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ORIGINAL RESEARCH

Derivations of the Schrödinger algebra and their applications

Yu Yang[1] · **Xiaomin Tang**[1]

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Abstract The Schrödinger algebra is a non-semisimple Lie algebra and plays an important role in mathematical physics and its applications. In this paper, all derivations of the Schrödinger algebra are determined. As applications, all biderivations, linear commuting maps and commutative post-Lie algebra structures on the Schrödinger algebra are obtained.

Keywords Derivation · Biderivation · Schrödinger algebra · Linear commuting map · Post-Lie algebra

Mathematics Subject Classification 16W10 · 16W25

1 Introduction

The Schrödinger Lie group describes symmetries of the free particle Schrödinger equation in [17]. The Lie algebra $\mathbb{S}(n)$ in $(n+1)$-dimensional space-time of the Schrödinger Lie group is called the Schrödinger algebra, see [8]. The Schrödinger algebra is a non-semisimple Lie algebra and plays an important role in mathematical physics. The Lie algebra $\mathbb{S}(1)$ is one of the most essential case for $n = 1$ and admits a universal 1-dimensional central extension which is called the centrally extended Schrödinger algebra or, simply, the Schrödinger algebra, abusing the language. We denote $\mathbb{S}(1)$ by $\mathcal{S}$ in this paper. Let $\mathbb{C}$ be the complex number field. Recall that the Schrödinger algebra $\mathcal{S}$ is a Lie algebra with a $\mathbb{C}$-basis $\{f, q, h, c, p, e\}$ and brackets

✉ Xiaomin Tang
x.m.tang@163.com

1 Department of Mathematics, Heilongjiang University, Harbin 150080, People's Republic of China

$$[h,e]=2e,\ [h,f]=-2f,\ [e,f]=h,$$
$$[h,p]=p,\ [h,q]=-q,\ [p,q]=c,$$
$$[e,q]=p,\ [p,f]=-q,\ [f,q]=0,$$
$$[e,p]=0,\ [c,\mathcal{S}]=0.$$

The Schrödinger algebra $\mathcal{S}$ can be viewed as a semidirect product

$$\mathcal{S}=\mathcal{H}\rtimes sl_2$$

of two subalgebras: a Heisenberg subalgebra $\mathcal{H} = \text{span}\{p,q,c\}$ and $sl_2 = \text{span}\{e,h,f\}$.

Recently there appeared a number of papers studying various aspects of structure and representation theory of the Schröinger algebra $\mathcal{S}$. In particular, the authors in [7,8] describe the simple highest weight modules for $\mathcal{S}$; the authors in [10] classify all simple modules over $\mathcal{S}$ which are weight and have finite dimensional weight spaces; the authors in [11] study the category $\mathcal{O}$ for $\mathcal{S}$; the authors in [25] describe the simple weight modules of $\mathcal{S}$ and the authors in [27] classify all simple Whittaker modules for $\mathcal{S}$. As far as we know, there are few researches about the structure theory of the Schrödinger algebra $\mathcal{S}$. In particular, [1] determines all Lie bialgebra structures for $\mathcal{S}$. In this paper, in order to characterize the biderivations, the linear commuting maps and the commutative post-Lie algebra structures on the Schrödinger algebra $\mathcal{S}$, we need first know the derivations of $\mathcal{S}$. But its derivations has not been found until now. For this purpose, we first compute the derivations of the Schrödinger algebra and then give some applications.

2 Derivations of the Schrödinger algebra

In this section, we will calculate the derivations of $\mathcal{S}$. Now let us review some details about the derivation of a Lie algebra.

Definition 2.1 A linear map D from a Lie algebra L into itself is called a derivation of L if it satisfies that

$$D([x,y])=[D(x),y]+[x,D(y)] \tag{1}$$

for all $x,y\in L$.

