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ORIGINAL RESEARCH

A numerical study of Asian option with high-order compact finite difference scheme

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Abstract In this paper, an unconditionally stable compact finite difference scheme is proposed for the solution of Asian option partial differential equation. Second derivative approximations of the unknowns are eliminated with the unknowns itself and their first derivative approximations while retaining the fourth order accuracy and tri-diagonal nature of the scheme. Proposed compact finite difference scheme is fourth order accurate in spatial variable and second order accurate in temporal variable. Moreover, consistency, stability and convergence of the proposed compact finite difference scheme is proved and it is shown that proposed compact finite difference scheme is unconditionally stable. It is shown that for a given accuracy, proposed compact finite difference scheme is significantly efficient as compared to the central difference scheme. Numerical results are given to validate the theoretical results.

Keywords Compact finite difference scheme · Option pricing · Asian option

1 Introduction

Besides the plain vanilla options (European [1] and American [2]), there are some other types of options known as exotic options [3]. Asian options [4,5], as one of the example of exotic options, first appeared in 1987 when the Bankers Trust Tokyo (hence the name Asian options) office developed a commercially used pricing formula for options on the average crude oil price. Asian options are securities whose payoffs

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depend on the average value of the underlying asset price over some time period. Because of this averaging feature, Asian options are difficult to price as compared to the standard European options. A comparison between Asian options and plain vanilla options is given in [6]. Asian options can be divided into two main categories on the basis of taking average namely geometric Asian options and arithmetic Asian options. Geometric Asian options [7] have closed form solution because geometric mean of random variables also follows log-normal distribution. In practice, most Asian options use the arithmetic average. Since the probability distribution of the sum of log-normally distributed random variables is analytically intractable, the problem of pricing arithmetic Asian options does not have a closed form solution. Therefore various numerical techniques have been applied to price the Asian options.

Let us review some existing literature for Asian options. Zvan et al. [8] demonstrated that numerical PDE techniques commonly used in finance for standard options are inaccurate in the case of Asian options. They have used flux limiter to retain accuracy while preventing oscillations. Vecer [9] has characterized Asian options by a one dimensional PDE (by considering Asian options as a option on a traded account) which could be applied to both discrete and continuous average Asian options. He applied classical finite difference method to solve the Asian options PDE. Nothing is said about the rate of convergence of the proposed method. Marcozzi [10] provided variational methods for pricing the Asian options. A theoretical framework is given by Marcozzi in his paper as numerical analysis of a finite element implementation. He provided the method to find the price of Asian options which has early exercise feature. Dubois et al. [11] has applied classical finite difference method to the Asian options PDE on a moving grid and showed that their method is much faster than method presented in [9]. They have not discussed about the rate of convergence of their method in their paper. D'halluin et al. [12] gave a semi-Lagrangian approach to price continuously observed fixed strike Asian options. A one-dimensional partial integro differential equations (PIDEs) has been solved in this paper at each time step and solution is updated using semi-Lagrangian time stepping. Rogers and Shi [5] has obtained lower-bounds for both types of Asian options. Lower bound formulas in [5] restrict the options maturity to exactly 1 year. This limitation has been removed by Chen et al. [13] and Rogers–Shi formula is extended to general maturities. There are some other methods also which has been applied to different types of Asian options. For eg. Fusai et al. [14] presented a new methodology based on maturity randomization to price discretely monitored arithmetic Asian options when the underlying asset evolves according to a generic Levy process. Foschi et al. [15] developed approximations for the density, the price and the Greeks of path dependent options of Asian style, in a general local volatility model. Zhang et al. [16] proposed an efficient pricing method for arithmetic and geometric Asian options under exponential Levy processes based on Fourier cosine expansions and Clenshaw–Curtis quadrature. Recently, Kumar et al. [17] presented a numerical study of Asian options with radial basis functions based on finite differences method.

The growing popularity of compact finite different schemes in recent years have brought about a renewed interest towards the finite difference approach. High-order compact finite difference schemes which consider not only the value of the function but also those of its first or higher derivatives as unknowns at each discretization point have been extensively studied and widely used to solve the PDEs arising in computational fluid dynamics and many areas of applied mathematics. In compact finite difference schemes, high-order accuracy is obtained even for small number of grid points. High-order compact schemes leads to a system of equations with coefficient matrix having smaller band width as compared to classical finite difference schemes. Various efforts towards numerical approximation of convection-diffusion equation using high-order compact finite difference approach can now be seen in literature [18–20]. For eg. in [18,19], modified differential equation approach is used to derive the high-order accurate first and second derivative approximations and the truncation error is compactly approximated. In these paper, original second order differential equation is considered as an auxiliary relation and original equation is differentiated in order to get the expressions for higher derivatives. Deriving compact schemes using modified differential approach for variable coefficient PDEs is difficult in general. A fourth order accurate compact finite difference scheme for convection-diffusion equation is proposed by Rigal [20]. He eliminated the highest order term in Taylor series in order to get a fourth order accurate compact finite difference scheme. High-order compact schemes have also been used in computational finance in order to compute the option prices. In 2004, During et al. [21] discussed the convergence of high-order compact finite difference scheme for nonlinear Black-Scholes equation. Zhao et al. [22] discussed the high-order compact schemes for pricing American options. Tangman et al. [23] in 2008 discussed the high-order compact scheme for numerical pricing of European and American options under Black-Scholes model.

In this paper, we propose an unconditionally stable compact finite difference scheme for the solution of Asian option PDE. The main advantages of the proposed compact finite difference scheme are as follows:

- Proposed compact finite difference scheme does not require the original equation as an auxiliary equation. In proposed compact finite difference scheme, second derivative approximations of the unknowns are eliminated with the unknowns itself and their first derivative approximations while retaining the fourth order accuracy and tri-diagonal nature of the scheme. Moreover the proposed compact finite difference scheme can be used for constant as well as variable coefficient PDEs without any modification.
- Proposed compact finite difference scheme is compared with the classical finite difference schemes and Padé schemes for second derivative using Fourier analysis and it is observed that proposed compact finite difference scheme has better resolution characteristics.
- We apply the proposed compact finite difference scheme to the Asian option PDE with various parameters and it is observed that proposed compact finite difference scheme is fourth order accurate. Consistency, stability and convergence of the proposed compact finite difference scheme is proved and it is shown that proposed compact finite difference scheme is unconditionally stable.
- Moreover, efficiency of the proposed compact finite difference scheme is compared to the central difference scheme by calculating the CPU time for a given accuracy and it is observed that proposed compact finite difference scheme is more efficient than central difference scheme.

The rest of the paper is organized as follows. In Sect. 2, arithmetic Asian options PDE is given. In Sect. 3, high-order compact finite difference approximations for first and second derivatives are discussed. Fourier analysis for different finite difference schemes is also discussed in this sections. In Sect. 4, temporal and spatial discretization for Asian option PDE is given. Consistency, stability and convergence of the proposed compact finite difference scheme is also proved in this section. In Sect. 5, numerical results for arithmetic Asian options with high-order compact finite difference method are given and obtained results are compared with the existing literature. In Sect. 6, conclusion of this paper is given and future work is proposed.

