i^T K_i x \leq u_{max}^2, \quad \forall x \in \mathcal{E}(P, r).$$

Applying S-Lemma to the above inequality gives,

$$u_{max}^2 - \tau_{6i} r \geq 0. \tag{35}$$

and,

$$\tau_{6i} P - K_i^T K_i \geq 0. \tag{36}$$

where τ_{6i} is a positive real scalar. Again, according to Schur complement, (36) can be written as,

$$\begin{bmatrix} P & K_i^T \\ * & \tau_{6i} I \end{bmatrix} \geq 0. \tag{37}$$

Now, pre- and post-multiplying $diag\left(P^{-1}, I\right)$ to (37) yields,

$$\begin{bmatrix} P^{-1} & P^{-1} K_i^T \\ * & \tau_{6i} I \end{bmatrix} \geq 0, \tag{38}$$

So, the norm-boundedness of switching control input (19) is guaranteed if (36) and (38) are met for all $i \in \{1, 2, \ldots, m\}$. We summarize the above discussion in Theorem 3.

Theorem 3 *Given positive real scalars* $0 < \delta < r$ *and* u_{max}. *Suppose there exist an n-element subset* $\mathcal{P} \subseteq \{1, 2, \ldots, m\}$ *of the subsystems where* $n \leq m$, *a real symmetric positive definite matrix L, real matrices* W_i, *real positive scalars* $r', \delta'', \rho_{on}, \rho_{off}, \mu \geq 1, \delta' > \delta, \zeta_i, \tau_{ji}, \gamma_{ai}, \gamma_{bi}, \gamma_{\phi i}$ *where* $i \in \{1, 2, \ldots, m\}$ *and* $j \in \{1, 2, \ldots, 6\}$ *such that the following conditions hold for all* $i \in \{1, 2, \ldots, m\}$,

$$\left(\rho_{on} + \rho_{off}\right) \frac{n\tau_{min}}{\tau_{pmax}} - \frac{ln\,\mu}{\tau_{min}} - \rho_{off} > 0, \tag{39}$$

$$\mu^{m-n} \delta' exp\left(\rho_{off}\left(\tau_{pmax} - n\tau_{min}\right)\right) \leq \delta'' \leq r, \tag{40}$$

$$\mu^{m-n} r' exp\left(\rho_{off}\left(\tau_{pmax} - n\tau_{min}\right)\right) \leq r, \tag{41}$$

$$\begin{bmatrix} -\tau_{1i}r + \tau_{2i}\delta & E_{\phi i}^T \\ * & \gamma_{\phi i} I \end{bmatrix} \geq 0, \tag{42}$$

$$\begin{bmatrix} \tau_{1i}L - \tau_{2i}L - \psi_i & LM_i^T & LE_{ai}^T & W_i^T E_{bi}^T \\ * & \zeta_i I & 0 & 0 \\ * & * & \gamma_{ai} I & 0 \\ * & * & * & \gamma_{bi} I \end{bmatrix} \geq 0, \tag{43}$$

$$-\tau_{3i}r + \tau_{4i}\delta \geq 0, \tag{44}$$

$$\begin{bmatrix} \tau_{3i}L - \tau_{4i}L + \mu L & LC_i^T \\ * & L \end{bmatrix} \geq 0, \tag{45}$$

$$\delta' - \tau_{5i}\delta \geq 0, \tag{46}$$

$$\begin{bmatrix} \tau_{5i}L & LC_i^T \\ * & L \end{bmatrix} \geq 0, \tag{47}$$

$$u_{max}^2 - \tau_{6i}r \geq 0, \tag{48}$$

$$\begin{bmatrix} L & W_i^T \\ * & \tau_{6i} I \end{bmatrix} \geq 0, \tag{49}$$

where,

$$\psi_i = LA_i^T + A_i L + W_i^T B_i^T + B_i W_i + \zeta_i I + \left(\gamma_{\phi i} + \gamma_{ai} + \gamma_{bi}\right) D_i^T D_i + \lambda_i L,$$

and,

$$\lambda_i = \begin{cases} \rho_{on}, & i \in \mathcal{P} \\ -\rho_{off}, & i \notin \mathcal{P} \end{cases}.$$

