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Mathematical analysis of an eco-epidemiological predator–prey model with stage-structure and latency

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Abstract In this paper, an eco-epidemiological predator–prey model with stage structure for the prey and a time delay describing the latent period of the disease is investigated. By analyzing corresponding characteristic equations, the local stability of the trivial equilibrium, the predator-extinction equilibrium, the disease-free equilibrium and the endemic equilibrium is addressed. The existence of Hopf bifurcations at the endemic equilibrium is established. By using Lyapunov functionals and LaSalle's invariance principle, sufficient conditions are obtained for the global asymptotic stability of the trivial equilibrium, the predator-extinction equilibrium, the disease-free equilibrium and the endemic equilibrium of the model.

Keywords Eco-epidemiological model · Stage structure · Latent period · Stability · Hopf bifurcation

Mathematics Subject Classification 34K18 · 34K20 · 34K60 · 92D25

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1 Introduction

Since the pioneering work of Anderson and May [1], great attention has been paid to developing realistic mathematical models for transmission dynamics of infectious diseases(see, for example [3,4,7–12,14,15,17–20]). An increasing number of works are devoted to the study of the relationsips between demographic processes among different populations and diseases. There are many works on dealting with predator–prey models with disease in the predator or the prey (see, for example [3,10,11,13,14, 17–19]). In [17], Zhang et al. considered the following delayed eco-epidemiological predator–prey model:

$$
\begin{aligned}
\dot{x}(t) &= rx(t) - ax^2(t) - a_1 x(t)S(t), \\
\dot{S}(t) &= a_2 x(t-\tau)S(t-\tau) - d_3 S(t) - \beta S(t)I(t), \\
\dot{I}(t) &= \beta S(t)I(t) - d_4 I(t),
\end{aligned}
\tag{1.1}
$$

where $x(t)$, $S(t)$ and $I(t)$ represent the densities of the prey, susceptible (sound) predator and infected predator population at time t, respectively. The parameters a, a_1, a_2, d_3, d_4, r and β are positive constants(see [17]). In system (1.1), the authors assumed that the infectious predator would die of diseases and only the healthy predator had predation capacity, but once infected with the disease, the predator would not be able to recover.

In most of the literrtures, it was frequently asumed that the disesse incubation is negligible. However, for some disesse, once infected, each susceptible individual becomes infectious instantaneously and later recovers with a permanent acquired immunity. An epidemic model based on these assumptions is called SIR (susceptible, infectious, recovered) model. In [4], Cooke formulated an epidemic model with time delay effect for the spread of a communicable disease carried by a vector by assuming that the force of infection at time t is given by $\beta S(t)I(t-\tau)$, where β is the average number of contacts per infective per day and $\tau > 0$ is a fixed time during which the infectious agents develop in the vector and it is only after that time that the infected vector can infect a susceptible human. Cooke considered the following model:

$$
\begin{aligned}
\dot{S}(t) &= A - d_1 S(t) - \beta S(t)I(t-\tau), \\
\dot{I}(t) &= \beta S(t)I(t-\tau) - (d_2 + \gamma)I(t), \\
\dot{R}(t) &= \gamma I(t) - d_3 R(t),
\end{aligned}
\tag{1.2}
$$

In system (1.2), $S(t)$ represents the number of individuals who are susceptible to the disease, that is, who are not yet infected at time t; $I(t)$ represents the number of infected individuals who are infectious and are able to spread the disease by contact with susceptible individuals; $R(t)$ represents the number of individuals who have been infected and then removed from the possibility of being infected again or of spreading at time t, respectively. The parameters d_1, d_2 and d_3 are positive constants representing the death rates of susceptibles, infectives and recovered, respectively. In [7], McCluskey further discussed system (1.2) and completely proved that if the basic

reproduction number is greater than unity, the endemic equilibrium is globally stable by constructing a suitable Lyapunov function.

Stage-structure is a natural phenomenon and represents, for example, the division of a population into immature and mature individuals. Predator–prey systems where only immature individuals are consumed by their predator are well known in nature. One example is described in [12], where Chinese fire-bellied newt, which is unable to feed on the mature Rana chensinensis, can only feed on the immature one. Stage-structured models have received great attention in recent years (see, for example, [8,12,13,16]).

Based on above discussions, in this paper, we study the following differential equations

$$
\begin{aligned}
\dot{x}_1(t) &= rx_2(t) - (r_1 + d_1)x_1(t) - ax_1^2(t) - a_1x_1(t)S(t), \\
\dot{x}_2(t) &= r_1x_1(t) - d_2x_2(t), \\
\dot{S}(t) &= a_2x_1(t)S(t) - bS^2(t) - d_3S(t) - \beta S(t)I(t), \\
\dot{I}(t) &= \beta e^{-\mu\tau}S(t-\tau)I(t-\tau) - (\mu + \gamma)I(t), \\
\dot{R}(t) &= \gamma I(t) - \mu R(t),
\end{aligned}
\tag{1.3}
$$

where $x_1(t)$ and $x_2(t)$ represent the densities of the immature and the mature prey population at time t, respectively. The parameters $a, a_1, a_2, b, d_1, d_2, d_3, r, r_1, \beta, \mu$ and γ are positive constants, in which r is the birth rate of the prey; a and b are the intra-specific competition rates of the immature prey and susceptible predator , respectively; d_1 and d_2 are the death rates of the immature prey and mature prey, respectively; r_1 is the transformation rate from the immature individuals to mature individuals for the prey; a_1 is the capturing rate of the predator, a_2/a_1 is the conversion rate of nutrients into the reproduction of the predator. $\tau \geq 0$ represents the latent period of the disease; γ is the recovery rate of infectious individuals. In system (1.3), we assume that only the susceptible predators have ability to capture immature prey.

The initial conditions for system (1.3) take the form

$$
\begin{aligned}
&x_1(\theta) = \varphi_1(\theta) \geq 0, \quad x_2(\theta) = \varphi_2(\theta) \geq 0, \quad S(\theta) = \phi_1(\theta) \geq 0, \quad I(\theta) = \phi_2(\theta) \geq 0, \\
&R(\theta) = \phi_3(\theta) \geq 0, \quad \varphi_1(0) > 0, \quad \varphi_2(0) > 0, \quad \phi_1(0) > 0, \quad \phi_2(0) > 0, \quad \phi_3(0) > 0, \\
&(\varphi_1(\theta), \varphi_2(\theta), \phi_1(\theta), \phi_2(\theta), \phi_3(\theta)) \in C\left([-\tau, 0], R_{+0}^5\right), \quad \theta \in [-\tau, 0),
\end{aligned}
\tag{1.4}
$$

where $R_{+0}^5 = \{(y_1, y_2, y_3, y_4, y_5) : y_i \geq 0, i = 1, 2, 3, 4, 5\}$.