2 Mathematical model

Asian options, introduced by Ingersoll [4], can be expressed as the solution of a two dimensional PDE for pricing of path dependent options which can be written as:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} + S\frac{\partial V}{\partial I} - rV = 0, \tag{2.1}$$

with the final condition.

$$V(S, I, t) = max\left(\frac{I}{T} - K, 0\right), \tag{2.2}$$

where r is interest rate, T is the time to expiration, K is the stock price and σ is the volatility of underlying asset and

$$I(t) = \int_0^t S(\xi)d\xi.$$

As we know that it is numerically expensive to solve a two dimensional PDE, some authors (eg. Rogers and Shi [5]) have tried to convert the above equation into one dimensional PDE by using the transformations

$$V = Su(z, t), \quad z = \frac{K - \frac{1}{T}\int_0^t S(\xi)d\xi}{S},$$

the above two-dimensional PDE (2.1) can be reduced to following one dimensional PDE:

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2 z^2 \frac{\partial^2 u}{\partial z^2} + \left(-\frac{1}{T} - rz\right)\frac{\partial u}{\partial z} = 0, \quad (z, t) \in (0, \infty) \times [0, T), \tag{2.3}$$

with the following terminal and boundary conditions

$$
\begin{aligned}
u(z,T) &= max(-z,0), \quad z \in (0,\infty),\\
u(0,t) &= \frac{1}{rT}(1-e^{-r(T-t)}), \quad t \in [0,T], \qquad (2.4)\\
\lim_{z\to\infty} u(z,t) &= 0, \quad t \in [0,T].
\end{aligned}
$$

Asian option PDE (2.3) with terminal and boundary condition (2.4) is backward in time, so now take $x = e^{-z}$ and $\tau = T - t$, the following initial boundary value problem for Asian option is obtained:

$$
\begin{aligned}
\frac{\partial u}{\partial \tau} &= \frac{1}{2}\sigma^2 x^2 (\ln x)^2 \frac{\partial^2 u}{\partial x^2} + \left[\left(\frac{1}{T} - r\ln x\right)x\right.\\
&\left. + \frac{1}{2}\sigma^2 x (\ln x)^2\right]\frac{\partial u}{\partial x}, \quad (x,\tau) \in (0,1)\times(0,T], \qquad (2.5)
\end{aligned}
$$

$$
\begin{aligned}
u(x,0) &= 0, \quad x \in (0,1),\\
u(0,\tau) &= 0, \quad \tau \in [0,T],\\
u(1,\tau) &= \frac{1}{rT}(1-e^{-r\tau}), \quad \tau \in [0,T]. \qquad (2.6)
\end{aligned}
$$

The above Asian option PDE (2.5) is solved by compact finite difference scheme which is discussed in the following sections. The price of arithmetic average Asian call option is obtained from the solution of above PDE by the relation $V(x, I, t) = Su(z, \tau)$.

3 High-order compact finite difference approximations for first and second derivatives

In this section, we derive fourth order accurate second derivative approximations of unknowns with the help of unknowns itself and their first derivative approximations. From Taylor series, second order accurate central difference approximation for first derivative can be written as follows

$$\Delta_x f_i = \frac{f_{i+1} - f_{i-1}}{2h}, \qquad (3.1)$$

and similarly second order accurate central difference approximation for second derivative can be written as

$$\Delta_x^2 f_i = \frac{f_{i+1} - 2f_i + f_{i-1}}{h^2}, \qquad (3.2)$$

where h is the grid size along x-direction and f_i is the value of $f(x_i)$ at a typical grid point x_i. Now, fourth-order accurate compact finite difference approximation for first derivative (Padé scheme for first derivative) can be written as

$$\frac{1}{4}f'_{i-1} + f'_i + \frac{1}{4}f'_{i+1} = \frac{1}{h}\left[-\frac{3}{4}f_{i-1} + \frac{3}{4}f_{i+1}\right], \qquad (3.3)$$

where f_i' is first derivative approximation of unknown f at grid point x_i. Similarly, fourth order accurate compact finite difference approximation for second derivative (Pad $\acute{e}$ scheme for second derivative) can be written as

$$\frac{1}{10}f''_{i-1} + f''_i + \frac{1}{10}f''_{i+1} = \frac{1}{h^2}\left[\frac{6}{5}f_{i-1} - \frac{12}{5}f_i + \frac{6}{5}f_{i+1}\right], \tag{3.4}$$

where f_i'' is second derivative approximation of unknown f at grid point x_i. Second derivative approximations of unknowns are eliminated using the unknowns itself and their first order derivative approximations while preserving the tri-diagonal nature of the scheme. If f_i' are also considered as a variable then from Eq. (3.3), we obtain

$$\frac{1}{4}f''_{i-1} + f''_i + \frac{1}{4}f''_{i+1} = \frac{1}{h}\left[-\frac{3}{4}f'_{i-1} + \frac{3}{4}f'_{i+1}\right], \tag{3.5}$$

Eliminating f''_{i-1} and f''_{i+1} from Eqs. (3.4) and (3.5), we obtain second derivative approximation as follows

$$f''_i = 2\frac{f_{i+1} - 2f_i + f_{i-1}}{h^2} - \frac{f'_{i+1} - f'_{i-1}}{2h}. \tag{3.6}$$

Using Eqs. (3.1) and (3.2) in above Eq. (3.6), we obtain

$$f''_i = 2\Delta_x^2 f_i - \Delta_x f'_i. \tag{3.7}$$

In Eq. (3.7), f_i' is obtained from Eq. (3.3). It is observed that Eqs. (3.3) and (3.7) provides fourth order accurate compact approximation to the first and second derivatives. The elimination of second order derivatives was initially proposed by Adam [24,25]. In case of non-periodic boundary conditions, additional compact relations are required at the boundary points. For the additional boundary formulations of various orders, one can see [25,26]. We compare the proposed schemes with classical finite difference scheme and other compact finite difference schemes as follows.

3.1 Fourier analysis

Fourier analysis is a classical technique to compare two difference schemes in numerical analysis. Fourier analysis of a finite difference scheme quantifies the resolution characteristics of the difference approximation. By resolution characteristics, we mean that the accuracy with which difference schemes represents the exact value over the full grid. For more details about the Fourier analysis of finite difference schemes, one can see [27].

Fourier analysis for first derivative approximation If we denote the wave number by ω and modified wave number for first derivative approximation by ω', then for fourth order accurate compact finite difference approximation Eq. (3.3)

$$\omega' = \frac{3sin(\omega)}{2 + cos(\omega)}. \tag{3.8}$$

For second order accurate classical finite difference approximation Eq. (3.1)

$$\omega' = sin(\omega), \tag{3.9}$$

and for fourth order accurate classical finite difference approximation

$$\omega' = \frac{-sin\,(2\omega)}{6} + \frac{4sin(\omega)}{3}. \tag{3.10}$$

In Fig. 1a modified wave numbers are plotted with respect to the wave numbers for exact differentiation ($\omega' = \omega$), fourth order accurate compact finite difference approximation Eq. (3.8), second order accurate classical finite difference approximation Eq. (3.9) and fourth order accurate classical finite difference approximation Eq. (3.10). It can be seen from Fig. 1a that fourth order accurate compact finite difference approximation has better resolution characteristics as compared to the classical finite difference approximations.

Fourier analysis for second derivative approximation If we denote modified wave number for second derivative approximation by ω'', then for fourth order accurate compact finite difference approximation Eq. (3.7)

$$\omega'' = \frac{5 - 4cos(\omega) - cos^2(\omega)}{2 + cos(\omega)}. \tag{3.11}$$

For fourth order accurate compact finite difference Padé approximation Eq. (3.4)

$$\omega'' = \frac{12\,(1 - cos(\omega))}{2 + cos(\omega)}. \tag{3.12}$$

For second order accurate classical finite difference approximation Eq. (3.2)

$$\omega'' = 2 - 2cos(\omega), \tag{3.13}$$

and for fourth order accurate classical finite difference approximation

$$\omega'' = \frac{cos(2\omega)}{6} - \frac{8cos(\omega)}{3} + \frac{5}{2}. \tag{3.14}$$

In Fig. 1b modified wave numbers are plotted with respect to the wave numbers for exact differentiation ($\omega'' = \omega^2$), compact finite difference approximation for second derivative Eq. (3.12), classical finite difference second order approximation Eq. (3.13) and classical finite difference fourth order approximation Eq. (3.14). It can be seen from Fig. 1b that compact finite difference approximation has better resolution characteristics as compared to the classical finite difference approximation.