Then, the closed-loop system (1), under the control law (19) with $u_i = K_i x = W_i L^{-1} x$, exponentially converges to $\mathcal{E}\left(L^{-1}, \delta''\right)$ for all initial conditions $x(t_0) \in \mathcal{E}\left(L^{-1}, r'\right)$. In addition, the control signal is norm-bounded for all $x(t) \in \mathcal{E}\left(L^{-1}, r\right)$ such that $\|u\|^2 = u^T u \leq u_{max}^2$.

Proof Let $L = P^{-1}$ and $W_i = K_i P^{-1} = K_i L$, and then substitute them into (28), (31), (34) and (38). This completes the proof. □

Remark To achieve the smallest attractive ellipsoid (ultimate bound), we can minimize δ'' and $trace\left\{P^{-1}\right\}$. To reduce the computational load, without loss of generality, we choose $\delta = 1$, and then try to reduce the size of $\mathcal{E}(P, \delta)$ by minimizing the trace of matrix $L = P^{-1}$, which reduces the sum of the squares of the ellipsoid's semiaxes [37]. However, reducing the trace of matrix L reduces the size of outer set $\mathcal{E}(P, r)$ when r is constant. Therefore, in order to maximize the outer ellipsoid contained in the region of validity $\mathbb{D}$, we maximize r, too. Hence, we propose the following optimization problem:

$$\begin{aligned} &minimize: \quad trace\{L\} + \omega_1\delta'' - \omega_2 r \\ &subject\ to: \quad r \geq \delta = 1, \quad \mu \geq 1, \quad \delta' > \delta = 1, \quad (39)\text{–}(49) \end{aligned} \tag{50}$$

where $\omega_1, \omega_2 > 0$ are real positive scalar as tradeoff coefficients.

Remark In practical stabilization, it must be ensured that all trajectories converge to the predetermined ellipsoid $\mathcal{E}(Q_{in}, 1)$, where $Q_{in} > 0$ is a given symmetric positive definite matrix. Considering the following implications,

$$\delta'' P^{-1} \leq Q_{in}^{-1} \leq r P^{-1} \Rightarrow \mathcal{E}\left(P,\ \delta''\right) \subseteq \mathcal{E}\left(Q_{in},\ 1\right) \subseteq \mathcal{E}\left(P,\ r\right),$$

we also involve the following linear matrix inequalities into the optimization problem (50),

$$\delta'' L \leq Q_{in}^{-1}, \quad \begin{bmatrix} L & X_{in}^T \\ * & rI \end{bmatrix} \geq 0. \tag{51}$$

where $X_{in}^T X_{in} = Q_{in}^{-1}$.

Remark In some nonlinear systems, the Lipschitz condition is not valid on $\mathbb{R}^n$ while may be met on a domain $\mathbb{D} \subseteq \mathbb{R}^n$, as assumed in (A2). In order to ensure the validity of (21), the following condition shall apply,

$$\mathcal{E}(P, r) \subseteq \mathcal{E}(Q_{out}, 1) \subseteq \mathbb{D},$$

where $Q_{out} > 0$ is a desired symmetric positive definite matrix. The above condition results from,

$$r P^{-1} \leq Q_{out}^{-1}.$$

Hence, let's consider the following bilinear matrix inequality along with other constraint,

$$Q_{out}^{-1} - rL \geq 0. \tag{52}$$

Remark Most of constraints in Theorem 3 are linear or bilinear that motivate us to use an augmented Lagrangian solver such as PENBMI. PENBMI allows one to resolve optimization problems with quadratic objective and bilinear matrix inequality (BMI) constraints [40]. Nevertheless, (39), (40) and (41) are nonlinear constraint that should be expressed in bilinear or linear forms. Since $\mu \geq 1$, we have $ln(\mu) \leq \mu - 1$. Therefore, we replace (39) with the following conservative but linear constraint,

$$\left(\rho_{on} + \rho_{off}\right) \frac{\tau_{min}}{\tau_{pmax}} - \frac{\mu - 1}{\tau_{min}} - \rho_{off} > 0, \tag{53}$$