It is well known by the fundamental theory of functional differential equations [5] that system (1.3) has a unique solution $(x_1(t), x_2(t), S(t), I(t), R(t))$ satisfying initial conditions (1.4). It is easy to show that all solutions of system (1.3) with initial conditions (1.4) are defined on $[0, +\infty)$ and remain positive for all $t \geq 0$.

The organization of this paper is as follows. In the next section, we show the permanence of solutions of system (1.3) with initial conditions (1.4). In Sect. 3, by using the theory on characteristic equation of delay differential equations with delay-dependent parameters developed by Beretta and Kuang [2], we discuss the local stability of each of feasible equilibria of system (1.3). We establish the existence of Hopf bifurcations

at the endemic-coexistence equilibrium. In Sect. 4, by means of Lyaponov functionals and LaSalle's invariance principle, we obtain sufficient conditions for the global stability of each of feasible equilibria of system (1.3).

2 Permanence

In this section, we give a result on the upper bound of positive solutions of system (1.3) with initial condition (1.4).

Lemma 2.1 *Positive solutions of system (1.3) with initial conditions (1.4) are ultimately bounded.*

Proof Let $(x_1(t), x_2(t), S(t), I(t), R(t))$ be any positive solution of system (1.3) with initial conditions (1.4). By the first and second equations of system (1.3), we can obtain

$$\dot{x}_1(t) \leq r x_2(t) - (r_1 + d_1)x_1(t) - a x_1^2(t),$$
$$\dot{x}_2(t) = r_1 x_1(t) - d_2 x_2(t).$$

which yields $\lim\sup_{t \to +\infty} x_1(t) \leq \frac{|rr_1 - d_2(r_1 + d_1)|}{ad_2} := M_1$, $\lim\sup_{t \to +\infty} x_2(t) \leq \frac{r_1}{d_2} M_1 := M_2$.

By the last three equations of system (1.3), we can obtain

$$\dot{S}(t) \leq a_2 M_1 S(t) - b S^2(t) - d_3 S(t) - \beta S(t) I(t),$$
$$\dot{I}(t) = \beta e^{-\mu \tau} S(t - \tau) I(t - \tau) - (\mu + \gamma) I(t), \tag{2.1}$$
$$\dot{R}(t) = \gamma I(t) - \mu R(t).$$

Define

$$N(t) = S(t - \tau) + e^{\mu \tau} I(t) + e^{\mu \tau} R(t).$$

Calculating the derivative of $N(t)$ along positive solutions of system (2.1), it follows that

$$\dot{N}(t) \leq -d N(t) - b \left[S(t - \tau) - \frac{a_2 M_1}{2b} \right]^2 + \frac{(a_2 M_1)^2}{4b}$$
$$\leq -d V(t) + \frac{(a_2 M_1)^2}{4b},$$

where $d = \min\{d_3, \mu\}$. If we choose $M_3 = (a_2 M_1)^2/(4bd)$, then

$$\lim\sup_{t \to \infty} S(t) \leq M_3, \quad \lim\sup_{t \to \infty} I(t) \leq e^{-\mu \tau} M_3, \quad \lim\sup_{t \to \infty} R(t) \leq e^{-\mu \tau} M_3.$$

This completes the proof. □

Lemma 2.2 *For any positive solution $(x_1(t), x_2(t), S(t), I(t), R(t))$ of system (1.3) with initial conditions (1.4), we have*

$$\liminf_{t \to +\infty} x_1(t) \geq \frac{rr_1 - d_2(r_1 + d_1 + a_1 M_3)}{ad_2} := \underline{x}_1,$$

$$\liminf_{t \to +\infty} S(t) \geq \frac{a_2 \underline{x}_1 - (d_3 + \beta e^{-\mu\tau} M_3)}{b} := \underline{S},$$

where M_3 is defined in Lemma 2.1.

Proof Let $(x_1(t), x_2(t), S(t), I(t), R(t))$ be any positive solution of system (1.3) with initial conditions (1.4). By Lemma 2.1, it follows that $\limsup_{t \to +\infty} S(t) \leq M_3$, $\limsup_{t \to +\infty} I(t) \leq e^{-\mu\tau} M_3$. Hence, for $\varepsilon > 0$ being sufficiently small, there is a $T_0 > 0$ such that if $t > T_0$, $S(t) < M_3 + \varepsilon$, $I(t) < e^{-\mu\tau} M_3 + \varepsilon$. Accordingly, for $\varepsilon > 0$ being sufficiently small, we derive from the first and the second equations of system (1.3) that, for $t > T_0$,

$$\dot{x}_1(t) \geq rx_2(t) - (r_1 + d_1)x_1(t) - ax_1^2(t) - a_1(M_3 + \varepsilon)x_1(t),$$
$$\dot{x}_2(t) = r_1 x_1(t) - d_2 x_2(t), \tag{2.2}$$

which leads to

$$\liminf_{t \to +\infty} x_1(t) \geq \frac{rr_1 - d_2(r_1 + d_1 + a_1 M_3)}{ad_2} := \underline{x}_1. \tag{2.3}$$

By the third equation of system (1.3), we can obtain

$$\dot{S}(t) \geq a_2 \underline{x}_1 S(t) - bS^2(t) - d_3 S(t) - (e^{-\mu\tau} M_3 + \varepsilon)S(t), \tag{2.4}$$

By comparison, we have

$$\liminf_{t \to +\infty} S(t) \geq \frac{a_2 \underline{x}_1 - (d_3 + \beta e^{-\mu\tau} M_3)}{b} := \underline{S}. \tag{2.5}$$

This completes the proof.   □

3 Local stability and Hopf bifurcations

In this section, we discuss the local stability of each of feasible equilibria of system (1.3) by analyzing the corresponding characteristic equations.