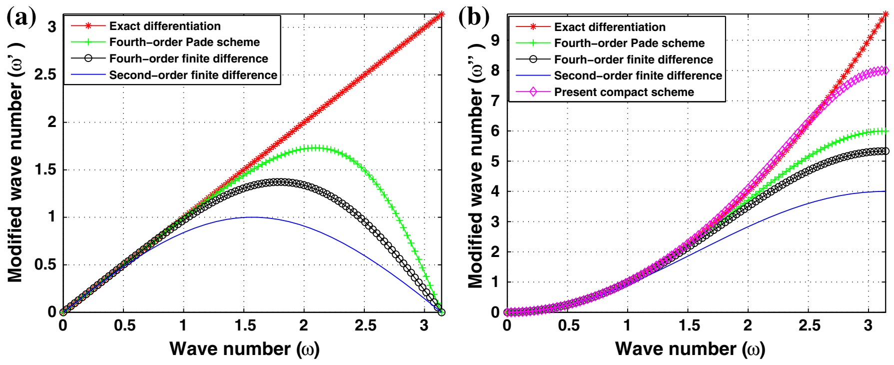

Fig. 1 Modified wave number versus wave number for various finite difference schemes **a** first derivative approximation, **b** second derivative approximation

4 Fully discrete problem

Asian option PDE (2.5) can be written as follows

$$\frac{\partial u}{\partial \tau} = \mathbb{L}u, \tag{4.1}$$

with the given initial and boundary conditions in Eq. (2.6), where

$$\mathbb{L}u = p(x,\tau)\frac{\partial u}{\partial x} + q(x,\tau)\frac{\partial^2 u}{\partial x^2}, \tag{4.2}$$

and

$$p(x,\tau) = \left[\left(\frac{1}{T} - r\ln x\right)x + \frac{1}{2}\sigma^2 x(\ln x)^2\right], \tag{4.3}$$

and

$$q(x,\tau) = \frac{1}{2}\sigma^2 x^2 (\ln x)^2. \tag{4.4}$$

Now we discretize the Eq. (4.1) in finite domain $(x,\tau) \in (\Omega = [0,1]) \times (0,T]$ as follows.

4.1 Temporal semi-discretization

To study the convergence analysis of Eq. (2.5), Crank–Nicolson scheme is used to discretize the temporal variable, keeping the space variable x continuous for a fixed step-size $\delta\tau = \frac{T}{M}$, where M is the number of grid points in time direction. Time semi-discretization of Eq. (2.5) gives:

$$\frac{u^{m+1} - u^m}{\delta\tau} = \frac{1}{2}\mathbb{L}_x u^m + \frac{1}{2}\mathbb{L}_x u^{m+1},$$

or

$$
\begin{aligned}
&(I-\frac{1}{2}\delta\tau\mathbb{L}_x)u^{m+1}=(I+\frac{1}{2}\delta\tau\mathbb{L}_x)u^m,\\
&u^m(0)=0,\quad u^m(1)=\frac{1}{rT}(1-e^{-r\tau_m}),\\
&u^0(x)=0,\qquad \text{for}\quad m=1,...,M,
\end{aligned}
\tag{4.5}
$$

where

$$
\mathbb{L}_x u^m(x)=p(x,\tau_m)\frac{\partial u^m}{\partial x}+q(x,\tau_m)\frac{\partial^2 u^m}{\partial x^2},
$$

and $u^m(x)$ is the approximation of the solution $u(x,\tau)$ at any time level τ_m. Now we discuss the stability for time semi-discrete Eq. (4.5) as follows:

4.1.1 Stability analysis

Now suppose $\hat{u}^m(x)$ is the solution of the following problem with a perturbation of the data

$$
\begin{aligned}
&(I-\frac{1}{2}\delta\tau\mathbb{L}_x)\hat{u}^{m+1}=(I+\frac{1}{2}\delta\tau\mathbb{L}_x)\hat{u}^m+\delta\tau\varepsilon^{m+1/2}(x),\\
&\hat{u}^m(0)=0,\quad \hat{u}^m(1)=\frac{1}{rT}(1-e^{-r\tau_m}),\\
&\hat{u}^0(x)=\hat{\varepsilon}^0(x),\qquad \text{for}\quad m=1,...,M,
\end{aligned}
\tag{4.6}
$$

where $\varepsilon^{m+1/2}(x)=\varepsilon(x,\tau_m+\frac{1}{2}\delta\tau)$ and $\hat{u}^0=u^0+\hat{\varepsilon}^0(x)$. A temporal semi-discretization scheme to the problem (2.5)– (2.6) is stable, if $\|u^m-\hat{u}^m\|_\infty$ is bounded by the upper bound of perturbation,i.e.

$$
\|u^m-\hat{u}^m\|_\infty\le C\left(\hat{\varepsilon}^0+\max_{0\le j\le m-1}\|\varepsilon^{j+1/2}\|_\infty\right).
\tag{4.7}
$$

In order to prove the above result for time semi-discrete Eq. (4.5), we need following Lemmas:

Lemma 1 *(Maximum and Minimum principles) The operator* $(I-\frac{1}{2}\delta\tau\mathbb{L}_x)\equiv\hat{\mathbb{L}}_x$ *satisfies maximum and minimum principle at any time level m, i. e. u can not attain its maximum and minimum at an interior point unless it is constant.*

Proof For proof, one can see [28]. □

With the help of Lemma. 1, it is observed that operator $\hat{\mathbb{L}}_x$, defined by

$$
\hat{\mathbb{L}}_x(w)=v,\quad in\quad \Omega,\quad w=0,\quad on\quad \bar{\Omega}\setminus\Omega.
$$

is a bijective linear operator.

Lemma 2 *Let the operator* $(I - \frac{1}{2}\delta\tau\mathbb{L}_x)^{-1} \equiv \hat{\mathbb{L}}_x^{-1}$ *be such that* $\hat{\mathbb{L}}_x^{-1}(v)$ *is solution* 'w' *of the following problem*

$$\hat{\mathbb{L}}_x(w) = v, \quad w = 0, \quad on \quad \bar{\Omega} \setminus \Omega.$$

then

$$\|\hat{\mathbb{L}}_x^{-1}\|_\infty \leq \frac{1}{1 + \frac{1}{2}\delta\tau\hat{\gamma}}.$$

Proof For proof of above Lemma, one can see [29]. □

In order to prove the stability of Crank-Nicolson scheme, we introduce the notations as follows: $\zeta^m = \hat{u}^m - u^m, \quad m = 0, 1, ..., M-1$ So from Eq. (4.5), we obtain

$$\begin{aligned} \zeta^0 &= \hat{\varepsilon}^0, \\ (I - \frac{1}{2}\delta\tau\mathbb{L}_x)\zeta^{m+1} &= (I + \frac{1}{2}\delta\tau\mathbb{L}_x)\zeta^m + \delta\tau\varepsilon^{m+1/2}, \\ \zeta^m(0) &= 0, \\ \zeta^m(1) &= 0, \end{aligned} \tag{4.8}$$

This implies

$$\zeta^{m+1} = (I - \frac{1}{2}\delta\tau\mathbb{L}_x)^{-1}\left[(I + \frac{1}{2}\delta\tau\mathbb{L}_x)\zeta^m + \delta\tau\varepsilon^{m+1/2}\right]. \tag{4.9}$$