For $x\in L$, it is easy to see that $\text{ad}x : L\to L, y\mapsto \text{ad}x(y)=[x,y]$ for all $y\in L$ is a derivation of L, which is called an inner derivation. Denote by Der L the vector space of all derivations, Inn L the vector space of all inner derivations. The first cohomology group of L with coefficients in L is the quotient space

$$H^1(L,L)=\text{Der } L/\text{Inn } L.$$

Lemma 2.2 *Let D be a linear map from $\mathcal{S}$ into itself. Then $D \in \operatorname{Der}\mathcal{S}$ if and only if the following* 10 *equations hold:*

$$0 = [D(f), q] + [f, D(q)], \tag{2}$$
$$0 = [D(e), p] + [e, D(p)], \tag{3}$$
$$2D(e) = [D(h), e] + [h, D(e)], \tag{4}$$
$$-2D(f) = [D(h), f] + [h, D(f)], \tag{5}$$
$$D(h) = [D(e), f] + [e, D(f)], \tag{6}$$
$$D(p) = [D(h), p] + [h, D(p)], \tag{7}$$
$$-D(q) = [D(h), q] + [h, D(q)], \tag{8}$$
$$D(c) = [D(p), q] + [p, D(q)], \tag{9}$$
$$D(p) = [D(e), q] + [e, D(q)], \tag{10}$$
$$-D(q) = [D(p), f] + [p, D(f)]. \tag{11}$$

Proof The "if" direction is easy to verify by a direct computation. Conversely, it must be satisfied with (1) for every pair of x, y from the basis $\{f, q, h, c, p, e\}$, which yields 36 equations. But some of them can be ignored. First, by letting $x = y$ in (1) we have that the left and right sides are equal to 0. Next, exchange the location of x, y ($x \neq y$) in (1), there are some equations and half of them are linear dependence, they should be ignored. On the other hand, if $x = c$ or $y = c$ then $[D(c), y] = 0$ or $[x, D(c)] = 0$. This means that when $x = c$ or $y = c$, $D(c)$ lies in the center of $\mathcal{S}$. By ignoring these equations, we see that Eqs. (2)–(11) are enough to claim that $D \in \operatorname{Der}\mathcal{S}$. The proof is completed. □

Let δ be an outer derivation of $\mathcal{S}$ determined by

$$\delta(h) = \delta(e) = \delta(f) = 0,\ \delta(c) = c,\ \delta(p) = \frac{1}{2}p,\ \delta(q) = \frac{1}{2}q. \tag{12}$$

We have the following main result in this section.

Theorem 2.3 $\operatorname{Der}\mathcal{S} = \operatorname{Inn}\mathcal{S} \oplus \mathbb{C}\delta$. *Furthermore,* $H^1(\mathcal{S}, \mathcal{S}) = \mathbb{C}\delta$.

Proof Assume that $D \in \operatorname{Der}\mathcal{S}$ and $A = (a_{ij})_{6\times 6}$ is the matrix of D under the basis $\{f, q, h, c, p, e\}$, i.e.,

$$\big(D(f), D(q), D(h), D(c), D(p), D(e)\big) = \big(f, q, h, c, p, e\big)\, A. \tag{13}$$

By (2) and (13), we have

$$\begin{aligned}&[a_{11}f + a_{21}q + a_{31}h + a_{41}c + a_{51}p + a_{61}e, q]\\ &\quad + [f, a_{12}f + a_{22}q + a_{32}h + a_{42}p + a_{52}p + a_{62}e] = 0.\end{aligned}$$

It follows by Definition 2.1 that $-a_{31}q + a_{51}c + a_{61}e + 2a_{32}f + a_{52}q - a_{62}h = 0$. This implies

$$a_{31} = a_{52},\ a_{51} = a_{61} = a_{32} = a_{62} = 0. \tag{14}$$

Similarly, by (3) and (4) we deduce that

$$\begin{aligned} &a_{25} = -a_{36},\ a_{15} = a_{35} = a_{16} = a_{26} = 0, a_{13} = -2a_{36},\\ &a_{23} = -a_{56},\ a_{33} = a_{46} = 0. \end{aligned} \tag{15}$$