Clearly, system (1.3) always has a trivial equilibrium $E_0(0, 0, 0, 0, 0)$. If $rr_1 > d_2(r_1 + d_1)$, then system (1.3) has a predator-extinction equilibrium $E_1(x_1^0, x_2^0, 0, 0, 0)$, where

$$x_1^0 = \frac{rr_1 - d_2(r_1 + d_1)}{ad_2}, \quad x_2^0 = \frac{r_1[rr_1 - d_2(r_1 + d_1)]}{ad_2^2}.$$

If the following condition holds:

$$(H_1) \quad \gamma_0 = \frac{rr_1 - d_2(r_1 + d_1)}{ad_2} - \frac{d_3}{a_2} > 0,$$

system (1.3) has a disease-free equilibrium $E_2(x_1^+, x_2^+, S^+, 0, 0)$, where

$$x_1^+ = \frac{a_1 d_3 + abx_1^0}{a_1 a_2 + ab}, \qquad x_2^+ = \frac{r_1}{d_2} x_1^+, \qquad S^+ = \frac{a}{a_1}\left(x_1^0 - x_1^+\right).$$

Further, if the following condition holds:

$$(H_2) \quad \gamma_1 = \frac{aa_2 \beta \gamma_0 e^{-\mu\tau}}{(ab + a_1 a_2)(\mu + \gamma)} > 1,$$

then system (1.3) has a unique positive equilibrium $E_*(x_1^*, x_2^*, S^*, I^*, R^*)$, where

$$x_1^* = \frac{rr_1 - d_2(r_1 + d_1)}{ad_2} - \frac{a_1(\mu + \gamma)}{a\beta} e^{\mu\tau}, \qquad x_2^* = \frac{r_1}{d_2} x_1^*,$$

$$S^* = \frac{(\mu + \gamma) e^{\mu\tau}}{\beta}, \qquad I^* = \frac{a_2 x_1^* - bS^* - d_3}{\beta}, \qquad R^* = \frac{\gamma}{\mu} I^*.$$

Note that the variable $R(t)$ does not appear in the first four equations of system (1.3). Therefore, we first consider the following subsystem of system (1.3):

$$\begin{aligned}
\dot{x}_1(t) &= rx_2(t) - (r_1 + d_1)x_1(t) - ax_1^2(t) - a_1 x_1(t)S(t), \\
\dot{x}_2(t) &= r_1 x_1(t) - d_2 x_2(t), \\
\dot{S}(t) &= a_2 x_1(t)S(t) - bS^2(t) - d_3 S(t) - \beta S(t)I(t), \\
\dot{I}(t) &= \beta e^{-\mu\tau} S(t - \tau)I(t - \tau) - (\mu + \gamma)I(t).
\end{aligned} \tag{3.1}$$

Accordingly, system (3.1) has the equilibria $E_0^1(0, 0, 0, 0)$, $E_1^1(x_1^0, x_2^0, 0, 0)$, $E_2^1(x_1^+, x_2^+, S^+, 0)$ and $E_*^1(x_1^*, x_2^*, S^*, I^*)$.

The characteristic equation of system (3.1) at the equilibrium E_0^1 takes the form

$$(\lambda + d_3)(\lambda + \mu + \gamma)\left[\lambda^2 + (r_1 + d_1 + d_2)\lambda + d_2(r_1 + d_1) - rr_1\right] = 0, \tag{3.2}$$

Clearly, Eq. (3.2) always has two negative real roots: $\lambda_1 = -d_3$, $\lambda_2 = -(\mu + \gamma)$. If $rr_1 < d_2(r_1 + d_1)$, then all roots of (3.2) are negative. If $rr_1 > d_2(r_1 + d_1)$, then (3.2) has one positive real root. Hence, E_0^1 is locally asymptotically stable when $rr_1 < d_2(r_1 + d_1)$ and unstable when $rr_1 > d_2(r_1 + d_1)$.

The characteristic equation of system (3.1) at the equilibrium E_1^1 is of the form

$$\begin{aligned}
(\lambda + \mu + \gamma)\left(\lambda + d_3 - a_2 x_1^0\right)&\left[\lambda^2 + \left(r_1 + d_1 + d_2 + 2ax_1^0\right)\lambda \right. \\
&\left. + rr_1 - d_2(r_1 + d_1)\right] = 0.
\end{aligned} \tag{3.3}$$

Clearly, Eq. (3.3) always has a negative real root $\lambda = -(\mu + \gamma)$. All the roots of the following equation

$$\lambda^2 + \left(r_1 + d_1 + d_2 + 2ax_1^0\right)\lambda + rr_1 - d_2(r_1 + d_1) = 0$$

are negative as $rr_1 > d_2(r_1 + d_1)$. The other root of (3.3) is $\lambda = a_2x_1^0 - d_3 = a_2\gamma_0$. Hence, the equilibrium E_1^1 is locally asymptotically stable when $\gamma_0 < 0$ and unstable when $\gamma_0 > 0$.

The characteristic equation of system (3.1) at the equilibrium E_2^1 takes the form

$$\left(\lambda^3 + e_2\lambda^2 + e_1\lambda + e_0\right)\left(\lambda + \mu + \gamma - \beta S^+ e^{-(\mu+\lambda)\tau}\right) = 0, \qquad (3.4)$$

where

$$e_2 = \frac{rr_1}{d_2} + ax_1^+ + d_2 + bS^+,$$

$$e_1 = (ad_2 + a_1a_2)x_1^+ S^+ + bS^+ \left(\frac{rr_1}{d_2} + ax_1^+ + d_2\right),$$

$$e_0 = d_2(ab + a_1a_2)x_1^+ S^+.$$

We first consider the following equation:

$$\lambda^3 + e_2\lambda^2 + e_1\lambda + e_0 = 0. \qquad (3.5)$$

A direct calculation shows that

$$\Delta_1 = e_2 > 0,$$

$$\Delta_2 = e_1e_2 - e_0 = \left(\frac{rr_1}{d_2} + ax_1^+ + d_2 + bS^+\right)\left[ad_2x_1^+ + bS^+\left(\frac{rr_1}{d_2} + ax_1^+\right)\right]$$

$$+ a_1x_1^+a_2S^+\left(\frac{rr_1}{d_2} + ax_1^+ + bS^+\right) + d_2bS^+\left(\frac{rr_1}{d_2} + d_2 + bS^+\right) > 0,$$

$$\Delta_3 = e_0\Delta_2 > 0.$$

Hence, by the Routh–Huiwitz criterion, we see that the Eq. (3.5) has no positive roots. Other roots of Eq. (3.4) are determined by the following equation:

$$f_1(\lambda) := \lambda + \mu + \gamma - \beta S^+ e^{-(\mu+\lambda)\tau} = 0. \qquad (3.6)$$

If $\gamma_1 > 1$, for λ real, it is easy to show that,

$$f_1(0) = \mu + \gamma - \beta S^+ e^{-\mu\tau} = (\mu + \gamma)(1 - \gamma_1) < 0, \quad \lim_{\lambda\to+\infty} f_1(\lambda) = +\infty.$$

Hence, $f_1(\lambda) = 0$ has at least one positive real root in this case. If $\gamma_1 < 1$, one has

$$Re(\lambda) = \beta S^+ e^{-\mu\tau} \cos(\tau Im\lambda) - (\mu + \gamma) < \beta S^+ e^{-\mu\tau} - (\mu + \gamma)$$
$$= -(\mu + \gamma)(1 - \gamma_1) < 0.$$

Accordingly, by Theorem 3.4.1 in Kuang [6], we see that if $\gamma_1 < 1$, the equilibrium E_2^1 is locally asymptotically stable. If $\gamma_1 > 1$, the equilibrium E_2^1 is unstable.

