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ORIGINAL RESEARCH

A class of linear codes with two weights or three weights from some planar functions

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Abstract Let $\mathbb{F}_q$ be a finite field with $q = p^m$ elements, where p is an odd prime and m is a positive integer. In this paper, let $D = \{(x_1, x_2, \ldots, x_n) \in \mathbb{F}_q^n \backslash \{(0, 0, \ldots)\} : Tr(x_1^{p^{k_1}+1} + x_2^{p^{k_2}+1} + \cdots + x_n^{p^{k_n}+1}) = c\}$, where $c \in \mathbb{F}_p$, Tr is the trace function from $\mathbb{F}_q$ to $\mathbb{F}_p$ and each $m/(m, k_i)$ ($1 \le i \le n$) is odd. we define a p-ary linear code $C_D = \{c(a_1, a_2, \ldots, a_n) : (a_1, a_2, \ldots, a_n) \in \mathbb{F}_q^n\}$, where $c(a_1, a_2, \ldots, a_n) = (Tr(a_1x_1 + a_2x_2 + \cdots + a_nx_n))_{(x_1,x_2,\ldots,x_n)\in D}$. We present the weight distributions of the classes of linear codes which have at most three weights.

Keywords Linear codes · Weight distribution · Guass sum · Weil sum

1 Introduction

Throughout this paper, let $\mathbb{F}_q$ be a finite field with $q = p^m$ elements, where p is an odd prime and m is a positive integer, and let Tr be the trace function from $\mathbb{F}_q$ to $\mathbb{F}_p$. An $[n, k, d]$ p-ary linear code $\mathscr{C}$ is a k-dimensional subspace of $\mathbb{F}_p^n$ and has minimum Hamming distance d. Let A_i denote the number of codewords with Hamming weight i in a code $\mathscr{C}$ of length n. The weight enumerator is defined by

$$1 + A_1z + \cdots + A_nz^n.$$

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The sequence $(1, A_1, \ldots, A_n)$ is called the weight distribution of the code $\mathscr{C}$. A code $\mathscr{C}$ is said to be a t-weight code if the number of nonzero A_i is equal to t. Weight distribution is an interesting topic and was investigated in [1–15]. The weight distribution of a code can not only give the error correcting ability of the code, but also allow the computation of the error probability of error detection and correction.

For a set $D = \{d_1, d_2, \ldots, d_n\} \subseteq \mathbb{F}_q$, define a linear code of length n over $\mathbb{F}_p$ by

$$\mathscr{C}_D = \{(Tr(xd_1), Tr(xd_2), \ldots, Tr(xd_n)) : x \in \mathbb{F}_q\}.$$

We call D the *defining set* of $\mathscr{C}_D$. Many known linear codes could be produced by selecting the defining set. For details of these known codes, the reader is referred to [3,13,14].

In this paper, we always assume that $n, m, k_1, \ldots, k_n$ are positive integers with each $m/\gcd(m, k_i)$ odd. Then each $f_i(x) = x^{p^{k_i}+1}$ is a planar function over $\mathbb{F}_q$ (see [16]). Fixing $c \in \mathbb{F}_p$, we define

$$D = \{(x_1, \ldots, x_n) \in \mathbb{F}_q^n \backslash \{(0, 0, \ldots)\} : Tr\left(x_1^{p^{k_1}+1} + \cdots + x_n^{p^{k_n}+1}\right) = c\},$$
$$\mathscr{C}_D = \{c(a_1, \ldots, a_n) : (a_1, a_2, \ldots, a_n) \in \mathbb{F}_q^n\},$$

where

$$c(a_1, \ldots, a_n) = (Tr(a_1x_1 + \cdots + a_nx_n))_{(x_1,\ldots,x_n)\in D}.$$

In fact, we have some well-known results as follows. If $n = 1$ and either $k_1 = 0$ or $m/\gcd(m, k_1)$ is odd, then it is just the result in [17,18].

In the paper, we will determine the weight distribution of the linear codes $\mathscr{C}_D$ in three cases: (1) $c = 0$, (2) $c \in \mathbb{F}_p^{*2}$, (3) $c \in \mathbb{F}_p^* \setminus \mathbb{F}_p^{*2}$. In the cases (2) and (3), we use the cyclotomic numbers of order 2 to get their distributions.

2 Preliminaries

Let $\mathbb{F}_q$ be a finite fields with q elements, where q is a power of a prime p. We define the additive character of $\mathbb{F}_q$ as follows:

$$\chi : \mathbb{F}_q \longrightarrow \mathbb{C}^*, x \longmapsto \zeta_p^{Tr(x)},$$

where ζ_p is a complex p-th primitive root of unity and Tr denotes the trace function from $\mathbb{F}_q$ to $\mathbb{F}_p$. The orthogonal property of additive characters [19] is given by

$$\sum_{x\in\mathbb{F}_q} \chi(ax) = \begin{cases} 0, & \text{if } a \in \mathbb{F}_q^*; \\ q, & \text{if } a = 0. \end{cases}$$

Let $\lambda : \mathbb{F}_q^* \longrightarrow \mathbb{C}^*$ be a multiplicative character of $\mathbb{F}_q^*$. The trivial character λ_0 defined by $\lambda_0(x) = 1$ for all $x \in \mathbb{F}_q^*$. The orthogonal property of multiplicative characters is given by

$$\sum_{x\in\mathbb{F}_q^*}\lambda(x)=\begin{cases} q-1, & \text{if } \lambda=\lambda_0;\\ 0, & \text{otherwise.}\end{cases}$$

Let $\bar{\lambda}$ be the conjugate character of λ defined by $\bar{\lambda}(x)=\overline{\lambda(x)}$. It is easy to obtain that $\lambda^{-1}=\bar{\lambda}$. The multiplicative group $\widehat{\mathbb{F}_q^*}$ is isomorphic to $\mathbb{F}_q^*$. For $\mathbb{F}_q^*=\langle\alpha\rangle$, define a multiplicative character by $\psi(\alpha)=\zeta_{q-1}$, where ζ_{q-1} denotes the primitive $q-1$-th root of complex unity. Then we have $\widehat{\mathbb{F}_q^*}=\langle\psi\rangle$. Set $\eta=\psi^{\frac{q-1}{2}}$ be the quadratic character of $\mathbb{F}_q$.

Define the Gauss sum over $\mathbb{F}_q$ by

$$G(\lambda)=\sum_{x\in\mathbb{F}_q^*}\lambda(x)\chi(x).$$

Let $(\frac{\cdot}{p})$ denote the Legendre symbol. The quadratic Gauss sums are known and given in the following.

Lemma 1 [19] *Suppose that $q=p^m$ and η is the quadratic multiplicative character of $\mathbb{F}_q$, where p is odd prime. Then*

$$G(\eta)=(-1)^{m-1}\sqrt{(p^*)^m}=\begin{cases}(-1)^{m-1}\sqrt{q}, & \text{if } p\equiv 1 \pmod 4,\\ (-1)^{m-1}(\sqrt{-1})^m\sqrt{q}, & \text{if } p\equiv 3 \pmod 4).\end{cases}$$

where $p^*=(-1)^{\frac{p-1}{2}}p$ is the discriminant of a prime p.