Since $(I - \frac{1}{2}\delta\tau\mathbb{L}_x)^{-1}$ is linear, Eq. (4.9) yields

$$\begin{aligned} \zeta^m = &\left(\prod_{j=0}^{m-1} R_{m-j}\right)\hat{\varepsilon}^0 + \delta\tau(I - \frac{1}{2}\delta\tau\mathbb{L}_x)^{-1}\varepsilon^{m-1/2} \\ &+ \delta\tau\sum_{i=1}^{m-1}\left(\prod_{j=0}^{i-1} R_{m-j}\right)(I - \frac{1}{2}\delta\tau\mathbb{L}_x)^{-1}\varepsilon^{m-1/2-i}, \end{aligned} \tag{4.10}$$

where

$$\prod_{j=0}^{m-1} R_{m-j} = R_m R_{m-1} ... R_1.$$

The operator, $R_m \equiv R(\delta\tau\mathbb{L}_x)$, $1 \leq m \leq M$, is defined in such a way that $u_1 \equiv R_m u$ is the solution of

$$(I - \frac{1}{2}\delta\tau\mathbb{L}_x)u_1 = (I + \frac{1}{2}\delta\tau\mathbb{L}_x)u_1, \quad in \quad \Omega,$$

$$u_1 = 0, \quad on \quad \bar{\Omega} \setminus \Omega.$$

Now, following the argument of [30] for the operator R_m, $\exists$ a constant $C > 0$ such that

$$\Big\| \prod_{j=0}^{n-1} R_{m-j} \Big\|_\infty \leq C \quad \forall \quad n = 1, 2, ..., m, \quad 1 \leq m \leq M. \tag{4.11}$$

Now using Eq. (4.11) and Lemma 2 in Eq. (4.10), we get

$$\|\zeta^m\|_\infty = \|u^m - \hat{u}^m\|_\infty \leq C \left(\hat{\varepsilon}^0 + \max_{0 \leq j \leq m-1} \|\varepsilon^{j+1/2}\|_\infty \right).$$

Hence we get the following result:

Theorem 1 *The temporal semi-discretization method* (4.5) *is unconditionally stable.*

4.1.2 Truncation error analysis and convergence

In order to discuss the error introduced in time semi-discretization, the local truncation error e^{m+1} is defined as follows

$$e^{m+1} = u(x, \tau_{m+1}) - \tilde{u}^{m+1},$$

where, $\tilde{u}^{m+1}$ is the solution obtained from semi-discrete scheme after one time step taking exact value as initial condition, i.e.

$$\begin{aligned} (I - \frac{1}{2}\delta\tau \mathbb{L}_x)\tilde{u}^{m+1}(x) &= (I + \frac{1}{2}\delta\tau \mathbb{L}_x)u(x, \tau_m) \\ \tilde{u}^{m+1}(0) = 0, \quad \tilde{u}^{m+1}(1) &= \frac{1}{rT}(1 - e^{-r\tau_{m+1}}). \end{aligned} \tag{4.12}$$

The global truncation error at any point τ_m is given as:

$$E^m = u(x, \tau_m) - u^m.$$

Lemma 3 *If*

$$\left| \frac{\partial^{i+j} u(x, \tau)}{\partial x^i \partial \tau^j} \right| \leq C, \quad 0 \leq j \leq 3, \quad 0 \leq i \leq 4,$$

then local truncation error for semi-discrete scheme Eq. (4.12) *satisfies*

$$\left\| \mathbf{e^m} \right\|_\infty \leq C(\delta\tau)^3,$$

where C is a constant independent of M.

Proof Since u is sufficiently smooth and $|u_{\tau\tau\tau}| \leq C$, therefore we can write from the Taylor series expansion

$$u(x, \tau_{m+1}) = u\left(x, \tau_{m+\frac{1}{2}}\right) + \frac{1}{2}\delta\tau u_t\left(x, \tau_{m+\frac{1}{2}}\right) + \frac{\delta\tau^2}{8}u_{\tau\tau}\left(x, \tau_{m+\frac{1}{2}}\right) + O(\delta\tau^3),$$

$$u(x, \tau_m) = u\left(x, \tau_{m+\frac{1}{2}}\right) - \frac{1}{2}\delta\tau u\left(x, \tau_{m+\frac{1}{2}}\right) + \frac{\delta\tau^2}{8}u_{\tau\tau}\left(x, \tau_{m+\frac{1}{2}}\right) + O(\delta\tau^3).$$

From above expressions, we get

$$\frac{u(x, \tau_{m+1}) - u(x, \tau_m)}{\delta\tau} = u_\tau\left(x, \tau_{m+\frac{1}{2}}\right) + O(\delta t^2). \tag{4.13}$$

Now using Eq. (2.5) in Eq. (4.13), we get

$$\frac{u(x, \tau_{m+1}) - u(x, \tau_m)}{\delta\tau} = \mathbb{L}_x u\left(x, \tau_{m+\frac{1}{2}}\right) + O(\delta\tau^2).$$

Since

$$\frac{1}{2}\left(u(x, \tau_{m+1}) + u(x, \tau_m)\right) = u\left(x, \tau_{m+\frac{1}{2}}\right) + \frac{\delta t^2}{8}u_{\tau\tau}\left(x, \tau_{m+\frac{1}{2}}\right) + O(\delta\tau^3),$$

and we have

$$\mathbb{L}_x\left[\frac{1}{2}u(x, \tau_{m+1}) + \frac{1}{2}u(x, \tau_m)\right] = \mathbb{L}_x u\left(x, \tau_{m+\frac{1}{2}}\right) + O(\delta\tau^2),$$

which implies

$$\mathbb{L}_x u\left(x, \tau_{m+\frac{1}{2}}\right) = \mathbb{L}_x\left[\frac{1}{2}u(x, \tau_{m+1}) + \frac{1}{2}u(x, \tau_m)\right] + O(\delta\tau^2).$$

From above relation and Eq. (4.13), we get

$$(I - \frac{1}{2}\delta\tau\mathbb{L}_x)u(x, \tau_{m+1}) = \left[I + \frac{1}{2}\delta\tau\mathbb{L}_x\right]u(x, \tau_m) + O(\delta\tau^3).$$

Using Eq (4.12) and above relation, we get

$$(I - \frac{1}{2}\delta\tau\mathbb{L}_x)e^{m+1} = O(\delta\tau^3),$$
$$e^{m+1}(0) = 0, \quad e^{m+1}(1) = 0,$$

By using Lemma 2, we get

$$\|\mathbf{e^m}\|_\infty \leq C(\delta\tau)^3,$$

which completes the proof. □

Lemma 4 *If* $\|\mathbf{E^m}\|_\infty$ *represents the maximum norm of the global truncation error, then global truncation error for the scheme* (4.12) *satisfies:*

$$\|\mathbf{E^m}\|_\infty \leq C(\delta\tau)^2, \tag{4.14}$$

where C *is a constant independent of* M.

Proof The global truncation error at any time τ_m can be written as follows:

$$E^m = u(x,\tau_m) - u^m = u(x,\tau_m) - \tilde{u}^m + \tilde{u}^m - u^m. \tag{4.15}$$

From Eq. (4.5), we get

$$(I - \frac{1}{2}\delta\tau\mathbb{L}_x)(\tilde{u}^m - u^m) = (I + \frac{1}{2}\delta\tau\mathbb{L}_x)(u(x,\tau_{m-1}) - u^{m-1}). \tag{4.16}$$

A recurrence relation in terms of global truncation error is obtained from the Eqs. (4.15)–(4.16) as follows:

$$E^m = E^0\left(\prod_{j=0}^{m-1} R_{m-j}\right) + e^m + \sum_{i=1}^{m-1}\left(\prod_{j=0}^{i-1} R_{m-j}\right)e^{m-i},$$
$$E^0 = u(x,\tau_0) - u^0 = 0.$$

From Eq. (4.11), we have

$$\begin{aligned}\|E^m\|_\infty &\leq m.C.\max_{1\leq i\leq m}\|e^i\|_\infty,\\ &= m\delta\tau.\frac{C}{\delta\tau}.\max_{1\leq i\leq m}\|e^i\|_\infty,\\ &\leq TC.\frac{1}{\delta\tau}\max_{1\leq i\leq m}\|e^i\|_\infty.\end{aligned} \tag{4.17}$$

Using Lemma 3, we get the desired result

$$\|E^m\|_\infty \leq C(\delta\tau)^2.$$

□

Now by using Theorem. 1, Lemma. 4 and Lax equivalence theorem [31], we get the following result.