In conclusion, we have the following results.

Theorem 3.1 *For system (3.1), we have the following:*

(i) *If $rr_1 < d_2(r_1 + d_1)$, then the equilibrium E_0^1 is locally asymptotically stable; if $rr_1 > d_2(r_1 + d_1)$, then E_0^1 is unstable.*

(ii) *Assume that $rr_1 > d_2(r_1 + d_1)$. If $\gamma_0 < 0$, then the equilibrium E_1^1 is locally asymptotically stable; if $\gamma_0 > 0$, then E_1^1 is unstable.*

(iii) *Assume that $\gamma_0 > 0$. If $\gamma_1 < 1$, then the equilibrium E_2^1 is locally asymptotically stable; if $\gamma_1 > 1$, then E_2^1 is unstable.*

By Theorem 3.1, for system (1.3), we have the following conclusions.

Corollary 3.1 *For system (1.3), we have the following:*

(i) *If $rr_1 < d_2(r_1 + d_1)$, then the equilibrium E_0 is locally asymptotically stable; if $rr_1 > d_2(r_1 + d_1)$, then E_0 is unstable.*

(ii) *Assume that $rr_1 > d_2(r_1 + d_1)$. If $\gamma_0 < 0$, then the equilibrium E_1 is locally asymptotically stable; if $\gamma_0 > 0$, then E_1 is unstable.*

(iii) *Assume that $\gamma_0 > 0$. If $\gamma_1 < 1$, then the equilibrium E_2 is locally asymptotically stable; if $\gamma_1 > 1$, then E_2 is unstable.*

The characteristic equation of system (3.1) at the endemic-coexistence equilibrium E_*^1 is of the form

$$\lambda^4 + u_3\lambda^3 + u_2\lambda^2 + u_1\lambda + u_0 + \left(v_3\lambda^3 + v_2\lambda^2 + v_1\lambda + v_0\right)e^{-\lambda\tau} = 0, \quad (3.7)$$

where

$$u_3 = \mu + \gamma + \frac{rr_1}{d_2} + ax_1^* + d_2 + bS^*,$$

$$u_2 = (\mu + \gamma)\left(\frac{rr_1}{d_2} + ax_1^* + d_2\right) + bS^*\left(\mu + \gamma + \frac{rr_1}{d_2} + ax_1^* + d_2\right)$$
$$+ ad_2x_1^* + a_1x_1^*a_2S^*,$$

$$u_1 = (\mu + \gamma)(ad_2 + a_1a_2S^*)x_1^* + d_2a_1a_2x_1^*S^* + bS^*\left[(\mu + \gamma)\left(\frac{rr_1}{d_2} + ax_1^* + d_2\right)\right.$$
$$\left. + ad_2x_1^*\right],$$

$$u_0 = d_2x_1^*S^*(\mu + \gamma)(a_1a_2 + ab),$$

$$v_3 = -(\mu + \gamma),$$

$$v_2 = -(\mu + \gamma)\left(\frac{rr_1}{d_2} + ax_1^* + d_2 + bS^* - \beta I^*\right),$$

$$v_1 = -(\mu + \gamma)\left[\left(\frac{rr_1}{d_2} + ax_1^* + d_2\right)(bS^* - \beta I^*) + ad_2x_1^* + a_1a_2x_1^*S^*\right]$$

$$v_0 = -d_2x_1^*(\mu + \gamma)[a(bS^* - \beta I^*) + a_1a_2S^*].$$

When $\tau = 0$, Eq. (3.7) becomes

$$\lambda^4 + (u_3 + v_3)\lambda^3 + (u_2 + v_2)\lambda^2 + (u_1 + v_1)\lambda + u_0 + v_0 = 0. \qquad (3.8)$$

By calculation, it follows that

$$\triangle_1 = u_3 + v_3 = \frac{rr_1}{d_2} + ax_1^* + d_2 + bS^* > 0,$$

$$\triangle_2 = (u_3 + v_3)(u_2 + v_2) - (u_1 + v_1) = a_1a_2x_1^*S^*\left(\frac{rr_1}{d_2} + ax_1^* + bS^*\right)$$

$$+ ad_2x_1^*\left(\frac{rr_1}{d_2} + ax_1^* + d_2\right)$$

$$+ \beta bS^*I^*(\mu + \gamma) + bS^*\left(\frac{rr_1}{d_2} + ax_1^* + d_2\right)\left(\frac{rr_1}{d_2} + ax_1^* + d_2 + bS^*\right) > 0,$$

$$\triangle_3 = (u_1 + v_1)\triangle_2 - (u_0 + v_0)(u_3 + v_3)^2 = bS^*\left(\frac{rr_1}{d_2} + ax_1^* + d_2\right)[\beta I^*(\mu + \gamma)]^2$$

$$+ d_2x_1^*S^*(a_1a_2 + ab)\left(\frac{rr_1}{d_2} + ax_1^* + d_2\right)$$

$$\times \left[ad_2x_1^* + bS^*\left(\frac{rr_1}{d_2} + ax_1^* + d_2 + bS^*\right)\right]$$

$$+ a_1x_1^*a_2S^*\left(\frac{rr_1}{d_2} + ax_1^* + bS^*\right)\left[\beta I^*(\mu + \gamma)\right.$$

$$\times \left(\frac{rr_1}{d_2} + ax_1^* + d_2\right) + d_2x_1^*S^*(a_1a_2 + ab)\right]$$

$$+ bS^*\beta I^*(\mu + \gamma)\left(\frac{rr_1}{d_2} + ax_1^* + d_2 + bS^*\right)$$

$$\times \left[\left(\frac{rr_1}{d_2}\right)^2 + 2\frac{rr_1}{d_2}(ax_1^* + d_2) + (ax_1^*)^2 + (d_2)^2\right]$$

$$+ \beta I^*d_2x_1^*b(S^*)^2(\mu + \gamma)(a_1a_2 + 2ab) > 0,$$
$$\triangle_4 = (u_0 + v_0)\triangle_3 = ad_2x_1^*\beta^2I^*(\mu + \gamma)\triangle_3 > 0.$$

By the Routh–Hurwitz criterion, the equilibrium E_*^1 of system (3.1) is locally asymptotically stable when $\tau = 0$.