Let χ' be the canonical additive character of $\mathbb{F}_p$ such that $\chi(x)=\chi'(\mathrm{Tr}(x))$ for $x\in\mathbb{F}_q$. Let η' be a quadratic character of $\mathbb{F}_p$, then $\eta(x)=\eta'(N_{q/p}(x))$ for $x\in\mathbb{F}_q^*$.

Lemma 2 [20] *Let $x\in\mathbb{F}_p^*$ and $q=p^m$, where p is odd prime.*

If m is even, then $\eta(x)=1$.

If m is odd, then $\eta(x)=\eta'(x)$.

Moreover, $G(\eta)=(-1)^{m-1}G(\eta')^m$, where $G(\eta)$ and $G(\eta')$ are the Gauss sums over $\mathbb{F}_q$ and $\mathbb{F}_p$, respectively.

We now give a brief introduction to the theory of quadratic forms over finite fields. Quadratic forms have been well studied and have applications in sequence design [11,21] and coding theory [7,22].

Lemma 3 *Let $d=gcd(k,m)$. Then*

$$(p^k+1,p^m-1)=\begin{cases}2, & \text{if } m/d \text{ is odd,}\\ p^d+1, & \text{if } m/d \text{ is even.}\end{cases}$$

In this paper, we assume that k an integer and $m/\gcd(k,m)$ odd. Then it is well-known that $f(x)=x^{p^k+1}$ is a planar function from $\mathbb{F}_q$ to $\mathbb{F}_q$. In [23,24], Coulter gave the valuations of the following Weil sums:

$$S_k(a,b)=\sum_{x\in\mathbb{F}_q}\chi(ax^{p^k+1}+bx),\quad a,b\in\mathbb{F}_q.$$

Lemma 4 [23, Theorem 1] *Let m/d be odd. Then*

$$S_k(a,0)=\eta(a)G(\eta)=\begin{cases}(-1)^{m-1}\sqrt{q}\eta(a), & \text{if } p\equiv 1 \pmod 4,\\ (-1)^{m-1}i^m\sqrt{q}\eta(a), & \text{if } p\equiv 3 \pmod 4.\end{cases}$$

Lemma 5 [24, Theorem 1] *Let q be odd and suppose $f(X)=a^{p^k}X^{p^{2k}}+aX$ is a permutation polynomial over F_q. Let x_0 be the unique solution of the equation $f(X)=-b^{p^k}$. The evaluation of $S_k(a,b)$ partitions into the following two cases:*

(1) *If m/d is odd, then*

$$\begin{aligned}S_k(a,b)&=\eta(a)G(\eta)\bar{\chi}\left(ax_0^{p^k+1}\right)\\&=\begin{cases}(-1)^{m-1}\sqrt{q}\eta(a)\bar{\chi}\left(ax_0^{p^k+1}\right), & \text{if } p\equiv 1 \pmod 4,\\ (-1)^{m-1}i^m\sqrt{q}\eta(a)\bar{\chi}\left(ax_0^{p^k+1}\right), & \text{if } p\equiv 3 \pmod 4.\end{cases}\end{aligned}$$

(2) *If m/d is even, then* $m=2e$, $a^{\frac{q-1}{p^d+1}}\neq(-1)^{m/d}$ *and*

$$S_k(a,b)=(-1)^{m/d}p^e\bar{\chi}\left(ax_0^{p^k+1}\right).$$

In fact, Lemma 4 is made of some revision in [24, Theorem 1].

3 Linear codes

Let $\mathbb{F}_q$ be the finite field with $q=p^m$ elements, where p is an odd prime and m is an positive integer. Let Tr denote the trace function from $\mathbb{F}_q$ to $\mathbb{F}_p$. In this section, we always assume that $n,k_1,\ldots,k_n$ are positive integers with each $m/\gcd(m,k_i)$ odd. Let $f_i(x)=x^{p^{k_i}+1}$, $x\in\mathbb{F}_q$, $i=1,\ldots,n$. It is known from [16] that f_i, $1\le i\le n$, are planar functions from $\mathbb{F}_q$ to $\mathbb{F}_q$.

3.1 The first case

Define

$$\begin{aligned}D_0=\{(x_1,\ldots,x_n)\in\mathbb{F}_q^n\backslash\{(0,\ldots,0)\}:Tr\left(x_1^{p^{k_1}+1}+\cdots+x_n^{p^{k_n}+1}\right)=0\},\\ \mathscr{C}_{D_0}=\{c(a_1,\ldots,a_n):(a_1,\ldots,a_n)\in\mathbb{F}_q^n\},\end{aligned}\tag{3.1}$$

where

$$c(a_1,\ldots,a_n)=(Tr(a_1x_1+\cdots+a_nx_n))_{(x_1,\ldots,x_n)\in D_0}.$$

Lemma 6 *Let $n_0 = |D_0|$. Suppose that mn is even, then*

$$n_0 = \frac{q^n - p}{p} + \frac{p-1}{p} G(\eta)^n$$
$$= \begin{cases} p^{mn-1} - 1 + (-1)^{(m-1)n}(p-1)p^{\frac{mn}{2}-1}, & \text{if } p \equiv 1 \pmod 4, \\ p^{mn-1} - 1 + (-1)^{(m-1)n+\frac{mn}{2}}(p-1)p^{\frac{mn}{2}-1}, & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

Suppose that mn is odd, then

$$n_0 = p^{mn-1} - 1.$$

Proof By Lemma 3, we have that

$$n_0 + 1 = \frac{1}{p} \sum_{y \in \mathbb{F}_p} \sum_{(x_1,\ldots,x_n) \in \mathbb{F}_q^n} \chi\left(y\left(x_1^{p^{k_1}+1} + \cdots + x_n^{p^{k_n}+1}\right)\right)$$
$$= \frac{q^n}{p} + \frac{1}{p} \sum_{y \in \mathbb{F}_p^*} \sum_{x_1 \in \mathbb{F}_q} \chi\left(yx_1^{p^{k_1}+1}\right) \cdots \sum_{x_n \in \mathbb{F}_q} \chi\left(yx_n^{p^{k_n}+1}\right)$$
$$= \frac{q^n}{p} + \frac{1}{p} \sum_{y \in \mathbb{F}_p^*} G(\eta)^n \eta(y)^n.$$

If m is even or m is odd and n is even, then $\eta(y)^n = 1$ for each $y \in \mathbb{F}_p^*$. Hence $n_0 = \frac{q^n - p}{p} + \frac{p-1}{p} G(\eta)^n$.

If mn is odd, then m is odd. Let η' be a quadratic character of $\mathbb{F}_p^*$, then $\eta(y) = \eta'(y)$ for each $y \in \mathbb{F}_p^*$. Hence $\sum_{y \in \mathbb{F}_p^*} \eta(y)^n = \sum_{y \in \mathbb{F}_p^*} \eta'(y) = 0$, so $n_0 = \frac{q^n - p}{p}$.

By Lemma 1, we can get the exact value of n_0.

Theorem 1 *Let $\mathscr{C}_{D_0}$ be the linear code defined as (3.1).*

If mn is even, then $\mathscr{C}_{D_0}$ is a two-weight code with the Hamming weight distribution in Table 1.

If mn is odd, then $\mathscr{C}_{D_0}$ is a three-weight code with the Hamming weight distribution in Table 2.

