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ORIGINAL RESEARCH

Residual power series method for solving time-space-fractional Benney-Lin equation arising in falling film problems

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Abstract In this paper, a recent analytic iterative technique, named as residual power series method is implemented to find the approximate solution of the nonlinear time-space-fractional Benney-Lin equation. The convergence analysis of the proposed scheme is also discussed. To test the validity, potentiality, and practical usefulness of the proposed method in solving such a complicated equation, several numerical examples with various initial conditions are considered. The analysis of the obtained approximate solution results reveal that the proposed method is a significant addition for exploring nonlinear fractional models in fractional theory and its computations.

Keywords Fractional Benney-Lin equation · Caputo derivative · Residual power series method

Mathematics Subject Classification 34A08 · 26A33 · 41-XX

1 Introduction

Recent decades have witnessed a fast growing applications of fractional calculus in diverse and widespread fields of science and engineering such as mechanics, medicine, electrical engineering, ecology, biology and many others. The list of applications of fractional calculus is still growing; perhaps "the fractional calculus is the calculus of twenty-first century". Many mathematicians provide the brief history and a compre-

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hensive treatment of fractional calculus and references are summarized in monographs [1–3]. In recent times, the Mittag-Leffler function becomes an important function due to its widespread use in the world of fractional calculus. The properties of Mittag-Leffler function, its numerous generalization and their applications are discussed in monograph [4]. A fractional derivative gives a perfect aid to characterize the memory and hereditary properties of various processes and materials. In this context, considerable attention has been given to the physical and engineering problems that conducted with the description of memory and hereditary characteristics of different materials and processes due to their non-locality characteristics [5–7] and developing tools which allow analyzing and controlling the dynamical behaviors of multi-soliton solutions for equations with fractal derivatives [8,9].

Nonlinear fractional differential equations have gained much interest due to the exact description of nonlinear phenomena. For the sake of better understanding of these phenomena in practical scientific research, there is a need to find their solutions.

The residual power series method (RPSM) was proposed by the Jordan mathematician Abu Arqub [10] as an efficient method for determining the values of the coefficients of the power series solution of the differential equations. The RPSM has been successfully applied in constructing the numerical solution of the generalized Lane-Emden equation, which is a highly nonlinear singular differential equation [11], in predicting the solitary pattern solutions for nonlinear time-fractional dispersive partial differential equations [12], in obtaining the approximate solution of the nonlinear fractional KdV-Burgers equation [13], and in the numerical solution of fractional foam drainage equation [14]. This technique is also applied in the numerical solution of different kinds of problems [15,16]. Moreover, the RPSM has been successfully applied in the numerical solutions of strongly linear and nonlinear fractional differential equations without implementing linearization, perturbation or discretization techniques, showing the simplicity and effectiveness of the method [17–19]. In fact, the application of RPSM in the numerical analysis field is not new and on the other side it possesses some of the well known advantages such as:

- It is accurate, needless effort to achieve the results.
- In the proposed method, it is possible to pick any point in the interval of integration and as well the approximate solutions and their derivatives will be applicable.
- The method does not require discretization of the variables, and it is not affected by computation round off errors and one is not faced with necessity of large computer memory and time.
- It is of global nature in terms of the solutions obtained as well as its ability to solve other mathematical, physical, and engineering problems.

In 1966, Benney [20] considered the long waves on the liquid films and found some attractive results and introduced the Benney-Lin equation, afterward modified by Lin [21]. This general equation arises in falling film problems. In many research articles the analytical or the numerical methods of Benney-Lin equation are discussed [22–24]. The little consideration has been given to the nonlinear fractional Benney-Lin equation with time-fractional derivatives [25]. Also, the application of nonlinear time-space-fractional Benney-Lin equations is missing in the literature. To fill this gap, we consider the initial value problem (IVP) of time-space-fractional Benney-Lin equation as

$$\begin{cases} D_t^{\alpha}u + uD_x^{\beta}u + D_x^{3\beta}u + \eta(D_x^{2\beta}u + D_x^{4\beta}u) + \mu D_x^{5\beta}u + D_x^{\beta}u = 0, \quad 0<\alpha,\beta<1,\ \eta>0,\ \mu\in\mathbb{R}, \\ u(x,0) = \phi(x). \end{cases} \tag{1}$$