In order to overcome the nonlinearity of (40) and (41), we give some relaxation. Suppose there exist a predetermined scalar $\pi \geq 1$ such that,

$$\mu^{m-n} exp\left(\rho_{off}\left(\tau_{pmax} - n\tau_{min}\right)\right) \leq \pi.$$

The above inequality holds if,

$$exp\left((m-n)\,ln\,\mu+\rho_{off}\left(\tau_{pmax}-n\tau_{min}\right)\right) \leq exp\left((m-n)\,(\mu-1)+\rho_{off}\left(\tau_{pmax}-n\tau_{min}\right)\right)\leq\pi,$$

In this way, we can replace (40) and (41) by the following linear constraints,

$$\pi\delta'\leq\delta''\leq r, \tag{54}$$

$$\pi r'\leq r, \tag{55}$$

$$(m-n)(\mu-1)+\rho_{off}(\tau_{pmax}-n\tau_{min})\leq ln\,\pi. \tag{56}$$

Due to the presence of the logarithmic function in (56), they are not bilinear with respect to π. Therefore, in the solving algorithm, we suggest the augmentation of parameter π taking,

$$\pi_k=\pi_{k-1}+\Delta\pi,$$

where $\pi_0=1$ and $0<\Delta\pi\ll 1$.

Remark The efficiency of PENBMI is critical to initial point selection. Fortunately, in Theorem 3, all bilinear terms contain the scalar variables. In this case, some other suitable LMI-based algorithms can be used to solve the optimization problem (50), for example see [37].

Considering the above remarks, we redefine the optimization problem (50) as follows,

$$\begin{aligned} &minimize: \quad trace\,\{L\}+\omega_1\delta''-\omega_2 r\\ &subject\ to: \quad r\geq 1,\quad \mu\geq 1,\quad \delta'>1,\quad (42)\text{–}(49),\ (51)\text{–}(56)\end{aligned} \tag{57}$$

and propose the following algorithm,

1. *Determine desired* $\mathcal{E}(Q_{in},1)$ *where* $Q_{in}>0$ *is symmetric.*
2. $X_{in}:=\left(Q_{in}^{-1}\right)^{\frac{1}{2}}$.
3. *For all* $\mathcal{P}$ *in power set of* $\{1,2,\ldots,m\}$
    a. n *is the number of elements in* $\mathcal{P}$.
    b. $\delta=1,\ \pi=1,\ 0<\Delta\pi\ll 1$.
    c. *while* $\pi\leq\pi_{max}$ *do*
        i. *Solve (57) using PENBMI or other reliable algorithms.*
        ii. *Save the results.*
        iii. $\pi=\pi+\Delta\pi$.

        End of while.

    End of for.
4. *Compare the results and choose the best controller gains.*

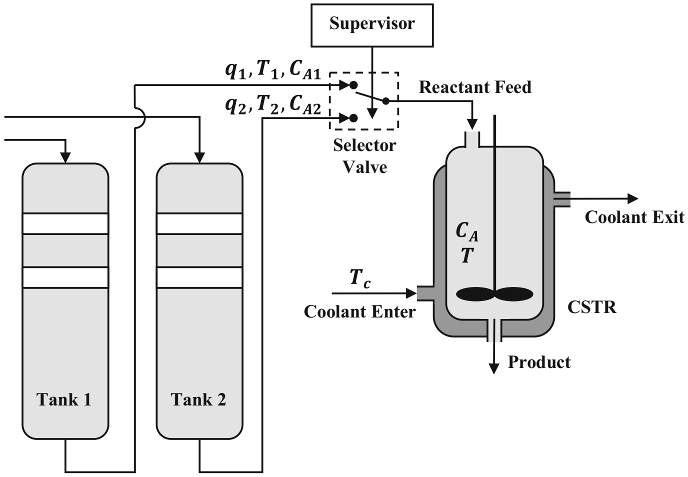

Fig. 4 Schematic diagram of CSTR with two input resources [42]