Theorem 2 *The time semi-discrete method* (4.5) *is second order convergent.*

4.2 Spatial discretization

High-order compact finite difference approximations as discussed in Sect. 3 are used for the spatial discretization of Eq. (4.5). If U_n^m represents the numerical solution of Eq. (4.1) at (x_n, τ_m) then Eq. (4.5) can be written in discrete form as follows

$$\frac{U_n^{m+1} - U_n^m}{\delta\tau} = \frac{1}{2}\mathbb{L}_\Delta U_n^m + \frac{1}{2}\mathbb{L}_\Delta U_n^{m+1}, \tag{4.18}$$

where

$$\mathbb{L}_\Delta U_n^m = p(x_n, \tau_m)U_{x_n}^m + q(x_n, \tau_m)U_{xx_n}^m, \tag{4.19}$$

and $U_{x_n}^m$, $U_{xx_n}^m$ represents fourth order accurate first and second derivative compact finite difference approximation of U at (x_n, τ_m). Now using the compact finite difference approximation (3.7) for second derivative in Eq. (4.18), we get

$$\begin{aligned}\frac{U_n^{m+1} - U_n^m}{\delta\tau} &= \left[\frac{1}{2}q_n^m(2\Delta_x^2 U_n^m - \Delta_x U_{x_n}^m) + \frac{1}{2}q_n^{m+1}(2\Delta_x^2 U_n^{m+1} - \Delta_x U_{x_n}^{m+1})\right] \\ &\quad + \left[\frac{1}{2}p_n^m U_{x_n}^m + \frac{1}{2}p_n^{m+1}U_{x_n}^{m+1}\right],\end{aligned} \tag{4.20}$$

where $1 \le n \le N$, $1 \le m \le M$ and N, M are the number of grid points in space and time direction respectively. Hence proposed compact finite difference scheme for Asian option PDE results to the fully-discrete problem as follows:

$$\begin{aligned}[1 - q_n^{m+1}\delta\tau\Delta_x^2]U_n^{m+1} &= [1 - q_n^m\delta\tau\Delta_x^2]U_n^m \\ &\quad + \frac{1}{2}\delta\tau[p_n^{m+1} - q_n^{m+1}\Delta_x]U_{x_n}^{m+1} + \frac{1}{2}\delta\tau[p_n^m - q_n^m\Delta_x]U_{x_n}^m. \\ U_0^m = 0, \quad U_N^m &= \frac{1}{rT}(1 - e^{-r\tau_m}).\end{aligned} \tag{4.21}$$

We discuss the consistency, stability and convergence of the scheme (4.21) in the following section. We prove that proposed compact finite difference scheme is unconditionally stable and second and fourth order accurate in time and spatial variable respectively.

4.2.1 Consistency, stability and convergence analysis

At first, we discuss the consistency of the proposed compact finite difference scheme (4.21).

Theorem 3 *(Consistency)*
Let $v \in C^\infty([0, 1] \times (0, T])$ satisfy the initial and boundary conditions (2.6). Then as $h, \delta\tau \to 0$,

$$\frac{\partial v}{\partial \tau}(x_n, \tau_m) - \mathbb{L}v(x_n, \tau_m) - \left(\frac{v(x_n, \tau_{m+1}) - v(x_n, \tau_m)}{\delta\tau} - \mathbb{L}_\Delta v(x_n, \tau_m)\right)$$
$$= O(\delta\tau^2 + h^4), \tag{4.22}$$

where $\mathbb{L}$ *and* $\mathbb{L}_\Delta$ *are defined in Eqs.* (4.2) *and* (4.19) *respectively and* $(x_n, \tau_m) \in (0, 1) \times (0, T]$.

Proof From the fact that compact finite difference approximations for first and second derivatives discussed in Sect. 3 are $O(h^4)$ accurate and by using the Lemma. 4, result follows. □

Stability analysis is very crucial aspect for the solution of time dependent problems using numerical algorithms. Since the coefficients of the Eq. (2.5) will always be bounded in sup norm for a discrete problem, principle of frozen coefficients can be used to prove the stability for compact finite difference scheme (4.21). Stability analysis for central difference schemes for variable coefficient problems is discussed in [32] using the principle of frozen coefficients. We carry out von-Neumann stability analysis for the compact finite difference scheme (4.21) as follows:

Theorem 4 *(Stability) The compact finite difference scheme* (4.21) *is unconditionally stable.*

Proof Let $U_n^m = b^m e^{im\omega}$ where $\omega = 2\pi h/\lambda$ is the phase angle with wavelength λ and b^m is the amplitude at time level m then from Eqs. (3.9), (3.13) and (3.8) we can write

$$\Delta_x U_n^m = i\frac{sin(\omega)}{h}U_n^m, \tag{4.23}$$

$$\Delta_x^2 U_n^m = \frac{2cos(\omega) - 2}{h^2}U_n^m, \tag{4.24}$$

$$U_{x_n}^m = i\frac{3sin(\omega)}{h(2 + cos(\omega))}U_n^m. \tag{4.25}$$

Using relation (4.23), (4.24) and (4.25) in the difference scheme (4.21), we get

$$\begin{aligned}
&\left[1 - 2q\delta\tau\left(\frac{cos(\omega) - 1}{h^2}\right)\right]U_n^{m+1} \\
&= \left[1 + 2q\delta\tau\left(\frac{cos(\omega) - 1}{h^2}\right)\right]U_n^m \\
&\quad + \frac{1}{2}\delta\tau\left[\left(q\frac{sin(\omega)}{h} + ip\right)\frac{3sin(\omega)}{h(2+cos(\omega))}\right]U_n^{m+1} \\
&\quad + \frac{1}{2}\delta\tau\left[\left(q\frac{sin(\omega)}{h} + ip\right)\frac{3sin(\omega)}{h(2+cos(\omega))}\right]U_n^m,
\end{aligned} \tag{4.26}$$

Then amplification factor A_F can be written as

$$A_F = \frac{1 + \frac{1}{2}\delta\tau \left[\left(q \frac{cos^2(\omega)+4cos(\omega)-5}{h^2(2+cos(\omega))} \right) + i \left(p \frac{3sin(\omega)}{h(2+cos(\omega))} \right) \right]}{1 - \frac{1}{2}\delta\tau \left[\left(q \frac{cos^2(\omega)+4cos(\omega)-5}{h^2(2+cos(\omega))} \right) + i \left(p \frac{3sin(\omega)}{h(2+cos(\omega))} \right) \right]}. \tag{4.27}$$

If

$$P = \delta\tau q \left(\frac{cos^2(\omega) + 4cos(\omega) - 5}{h^2(2 + cos(\omega))} \right),$$

$$Q = \delta\tau p \left(\frac{3sin(\omega)}{h(2 + cos(\omega))} \right),$$

then

$$A_F = \frac{1 + \frac{1}{2}(P + iQ)}{1 - \frac{1}{2}(P + iQ)}. \tag{4.28}$$

This implies

$$|A_F|^2 = \frac{(1 + \frac{P}{2})^2 + \frac{1}{4}Q^2}{(1 - \frac{P}{2})^2 + \frac{1}{4}Q^2}. \tag{4.29}$$