Proof Firstly, we determine the weight distribution of the code $\mathscr{C}_{D_0}$. Define the following parameter

$$N_a = \left|\left\{(x_1, \ldots, x_n) \in \mathbb{F}_q^n : \mathrm{Tr}\left(x_1^{p^{k_1}+1} + \cdots + x_n^{p_n^{k_n}+1}\right) = 0, \right.\right.$$
$$\left.\left. \mathrm{Tr}(a_1 x_1 + \cdots + a_n x_n) = 0 \right\}\right| - 1,$$

where $a = (a_1, \ldots, a_n) \in \mathbb{F}_q^n$. By definition and the basic facts of additive characters, for each $a = (a_1, \ldots, a_n) \in \mathbb{F}_q^n \setminus \{(0, \ldots, 0)\}$, we have

Table 1 *mn* is even

Weight	Multiplicity
$p \equiv 1 \pmod 4$	
0	1
$(p-1)p^{mn-2}$	$p^{mn-1} - 1 + (-1)^{(m-1)n}(p-1)p^{\frac{mn}{2}-1}$
$(p-1)\left(p^{mn-2} + (-1)^{(m-1)n} p^{\frac{mn}{2}-1}\right)$	$p^{mn} - p^{mn-1} - (-1)^{(m-1)n}(p-1)p^{\frac{mn}{2}-1}$
$p \equiv 3 \pmod 4$	
0	1
$(p-1)p^{mn-2}$	$p^{mn-1} - 1 + (-1)^{(m-1)n+\frac{mn}{2}}(p-1)p^{\frac{mn}{2}-1}$
$(p-1)\left(p^{mn-2} + (-1)^{(m-1)n+\frac{mn}{2}} p^{\frac{mn}{2}-1}\right)$	$p^{mn} - p^{mn-1} - (-1)^{(m-1)n+\frac{mn}{2}}(p-1)p^{\frac{mn}{2}}$

Table 2 *mn* is odd

Weight	Multiplicity
$p \equiv 1 \pmod 4$	
0	1
$(p-1)p^{mn-2}$	$p^{mn-1} - 1 + (-1)^{(m-1)n}(p-1)p^{\frac{mn}{2}-1}$
$(p-1)\left(p^{mn-2} + (-1)^{(m-1)n} p^{\frac{mn-3}{2}}\right)$	$\frac{p-1}{2}p^{mn-1} - (-1)^{(m-1)n}\frac{(p-1)}{2}p^{\frac{nm-1}{2}}$
$(p-1)\left(p^{mn-2} - (-1)^{(m-1)n} p^{\frac{mn-3}{2}}\right)$	$\frac{p-1}{2}p^{mn-1} - (-1)^{(m-1)n}\frac{(p-1)}{2}p^{\frac{nm-1}{2}}$
$p \equiv 3 \pmod 4$	
0	1
$(p-1)p^{mn-2}$	$p^{mn-1} - 1$
$(p-1)\left(p^{mn-2} - (-1)^{(m-1)n+\frac{mn+1}{2}} p^{\frac{mn-3}{2}}\right)$	$\frac{(p-1)}{2}p^{mn-1} - (-1)^{(m-1)n+\frac{mn+1}{2}}\frac{(p-1)}{2}p^{\frac{nm-1}{2}}$
$(p-1)\left(p^{mn-2} + (-1)^{(m-1)n+\frac{mn+1}{2}} p^{\frac{mn-3}{2}}\right)$	$\frac{(p-1)}{2}p^{mn-1} + (-1)^{(m-1)n+\frac{mn+1}{2}}\frac{(p-1)}{2}p^{\frac{nm-1}{2}}$

$$
\begin{aligned}
N_a &= \frac{1}{p^2}\sum_{(x_1,\ldots,x_n)\in\mathbb{F}_q^n}\sum_{y\in\mathbb{F}_p}\chi\left(y\left(x_1^{p^{k_1}+1}+\cdots+x_n^{p^{k_n}+1}\right)\right)\sum_{z\in\mathbb{F}_p}\chi\left(z\left(a_1x_1+\cdots+a_nx_n\right)\right)-1\\
&= \frac{q^n-p^2}{p^2}+\frac{1}{p^2}\sum_{(x_1,\ldots,x_n)\in\mathbb{F}_q^n}\sum_{y\in\mathbb{F}_p^*}\chi\left(y\left(x_1^{p^{k_1}+1}+\cdots+x_n^{p^{k_n}+1}\right)\right)\\
&\quad+\frac{1}{p^2}\sum_{(x_1,\ldots,x_n)\in\mathbb{F}_q^n}\sum_{z\in\mathbb{F}_p^*}\chi(z(a_1x_1+\cdots+a_nx_n))\\
&\quad+\frac{1}{p^2}\sum_{(x_1,\ldots,x_n)\in\mathbb{F}_q^n}\sum_{y,z\in\mathbb{F}_p^*}\chi\left(y\left(x_1^{p^{k_1}+1}+\cdots+x_n^{p^{k_n}+1}\right)+z(a_1x_1+\cdots+a_nx_n)\right)\\
&= \frac{q^n-p^2}{p^2}+\Omega_1+\Omega_2+\Omega_3.
\end{aligned}
$$

By Lemmas 2 and 4, we have that

$$\Omega_1 = \frac{1}{p^2}\sum_{y\in\mathbb{F}_p^*}\sum_{x_1\in\mathbb{F}_q}\chi\left(yx_1^{p^{k_1}+1}\right)\cdots\sum_{x_n\in\mathbb{F}_q}\chi\left(yx_n^{p^{k_n}+1}\right)$$
$$= \frac{1}{p^2}\sum_{y\in\mathbb{F}_p^*}\eta(y)^n G(\eta)^n$$
$$= \begin{cases} \frac{p-1}{p^2}G(\eta)^n, & \text{if } mn \text{ is even,} \\ 0, & \text{if } mn \text{ is odd.} \end{cases}$$

By $a=(a_1,\ldots,a_n)\in\mathbb{F}_q^n\setminus\{(0,\ldots,0)\}$, we have that

$$\Omega_2 = \frac{1}{p^2}\sum_{z\in\mathbb{F}_p^*}\sum_{x_1\in\mathbb{F}_q}\chi(za_1x_1)\ldots\sum_{x_n\in\mathbb{F}_q}\chi(za_nx_n) = 0.$$

To compute N_a, it is sufficient to determine the value of the exponential sum

$$\Omega_3 = \frac{1}{p^2}\sum_{y,z\in\mathbb{F}_p^*}\sum_{x_1\in\mathbb{F}_q}\chi\left(yx_1^{p^{k_1}+1}+za_1x_1\right)\cdots\sum_{x_n\in\mathbb{F}_q}\chi\left(yx_n^{p^{k_n}+1}+za_nx_n\right).$$

For each k_i and $d_i=\gcd(k_i,m)$, m/d_i is odd. Hence for $y\in\mathbb{F}_p^*$, the polynomial $f_i(x)=y^{p^{k_i}}x^{p^{2k_i}}+yx=y(x^{p^{2k_i}}+x)$ must be a permutation polynomial over $\mathbb{F}_q$. In fact, suppose that there is $0\neq b\in\mathbb{F}_q$ such that $f_i(b)=0$. Then $b^{p^{2k_i}-1}=-1$. Let α be a primitive element of $\mathbb{F}_q^*$ and $b=\alpha^t$, then

$$t(p^{2k_i}-1)\equiv\frac{p^m-1}{2} \pmod{p^m-1}. \tag{3.2}$$

Let $d_i=\gcd(m,k_i)$, then $\gcd(2k_i,m)=d_i$ by m/d_i odd. Hence $\gcd(p^{2k_i}-1,p^m-1)=(p^{d_i}-1)$ and $(p^{d_i}-1)\nmid\frac{p^m-1}{2}$, so (3.2) is contradictory.