4 Illustrative examples

Example 1 In this example, the proposed approach is applied to the highly nonlinear model of a continuous stirred tank reactor (CSTR) fed by two different resource streams. The supervisor determines that which of these resources will selected as feeding stream (see Fig. 4). In this way, the reactor operates in two different modes [37,44]. Assuming constant liquid volume, perfect mixing and negligible heat loss, the irreversible exothermic reaction $A \rightarrow B$ is described as follows,

$$\frac{dC_A}{dt} = \frac{q_i}{V}(C_{Ai} - C_A) - a_0 exp\left(-\frac{E}{RT}\right) C_A,$$
$$\frac{dT}{dt} = \frac{q_i}{V}(T_i - T) - a_1 exp\left(-\frac{E}{RT}\right) C_A + a_2 (T_c - T).$$

where the concentration of the reactant A (C_A) and the reactor temperature T should be regulated to their nominal values by manipulating the temperature of the coolant stream T_c under an arbitrary periodic switching. It is assumed that the flow rate of coolant stream is constant. The nominal value of parameters are given in Table 1. The nominal operating conditions corresponding to an unstable equilibrium point are $T_c^* = 300°\text{K}$, $C_A^* = 0.5\,\text{mol/L}$, $T^* = 350°\text{K}$ for both modes [37,44].

By defining $x_1 = C_A - C_A^*$, $x_2 = T - T^*$, and $u = T_c - T_c^*$, the nonlinear model of CSTR can be obtained in the form of (1) with,

Table 1 Nominal parameters of the process [44]

Parameter	Unit	Value
V	L	100
ρ	$\mathrm{g\,L^{-1}}$	1000
C_ρ	$\mathrm{J\,g^{-1}\,K^{-1}}$	0.239
ΔH	$\mathrm{J\,mol^{-1}}$	-5×10^4
E/R	K	8750
UA	$\mathrm{J\,min^{-1}\,K^{-1}}$	5×10^4
$a_0 = k_0$	$\mathrm{min^{-1}}$	7.2×10^4
$a_1 = \frac{\Delta H}{\rho C_\rho} k_0$	$\mathrm{mol^{-1}\,L\,K\,min^{-1}}$	-1.506×10^{13}
$a_2 = \frac{UA}{V\rho C_\rho}$	$\mathrm{J\ min^{-1}}$	2.092
q_1	$\mathrm{L\,min^{-1}}$	50
q_2	$\mathrm{L\,min^{-1}}$	200
C_{A1}	$\mathrm{mol\ L^{-1}}$	1.5
C_{A2}	$\mathrm{mol\ L^{-1}}$	0.75
$T_1 = T_2$	K	350

$$A_i = \begin{bmatrix} -\frac{q_i}{V} - a_0 C_e & -a_0 C_e C_A^* \frac{E/R}{(T^*)^2} \\ -a_1 C_e & -\frac{q_i}{V} - a_2 - a_0 C_e C_A^* \frac{E/R}{(T^*)^2} \end{bmatrix}, B_i = \begin{bmatrix} 0 \\ a_2 \end{bmatrix}, C_i = I,$$

$$f_i = -\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} exp\left(-\frac{E}{R\,(x_2 + T^*)}\right) x_1 + \begin{bmatrix} a_0 C_e & a_0 C_e C_A^* \frac{E/R}{(T^*)^2} \\ a_1 C_e & a_0 C_e C_A^* \frac{E/R}{(T^*)^2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

where $C_e = exp\left(-\frac{E/R}{T^*}\right)$. By separating the nonlinear portion of f_i form its linear part, it is easy to verify that the vector function f_i satisfies the Lipschitz condition (A2) on $\mathbb{R}^2$ with,

$$M_i = \left(N_1^T N_1 + N_2^T N_2\right)^{1/2},$$

in which,

$$N_1 = \begin{bmatrix} a_0 & 0 \\ a_1 & 0 \end{bmatrix}, \quad N_2 = \begin{bmatrix} a_0 C_e & a_0 C_e C_A^* \frac{E/R}{(T^*)^2} \\ a_1 C_e & a_0 C_e C_A^* \frac{E/R}{(T^*)^2} \end{bmatrix}.$$