Stability condition $|A_F| \leq 1$ implies

$$2P \leq 0. \tag{4.30}$$

Since q is always positive from Eq. (4.4) and

$$\left(\frac{cos^2(\omega) + 4cos(\omega) - 5}{h^2(2 + cos(\omega))} \right) \leq 0 \quad \forall \quad \omega \in [0, 2\pi],$$

which implies $P \leq 0$. Hence, inequality (4.30) is always satisfied and compact finite difference scheme (4.21) is unconditionally stable. □

Now, in order to prove the convergence of the fully discrete problem, we use the following Lemmas:

Lemma 5 *The matrix associated with $[1 - q_n^{m+1}\delta t \Delta_x^2]$ is an M-matrix.*

Proof In $[1-q_n^{m+1}\delta t\Delta_x^2]$, Δ_x^2 is the second derivative central difference approximation defined in Eq. (3.2). Let $a_{i,j}$ be the i^{th} row and j^{th} column entry of the matrix associated to $[1 - q_n^{m+1}\delta t \Delta_x^2]$, then

$$a_{i,i} = 1 + 2q_n^{m+1}\frac{\delta\tau}{h^2},$$

and for $i \neq j$

$$a_{i,j} = -q_n^{m+1}\frac{\delta\tau}{h^2}.$$

It is observed from Eq. (4.4) that q_n^{m+1} will always be positive. Using the tri-diagonal nature of the central difference approximation, it is clear that there will be only two off diagonal entries in each row. It gives that matrix associated to $[1 - q_n^{m+1}\delta\tau\Delta_x^2]$ is diagonally dominant and has non-positive off diagonal entries. Then it follows from [33] that matrix associated to $[1 - q_n^{m+1}\delta t\Delta_x^2]$ is an M-matrix.

Lemma 6 *(Discrete maximum principle) The operator $[1 - q_n^{m+1}\delta\tau\Delta_x^2]$ given in Eq. (4.21) satisfies discrete maximum principle, i.e. if v_i and w_i are two mesh function such that $v_0 \geq w_0$, $v_N \geq w_N$ and $[1 - q_n^{m+1}\delta\tau\Delta_x^2]v_i \geq [1 - q_n^{m+1}\delta\tau\Delta_x^2]w_i$ $\forall$ $1 \leq i \leq N-1$, then $v_i \geq w_i$ $\forall$ i.*

Proof We proved that matrix associated to $[1 - q_n^{m+1}\delta\tau\Delta_x^2]$ is an M-matrix. By using the same argument as Lemma 3.1 in [34], result follows. □

Let $\tilde{U}_n^m$ be the solution of fully discrete problem (4.21) after one time step taking exact solution as the initial condition i.e.

$$\begin{aligned}(I - \frac{1}{2}\delta\tau\mathbb{L}_\Delta)\tilde{U}_n^{m+1} &= (I + \frac{1}{2}\delta\tau\mathbb{L}_\Delta)u_n^m,\\ \tilde{U}_0^m &= 0,\\ \tilde{U}_N^m &= \frac{1}{rT}(1 - e^{-r\tau_m}).\end{aligned} \tag{4.31}$$

In order to prove the convergence of the proposed compact finite difference scheme (4.21), we use following Lemma.

Lemma 7 *Let $\tilde{u}^m(x)$ be the solution of (4.12) and $\tilde{U}_n^m$ be the solution of (4.31). Then*

$$|\tilde{u}_n^m - \tilde{U}_n^m| \leq C\delta\tau h^4, \quad 1 \leq n \leq N,$$

where C is independent of $\delta\tau$ and N.

Proof We proved in Lemma 6 that operator $[1 - q_n^{m+1}\delta\tau\Delta_x^2]$ given in Eq. (4.21) satisfies discrete maximum principle. Now using Lemma 5 of [35], result follows. □

Theorem 5 *(Convergence) Let $u(x,\tau)$ be the exact solution of (2.5) and U be the solution of fully discrete problem (4.21). Then for a constant C, independent of h and $\delta\tau$, such that*

$$|u(x_n,\tau_m) - U_n^m| \leq C[\delta\tau^2 + h^4], \quad 1 \leq n \leq N,\ 1 \leq m \leq M.$$

Proof Global error at any time level m can be written as follows

$$|u(x_n,\tau_m) - U_n^m| \leq |u(x_n,\tau_m) - \tilde{u}_n^m| + |\tilde{u}_n^m - \tilde{U}_n^m| + |\tilde{U}_n^m - U_n^m|.$$

Using Lemmas. 3 and 7 in the above expression, we get

$$|u(x_n, \tau_m) - U_n^m| \leq C\delta\tau(\delta\tau^2 + h^4) + |\tilde{U}_n^m - U_n^m|. \tag{4.32}$$

In order to find the bound for last term of above expression (4.32), we consider that $\tilde{U}_n^m - U_n^m$ can be written as the solution at one time step of fully discrete problem (4.21) with zero boundary condition and $u(x, \tau_{m+1}) - U^{m+1}$ as the final value. Then we can write

$$|\tilde{U}_n^m - U_n^m| \leq C\|u(x, \tau_{m+1}) - U^{m+1}\|_\infty. \tag{4.33}$$

From Eqs. (4.32) and (4.33), we get

$$\begin{aligned} |u(x_n, \tau_m) - U_n^m| &\leq C\delta\tau(\delta\tau^2 + h^4) + C\|u(x, \tau_{m+1}) - U^{m+1}\|_\infty, \\ &\leq 2C\delta\tau(\delta\tau^2 + h^4) + C\|u(x, \tau_m) - U^m\|_\infty, \\ &\leq \ldots \\ &\leq MC\delta\tau(\delta\tau^2 + h^4) + C\|u(x, \tau_0) - U^0\|_\infty, \\ &\leq C[\delta\tau^2 + h^4], \end{aligned} \tag{4.34}$$

which completes the proof. □

We conclude our result as a particular case of the Theorem 5 as follows:

Corollary 1 *Let all the assumptions in the Theorem 5 are satisfied. For $\delta\tau = h^2$, $\exists$ a positive constant C' independent of $\delta\tau$ and h such that*

$$|u(x_n, \tau_m) - U_n^m| \leq C'(h^4), \quad 1 \leq n \leq N, \; 1 \leq m \leq M.$$

4.3 Solution to algebraic system

Solution of algebraic system associated with the difference scheme (4.21) is discussed in this section. If we denote

$$\mathbf{U} = (U_1, U_2, \ldots, U_n)^T \text{ and } \mathbf{U}_x = (U_{x_1}, U_{x_2}, \ldots, U_{x_n})^T,$$

then system of equations corresponding to the difference scheme (4.21) can be written in matrix form as follows

$$A\mathbf{U}^{m+1} = F(\mathbf{U}^m, \mathbf{U}_x^m, \mathbf{U}_x^{m+1}). \tag{4.35}$$

It is observed that proposed compact finite difference scheme leads to a diagonally-dominant, tri-diagonal system of linear equations which can be efficiently solved by Thomas algorithm, requiring $O(N)$ operations. The value of $\mathbf{U}_x^m$ can be obtained by solving a tri-diagonal system of equations from Eq. (3.3). The main problem is due to the presence of $\mathbf{U}_x^{m+1}$ on the right hand side of the Eq. (4.35). For this we use correcting to convergence approach [36] which is summarized in the following algorithm.

Algorithm for correcting to convergence approach

1. Start with $\mathbf{U}^m$.
2. Obtain $\mathbf{U}_x^m$ using Eq. (3.3).
3. Take $\mathbf{U}_{old}^{m+1} = \mathbf{U}^m$, $\mathbf{U}_{x_{old}}^{m+1} = \mathbf{U}_x^m$.
4. Correct to $\mathbf{U}_{new}^{m+1}$ using Eq. (4.35).
5. If $\|\mathbf{U}_{new}^{m+1} - \mathbf{U}_{old}^{m+1}\| <$ tolerance, then $\mathbf{U}_{new}^{m+1} = \mathbf{U}_{old}^{m+1}$.
6. Obtain $\mathbf{U}_{x_{new}}^{m+1}$ using Eq. (3.3).
7. $\mathbf{U}_{old}^{m+1} = \mathbf{U}_{new}^{m+1}$, $\mathbf{U}_{x_{old}}^{m+1} = \mathbf{U}_{x_{new}}^{m+1}$ and go to step 4.