Next, substituting Eqs. (14) and (15) into Eqs. (5)–(11), in turn, we obtain

$$\begin{aligned} &a_{21} = -a_{42} = a_{53} = k_1,\ a_{63} = -2a_{31} = -2a_{52} = -2k_2,\\ &a_{13} = 2a_{25} = -2a_{36} = 2k_5,\ a_{23} = -a_{45} = -a_{56} = -k_6,\\ &a_{11} = -a_{66} = -k_3,\ a_{22} = -k_4, a_{55} = k_3 + k_4, a_{44} = k_3 + 2k_4, \end{aligned}$$

where $k_i \in \mathbb{C}$, $i = 1, \ldots, 6$, and the other elements of A are all 0. Therefore, we can deduce that

$$\begin{aligned} &D(f) = -k_3 f + k_1 q + k_2 h,\ \ D(q) = k_4 q - k_1 c + k_2 p,\\ &D(h) = 2k_5 f - k_6 q + k_1 p - 2k_2 e,\ \ D(c) = (k_3 + 2k_4)c,\\ &D(p) = k_5 q + k_6 c + (k_3 + k_4)p,\ \ D(e) = -k_5 h + k_6 p + k_3 e. \end{aligned}$$

Now we denote by $\lambda_D = k_3 + 2k_4$ and $x_D = k_5 f - k_6 q - k_1 p + k_2 e + \frac{k_3}{2}h + t_D c$ for some $t_D \in \mathbb{C}$ associated with D. Then it follows that $D(f) = \mathrm{ad}x_D(f)$, $D(q) = \mathrm{ad}x_D(q) + \frac{\lambda_D}{2}q$, $D(h) = \mathrm{ad}x_D(h)$, $D(c) = \mathrm{ad}x_D(c) + \lambda_D c$, $D(p) = \mathrm{ad}x_D(p) + \frac{\lambda_D}{2}p$ and $D(e) = \mathrm{ad}x_D(e)$. Let δ be the linear map from $\mathcal{S}$ into itself given by (12), then we have $D(y) = \mathrm{ad}x_D(y) + \lambda_D\delta(y)$ for all $y \in \mathcal{S}$. The proof is completed. □

3 Biderivations of the Schrödinger algebra

Biderivations are a subject of research in various areas, see [2,6,9,13,14,20,23,24,26]. In [2], Brešar et al. [3] showed that all biderivations on commutative prime rings are inner biderivations and determined the biderivations of semiprime rings. This theorem has proved to be useful in the study of commutating maps. More details regarding commuting maps, biderivations and their generalizations can be found in the survey article. The notion of biderivation of Lie algebras was introduced in [24]. And then, many authors began studying (super-)biderivations of some Lie (super-)algebras, such as [6,12,14,20,23,26]. For an arbitrary Lie algebra L, we recall that a bilinear map $g : L \times L \to L$ is a biderivation of L if it is a derivation with respect to both components. More precisely, one has

Definition 3.1 Assume that L is a Lie algebra. A bilinear map $g : L \times L \to L$ is called a biderivation if it satisfies

$$g([x, y], z) = [x, g(y, z)] + [g(x, z), y], \quad (16)$$
$$g(x, [y, z]) = [g(x, y), z] + [y, g(x, z)] \quad (17)$$

for all $x, y, z \in L$.

Let $\lambda \in \mathbb{C}$. The bilinear map $g : L \times L \to L$ given by $g(x, y) = \lambda[x, y]$ is a biderivation of L which is said to be inner.

Lemma 3.2 *Suppose that g is a biderivation of $\mathcal{S}$. Then there are two linear maps ϕ and ψ from $\mathcal{S}$ into itself such that*

$$g(x, y) = l_x\delta(y) + [\phi(x), y] = r_y\delta(x) + [x, \psi(y)] \quad (18)$$

for all $x, y \in L$, where l_x, r_x are complex numbers depend on x, and δ is given by Theorem 2.3.

Proof For the biderivation g of $\mathcal{S}$ and a fixed element $x \in \mathcal{S}$, we define a map $\phi_x : \mathcal{S} \to \mathcal{S}$ given by $\phi_x(y) = g(x, y)$. It easy to verify from (17) that ϕ_x is a derivation of $\mathcal{S}$. Thanks to Theorem 2.3, there is a map $\phi : \mathcal{S} \to \mathcal{S}$ such that $\phi_x = l_x\delta + \mathrm{ad}\phi(x)$, i.e., $g(x, y) = l_x\delta(y) + [\phi(x), y]$, where $l_x \in \mathbb{C}$. Since g is bilinear, it is easy to see that ϕ is linear. Similarly, if we define a map ψ_z from $\mathcal{S}$ into itself given by $\psi_z(y) = g(y, z)$ for all $y \in \mathcal{S}$, we can obtain a linear map ψ from $\mathcal{S}$ into itself such that $g(x, y) = r_y\delta(x) + \mathrm{ad}(-\psi(y))(x) = r_y\delta(x) + [x, \psi(y)]$. The proof is completed. □