Since $f_i(x)=y(x^{p^{2k_i}}+x)$ is a permutation polynomial over $\mathbb{F}_q$, for each $a_i\in\mathbb{F}_q$ there is the unique solution $b_i\in\mathbb{F}_q$ of the equation $x_i^{p^{2k_i}}+x_i+a_i^{p^{k_i}}=0$. In fact, there is a one-to-one correspondence between $a_i\in\mathbb{F}_q$ and $b_i\in\mathbb{F}_q$, and $a_i=0$ is correspond to $b_i=0$. Hence there is the unique solution $wb_i\in\mathbb{F}_q$ of the equation $y(x_i^{p^{2k_i}}+x_i+wa_i^{p^{k_i}})=0$, where $w=\frac{z}{y}\in\mathbb{F}_p^*$.

By Lemma 5, we have that

$$\Omega_3 = \frac{1}{p^2}\sum_{y,w\in\mathbb{F}_p^*}\sum_{x_1\in\mathbb{F}_q}\chi\left(yx_1^{p^{k_1}+1}+ywa_1x_1\right)\cdots\sum_{x_n\in\mathbb{F}_q}\chi\left(yx_n^{p^{k_n}+1}+ywa_nx_n\right)$$
$$= \frac{1}{p^2}\sum_{y\in\mathbb{F}_p^*}\eta(y)^nG(\eta)^n\sum_{w\in\mathbb{F}_p^*}\chi\left(y\sum_{i=1}^{n}(wb_i)^{p^{k_i}+1}\right)$$

$$= \frac{1}{p^2}\sum_{y\in\mathbb{F}_p^*}\eta(y)^n G(\eta)^n \sum_{w\in\mathbb{F}_p^*}\chi'\left(yw^2\,\mathrm{Tr}\left(\sum_{i=1}^{n}(b_i)^{p^{k_i}+1}\right)\right).$$

Set

$$\begin{aligned}
\Gamma_0 &= \left\{(b_1,\dots,b_n)\in\mathbb{F}_q^n\setminus\{(0,\dots,0)\}\,|\,\mathrm{Tr}\left(\sum_{i=1}^{n} b_i^{p^{k_i}+1}\right)=0\right\},\\
\Gamma_1 &= \left\{(b_1,\dots,b_n)\in\mathbb{F}_q^n\,|\,\mathrm{Tr}\left(\sum_{i=1}^{n} b_i^{p^{k_i}+1}\right)\in\mathbb{F}_p^*\setminus\mathbb{F}_p^{*\,2}\right\},\\
\Gamma_2 &= \left\{(b_1,\dots,b_n)\in\mathbb{F}_q^n\,|\,\mathrm{Tr}\left(\sum_{i=1}^{n} b_i^{p^{k_i}+1}\right)\in\mathbb{F}_p^{*\,2}\right\}.
\end{aligned}$$

To compute the value of Ω_3, we divide into two cases.

The first case: mn is even, i.e. either m is even or n is even. Then we have that $\eta(y)^n = 1$ for $y\in\mathbb{F}_p^*$.

If $(b_1,\dots,b_n)\in\Gamma_0$, then $\Omega_3 = \frac{(p-1)^2}{p^2}G(\eta)^n$,

$$N_a = \frac{q^n-p^2}{p^2} + \frac{p-1}{p^2}G(\eta)^n + \frac{(p-1)^2}{p^2}G(\eta)^n = \frac{q^n-p^2}{p^2} + \frac{p-1}{p}G(\eta)^n.$$

Hence by Lemma 6, the weight of $\mathscr{C}_{D_0}$ is

$$n_0 - N_a = p^{mn-1} - p^{mn-2}.$$

If $(b_1,\dots,b_n)\in\Gamma_1\cup\Gamma_2$, then

$$\Omega_3 = \frac{1}{p^2}G(\eta)^n\sum_{y,w\in\mathbb{F}_p^*}\chi'\left(yw^2\,\mathrm{Tr}\left(\sum_{i=1}^{n} b_i^{p^{k_i}+1}\right)\right) = -\frac{(p-1)}{p^2}G(\eta)^n,\ N_a = \frac{q^n-p^2}{p^2}.$$

Hence by Lemma 6, the weight of $\mathscr{C}_{D_0}$ is

$$\begin{aligned}
n_0 - N_a &= (p-1)\left(\frac{q^n}{p^2}+\frac{1}{p}G(\eta)^n\right)\\
&= \begin{cases}(p-1)\left(p^{mn-2}+(-1)^{(m-1)n}p^{\frac{mn}{2}-1}\right), & \text{if } p\equiv 1 \pmod 4,\\ (p-1)\left(p^{mn-2}+(-1)^{(m-1)n+\frac{mn}{2}}p^{\frac{mn}{2}-1}\right), & \text{if } p\equiv 3 \pmod 4.\end{cases}
\end{aligned}$$

The second case: mn is odd. Then we have that that $\eta(y)^n = \eta'(y)$ for $y\in\mathbb{F}_p^*$.

If $(b_1,\ldots,b_n)\in\Gamma_0$, then

$$\Omega_3=\frac{p-1}{p^2}G(\eta)^n\sum_{y\in\mathbb{F}_p^*}\eta'(y)=0,\ N_a=\frac{q^n-p^2}{p^2},\ n_0-N_a=(p-1)p^{mn-2}.$$

If $(b_1,\ldots,b_n)\in\Gamma_2$, so $\mathrm{Tr}(\sum_{i=1}^n b_i^{p^{k_i}+1})=c\in{\mathbb{F}_p^*}^2$, then

$$\Omega_3=\frac{p-1}{p^2}G(\eta)^n\sum_{y\in\mathbb{F}_p^*}\eta'(y)\chi'(y)=\frac{p-1}{p^2}G(\eta)^nG(\eta'),$$

$$N_a=\frac{q^n-p^2}{p^2}+\frac{p-1}{p^2}G(\eta)^nG(\eta'),$$

$$n_0-N_a=\begin{cases}(p-1)\left(p^{mn-2}-(-1)^{(m-1)n}p^{\frac{mn-3}{2}}\right), & \text{if } p\equiv 1 \pmod 4,\\ (p-1)\left(p^{mn-2}-(-1)^{(m-1)n+\frac{mn+1}{2}}p^{\frac{mn-3}{2}}\right), & \text{if } p\equiv 3 \pmod 4.\end{cases}$$

If $(b_1,\ldots,b_n)\in\Gamma_1$, so $\mathrm{Tr}(\sum_{i=1}^n b_i^{p^{k_i}+1})=c\in\mathbb{F}_p^*\setminus{\mathbb{F}_p^*}^2$, then

$$\Omega_3=\frac{p-1}{p^2}G(\eta)^n\sum_{y\in\mathbb{F}_p^*}\eta'(cy)\chi'(y)=-\frac{p-1}{p^2}G(\eta)^nG(\eta'),$$

$$N_a=\frac{q^n-p^2}{p^2}-\frac{p-1}{p^2}G(\eta)^nG(\eta').$$

$$n_0-N_a=\begin{cases}(p-1)\left(p^{mn-2}+(-1)^{(m-1)n}p^{\frac{mn-3}{2}}\right), & \text{if} p\equiv 1 \pmod 4,\\ (p-1)\left(p^{mn-2}+(-1)^{(m-1)n+\frac{mn+1}{2}}p^{\frac{mn-3}{2}}\right), & \text{if } p\equiv 3 \pmod 4.\end{cases}$$

Secondly, we determine the frequency of each nonzero weight of $\mathscr{C}_{D_0}$. It is sufficient to consider the values of $|\Gamma_i|$, $i=0,1,2$.