Stopping criterion for inner iteration is taken tolerance= 10^{-12} in above approach. In correcting to convergence approach, it is not known in advance that how many iterations will be required to achieve desired accuracy. Let n_m be the number of iterations required by correcting to convergence approach at fixed time level m and denote $n_s = \max_{1 \le m \le M} n_m$. A tri-diagonal system of equations is solved for each iteration with $O(N)$ operations. Therefore, maximum computational complexity of the proposed compact finite difference scheme will be of order $O(n_s N M)$, where M and N are the number of grid points in time and space direction respectively.

5 Numerical results

In this section, we verify the theoretical results obtained in the previous sections numerically. Numerical results for Asian options PDE Eq. (2.5) are given in this section. Since exact solution of our problem is not available, numerical results are compared with the existing literature [5,7,37,38].

We have plotted the solution of Asian option PDE at time $t = 0$ for various parameters. In Fig. 2a, solution is plotted for maturity $T = 1$ and various value of volatilities. Effect of various values of volatility can be seen in the figure. In Fig. 2b, solution is plotted for long maturity $T = 3$ and various value of volatilities. Value of Asian options as function of time and asset price is also plotted for various values of volatil-

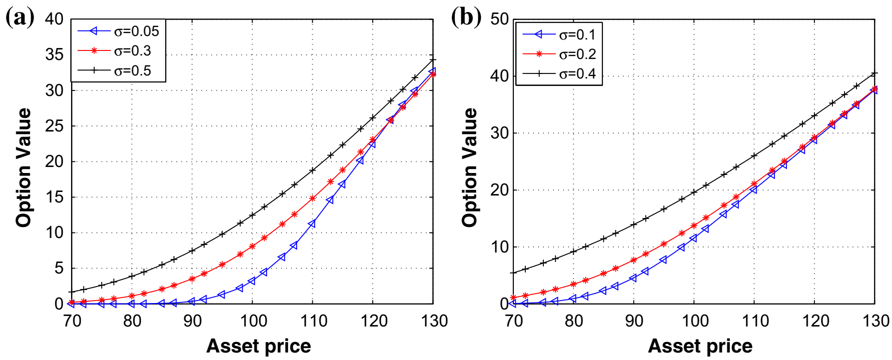

Fig. 2 Value of arithmetic average Asian call option at $t = 0$, $K = 100$, $r = 0.09$ and various values of σ for **a** $T = 1$, **b** $T = 3$

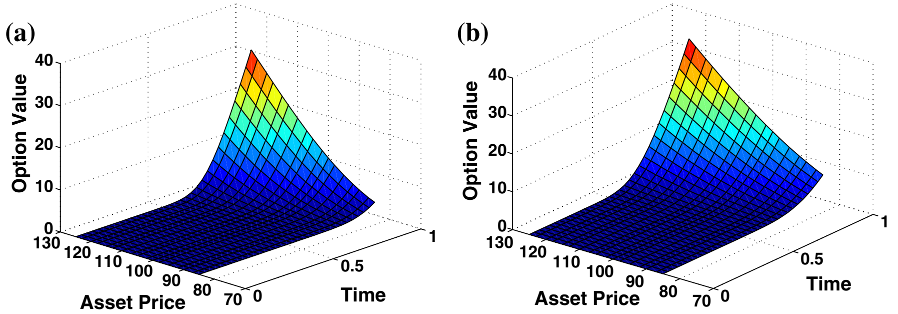

Fig. 3 Value of arithmetic average Asian call option as a function of time and asset price with the parameters $t = 0$, $K = 100$, $r = 0.15$, $T = 1$ and **a** $\sigma = 0.05$, **b** $\sigma = 0.5$

Table 1 Comparison of value of Asian options by various methods for different volatilities and interest rates for $S = 100$, $T = 1$, number of grid points $N = 256$ and $\delta\tau = h^2$

r	σ	K	Present scheme	Zhang [37]	Chen–Lyuu [38]	Lower bound [5]	Upper bound [5]
0.05	0.10	90	11.951022	11.9510927	11.951076	11.951	11.973
		100	3.641465	3.6413864	3.641344	3.641	3.663
		110	0.331530	0.3312030	0.331074	0.331	0.353
0.09		90	13.385286	13.3851974	13.385190	13.385	13.410
		100	4.915267	4.9151167	4.915075	4.915	4.942
		110	0.630486	0.6302713	0.630064	0.630	0.657
0.15		90	15.399433	15.3987687	15.398767	15.399	15.445
		100	7.029775	7.0277081	7.027678	7.028	7.066
		110	1.413228	1.4136149	1.413286	1.413	1.451
0.05	0.20	90	12.597643	12.5959916	12.595602	12.595	12.687
		100	5.763544	5.7630881	5.762708	5.762	5.854
		110	1.990346	1.9898945	1.989242	1.989	2.080
0.09		90	13.832664	13.8314996	13.831220	13.831	13.927
		100	6.788845	6.7773481	6.776999	6.777	6.872
		110	2.548323	2.5462209	2.545459	2.545	2.641
0.15		90	15.643654	15.6417575	15.641598	15.641	15.748
		100	8.413884	8.4088330	8.408519	8.408	8.515
		110	3.558633	3.5556100	3.554687	3.554	3.661
0.05	0.30	90	13.955422	13.9538233	13.952421	13.952	14.161
		100	7.946673	7.9456288	7.944357	7.944	8.153
		110	4.073546	4.0717942	4.070115	4.070	4.279
0.09		90	14.983689	14.9839595	14.982782	14.983	15.194
		100	8.829477	8.8287588	8.827548	8.827	9.039
		110	4.699342	4.6967089	4.694902	4.695	4.906
0.15		90	16.514844	16.5129113	16.512024	16.512	16.732
		100	10.209544	10.2098305	10.208724	10.208	10.429
		110	5.729678	5.7301225	5.728161	5.728	5.948

Table 2 Comparison of value of Asian options by various methods for different volatilities for $S = 100$, $r = 0.09$ and $T = 1$, number of grid points $N = 256$ and $\delta\tau = h^2$

σ	K	Present scheme	Zhang [37]	Zhang-AA2 [7]	Zhang-AA3 [7]	Chen–Lyuu [38]
0.05	95	8.808794	8.808839	8.80884	8.80884	8.808839
	100	4.308244	4.3082350	4.30823	4.30823	4.308231
	105	0.958352	0.9583841	0.95838	0.95838	0.958331
0.1	95	8.911823	8.9118509	8.91171	8.91184	8.911836
	100	4.915096	4.9151167	4.91514	4.91512	4.915075
	105	2.070020	2.0700634	2.07006	2.07006	2.069930
0.2	95	9.995385	9.9956567	9.99597	9.99569	9.995362
	100	6.776743	6.7773481	6.77758	6.77738	6.776999
	105	4.295972	4.2965626	4.29643	4.29649	4.295941
0.3	95	11.654643	11.6558858	11.65747	11.65618	11.654758
	100	8.827461	8.8287588	8.82942	8.82900	8.827548
	105	6.516544	6.5177905	6.51763	6.51802	6.516355
0.4	95	13.508865	13.5107083	13.51426	13.51182	13.507892
	100	10.921764	10.9237708	10.92507	10.92474	10.920891
	105	8.728562	8.7199362	8.72936	8.73089	8.726804
0.5	95	15.439622	15.4427163	15.44890	15.44587	15.437069
	100	13.029882	13.0281555	13.03015	13.03017	13.022532
	105	10.924875	10.9296247	10.92800	10.93253	10.923750

ities. For small value of volatility ($\sigma = 0.05$), the value of Asian option as function of time and asset price is plotted in Fig. 3a. In Fig. 3b, the value of Asian option as function of time and asset price is plotted for large value of volatility ($\sigma = 0.5$). It is observed that proposed compact finite difference scheme is accurate for both small and large volatilities.