In purely dispersive form, for $\alpha = 1$, $\beta = 1$ and $\eta = 0$, Eq. (1) is reduces to the standard Kawahara equation (or the fifth-order Kortewegde Vries equation) that describes the water waves with surface tension [26]. In the purely dissipative form, Eq. (1) reduces to the long wave simplification of the Navier-Stokes equation and has been used to describe different phenomena such as spatial patterns of the Belousov-Zhabotinsky reaction, surface-tension-driven convection in a liquid film, and unstable flame fronts. Also, for $\alpha = 1$, $\beta = 1$ and $\mu = 0$, i.e., the dissipative-dispersive equation, Eq. (1) is reduced to the generalized Kuramoto-Sivashinsky equation that explains the waves in the inclined and vertical falling film, in liquid films that are subjected to interfacial stress from adjacent gas flow, unstable drift waves in plasma, interfacial instability between two concurrent viscous fluids and phase evolution for the complex Ginzburg-Landau equation.

The paper is organized as follows: Sect. 2, presents some preliminaries of fractional calculus and the residual power series method. In Sect. 3, a power series solution for time-space-fractional Benney-Lin equation by its power series expansion among its truncated residual function is constructed. Convergence analysis of proposed method is discussed in Sect. 4. In Sect. 5, some numerical results are presented which demonstrate the effectiveness of the numerical scheme.

2 Preliminaries

This section provides the operational properties for elucidating sufficient fractional calculus theory, to enable us to follow the solutions of time-space-fractional Benney-Lin equation. In recent times, the fractional differential equations have gained much attention due to the fact that they can generate fractional Brownian motion, which is generalization of Brownian motion. There are several definitions for fractional derivatives and integrals, like Riemann-Liouville, Caputo, Riesz, Weyl, Grunwald-Letnikov, Hadamard, etc. The Riemann-Liouville and Caputo's are the most common definitions. But Caputo's approach is suitable for real world physical problems because it defines integer order initial conditions for fractional differential equations.

Throughout this paper, N is the set of integer numbers, R is the set of real numbers, and Γ is the Gamma function.

Definition 2.1 The Riemann-Liouville time-fractional integral of order α of $u(x,t)$ is defined as

$$I_t^{\alpha}u(x,t) = \begin{cases} \frac{1}{\Gamma(\alpha)}\int_s^t (t-\xi)^{\alpha-1}u(x,\xi)d\xi, & \alpha > 0,\ x\in I,\ t>\xi>s\geq 0, \\ u(x,t), & \alpha = 0. \end{cases}$$

Definition 2.2 The Caputo's time fractional derivative of order α of $u(x,t)$ is defined as

$$D_t^{\alpha} u(x,t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_s^t (t-\xi)^{m-\alpha-1} \frac{\partial^m u(x,\xi)}{\partial \xi^m} d\xi, & 0 \le m-1 < \alpha < m,\ t > \xi > s \ge 0, x \in I, \\ \frac{\partial^m u(x,t)}{\partial t^m}, & \alpha = m \in N. \end{cases}$$

Theorem 2.1 *If $m-1 < \alpha \le m$, $m \in N$, then*

(a) $D_t^{\alpha} I_t^{\alpha} u(x,t) = u(x,t)$,
(b) $I_t^{\alpha} D_t^{\alpha} u(x,t) = u(x,t) - \sum_{i=0}^{n-1} \frac{\partial^i u(x,s^+)}{\partial t^i} \frac{t^i}{i!}$.