Since the value of parameters a_0 and a_1 are too high, the above Lipschitz constants matrix may results in infeasibility during the solving optimization problem (57) or a high gain in controller (19). On the other hand, due to limitation of temperature changes in the reactor (see [45]), we simply consider the subspace $\mathbb{D} = \left\{x \in \mathbb{R}^2 \middle| |x_2| \leq 20\right\}$ and redefine the Lipschitz constants matrix with,

$$N_1 = \begin{bmatrix} a_0 C_{max} & 0 \\ a_1 C_{max} & 0 \end{bmatrix}.$$

where $C_{max} = exp\left(-\frac{E}{R(T^*+20)}\right)$.

Furthermore, due to the uncertainty in physical parameters of CSTR (such as reaction enthalpy, pre-exponential factor and overall heat transfer coefficient, see [46]) and in order to simulation, we consider the following uncertainties,

$$D_i = 0.1\, I, \quad F_i = \begin{bmatrix} sin\,(t) & 0 \\ 0 & cos\,(t) \end{bmatrix},$$
$$E_{ai} = \begin{bmatrix} 0.1 & 0.1 \\ 0.1 & 0.1 \end{bmatrix}, \quad E_{bi} = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}, \quad E_{\phi i} = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}.$$

Now, by considering the desired outer set $\mathcal{E}\,(Q_{out}, 1) \subset \mathbb{D}$ with $Q_{out} = diag\,\{1, 0.0025\}$, the desired inner set $\mathcal{E}\,(Q_{in}, 1)$ where $Q_{in} = diag\,\{10, 10\}$ and considering $u_{max} = 275$, $\tau_{min} = 0.1\, min$ and $\tau_{pmax} = 1\, min$, the gain matrices are obtained as follows,

$$K_1 = \begin{bmatrix} -77.4904 & -6.9830 \end{bmatrix}, \quad K_2 = \begin{bmatrix} -81.4530 & -6.6070 \end{bmatrix},$$
$$P = \begin{bmatrix} 1000 & -0.0000 \\ -0.0000 & 2.5000 \end{bmatrix}, \quad r = r' = 1 \times 10^3$$

For time domain simulations, let us consider a case that the first system is active for 0.5 minute, and then the second system is activated for 0.1 minute. This switching scheme continuous to the end. The trajectories for different initial conditions along with the outer region of attraction and the attractive ellipsoid are shown in Fig. 5. In addition, Fig. 6 compares the magnitude of the control signal designed by the proposed approach with the controller obtained in [37]. As seen, due to the lower conservatism in choice of the Lipschitz constant matrix, the magnitude of control signal decreases. Also, the volume of the ultimate bound found by the proposed approach is less than the volume found by [37] (see Fig. 7). With regard to Figs. 6 and 7, it can be concluded that our proposed method makes convergence to a smaller region containing the origin with lower energy consumption.

Furthermore, to show the efficiency of the proposed approach in presence of impulse, we also consider the jump function when a subsystem switched to another one, with,

$$C_1 = -C_2 = \begin{bmatrix} 1.25 & 0 \\ 0 & 1.25 \end{bmatrix}.$$

By applying the proposed approach, we can obtain the gain matrices as follows,

$$K_1 = \begin{bmatrix} -78.6793 & -7.1031 \end{bmatrix}, \quad K_2 = \begin{bmatrix} -81.2906 & -7.0395 \end{bmatrix},$$
$$P = \begin{bmatrix} 1000 & -0.0000 \\ -0.0000 & 2.5000 \end{bmatrix}, \quad r = 1000, \quad r' = 639.9960, \quad \delta' = 1.5625$$