Suppose that Eq. (3.7) has a pair of conjugate purely imaginary roots $\pm i\omega (\omega > 0)$. Substituting $\lambda = i\omega$ into (3.7) and separating the real and imaginary parts, one obtains

$$
\begin{aligned}
(v_3\omega^3 - v_1\omega)\sin \omega\tau + (v_2\omega^2 - v_0)\cos \omega\tau &= \omega^4 - u_2\omega^2 + u_0, \\
(v_3\omega^3 - v_1\omega)\cos \omega\tau - (v_2\omega^2 - v_0)\sin \omega\tau &= -u_3\omega^3 + u_1\omega.
\end{aligned}
\tag{3.9}
$$

Squaring and adding the two equations of (3.9), we derive that

$$
\omega^8 + s_3\omega^6 + s_2\omega^4 + s_1\omega^2 + s_0 = 0,
\tag{3.10}
$$

where

$$
\begin{aligned}
s_3 &= u_3^2 - 2u_2 - v_3^2, & s_2 &= u_2^2 + 2u_0 - 2u_1u_3 - v_2^2 + 2v_1v_3, \\
s_1 &= u_1^2 - 2u_0u_2 - v_1^2 + 2v_0v_2, & s_0 &= u_0^2 - v_0^2.
\end{aligned}
$$

Let $z = \omega^2$, then Eq. (3.10) can be rewritten as

$$
s(z) := z^4 + s_3z^3 + s_2z^2 + s_1z + s_0 = 0.
\tag{3.11}
$$

If $s_3 > 0$ and $s_2^2 - 4s_1s_3 < 0$, then $s(z)$ has always no positive roots. Hence, under these conditions, Eq. (3.7) has no purely imaginary roots for any $\tau > 0$ and accordingly, the equilibrium E_*^1 is locally asymptotically stable for all $\tau \geq 0$.

If Eq. (3.11) has at least one positive root, without loss of generality, we assume that (3.11) has four positive roots, namely, z_1, z_2, z_3 and z_4, respectively. Accordingly, Eq. (3.10) has four positive roots $\omega_k = \sqrt{z_k}(k = 1, 2, 3, 4)$.

For $k = 1, 2, 3, 4$, from (3.9) one can get the corresponding $\tau_k^j > 0$ such that (3.7) has a pair of purely imaginary roots $\pm i\omega_k$ given by

$$
\tau_k^j = \frac{2\pi j}{\omega_k} + \frac{1}{\omega_k}\arccos \frac{v_2\omega_k^2\left(\omega_k^4 - u_2\omega_k^2 + u_0\right) + \left(v_3\omega_k^3 - v_1\omega_k\right)\left(u_1\omega_k - u_3\omega_k^3\right)}{\left(v_2\omega_k^2\right)^2 + \left(v_3\omega_k^3 - v_1\omega_k\right)^2},
$$
$$
j = 0, 1, 2, \ldots
$$

Differentiating the two sides of (3.7) with respect to τ, it follows that

$$
\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{4\lambda^3 + 3u_3\lambda^2 + 2u_2\lambda + u_1}{-\lambda(\lambda^4 + u_3\lambda^3 + u_2\lambda^2 + u_1\lambda + u_0)} + \frac{3v_3\lambda^2 + 2v_2\lambda + v_1}{\lambda(v_3\lambda^3 + v_2\lambda^2 + v_1\lambda + v_0)}
$$
$$
- \frac{\tau}{\lambda}.
$$

After some algebra, one obtains that

$$
\begin{aligned}
sign\left\{\frac{dRe\lambda}{d\tau}\right\}_{\tau=\tau_k^j} &= sign\left\{Re\left(\frac{d\lambda}{d\tau}\right)^{-1}\right\}_{\tau=\tau_k^j} \\
&= sign\left\{-\frac{\left(u_1-3u_3\omega_k^2\right)\left(u_3\omega_k^2-u_1\right)+2\left(u_2-2\omega_k^2\right)\left(\omega_k^4-u_2\omega_k^2+u_0\right)}{\omega_k^2\left(u_1-u_3\omega_k^2\right)^2+\left(\omega_k^4-u_2\omega_k^2+u_0\right)^2}\right. \\
&\quad\left.+\frac{\left(v_1-3v_3\omega_k^2\right)\left(v_3\omega_k^2-v_1\right)-2v_2^2\omega_k^2}{(v_2\omega_k)^2+\left(v_1\omega_k-v_3\omega_k^3\right)^2}\right\}
\end{aligned}
$$

We derive from (3.9) that

$$
\omega_k^2\left(u_1-u_3\omega_k^2\right)^2+\left(\omega_k^4-u_2\omega_k^2+u_0\right)^2=(v_2\omega_k)^2+\left(v_1\omega_k-v_3\omega_k^3\right)^2.
$$

Hence, it follows that

$$
\begin{aligned}
sign\left\{\frac{dRe\lambda}{d\tau}\right\}_{\tau=\tau_k^j} &= sign\left\{\frac{4\omega_k^6+3s_3\omega_k^4+2s_2\omega_k^2+s_1}{(v_2\omega_k)^2+\left(v_1\omega_k-v_3\omega_k^3\right)^2}\right\} \\
&= sign\left\{\frac{s'(z_k)}{(v_2\omega_k)^2+\left(v_1\omega_k-v_3\omega_k^3\right)^2}\right\}.
\end{aligned}
$$

Based on the theory on characteristic equation of delay differential equations with delay-dependent parameters developed by [6], one can obtain the following result.

Theorem 3.2 *Let $\gamma_1 > 1$ hold. For system (1.3), we have*

(i) *If $s_3 > 0$, and $s_2^2 - 4s_1s_3 < 0$, then the coexistence equilibrium E_* is locally asymptotically stable for all $\tau \geq 0$.*

(ii) *If $s(z) = 0$ has at least one positive root z_k, then all roots of (3.7) have negative real parts for $\tau \in [0, \tau_k^0)$, and the equilibrium E_* of system (1.3) is locally asymptotically stable for $\tau \in [0, \tau_k^0)$.*

(iii) *If all conditions as stated in (ii) hold true and $s'(z_k) > 0$, then system (1.3) undergoes a Hopf bifurcation at E_* when $\tau = \tau_k^j (j = 0, 1, \ldots)$.*

4 Global stability

In this section, we study the global stability of the trivial equilibrium E_0, the predator-extinction equilibrium E_1, the disease-free equilibrium E_2 and the endemic-coexistence equilibrium E_*, respectively. The strategy of proofs is to use Lyapunov functions and LaSalle's invariance principle.