Lemma 3.3 *Suppose g is a biderivation of $\mathcal{S}$. Then for any $u, v \in \{f, q, h, p, e\}$ we have $g(c, u) = r_u c$, $g(v, c) = l_v c$, $g(c, c) = 0$, where $r_u, l_v \in \mathbb{C}$ are defined by Lemma 3.2.*

Proof By Theorem 2.3 and (12), it follows that $g(c, u) = r_u\delta(c) = r_u c$. Similarly, $g(v, c) = r_v\delta(c) = l_v c$. Note that (16) implies that $g(c, c) = g([p, q], c) = [p, l_q c] + [l_p c, q] = 0$. The proof is completed. □

We now state our main result in this section as follows.

Theorem 3.4 *Any biderivation of $\mathcal{S}$ is inner.*

Proof Suppose g is a biderivation of $\mathcal{S}$. By Lemma 3.2, there exist two matrices $B = (b_{ij})_{6\times5}$ and $C = (c_{ij})_{6\times5}$ such that

$$\big(\phi(f), \phi(q), \phi(h), \phi(p), \phi(e)\big) = \big(f, q, h, c, p, e\big)\, B, \quad (19)$$
$$\big(\psi(f), \psi(q), \psi(h), \psi(p), \psi(e)\big) = \big(f, q, h, c, p, e\big)\, C. \quad (20)$$

Take $x, y \in \{f, q, h, p, e\}$ in (18), then we can obtain 25 equations. Firstly, let $x = f$ or $y = f$ in (18), one has

$$g(f, f) = [\phi(f), f] = [f, \psi(f)], \quad (21)$$

$$g(f,e)=[\phi(f),e]=[f,\psi(e)], \tag{22}$$
$$g(e,f)=[\phi(e),f]=[e,\psi(f)], \tag{23}$$
$$g(f,h)=[\phi(e),f]=[e,\psi(f)], \tag{24}$$
$$g(h,f)=[\phi(h),f]=[h,\psi(f)], \tag{25}$$
$$g(f,q)=\frac{1}{2}l_f q+[\phi(f),q]=[f,\psi(q)], \tag{26}$$
$$g(q,f)=[\phi(q),f]=\frac{1}{2}r_f q=[q,\psi(f)], \tag{27}$$
$$g(f,p)=\frac{1}{2}l_f p+[\phi(f),p]=[f,\psi(p)], \tag{28}$$
$$g(p,f)=[\phi(q),q]=\frac{1}{2}r_f p+[p,\psi(f)], \tag{29}$$

where ϕ, ψ, l_x, r_x are given by Lemma 3.2. According to (21),(19) and (20), it follows that

$$[b_{11}f+b_{21}q+b_{31}h+b_{41}c+b_{51}p+b_{61}e,f]$$
$$=[f,c_{11}f+c_{21}q+c_{31}h+c_{41}c+c_{51}p+c_{61}e],$$

which yields $-2b_{31}f-b_{51}q+b_{61}h=2c_{31}f+c_{51}q+-c_{61}h$. Therefore, we have $b_{31}=-c_{31}, b_{51}=-c_{51}, b_{61}=-c_{61}$. Similarly, by (22) we have $b_{11}=c_{65}, b_{21}=b_{31}=c_{35}=c_{55}=0$. This indicates that $c_{31}=0$. By (23) we obtain $c_{11}=c_{65}, b_{35}=b_{55}=c_{21}=0$ and by (24) we obtain $b_{11}=c_{33}$, $b_{51}=0=c_{51}$, $b_{61}=0=c_{61}$. According to (21), we can also obtain that $c_{53}=c_{63}=0$. In view of (25)–(29), it follows that $c_{11}=b_{33}=b_{54}$ and $b_{32}=b_{62}=c_{32}=c_{62}=b_{34}=b_{64}=c_{34}=c_{64}=b_{53}=b_{63}=b_{52}=c_{52}=l_f=r_f=0$.