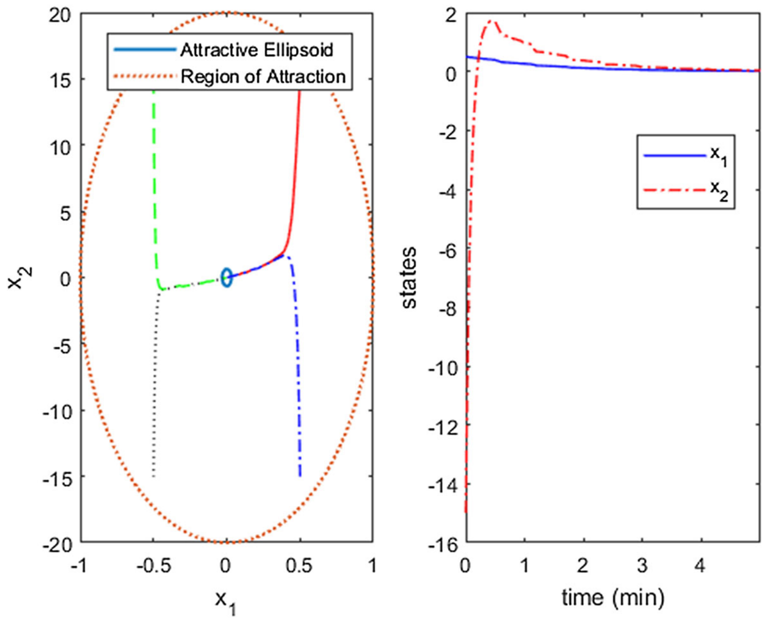

Fig. 5 (left) Region of attraction and attractive ellipsoid determined by the proposed approach , and convergence of trajectories for some initial conditions. (right) State variables of the switched CSTR for initial condition $x\,(0) = \begin{bmatrix} +0.5 & -15 \end{bmatrix}$

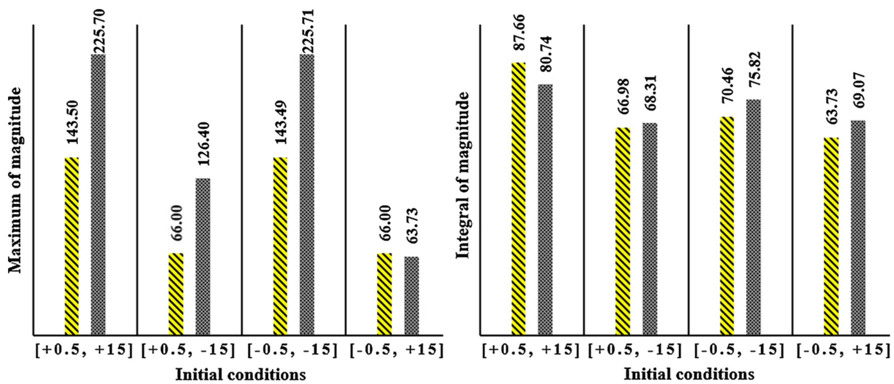

Fig. 6 Comparison of the control signals designed using our approach (dashed) and the proposed method in [37] (dotted) for different initial conditions

Figure 8 shows simulation results for initial condition $x\,(0) = \begin{bmatrix} +\,0.5 & +15 \end{bmatrix}$. Due to the jump in state variables, the Lyapunov function candidate increases at impulse instants but this increase is compensated by further reduction during the next continuous flow.