Theorem 4.1 *If $rr_1 < d_2(r_1 + d_1)$, then the trivial equilibrium $E_0(0, 0, 0, 0, 0)$ of system (1.3) is globally asymptotically stable.*

Proof By Theorem 3.1, we see that if $rr_1 < d_2(r_1 + d_1)$, then E_0 is locally asymptotically stable. Hence, we only prove that all positive solutions of system (1.3) with initial conditions (1.4) converge to E_0. Let $(x_1(t), x_2(t), S(t), I(t), R(t))$ be any positive solution of system (1.3) with initial conditions (1.4). Define

$$V_0(t) = x_1(t) + \frac{r}{d_2}x_2(t) + \frac{a_1}{a_2}S(t) + \frac{a_1}{a_2}e^{\mu\tau}I(t) + \frac{a_1}{a_2}e^{\mu\tau}R(t)$$

$$+ \frac{a_1}{a_2}\beta \int_{t-\tau}^{t} S(u)I(u)du.$$

Calculating the derivative of $V_0(t)$ along positive solutions of system (1.3), it follows that

$$\dot{V}_0(t) = -\frac{d_2(r_1 + d_1) - rr_1}{d_2}x_1(t) - ax_1^2(t) - \frac{a_1}{a_2}bS^2(t) - \frac{a_1}{a_2}d_3S(t)$$

$$- \frac{a_1}{a_2}e^{\mu\tau}\mu I(t) - \frac{a_1}{a_2}e^{\mu\tau}\mu R(t). \tag{4.1}$$

If $rr_1 < d_2(r_1 + d_1)$, it then follows from (4.1) that $\dot{V}_0(t) \leq 0$. By Theorem 5.3.1 in [5], solutions limit to Λ, the largest invariant subset of $\{\dot{V}_0(t) = 0\}$. Clearly, we see from (4.1) that $\dot{V}_0(t) = 0$ if and only if $x_1(t) = 0$, $S(t) = 0$, $I(t) = 0$ and $R(t) = 0$. Noting that Λ is invariant, for each element in Λ, we have $x_1(t) = 0$. It therefore follows from the first equation of system (1.3) that

$$0 = \dot{x}_1(t) = rx_2(t),$$

which yields $x_2(t) = 0$. Hence, $\dot{V}_0(t) = 0$ if and only if $(x_1(t), x_2(t), S(t), I(t), R(t)) = (0, 0, 0, 0, 0)$. Accordingly, the global asymptotic stability of E_0 follows from LaSalle's invariant principle for delay differential systems. □

Theorem 4.2 *Assume that $rr_1 > d_2(r_1 + d_1)$ holds. If $\gamma_0 < 0$, then the predator-extinction equilibrium $E_1(x_1^0, x_2^0, 0, 0, 0)$ of system (1.3) is globally asymptotically stable.*

Proof By Theorem 3.1, we see that if $\gamma_0 < 0$, then E_1 is locally asymptotically stable. Hence, we only prove that all positive solutions of system (1.3) with initial conditions (1.4) converge to E_1. Let $(x_1(t), x_2(t), S(t), I(t), R(t))$ be any positive solution of system (1.3) with initial conditions (1.4). System (1.3) can be rewritten as

$$\dot{x}_1(t) = \frac{r}{x_1^0}\left[-x_2(t)\left(x_1(t) - x_1^0\right)\right.$$

$$\left. + x_1(t)\left(x_2(t) - x_2^0\right)\right] + x_1(t)\left[-a\left(x_1(t) - x_1^0\right)\right] - a_1x_1(t)S(t),$$

$$\dot{x}_2(t) = \frac{r_1}{x_2^0}\left[-x_1(t)\left(x_2(t) - x_2^0\right) + x_2(t)\left(x_1(t) - x_1^0\right)\right],$$

$$\dot{S}(t) = a_2x_1(t)S(t) - bS^2(t) - d_3S(t) - \beta S(t)I(t),$$

$$\dot{I}(t) = \beta e^{-\mu\tau} S(t-\tau)I(t-\tau) - (\mu+\gamma)I(t),$$
$$\dot{R}(t) = \gamma I(t) - \mu R(t). \tag{4.2}$$

Define

$$V_{11}(t) = x_1 - x_1^0 - x_1^0 \ln\frac{x_1}{x_1^0} + k_1\left(x_2 - x_2^0 - x_2^0 \ln\frac{x_2}{x_2^0}\right) + \frac{a_1}{a_2}S$$
$$+ \frac{a_1}{a_2}e^{\mu\tau}I + \frac{a_1}{a_2}e^{\mu\tau}R.$$

where $k_1 = rx_2^0/(r_1x_1^0)$. Calculating the derivative of $V_{11}(t)$ along positive solutions of (4.2), it follows that

$$\dot{V}_{11}(t) = \frac{x_1(t)-x_1^0}{x_1(t)}\dot{x}_1(t) + k_1\frac{x_2(t)-x_2^0}{x_2(t)}\dot{x}_2(t) + \frac{a_1}{a_2}\dot{S}(t)$$
$$+ \frac{a_1}{a_2}e^{\mu\tau}\dot{I}(t) + \frac{a_1}{a_2}e^{\mu\tau}\dot{R}(t)$$
$$= -\frac{r}{x_1^0}\left(\sqrt{\frac{x_2(t)}{x_1(t)}}(x_1(t)-x_1^0) - \sqrt{\frac{x_1(t)}{x_2(t)}}(x_2(t)-x_2^0)\right)^2 - a\left(x_1(t)-x_1^0\right)^2$$
$$- \frac{a_1}{a_2}bS^2(t) - \frac{a_1}{a_2}\left(d_3 - a_2x_1^0\right)S(t) - \frac{a_1}{a_2}e^{\mu\tau}\mu I(t) - \frac{a_1}{a_2}e^{\mu\tau}\mu R(t)$$
$$- \frac{a_1}{a_2}\beta S(t)I(t) + \frac{a_1}{a_2}\beta S(t-\tau)I(t-\tau). \tag{4.3}$$

Define

$$V_1(t) = V_{11}(t) + \frac{a_1}{a_2}\beta\int_{t-\tau}^t S(u)I(u)du.$$

By calculation, we have that

$$\dot{V}_1(t) = -\frac{r}{x_1^0}\left(\sqrt{\frac{x_2(t)}{x_1(t)}}(x_1(t)-x_1^0) - \sqrt{\frac{x_1(t)}{x_2(t)}}(x_2(t)-x_2^0)\right)^2 - a\left(x_1(t)-x_1^0\right)^2$$
$$- \frac{a_1}{a_2}bS^2(t) - \frac{a_1}{a_2}e^{\mu\tau}\mu I(t) - \frac{a_1}{a_2}e^{\mu\tau}\mu R(t) + a_1\gamma_0 S(t). \tag{4.4}$$