Next, let $x=q$ or $y=q$ in (18), we obtain

$$g(q,q)=\frac{1}{2}l_q q[\phi(q),q]=\frac{1}{2}r_q q+[q,\psi(q)], \tag{30}$$
$$g(q,h)=[\phi(q),h]=\frac{1}{2}r_h q+[q,\psi(h)], \tag{31}$$
$$g(h,q)=\frac{1}{2}l_h q+[\phi(h),q]=[h,\psi(q)], \tag{32}$$
$$g(q,p)=\frac{1}{2}l_q p+[\phi(q),p]=\frac{1}{2}r_p q+[q,\psi(p)], \tag{33}$$
$$g(p,q)=\frac{1}{2}l_p q+[\phi(p),q]=\frac{1}{2}r_q p+[p,\psi(q)], \tag{34}$$
$$g(q,e)=[\phi(q),e]=\frac{1}{2}r_e q+[q,\psi(e)], \tag{35}$$
$$g(e,q)=\frac{1}{2}l_e q+[q,\psi(e)]=[\phi(q),e]. \tag{36}$$

By (30)–(36) and the conclusion of (21)–(29), we have

$$b_{22} = c_{54} = c_{65}, c_{22} = b_{54} = b_{65}, \tag{37}$$
$$\frac{1}{2}l_h + c_{22} = b_{33}, \frac{1}{2}r_h + c_{33} = b_{22}, \tag{38}$$

and $b_{12} = c_{12} = r_p = r_q = r_e = l_p = l_q = l_e = 0$.

In addition, the remaining equations yield that

$$g(h, h) = [\phi(h), h] = [h, \psi(h)], \tag{39}$$
$$g(h, p) = \frac{1}{2}l_h p + [\phi(h), p] = [h, \psi(p)], \tag{40}$$
$$g(p, h) = [\phi(p), h] = \frac{1}{2}r_h p + [p, \psi(h)], \tag{41}$$
$$g(h, e) = [\phi(h), e] = [h, \psi(e)], \tag{42}$$
$$g(e, h) = [\phi(e), h] = [e, \psi(h)], \tag{43}$$
$$g(p, p) = \frac{1}{2}l_p p + [\phi(p), p] = \frac{1}{2}r_p p + [p, \psi(p)], \tag{44}$$
$$g(p, e) = [\phi(p), e] = \frac{1}{2}r_e p + [p, \psi(e)], \tag{45}$$
$$g(e, p) = \frac{1}{2}l_e p + [\phi(e), p] = [e, \psi(p)], \tag{46}$$
$$g(e, e) = [e, \psi(e)] = [\phi(e), e]. \tag{47}$$

By (39)–(47), we have $b_{33} = c_{65}, b_{65} = c_{33}$ and $b_{13} = c_{13} = b_{23} = c_{23} = b_{14} = c_{14} = b_{24} = c_{24} = b_{15} = b_{25} = c_{15} = c_{25} = 0$. By (37), (38), (40), (41), we have $l_h = r_h = 0$. Finally, according to the results above, we obtain that

$$b_{11} = b_{22} = b_{33} = b_{54} = b_{65} = c_{11} = c_{22} = c_{33} = c_{54} = c_{65} = \lambda, \tag{48}$$
$$l_f = l_q = l_h = l_p = l_e = r_f = r_q = r_h = r_p = r_e = 0 \tag{49}$$

for some $\lambda \in \mathbb{C}$ and b_{4i}, c_{4i} $(i = 1, 2, 3, 4, 5)$ are arbitrary, other elements of B and C are all 0. In other words, we have

$$B = \begin{pmatrix} \lambda & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 \\ b_{41} & b_{42} & b_{43} & b_{44} & b_{45} \\ 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}, \ C = \begin{pmatrix} \lambda & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 \\ c_{41} & c_{42} & c_{43} & c_{44} & c_{45} \\ 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}.$$