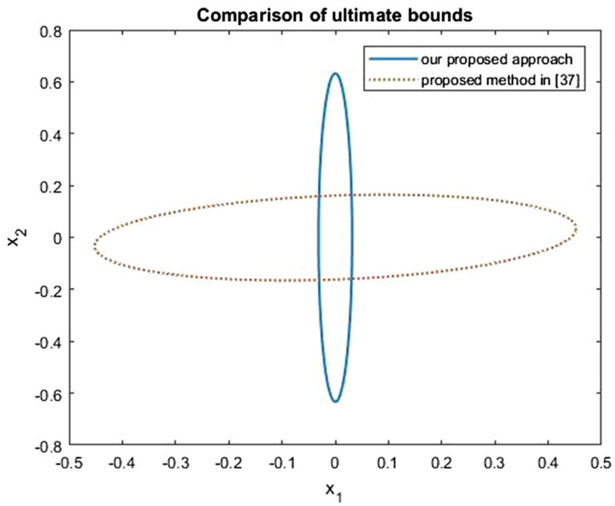

Fig. 7 Comparison of ultimate bounds obtained by our proposed approach with that designed in by [37]

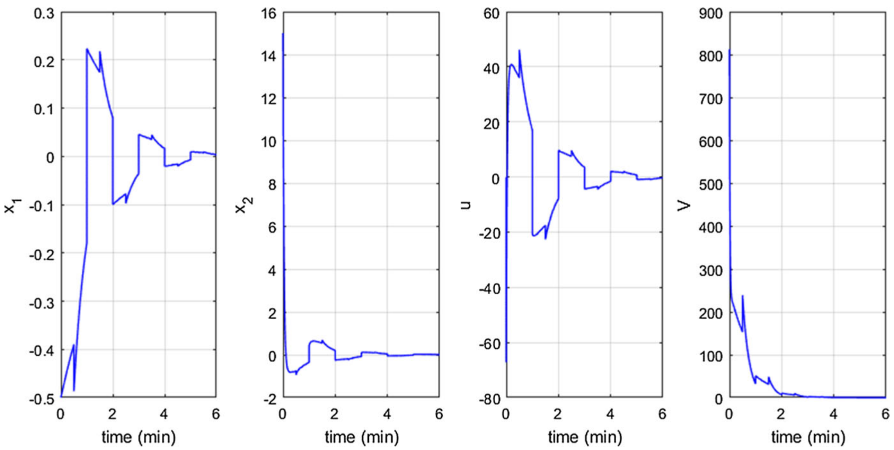

Fig. 8 (form left to right) The state variables, control signal and the value of Lyapunov function versus time for initial condition $x\,(0) = \left[+0.5 \quad +15\right]$

Example 2 Consider the impulsive switched system (1) with the known parameters given by,

$$A_1 = \begin{bmatrix} 0.5 & 3 \\ 10 & 0.5 \end{bmatrix}, B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$f_{c1}\,(x) = 0.5 \times \begin{bmatrix} tanh\,x_1 \\ tanh\,x_2 \end{bmatrix}, \quad f_{c2}\,(x) = \begin{bmatrix} 0.1925\,(cos\,x_1 - 1)^2 \\ 0.5sin^2 x_2 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 1.2 & 0 \\ 0 & 1.2 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1.2 & 0 \\ 0 & 1.2 \end{bmatrix}.$$

and uncertainties defined by the following matrices,

$$D_1 = D_2 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad F_1(t) = F_2(t) = \begin{bmatrix} sin(t) & 0 \\ 0 & cos(t) \end{bmatrix},$$
$$E_{\phi 1} = E_{\phi 2} = E_{b1} = E_{b2} = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, \quad E_{a1} = E_{a2} = \begin{bmatrix} 0.1 & 0.1 \\ 0.1 & 0.1 \end{bmatrix}.$$

It can be verified that the vector functions $f_{ci}(x)$ satisfy the Lipschitz condition (A2) with $M_i = 0.5 \times I_{2\times 2}$ on $\mathbb{R}^n$. Note that the second subsystem not only is unstable but also is uncontrollable. For time-domain simulations, the given impulsive switched system is simulated over the time interval [0, 5] with an impulse sequence $\{0.4, 0.6, 1.0, 1.2, 1.6, \ldots\}$ under the periodic switching signal $\sigma(t)$ with $\sigma(0) = 1$. This switching law satisfies the minimum dwell-time $\tau_{min} = 0.2\,sec$ and the maximum switching cycle $\tau_{pmax} = 0.6\,sec$. We also consider $u_{max} = 50$. Then, to obtain the control parameters, the optimization problem (57) is solved. The results are,

$$W_1 = \begin{bmatrix} -0.1761 & -0.0224 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 0 & 0 \end{bmatrix},$$
$$\delta'' = 7.3360, \quad r = 1000, \quad L = \begin{bmatrix} 0.0463 & -0.0415 \\ -0.0415 & 0.0563 \end{bmatrix}.$$