By Lemma 6, it is clear that

$$|\Gamma_0|=n_0=\begin{cases}\frac{q^n-p}{p}+\frac{p-1}{p}G(\eta)^n, & \text{if } mn \text{ is even},\\ \frac{q^n-p}{p}, & \text{if } mn \text{ is odd}.\end{cases}$$

If mn is even, then by Lemma 1,

$$|\Gamma_0|=\begin{cases}\frac{q^n-p}{p}+(-1)^{(m-1)n}(p-1)p^{\frac{mn}{2}-1}, & \text{if } p\equiv 1 \pmod 4,\\ \frac{q^n-p}{p}+(-1)^{(m-1)n+\frac{mn}{2}}(p-1)p^{\frac{mn}{2}-1}, & \text{if } p\equiv 3 \pmod 4.\end{cases}$$

Since $|\Gamma_0|<q^n$. Without loss of generality, suppose that $\Gamma_2\neq\emptyset$. For some $c\in{\mathbb{F}_p^*}^2$, there are $(x_1,\ldots,x_n)\in\mathbb{F}_q^n$ such that $\mathrm{Tr}(x_1^{p^{k_1}+1}+\cdots+x_n^{p^{k_n}+1})=c\in{\mathbb{F}_p^*}^2$.

By the property of the trace function, the values are presented averagely from ${\mathbb{F}_p^*}^2$. Hence

$$\begin{aligned}
|\Gamma_2| &= \frac{p-1}{2p}\sum_{y\in\mathbb{F}_p}\sum_{(x_1,\ldots,x_n)\in\mathbb{F}_q^n}\chi'\left(y\,\mathrm{Tr}\left(x_1^{p^{k_1}+1}+\cdots+x_n^{p^{k_n}+1}\right)-cy\right)\\
&= \frac{q^n(p-1)}{2p}+\frac{p-1}{2p}\sum_{y\in\mathbb{F}_p^*}\sum_{x_1\in\mathbb{F}_q}\chi\left(yx_1^{p_1^{k_1}+1}\right)\ldots\sum_{x_n\in\mathbb{F}_q}\chi\left(yx_n^{p^{k_n}+1}\right)\chi'(-cy)\\
&= \frac{(p-1)q^n}{2p}+\frac{p-1}{2p}\sum_{y\in\mathbb{F}_p^*}G(\eta)^n\eta(y)^n\chi'(-cy)\\
&= \begin{cases}\frac{(p-1)q^n}{2p}-\frac{p-1}{2p}G(\eta)^n, & \text{if } mn \text{ is even},\\ \frac{(p-1)q^n}{2p}+\frac{p-1}{2p}\eta'(-1)G(\eta)^nG(\eta'), & \text{if } mn \text{ is odd}.\end{cases}
\end{aligned}$$

If mn is even, then by Lemma 1,

$$|\Gamma_2| = \begin{cases}\frac{p-1}{2p}\left(p^{mn}-(-1)^{(m-1)n}p^{\frac{mn}{2}}\right), & \text{if } p\equiv 1 \pmod 4,\\ \frac{p-1}{2p}\left(p^{mn}-(-1)^{(m-1)n+\frac{mn}{2}}p^{\frac{mn}{2}}\right), & \text{if } p\equiv 3 \pmod 4.\end{cases}$$

If mn is odd, then by Lemma 1,

$$|\Gamma_2| = \begin{cases}\frac{p-1}{2p}\left(p^{mn}+(-1)^{(m-1)n+\frac{mn+1}{2}}p^{\frac{mn+1}{2}}\right), & \text{if } p\equiv 1 \pmod 4,\\ \frac{p-1}{2p}\left(p^{mn}-(-1)^{(m-1)n+\frac{mn+1}{2}}p^{\frac{mn+1}{2}}\right), & \text{if } p\equiv 3 \pmod 4.\end{cases}$$

Since $|\Gamma_0|+|\Gamma_2|<q^n$, $\Gamma_1\neq\emptyset$. Similarly, the values of the trace function are presented averagely from $\mathbb{F}_p^*\setminus{\mathbb{F}_p^*}^2$ and

$$|\Gamma_1| = \begin{cases}\frac{(p-1)q^n}{2p}-\frac{p-1}{2p}G(\eta)^n, & \text{if } mn \text{ is even},\\ \frac{(p-1)q^n}{2p}-\frac{p-1}{2p}\eta'(-1)G(\eta)^nG(\eta'), & \text{if } mn \text{ is odd}.\end{cases}$$

If mn is even, then

$$|\Gamma_1| = \begin{cases}\frac{p-1}{2p}\left(p^{mn}-(-1)^{(m-1)n}p^{\frac{mn}{2}}\right), & \text{if } p\equiv 1 \pmod 4,\\ \frac{p-1}{2p}\left(p^{mn}-(-1)^{(m-1)n+\frac{mn}{2}}p^{\frac{mn}{2}}\right), & \text{if } p\equiv 3 \pmod 4.\end{cases}$$

If mn is odd, then by Lemma 1,

$$|\Gamma_1| = \begin{cases}\frac{p-1}{2p}\left(p^{mn}-(-1)^{(m-1)n+\frac{mn+1}{2}}p^{\frac{mn+1}{2}}\right), & \text{if } p\equiv 1 \pmod 4,\\ \frac{p-1}{2p}\left(p^{mn}+(-1)^{(m-1)n+\frac{mn+1}{2}}p^{\frac{mn+1}{2}}\right), & \text{if } p\equiv 3 \pmod 4.\end{cases}$$

Hence, we get the Tables 1 and 2. □

Example 1 Let $m = 3$, $n = 1$, $p = 3$, the code $\mathscr{C}_{D_0}$ has parameters [8,3,4] and weight enumerator $1 + 6z^4 + 8z^6 + 12z^8$. This code is almost optimal linear code,as the optimal one has parameters [8,3,5] by the Griesmer bound.

Example 2 Let $m = 2$, $n = 2$, $p = 3$, the code $\mathscr{C}_{D_0}$ has parameters [44,4,18] and weight enumerator $1 + 32z^{18} + 36z^{24}$.

3.2 The second case

Fix $c \in \mathbb{F}_p^{*2}$ and define

$$D_2 = \left\{(x_1, \ldots, x_n) \in \mathbb{F}_q^n : \mathrm{Tr}\left(x_1^{p^{k_1}+1} + \cdots + x_n^{p^{k_n}+1}\right) = c\right\},$$

$$\mathscr{C}_{D_2} = \left\{c(a_1, \ldots, a_n) : (a_1, \ldots, a_n) \in \mathbb{F}_q^n\right\}, \tag{3.3}$$

where

$$c(a_1, \ldots, a_n) = (\mathrm{Tr}(a_1x_1 + \cdots + a_nx_n))_{(x_1,\ldots,x_n)\in D_2}.$$

Since $\mathscr{C}_{D_2}$ is a linear code over $\mathbb{F}_p$, it is independent of the choice of c. For convenience, we take $c = 1$.