We apply the proposed compact finite difference scheme to the Asian option PDE with the parameters, $S = 100$, $T = 1$ for various values of volatility (σ), interest rates (r) and strike prices (K). Results obtained from these parameters for Asian option PDE are given in Table 1 and compared with the results given in [37] and [38]. Lower and upper bounds for different Asian options are taken from [5]. It is observed from the Table 1 that results obtained from proposed compact finite difference scheme are in a good accuracy with the existing literature.

Now, proposed compact finite difference scheme is applied to the Asian option PDE for small and large volatilities at different strike prices (K) with the parameters, $S = 100$, $T = 1$, $r = 0.09$. The values of Asian option and their comparison with the results in [7,37] and [38] are given in Table 2. From the Table 2, it is observed that proposed compact finite difference scheme is accurate for small and large both type of volatilities.

In Table 3, values of Asian option obtained from proposed compact finite difference scheme for large maturity time ($T = 3$) at different strike prices (K) and volatilities with the parameters, $S = 100$, $T = 3$, $r = 0.09$ are presented. The values of Asian

Table 3 Comparison of value of Asian options by various methods for different volatilities for $S = 100$, $r = 0.09$ and $T = 3$ number of grid points $N = 256$ and $\delta\tau = h^2$

σ	K	Present scheme	Zhang [37]	Ju [39]	Hsu [40]	Kumar [17]
0.05	95	15.116316	15.1162646	15.11626	15.116230	15.116784
	100	11.304521	11.3036080	11.30360	11.304036	11.303619
	105	7.554853	7.5533233	7.55335	7.554073	7.550559
0.1	95	15.213952	15.2138005	15.21396	15.213921	15.214139
	100	11.637522	11.6376573	11.63798	11.637813	11.637450
	105	8.390654	8.3912219	8.39140	8.391189	8.390679
0.2	95	16.638433	16.6372081	16.63942	16.637276	16.637222
	100	13.767687	13.7669267	13.76770	13.767043	13.766921
	105	11.219965	11.2198706	11.21879	11.220047	11.219881
0.3	95	19.023244	19.0231619	19.02652	19.023263	19.023123
	100	16.585898	16.5861236	16.58509	16.586222	16.586118
	105	14.393766	14.3929780	14.38751	14.393083	14.392999
0.4	95	21.741821	21.7409242	21.74461	21.740973	21.740921
	100	19.585638	19.5882516	19.58355	19.588307	19.588251
	105	17.625127	17.6254416	17.61269	17.625501	17.625444
0.5	95	24.571934	24.5718705	24.57740	24.571913	24.571875
	100	22.630648	22.6307858	22.62276	22.630828	22.630790
	105	20.843231	20.8431853	20.82213	20.843226	20.843189

option are compared with the results in [17], [37,39], and [40]. From the Table 3, it is observed that proposed compact finite difference scheme is accurate for large maturity time also.

Rate of convergence We have proved in Sect. 4 that proposed compact finite difference scheme is fourth order accurate in spatial variable. In order to show the convergence rate numerically, we take $\delta\tau = h^2$. In Fig. 4, error in sup norm between the numerical solution at grid size h and $h/2$ ($Error = \|U_h - U_{\frac{h}{2}}\|_\infty$) for $K = 100$, $S = 100$, $\sigma = 0.30$, $r = 0.06$ and $T = 1$ is plotted with respect to the number of grid points. It can be observed from the Fig. 4 that proposed compact finite difference scheme exhibit approximately fourth order convergence rate.

Efficiency of proposed compact finite difference scheme for Asian option PDE In Fig. 5, error in sup norm between the numerical solution at grid size h and the reference solution ($Error = \|U_{ref} - U_h\|_\infty$) for $K = 100$, $S = 100$, $\sigma = 0.30$, $r = 0.15$ and $T = 1$ is plotted with respect to the CPU time. Reference solution (U_{ref}) is computed for the same parameters with $h = 9.765625e - 004$. We compute the error and corresponding CPU time at grid points N=8, 16, 32, 64, 128 using central difference scheme and proposed compact finite difference scheme. It can be observed from the figure that for a given accuracy, proposed compact finite difference scheme is significantly efficient as compared to the central difference scheme.

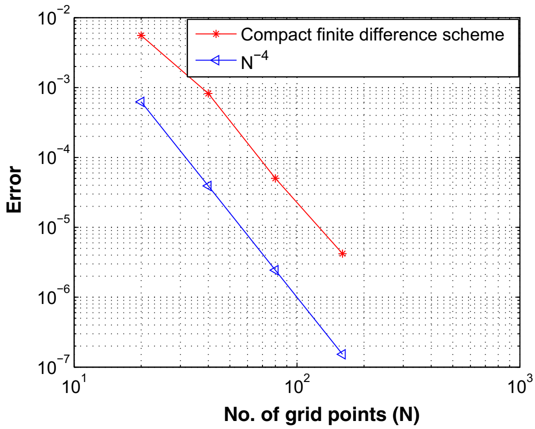

Fig. 4 Error in sup norm between the numerical solution at grid size h and $h/2$ for $K = 100$, $S = 100$, $\sigma = 0.30$, $r = 0.06$ and $T = 1$ versus N

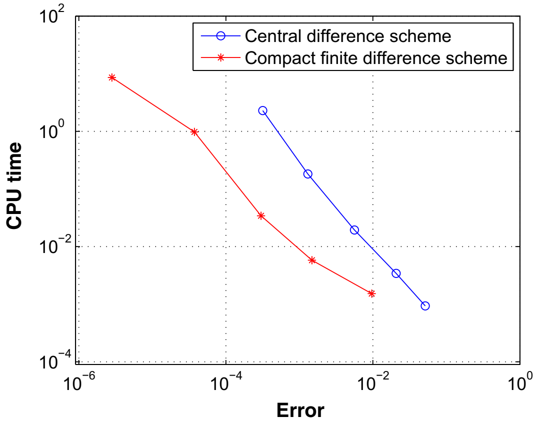

Fig. 5 Efficiency: CPU time and max error for central difference schemes and proposed compact finite difference scheme for Asian option PDE

6 Conclusion and future work

An unconditionally stable compact finite difference scheme is proposed to solve the Asian option PDE. In the proposed scheme, second derivative approximations of the unknowns are eliminated with the unknowns itself and their first derivative approximations while retaining the tri-diagonal nature of the scheme. Fourier analysis of the difference schemes is presented and it is concluded that proposed compact finite difference approximations have better resolution characteristics as compared to classical finite difference schemes. Proposed compact finite difference scheme results a diagonally dominant system of linear equations which can be solved using Thomas algorithm efficiently. Consistency, stability and convergence is proved for the fully discrete problem and it is shown that proposed compact finite difference scheme is second order accurate in time variable and fourth order accurate in space variable. Error with the reference solution and CPU time is also plotted and it is observed that for a given accuracy, proposed compact finite difference scheme is more efficient as compared to the central difference schemes. In future, we would also like to use the proposed compact finite difference scheme for other types of exotic options, for eg. Barrier options, Look-back options. Since the proposed compact finite difference scheme is easily extendable for the two dimensional problems in a similar manner, we would like to extend the proposed compact finite difference scheme for two dimensional problems.

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