The simulation results are shown in Fig. 9 where the trajectories converge to the attractive ellipsoid. It should be mentioned that approaches in which reduction of the value of Lyapunov function is considered for all subsystems during the continuous flow fails in solving this example.

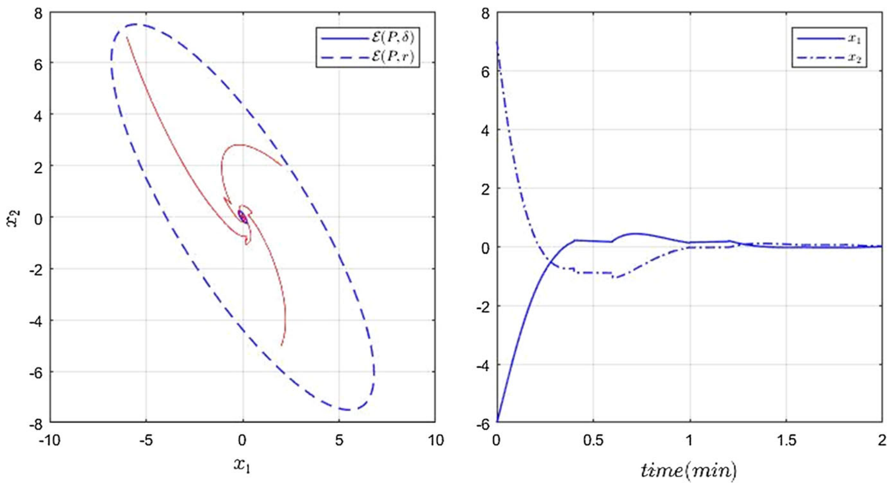

Fig. 9 (left) The attractive ellipsoid and the ellipsoid of attraction. All the trajectories starting in the region of attraction converge to the attractive zone. (right) The state variables for for initial condition $x(0) = \begin{bmatrix} -6 & +7 \end{bmatrix}$

5 Conclusion

Actuator saturation, i.e. norm-boundedness of control input, remains as a highly challenging problem for impulsive switched systems operating in uncertain environments. For such systems, a global state-feedback control law is either non-existent or is highly nontrivial. Instead, this paper proposes a local approach to stabilization of such switched systems. At first, we propose some local sufficient conditions, in terms of a Lyapunov function candidate, that investigate the stability for a general model of nonlinear impulsive periodic switched system. These conditions are also presented in forms of linear and bilinear matrix inequalities for the targeted system.

Various aspects of the proposed approach can be highlighted as follows:

- Considering challenging aspects of a real-world system, including the linear/nonlinear uncertainty, actuator saturation, known nonlinearity, switching and impulsive effect.
- Considering the local Lipschitz condition on a subspace rather than on $\mathbb{R}^n$ and developing local stabilization criteria that is useful when the nonlinearities are not globally Lipschitz, or when the Lipschitz constant matrix is too large on the whole of state space. It is also applicable in processes that the region of validity is limited due to physical issues.
- Applicable to the switched systems that have unstable and uncontrollable subsystems.
- Developing the stabilization criteria in the form of linear or bilinear matrix inequalities, which is solvable using reliable LMI-based algorithms.
- Finding the largest region of attraction along with smallest attractive ellipsoid via proposing an optimization problem.

As the next step in this research, we hope to reformulate the optimization problem to achieve a stabilizing state feedback controller for other types of known nonlinearities, jump functions, and unknown uncertainties. We also hope to use soft computing strategies to determine the optimal parameters of the optimization problem.

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