By Lemma 1 and the computation of Γ_2 as above, we can get the result.

Lemma 7 *Let $n_2 = |D_2|$. Suppose that mn is even, then*

$$n_2 = \frac{q^n}{p} - \frac{1}{p}G(\eta)^n = \begin{cases} p^{mn-1} - (-1)^{(m-1)n}p^{\frac{mn}{2}-1}, & \text{if } p \equiv 1 \pmod 4, \\ p^{mn-1} - (-1)^{(m-1)n+\frac{mn}{2}}p^{\frac{mn}{2}-1}, & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

Suppose that mn is odd, then

$$\begin{aligned} n_2 &= \frac{q^n}{p} + \frac{1}{p}\eta'(-1)G(\eta)^n G(\eta') \\ &= \begin{cases} p^{mn-1} + (-1)^{(m-1)n}p^{\frac{mn-1}{2}}, & \text{if } p \equiv 1 \pmod 4, \\ p^{mn-1} - (-1)^{(m-1)n+\frac{mn+1}{2}}p^{\frac{mn-1}{2}}, & \text{if } p \equiv 3 \pmod 4. \end{cases} \end{aligned}$$

Theorem 2 *Let $\mathscr{C}_{D_2}$ be the linear code defined as (3.3).*

If mn is even, then $\mathscr{C}_{D_2}$ is a two-weight code with the Hamming weight distribution in Table 3.

If mn is odd, then $\mathscr{C}_{D_1}$ is a two-weight code with the Hamming weight distribution in Table 4.

Table 3 *mn* is even

Weight	Multiplicity
$p \equiv 1 \pmod 4$	
0	1
$(p-1)p^{mn-2}$	$\frac{p+1}{2}p^{mn-1} - 1 + (-1)^{(m-1)n}\frac{p-1}{2}p^{\frac{mn}{2}-1}$
$(p-1)p^{mn-2} - 2(-1)^{(m-1)n}p^{\frac{mn}{2}-1}$	$\frac{p-1}{2}p^{mn-1} - (-1)^{(m-1)n}\frac{p-1}{2}p^{\frac{mn}{2}-1}$
$p \equiv 3 \pmod 4$	
0	1
$(p-1)p^{mn-2}$	$\frac{p+1}{2}p^{mn-1} - 1 + (-1)^{(m-1)n+\frac{mn}{2}}\frac{p-1}{2}p^{\frac{mn}{2}-1}$
$(p-1)p^{mn-2} - 2(-1)^{(m-1)n+\frac{mn}{2}}p^{\frac{mn}{2}-1}$	$\frac{p-1}{2}p^{mn-1} - (-1)^{(m-1)n+\frac{mn}{2}}\frac{p-1}{2}p^{\frac{mn}{2}-1}$

Table 4 *mn* is odd

Weight	Multiplicity
$p \equiv 1 \pmod 4$	
0	1
$(p-1)p^{mn-2}$	$\frac{p+1}{2}p^{mn-1} - 1 - (-1)^{(m-1)n+\frac{mn+1}{2}}\frac{p-1}{2}p^{\frac{mn-1}{2}}$
$(p-1)p^{mn-2} + 2(-1)^{(m-1)n}p^{\frac{mn-3}{2}}$	$\frac{p-1}{2}p^{mn-1} + (-1)^{(m-1)n+\frac{mn+1}{2}}\frac{p-1}{2}p^{\frac{mn-1}{2}}$
$p \equiv 3 \pmod 4$	
0	1
$(p-1)p^{mn-2}$	$\frac{p+1}{2}p^{mn-1} - 1 - (-1)^{(m-1)n+\frac{mn+1}{2}}\frac{p-1}{2}p^{\frac{mn-1}{2}}$
$(p-1)p^{mn-2} - (-1)^{(m-1)n+\frac{mn+1}{2}}p^{\frac{mn-3}{2}})$	$\frac{p-1}{2}p^{mn-1} + (-1)^{(m-1)n+\frac{mn+1}{2}}\frac{p-1}{2}p^{\frac{mn-1}{2}}$

Proof Firstly, we determine the weight distribution of the code $\mathscr{C}_{D_2}$. Fix $c = 1 \in \mathbb{F}_p^{*2}$. Define the following parameter

$$N'_a = \left|\left\{(x_1, \ldots, x_n) \in \mathbb{F}_q^n : \mathrm{Tr}\left(x_1^{p^{k_1}+1} + \cdots + x_n^{p_n^{k_n}+1}\right) = 1, \mathrm{Tr}(a_1x_1 + \cdots + a_nx_n) = 0\right\}\right|,$$

where $a = (a_1, \ldots, a_n) \in \mathbb{F}_q^n \setminus \{(0, \ldots, 0)\}$. We have

$$\begin{aligned} N'_a &= \frac{1}{p^2}\sum_{(x_1,\ldots,x_n)\in\mathbb{F}_q^n, y,z\in\mathbb{F}_p} \chi'\left(y(\mathrm{Tr}\left(x_1^{p^{k_1}+1} + \cdots + x_n^{p^{k_n}+1}\right) - 1)\right)\chi(z(a_1x_1 + \ldots + a_nx_n)) \\ &= \frac{q^n}{p^2} + \frac{1}{p^2}\sum_{(x_1,\ldots,x_n)\in\mathbb{F}_q^n}\sum_{y\in\mathbb{F}_p^*}\chi'\left(y\left(\mathrm{Tr}(x_1^{p^{k_1}+1} + \cdots + x_n^{p^{k_n}+1}\right) - 1\right) \\ &\quad + \frac{1}{p^2}\sum_{(x_1,\ldots,x_n)\in\mathbb{F}_q^n}\sum_{z\in\mathbb{F}_p^*}\chi(z(a_1x_1 + \cdots + a_nx_n)) \\ &\quad + \frac{1}{p^2}\sum_{(x_1,\ldots,x_n)\in\mathbb{F}_q^n}\sum_{y,z\in\mathbb{F}_p^*}\chi'\left(y\left(\mathrm{Tr}(x_1^{p^{k_1}+1} + \cdots + x_n^{p^{k_n}+1}\right) - 1\right)\chi(z(a_1x_1 + \cdots + a_nx_n)) \end{aligned}$$

$$=: \frac{q^n}{p^2} + \Omega_1' + \Omega_2' + \Omega_3'.$$

We have that

$$\begin{aligned}\Omega_1' &= \frac{1}{p^2} \sum_{y \in \mathbb{F}_p^*} \sum_{x_1 \in \mathbb{F}_q} \chi\left(y x_1^{p^{k_1}+1}\right) \cdots \sum_{x_n \in \mathbb{F}_q} \chi\left(y x_n^{p^{k_n}+1}\right) \sum \chi'(-y) \\ &= \frac{1}{p^2} \sum_{y \in \mathbb{F}_p^*} G(\eta)^n \eta(y)^n \chi'(-y) \\ &= \begin{cases} -\frac{1}{p^2} G(\eta)^n, & \text{if } mn \text{ is even}, \\ \frac{1}{p^2} \eta(-1) G(\eta)^n G(\eta'), & \text{if } mn \text{ is odd}. \end{cases}\end{aligned}$$

By $(a_1, \ldots, a_n) \in \mathbb{F}_p^n \setminus \{(0, \ldots, 0)\}$, we have that

$$\Omega_2' = 0.$$

Similarly, we have

$$\Omega_3' = \frac{1}{p^2} \sum_{y \in \mathbb{F}_p^*} \eta(y)^n G(\eta)^n \sum_{w \in \mathbb{F}_p^*} \chi'\left(y\left(w^2 \operatorname{Tr}\left(\sum_{i=1}^{n} (b_i)^{p^{k_i}+1}\right) - 1\right)\right),$$

where $(b_1, \ldots, b_n)$ is one-to-one correspondent to $(a_1, \ldots, a_n)$, and $(0, \ldots, 0)$ is correspondence to $(0, \ldots, 0)$.