It follows from (4.4) that if $\gamma_0 < 0$ holds, then $\dot{V}_1(t) \le 0$. By Theorem 5.3.1 in [5], solutions limit to Λ, the largest invariant subset of $\{\dot{V}_1(t) = 0\}$. Clearly, we see from (4.4) that $\dot{V}_1(t) = 0$ if and only if $x_1(t) = x_1^0, x_2(t) = x_2^0, S(t) = 0, I(t) = 0$ and $R(t) = 0$. Hence, the only invariant set $\Lambda = \{(x_1^0, x_2^0, 0, 0, 0)\}$. Using LaSalle's invariant principle for delay differential systems, the global asymptotic stability of E_1 follows. $\qquad\square$

Theorem 4.3 *Let $\gamma_0 > 0$ hold. If $\gamma_1 < 1$, then the disease-free equilibrium $E_2(x_1^+, x_2^+, S^+, 0, 0)$ is globally asymptotically stable.*

Proof By Theorem 3.1, we see that if $\gamma_0 > 0$ and $\gamma_1 < 1$ hold, then the equilibrium E_2 is locally asymptotically stable. Hence, it suffices to show that all positive solutions of system (1.3) with initial conditions (1.4) converge to E_2. We achieve this by constructing a global Lyapunov function. Let $(x_1(t), x_2(t), S(t), I(t), R(t))$ be any positive solution of system (1.3) with initial conditions (1.4). System (1.3) can be rewritten as

$$
\begin{aligned}
\dot{x}_1(t) &= \frac{r}{x_1^+}\left[-x_2(t)\left(x_1(t) - x_1^+\right) + x_1(t)\left(x_2(t) - x_2^+\right)\right] + x_1(t)\left[-a\left(x_1(t) - x_1^+\right)\right] \\
&\quad + a_1 S^+ x_1(t) - a_1 x_1(t) S(t), \\
\dot{x}_2(t) &= \frac{r_1}{x_2^+}\left[-x_1(t)\left(x_2(t) - x_2^+\right) + x_2(t)\left(x_1(t) - x_1^+\right)\right], \\
\dot{S}(t) &= a_2 x_1(t) S(t) - b S^2(t) - d_3 S(t) - \beta S(t) I(t), \\
\dot{I}(t) &= \beta e^{-\mu\tau} S(t - \tau) I(t - \tau) - (\mu + \gamma) I(t), \\
\dot{R}(t) &= \gamma I(t) - \mu R(t).
\end{aligned}
\tag{4.5}
$$

Define

$$
\begin{aligned}
V_{21}(t) &= x_1 - x_1^+ - x_1^+ \ln\frac{x_1}{x_1^+} + k_2\left(x_2 - x_2^+ - x_2^+ \ln\frac{x_2}{x_2^+}\right) \\
&\quad + \frac{a_1}{a_2}\left(S - S^+ - S^+ \ln\frac{S}{S^+}\right) + \frac{a_1}{a_2}e^{\mu\tau} I,
\end{aligned}
$$

where $k_2 = r x_2^+/(r_1 x_1^+)$. Calculating the derivative of $V_{21}(t)$ along positive solutions of system (4.5), it follows that

$$
\begin{aligned}
\dot{V}_{21}(t) &= \frac{x_1(t) - x_1^+}{x_1(t)}\dot{x}_1(t) + k_2\frac{x_2(t) - x_2^+}{x_2(t)}\dot{x}_2(t) + \frac{a_1(S(t) - S^+)}{a_2 S(t)}\dot{S}(t) + \frac{a_1}{a_2}e^{\mu\tau}\dot{I}(t) \\
&= -\frac{r}{x_1^+}\left(\sqrt{\frac{x_2(t)}{x_1(t)}}\left(x_1(t) - x_1^+\right) - \sqrt{\frac{x_1(t)}{x_2(t)}}\left(x_2(t) - x_2^+\right)\right)^2 - a\left(x_1(t) - x_1^+\right)^2 \\
&\quad - \frac{a_1}{a_2}b(S(t) - S^+)^2 - \frac{a_1}{a_2}\beta S(t) I(t) + \frac{a_1}{a_2}\beta S(t - \tau) I(t - \tau) \\
&\quad - \frac{a_1}{a_2}[e^{\mu\tau}(\mu + \gamma) - \beta S^+]I(t).
\end{aligned}
\tag{4.6}
$$

Define

$$
V_2(t) = V_{21}(t) + \frac{a_1}{a_2}\beta\int_{t-\tau}^{t} S(u) I(u)\,du.
$$

By calculation, we have that

$$\dot{V}_2(t) = -\frac{r}{x_1^+} \left[\sqrt{\frac{x_2(t)}{x_1(t)}} \left(x_1(t) - x_1^+\right) - \sqrt{\frac{x_1(t)}{x_2(t)}} \left(x_2(t) - x_2^+\right) \right]^2 - a \left(x_1(t) - x_1^+\right)^2$$

$$- \frac{a_1}{a_2} b(S(t) - S^+)^2 - \frac{a_1}{a_2} e^{\mu\tau} (\mu + \gamma)(1 - \gamma_1) I(t). \tag{4.7}$$

It follows from (4.7) that if $\gamma_1 < 1$ holds true, then $\dot{V}_2(t) \le 0$ with equality if and only if $x_1(t) = x_1^+$, $x_2(t) = x_2^+$, $S(t) = S^+$, $I(t) = 0$. Hence, the only invariant set in M is $\Lambda = \{(x_1^+, x_2^+, S^+, 0)\}$.

By the fifth equation of system (1.3), it follows that $\lim_{t\to+\infty} R(t) = 0$. Therefore, the global asymptotic stability of the equilibrium E_2 of system (1.3) follows from LaSalle's invariant principle for delay differential systems. □

Theorem 4.4 *Assume that $\gamma_1 > 1$ holds, then the endemic-coexistence equilibrium $E_*(x_1^*, x_2^*, S^*, I^*, R^*)$ of system (1.3) is globally attractive provided*

$$(H_3) \ \underline{S} > \frac{\beta}{b} I^*.$$

Here, $\underline{S}$ is the persistency constant for $S(t)$ as defined in Lemma 2.2.