The other properties of fractional derivatives can be seen in [1–4]. *It may be noted that in the further discussion, the fractional derivative is taken in Caputo's sense. From* [19], *some results which are essential for the RPSM are stated as follows:*

Definition 2.3 A power series expansion of the form

$$\sum_{k=0}^{\infty} c_k (t-t_0)^{k\alpha} = c_0 + c_1 (t-t_0)^{\alpha} + c_2 (t-t_0)^{2\alpha} + \ldots,\ 0 \le m-1 < \alpha \le m,\ t \ge t_0, \tag{2}$$

is called fractional power series about $t = t_0$.

Theorem 2.2 *Suppose that f has a fractional power series representation at $t = t_0$ of the form*

$$f(t) = \sum_{k=0}^{\infty} c_k (t-t_0)^{k\alpha}, \quad 0 \le m-1 < \alpha \le m,\ t_0 \le t < t_0 + R. \tag{3}$$

If $D^{k\alpha} f(t)$ are continuous on $(t_0, t_0 + R)$, $k = 0, 1, 2, \ldots,$ then the coefficients "c_k" appearing in the Eq. (3) *can be determined as follows:*

$$c_k = \frac{D^{k\alpha} f(t_0)}{\Gamma(k\alpha + 1)},$$

where R is the radius of convergence of the fractional power series.

Definition 2.4 For $0 \le m-1 < \alpha \le m$, a power series expansion of the form

$$\sum_{k=0}^{\infty} w_k(x)(t-t_0)^{k\alpha} = w_0(x) + w_1(x)(t-t_0)^{\alpha} + w_2(x)(t-t_0)^{2\alpha} + \ldots, \quad t \ge t_0, \tag{4}$$

is called multiple fractional power series about $t = t_0$, where w_k's are functions of x called the coefficients of the series.

3 Residual power series for space-time fractional Benney-Lin equation

Consider time-space-fractional Benney-Lin equation as

$$D_t^{\alpha}u + uD_x^{\beta}u + D_x^{3\beta}u + \eta(D_x^{2\beta}u + D_x^{4\beta}u) + \mu D_x^{5\beta}u + D_x^{\beta}u = 0,$$
$$0 < \alpha, \beta < 1, \ \eta > 0, \ \mu \in \mathbb{R}, \quad (5)$$

subject to the initial condition

$$u(x, 0) = \phi(x). \quad (6)$$

The purpose of this study is to construct a power series solution for Eqs. (5) and (6) by its power series expansion among its truncated residual function.

3.1 General procedure of the residual power series solution

The main steps of this procedure are described as follows:
Step 1 Suppose that the solution of Eqs. (5) and (6) is expressed in the form of fractional power series expansion about the initial point $t = 0$ as follows:

$$u(x, t) = \sum_{k=0}^{\infty} w_k(x) \frac{t^{k\alpha}}{\Gamma(k\alpha + 1)}, \quad 0 < \alpha \leq 1, \ x \in I, \ 0 \leq t < R. \quad (7)$$

The RPSM guarantees that the analytical approximate solution for Eqs. (5) and (6) are in the form of an infinite fractional power series. To get the numerical values from this series, let $u_m(x, t)$ denotes the m-th truncated series of $u(x, t)$. That is,

$$u_m(x, t) = \sum_{k=0}^{m} w_k(x) \frac{t^{k\alpha}}{\Gamma(k\alpha + 1)}, \quad 0 < \alpha \leq 1, \ x \in I, \ 0 \leq t < R. \quad (8)$$

Take $m = 0$ and by the initial condition, the 0th residual power series approximate solution of $u(x, t)$ can be written in the following form:

$$u_0(x, t) = w_0(x) = u(x, 0) = \phi(x). \quad (9)$$

The Eq. (8) can be rewritten as

$$u_m(x, t) = \phi(x) + \sum_{k=1}^{m} w_k(x) \frac{t^{k\alpha}}{\Gamma(k\alpha + 1)}, \ 0 < \alpha \leq 1, \ x \in I, \ 0 \leq t, \ m = 1, 2, 3, \ldots. \quad (10)$$