To compute the value of Ω_3', we divide into two cases.

The first case: mn is even. Then $\eta(y)^n = 1$ for $y \in \mathbb{F}_p^*$.

If $(b_1, \ldots, b_n) \in \Gamma_0$, then

$$\Omega_3' = \frac{p-1}{p^2} G(\eta)^n \sum_{y \in \mathbb{F}_p^*} \chi'(-y) = -\frac{p-1}{p^2} G(\eta)^n,$$

$$N_a' = \frac{q^n}{p^2} - \frac{1}{p^2} G(\eta)^n - \frac{p-1}{p^2} G(\eta)^n = \frac{q^n}{p^2} - \frac{1}{p} G(\eta)^n, n_2 - N_a' = (p-1)p^{mn-2}.$$

If $(b_1, \ldots, b_n) \in \Gamma_1$, then $w^2 \operatorname{Tr}((\sum_{i=1}^n (b_i)^{p^{k_i}+1}) \neq 1$ for any $w \in \mathbb{F}_p^*$. Hence

$$\Omega_3' = -\frac{p-1}{p^2} G(\eta)^n, N_a' = \frac{q^n}{p^2} - \frac{1}{p} G(\eta)^n, n_2 - N_a' = (p-1)p^{mn-2}.$$

If $(b_1, \ldots, b_n) \in \Gamma_2$, then there are only two $\pm w_0 \in \mathbb{F}_p^*$ such that w_0^2 $\operatorname{Tr}((\sum_{i=1}^n (b_i)^{p^{k_i}+1}) = 1$, so

$$\Omega_3' = \frac{1}{p^2} G(\eta)^n \left(\sum_{y, w \in \mathbb{F}_q^*, w \neq \pm w_0} \chi'\left(y\left(w^2 \operatorname{Tr}\left(\sum_{i=1}^{n} (b_i)^{p^{k_i}+1}\right) - 1\right)\right) + 2(p-1) \right)$$

$$\begin{aligned}
&= \frac{1}{p^2}G(\eta)^n(-(p-3)+2(p-1)) = \frac{p+1}{p^2}G(\eta)^n,\\
N_a' &= \frac{q^n}{p^2} - \frac{1}{p^2}G(\eta)^n + \frac{p+1}{p^2}G(\eta)^n = \frac{q^n}{p^2} + \frac{1}{p}G(\eta)^n,\\
n_2 - N_a' &= \frac{(p-1)q^n}{p^2} - \frac{2}{p}G(\eta)^n\\
&= \begin{cases} (p-1)p^{mn-2} - 2(-1)^{(m-1)n}p^{\frac{mn}{2}-1}, & \text{if } p \equiv 1 \pmod 4,\\ (p-1)p^{mn-2} - 2(-1)^{(m-1)n+\frac{mn}{2}}p^{\frac{mn}{2}-1}, & \text{if } p \equiv 3 \pmod 4.\end{cases}
\end{aligned}$$

The second case: mn is odd. Then $\eta(y)^n = \eta'(y)$ for any $y \in \mathbb{F}_p^*$.
If $(b_1, \ldots, b_n) \in \Gamma_0$, then

$$\Omega_3' = \frac{p-1}{p^2}\eta'(-1)G(\eta)^nG(\eta'),\ N_a' = \frac{q^2}{p^2} + \frac{1}{p}\eta'(-1)G(\eta)^nG(\eta').$$
$$n_2 - N_a' = (p-1)p^{mn-2}.$$

If $(b_1, \ldots, b_n) \in \Gamma_1$, then $w^2\,\mathrm{Tr}(\sum_{i=1}^n (b_i)^{p^{k_i}+1}) \neq 1$ for any $w \in \mathbb{F}_p^*$. let β be a primitive root of $\mathbb{F}_p^*$, then $C_0 =< \beta^2 >$ be a subgroup of $\mathbb{F}_p^*$ and $C_1 = \beta C_0$, so $\mathbb{F}_p^* = C_0 \cup C_1$. Define

$$(i, j)_2 = |(C_i + 1) \cap C_j|, \quad i, j = 0, 1,$$

all cyclotomic numbers of order 2. We have the following results [25]: If $p \equiv 1 \pmod 4$, then $(1,0)_2 = (0,1)_2 = (1,1)_2 = \frac{p-1}{4}$, $(0,0)_2 = \frac{p-5}{4}$. If $p \equiv 3 \pmod 4$, then $(1,0)_2 = (0,0)_2 = (1,1)_2 = \frac{p-3}{4}$, $(0,1)_2 = \frac{p+1}{4}$.
Hence

$$\begin{aligned}
\Omega_3' &= \frac{2}{p^2}\sum_{y\in\mathbb{F}_p^*} G(\eta)^n\eta'(y)\left(\sum_{s\in C_1, s-1\in C_0}\chi'((s-1)y) + \sum_{s\in C_1, s-1\in C_1}\chi'(s-1)y\right)\\
&= \frac{2}{p^2}G(\eta)^nG(\eta')((0,1)_2 - (1,1)_2),\\
N_a' &= \frac{q^2}{p^2} + \frac{1}{p^2}\eta(-1)G(\eta)^nG(\eta') + \frac{2}{p^2}G(\eta)^nG(\eta')((0,1)_2 - (1,1)_2),\\
n_2 - N_a' &= \begin{cases} (p-1)p^{mn-2}, & \text{if } p \equiv 1 \pmod 4,\\ (p-1)p^{mn-2} - 2(-1)^{(m-1)n+\frac{mn+1}{2}}p^{\frac{mn-3}{2}}, & \text{if } p \equiv 3 \pmod 4.\end{cases}
\end{aligned}$$

If $(b_1, \ldots, b_n) \in \Gamma_2$, then there are only two $\pm w_0 \in \mathbb{F}_p^*$ such that w_0^2 $\mathrm{Tr}((\sum_{i=1}^n (b_i)^{p^{k_i}+1}) = 1$. Hence

$$\Omega_3' = \frac{2}{p^2}\sum_{y\in\mathbb{F}_p^*}\eta(y)G(\eta)^n\left(\sum_{s\in C_0, s-1\in C_0}\chi'((s-1)y\right) + \sum_{s\in C_0, s-1\in C_1}\chi'((s-1)y) + 1)$$

$$
\begin{aligned}
&= \frac{2}{p^2} G(\eta)^n G(\eta')((0,0)_2 - (1,0)_2),\\
N'_a &= \frac{q^2}{p^2} + \frac{1}{p^2}\eta(-1)G(\eta)^n G(\eta') + \frac{2}{p^2} G(\eta)^n G(\eta')((0,0)_2 - (1,0)_2),\\
n_2 - N'_a &= \begin{cases} (p-1)p^{mn-2} + 2(-1)^{(m-1)n} p^{\frac{mn-3}{2}}, & \text{if } p \equiv 1 \pmod 4,\\ (p-1)p^{mn-2}, & \text{if } p \equiv 3 \pmod 4. \end{cases}
\end{aligned}
$$

By the computations of $\Gamma_0, \Gamma_1, \Gamma_2$, we get the Tables 3 and 4. □

Example 3 Let $m = 1, n = 2, p = 5$, the code $\mathscr{C}_{D_2}$ is a [4,2,2] almost optimal linear code, as the optimal one has parameters [4,2,3] by the Griesmer bound.