By Lemma 3.3 and (49), we have

$$g(c, u) = g(v, c) = l_c = r_c = 0. \tag{50}$$

Therefore, we deduce that

$$g(c, u) = \lambda[c, u], g(v, c) = \lambda[v, c]. \tag{51}$$

To summarize, we can know that $g(u, v) = \lambda[u, v]$ when $u, v \in \{f, q, h, p, e\}$. This, together with Lemmas 3.2, 3.3 and Eqs. (50), (51), yields that the biderivation of $\mathcal{S}$ is inner. □

4 Other applications

4.1 Linear commuting maps on Lie algebras

Recall that a linear commuting map ϕ on a Lie algebra L subject to $[\phi(x), x] = 0$ for any $x \in L$. The first important result on linear (or additive) commuting maps is Posnes theorem [18] from 1957. Then many scholars study commuting maps on all kinds of algebra structures, Brešar [3] briefly discuses various extensions of the notion of a commuting map. About the recent articles on commuting maps we can reference [3,6,14,23,26].

Obviously, if ϕ on L is such a map, then $[\phi(x), y] = [x, \phi(y)]$ for any $x, y \in L$. Define by $f(x, y) = [\phi(x), y] = [x, \phi(y)]$, then it is easy to check that f is a biderivation of L.

Using Theorem 3.4, we get the following result.

Theorem 4.1 *Any linear map ϕ on $\mathcal{S}$ is commuting if and only if there are $\lambda \in \mathbb{C}$ and a linear function $\sigma : \mathcal{S} \to \mathbb{C}$ such that*

$$\phi(x) = \lambda x + \sigma(x)c \text{ for all } x \in \mathcal{S}.$$

Proof The "if" part is easy to verify. We now prove the "only if" part. By the above discuss we see that $g(x, y) = [\phi(x), y]$, $x, y \in \mathcal{S}$ is a biderivation of $\mathcal{S}$. It follows by Theorem 3.4 that $[\phi(x), y] = [\lambda x, y]$ for some $\lambda \in \mathbb{C}$. Furthermore, we have $[\phi(x) - \lambda x, y] = 0$ and then $\phi(x) - \lambda x \in Z(\mathcal{S}) = \mathbb{C}c$. This means that there is a map σ from $\mathcal{S}$ into $\mathbb{C}$ such that

$$\phi(x) - \lambda x = \sigma(x)c.$$

It is easy to verify that σ is linear. The proof is completed. □

4.2 Post-Lie algebra

Post-Lie algebras have been introduced by Valette in connection with the homology of partition posets and the study of Koszul operads [22]. As [4] pointed out, post-Lie algebras are natural common generalization of pre-Lie algebras and LR-algebras in the geometric context of nil-affine actions of Lie groups. Recently, many authors study some post-Lie algebras and post-Lie algebra structures, see [4,5,15,16,19]. In particular, the authors in [4] study the commutative post-Lie algebra structure on Lie algebra. Let us recall the following definition of a commutative post-Lie algebra.

Definition 4.2 Let $(L, [,])$ be a complex Lie algebra. A commutative post-Lie algebra structure on L is a $\mathbb{C}$-bilinear product $x * y$ on L and satisfying the following identities:

$$x * y = y * x,$$
$$[x, y] * z = x * (y * z) - y * (x * z),$$
$$x * [y, z] = [x * y, z] + [y, x * z]$$

for all $x, y, z \in L$. We also say that $(L, [,], *)$ is a commutative post-Lie algebra.

Lemma 4.3 *[21] Let $(L, [,], *)$ be a commutative post-Lie algebra. If we define a bilinear map $g : L \times L \to L$ given by $g(x, y) = x * y$ for all $x, y \in L$, then g is a biderivation of L.*

Theorem 4.4 *Any commutative post-Lie algebra structure on $\mathcal{S}$ is trivial. Namely, $x * y = 0$ for all $x, y \in \mathcal{S}$.*

Proof Suppose that $(\mathcal{S}, [,], *)$ is a commutative post-Lie algebra. By Lemma 4.3 and Theorem 3.4, we know that there is $\lambda \in \mathbb{C}$ such that $x * y = \lambda[x, y]$ for all $x, y \in \mathcal{S}$. On the other hand, since the post-Lie algebra is commutative, so we have $\lambda[x, y] = \lambda[y, x]$. This implies $\lambda = 0$. The proof is completed. □

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