Proof Let $(x_1(t), x_2(t), S(t), I(t), R(t))$ be any positive solution of system (1.3) with initial conditions (1.4). System (1.3) can be rewritten as

$$\dot{x}_1(t) = \frac{r}{x_1^*} \left[-x_2(t) \left(x_1(t) - x_1^*\right) + x_1(t) \left(x_2(t) - x_2^*\right) \right] + x_1(t) \left[-a \left(x_1(t) - x_1^*\right) \right]$$

$$+ a_1 S^* x_1(t) - a_1 x_1(t) S(t),$$

$$\dot{x}_2(t) = \frac{r_1}{x_2^*} \left[-x_1(t) \left(x_2(t) - x_2^*\right) + x_2(t) \left(x_1(t) - x_1^*\right) \right],$$

$$\dot{S}(t) = a_2 x_1(t) S(t) - b S^2(t) - d_3 S(t) - \beta S(t) I(t),$$

$$\dot{I}(t) = \beta e^{-\mu\tau} S(t - \tau) I(t - \tau) - (\mu + \gamma) I(t),$$

$$\dot{R}(t) = \gamma I(t) - \mu R(t). \tag{4.8}$$

Define

$$V_{31}(t) = x_1 - x_1^* - x_1^* \ln \frac{x_1}{x_1^*} + k_3 \left(x_2 - x_2^* - x_2^* \ln \frac{x_2}{x_2^*} \right) + \frac{a_1}{a_2} \left(S - S^* - S^* \ln \frac{S}{S^*} \right)$$

$$+ \frac{a_1}{a_2} e^{\mu\tau} \left(I - I^* - I^* \ln \frac{I}{I^*} \right).$$

where $k_3 = r x_2^* / (r_1 x_1^*)$.

Calculating the derivative of $V_{31}(t)$ along positive solutions of system (4.8), it follows that

$$
\begin{aligned}
\dot{V}_{31}(t) &= \frac{x_1(t) - x_1^*}{x_1(t)} \dot{x}_1(t) + k_3 \frac{x_2(t) - x_2^*}{x_2(t)} \dot{x}_2(t) + \frac{a_1}{a_2} \frac{S(t) - S^*}{S(t)} \dot{S}(t) \\
&\quad + \frac{a_1}{a_2} e^{\mu\tau} \frac{I(t) - I^*}{I(t)} \dot{I}(t) \\
&= -\frac{r}{x_1^*} \left[\sqrt{\frac{x_2(t)}{x_1(t)}} (x_1(t) - x_1^*) - \sqrt{\frac{x_1(t)}{x_2(t)}} (x_2(t) - x_2^*) \right]^2 - a (x_1(t) - x_1^*)^2 \\
&\quad + \frac{a_1}{a_2} \beta I^* (S(t) - S^*) - \frac{a_1}{a_2} b(S(t) - S^*)^2 - \frac{a_1}{a_2} \beta S(t) I(t) \\
&\quad + \frac{a_1}{a_2} \beta S(t-\tau) I(t-\tau) \\
&\quad - \frac{a_1}{a_2} \beta I^* \frac{S(t-\tau) I(t-\tau)}{I(t)} + \frac{a_1}{a_2} \beta S^* I^*.
\end{aligned}
\tag{4.9}
$$

Define

$$
V_3(t) = V_{31}(t) + \frac{a_1}{a_2} \beta \int_{t-\tau}^{t} \left[S(u)I(u) - S^* I^* - S^* I^* \ln \frac{S(u)I(u)}{S^* I^*} \right] du.
$$

Direct calculation shows that

$$
\begin{aligned}
\dot{V}_3(t) &= -\frac{r}{x_1^*} \left[\sqrt{\frac{x_2(t)}{x_1(t)}} (x_1(t) - x_1^*) - \sqrt{\frac{x_1(t)}{x_2(t)}} (x_2(t) - x_2^*) \right]^2 - a (x_1(t) - x_1^*)^2 \\
&\quad - \frac{a_1}{a_2} \beta S^* I^* \left[\frac{S(t-\tau) I(t-\tau)}{S^* I(t)} - 1 - \ln \frac{S(t-\tau) I(t-\tau)}{S^* I(t)} \right] \\
&\quad - \frac{a_1}{a_2} \beta S^* I^* \left[\frac{S^*}{S(t)} - 1 - \ln \frac{S^*}{S(t)} \right] - \frac{a_1}{a_2} (S(t) - S^*)^2 \left[b - \frac{\beta I^*}{S(t)} \right].
\end{aligned}
\tag{4.10}
$$

Note that the function $g(x) = x - 1 - \ln x$ is always non-negative for any $x > 0$, and $g(x) = 0$ if and only if $x = 1$. Hence, if $S(t) > \frac{\beta}{b} I^*$ for $t \geq T$, we have $\dot{V}_3(t) \leq 0$ with equality if and only if $x_1(t) = x_1^*, x_2(t) = x_2^*, S(t) = S(t-\tau) = S^*$ and $I(t) = I(t-\tau)$. We now look for the invariant subset Λ within the set

$$
M = \left\{ (x_1, x_2, S, I) : x_1(t) = x_1^*, x_2(t) = x_2^*, S(t) = S^*, I(t) = I(t-\tau) \right\}.
$$

Since $S(t) = S^*$ on Λ and consequently, from the third equation of system (1.3) we obtain

$$
0 = \dot{S}(t) = a_2 x_1^* S^* - b(S^*)^2 - d_3 S^* - \beta S^* I(t).
\tag{4.11}
$$

This yields $I(t) = I^*$. Hence, the only invariant set in M is $\Lambda = \{(x_1^*, x_2^*, S^*, I^*)\}$.

By the fifth equation of system (1.3), it follows that $\lim_{t \to +\infty} R(t) = R^*$. Therefore, the global attractiveness of E_* follows from LaSalle invariant principle for delay differential systems. This completes the proof. □

5 Conclusion

In this paper, we have incorporated the disease for the predator and stage structure for the prey into an eco-epidemiology model. By analyzing the corresponding characteristic equation, the local stability of each of feasible equilibria has been established. It has been shown that, under some conditions, the time delay due to gestation of the predators may destabilize the endemic-coexistence equilibrium of system (1.3) and cause the population to fluctuate. From Theorem 3.2, we see that there is a threshold τ_k^0 for the time delay such that below it the endemic-coexistence equilibrium is stable, but if the delay is greater than the threshold, sustained oscillations arise. By means of Lyapunov functionals and LaSalle's invariant principle, sufficient conditions were obtained for the global stability of the endemic-coexistence equilibrium, the disease-free equilibrium, the predator-extinction equilibrium and the trivial equilibrium of system (1.2).

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