Example 4 Let $m = 2, n = 2, p = 3$, the code $\mathscr{C}_{D_2}$ has parameters [24,4,12] and weight enumerator $1 + 24z^{12} + 50z^{18}$.

Example 5 Let $m = 3, n = 3, p = 3$, the code $\mathscr{C}_{D_2}$ has parameters [6642,9,4374] and weight enumerator $1 + 13202z^{4374} + 6480z^{4401}$.

3.3 The third case

Fix $c \in \mathbb{F}_p^* \setminus \mathbb{F}_p^{*2}$ and define

$$
\begin{aligned}
D_1 &= \{(x_1, \ldots, x_n) \in \mathbb{F}_q^n : \mathrm{Tr}\left(x_1^{p^{k_1}+1} + \cdots + x_n^{p^{k_n}+1}\right) = c\},\\
\mathscr{C}_{D_1} &= \{c(a_1, \ldots, a_n) : (a_1, \ldots, a_n) \in \mathbb{F}_q^n\},
\end{aligned} \tag{3.4}
$$

where

$$c(a_1, \ldots, a_n) = (\mathrm{Tr}(a_1x_1 + \cdots + a_nx_n))_{(x_1,\ldots,x_n)\in D_1}.$$

Since $\mathscr{C}_{D_1}$ is a linear code over $\mathbb{F}_p$, it is independent of the choice of c.

By Lemma 1 and the computation of Γ_1 as above, we can get the result similarly with Lemma 7.

Lemma 8 *Let $n_1 = |D_1|$. Suppose that mn is even, then*

$$n_1 = \frac{q^n}{p} - \frac{1}{p}G(\eta)^n = \begin{cases} p^{mn-1} - (-1)^{(m-1)n} p^{\frac{mn}{2}-1}, & \text{if } p \equiv 1 \pmod 4,\\ p^{mn-1} - (-1)^{(m-1)n+\frac{mn}{2}} p^{\frac{mn}{2}-1}, & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

Suppose that mn is odd, then

$$
\begin{aligned}
n_1 &= \frac{q^n}{p} - \frac{1}{p}\eta'(-1)G(\eta)^n G(\eta')\\
&= \begin{cases} p^{mn-1} - (-1)^{(m-1)n} p^{\frac{mn-1}{2}}, & \text{if } p \equiv 1 \pmod 4,\\ p^{mn-1} + (-1)^{(m-1)n+\frac{mn+1}{2}} p^{\frac{mn-1}{2}}, & \text{if } p \equiv 3 \pmod 4. \end{cases}
\end{aligned}
$$

Table 5 *mn* is even

Weight	Multiplicity
$p \equiv 1 \pmod 4$	
0	1
$(p-1)p^{mn-2}$	$\frac{p+1}{2}p^{mn-1} - 1 + (-1)^{(m-1)n}\frac{p-1}{2}p^{\frac{mn}{2}-1}$
$(p-1)p^{mn-2} - 2(-1)^{(m-1)n}p^{\frac{mn}{2}-1}$	$\frac{p-1}{2}p^{mn-1} - (-1)^{(m-1)n}\frac{p-1}{2}p^{\frac{mn}{2}-1}$
$p \equiv 3 \pmod 4$	
0	1
$(p-1)p^{mn-2}$	$\frac{p+1}{2}p^{mn-1} - 1 + (-1)^{(m-1)n+\frac{mn}{2}}\frac{p-1}{2}p^{\frac{mn}{2}-1}$
$(p-1)p^{mn-2} - 2(-1)^{(m-1)n+\frac{mn}{2}}p^{\frac{mn}{2}-1}$	$\frac{p-1}{2}p^{mn-1} - (-1)^{(m-1)n+\frac{mn}{2}}\frac{p-1}{2}p^{\frac{mn}{2}-1}$

Table 6 *mn* is odd

Weight	Multiplicity
$p \equiv 1 \pmod 4$	
0	1
$(p-1)p^{mn-2}$	$\frac{p+1}{2}p^{mn-1} - 1 - (-1)^{(m-1)n+\frac{mn+1}{2}}\frac{p-1}{2}p^{\frac{mn-1}{2}}$
$(p-1)p^{mn-2} - 2(-1)^{(m-1)n}p^{\frac{mn-3}{2}}$	$\frac{p-1}{2}p^{mn-1} - (-1)^{(m-1)n+\frac{mn+1}{2}}\frac{p-1}{2}p^{\frac{mn-1}{2}}$
$p \equiv 3 \pmod 4$	
0	1
$(p-1)p^{mn-2}$	$\frac{p+1}{2}p^{mn-1} - 1 + (-1)^{(m-1)n+\frac{mn+1}{2}}\frac{p-1}{2}p^{\frac{mn-1}{2}}$
$(p-1)p^{mn-2} + 2(-1)^{(m-1)n+\frac{mn+1}{2}}p^{\frac{mn-3}{2}}$	$\frac{p-1}{2}p^{mn-1} - (-1)^{(m-1)n+\frac{mn+1}{2}}\frac{p-1}{2}p^{\frac{mn-1}{2}}$

Theorem 3 *Let $\mathscr{C}_{D_1}$ be the linear code defined as* (3.4).

If mn is even, then $\mathscr{C}_{D_1}$ is a two-weight code with the Hamming weight distribution in Table 5.

If mn is odd, then $\mathscr{C}_{D_1}$ is a two-weight code with the Hamming weight distribution in Table 6.

Proof By the process of proving Theorem 2, we can get the result. □

4 Concluding remarks

There is a recent survey on three-weight codes [3,9,17,18,22,26,27]. We did not find the the parameters of the binary three-weight codes of this paper in these literatures.

Linear codes can be used to construct secret sharing schemes [28]. Let $w_{\min}$ and $w_{\max}$ denote the minimum and maximum nonzero Hamming weights of a linear code $\mathscr{C}$. To obtain secret sharing schemes with interesting access structures, we would like to construct linear codes which have the property that

$$w_{\min}/w_{\max} > \frac{p-1}{p}.$$

We remark that the linear codes in this paper can be employed in secret sharing schemes using the framework in [28].

For the code of $\mathscr{C}_D$ the Theorem 3.2, we have

$$w_{\min}/w_{\max} = \frac{(p-1)p^{mn-2}}{(p-1)\left(p^{nm-2}+p^{\frac{nm-3}{2}}\right)} > \frac{p-1}{p}, \quad where\, n \geq 1, m > 4$$

For the code of $\mathscr{C}_D$ the Theorem 3.4, we have

$$w_{\min}/w_{\max} = \frac{(p-1)p^{mn-2}}{(p-1)p^{nm-2}+2p^{\frac{nm-3}{2}}} > \frac{p-1}{p}, \quad where\, n \geq 1, m > 4$$

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