By viewing the representations of $u_m(x, t)$, the mth residual power series approximate solution will be obtained after $w_k(x), \ k = 1, 2, 3, \ldots, m$, are available.
Step 2 Define the residual function for Eqs. (5) and (6) as follows:

$$Res(x, t) = D_t^{\alpha}u + uD_x^{\beta}u + D_x^{3\beta}u + \eta(D_x^{2\beta}u + D_x^{4\beta}u) + \mu D_x^{5\beta}u + D_x^{\beta}u, \quad (11)$$

and the mth residual function can be expressed as

$$Res_m(x,t) = \frac{\partial^\alpha u_m(x,t)}{\partial t^\alpha} + u_m(x,t)\frac{\partial^\beta u_m(x,t)}{\partial x^\beta} + \frac{\partial^{3\beta} u_m(x,t)}{\partial x^{3\beta}} + \eta\left(\frac{\partial^{2\beta} u_m(x,t)}{\partial x^{2\beta}} + \frac{\partial^{4\beta} u_m(x,t)}{\partial x^{4\beta}}\right) + \mu\frac{\partial^{5\beta} u_m(x,t)}{\partial x^{5\beta}} + \frac{\partial^\beta u_m(x,t)}{\partial x^\beta}, \quad m = 1, 2, 3, \ldots. \tag{12}$$

From [1–6], some useful results of $Res_m(x,t)$ which are essential in the residual power series solution are stated as follows:

$$\begin{aligned} &\text{(i)} \quad Res(x,t) = 0,\\ &\text{(ii)} \quad \lim_{m\to\infty} Res_m(x,t) = Res(x,t) \; for\ each\ x \in I\ and\ t \geq 0,\\ &\text{(iii)} \quad D_t^{i\alpha} Res(x,0) = D_t^{i\alpha} Res_m(x,0) = 0, \quad i = 0, 1, 2, \ldots, m. \end{aligned} \tag{13}$$

Step 3 Substitute the mth truncated series of $u(x,t)$ into Eq. (12) and calculate the fractional derivative $D_t^{(m-1)\alpha}$ of $Res_m(x,t)$, $m = 1, 2, 3, \ldots$ at $t = 0$, together with Eq. (13), the following algebraic system is obtained:

$$D_t^{(m-1)\alpha} Res_m(x,0) = 0, \quad 0 < \alpha \leq 1, \; m = 1, 2, 3, \ldots. \tag{14}$$

Step 4 After solving the system (14), the values of the coefficients $w_k(x)$, $k = 1, 2, 3, \ldots, m$ are obtained. Thus, the mth residual power series approximate solution is derived.

In the next discussion, the 1st, 2nd, 3rd and 4th residual power series approximate solutions are determined in detail by following the above steps.

For $\boldsymbol{m = 1}$, the 1st residual power series solution can be written in the form of

$$u_1(x,t) = \phi(x) + w_1(x)\frac{t^\alpha}{\Gamma(\alpha+1)}. \tag{15}$$

The 1st residual function can be written as follows:

$$Res_1(x,t) = \frac{\partial^\alpha u_1(x,t)}{\partial t^\alpha} + u_1(x,t)\frac{\partial^\beta u_1(x,t)}{\partial x^\beta} + \frac{\partial^{3\beta} u_1(x,t)}{\partial x^{3\beta}} + \eta\left(\frac{\partial^{2\beta} u_1(x,t)}{\partial x^{2\beta}} + \frac{\partial^{4\beta} u_1(x,t)}{\partial x^{4\beta}}\right) + \mu\frac{\partial^{5\beta} u_1(x,t)}{\partial x^{5\beta}} + \frac{\partial^\beta u_1(x,t)}{\partial x^\beta}. \tag{16}$$

Substitute the 1st truncated series, $u_1(x,t)$, of Eqs. (5) and (6) into the 1-st residual function as follows:

$$Res_1(x,t) = w_1(x) + \left(\phi(x) + w_1(x)\frac{t^\alpha}{\Gamma(\alpha+1)}\right)\left(D_x^\beta \phi(x) + \frac{t^\alpha}{\Gamma(\alpha+1)} D_x^\beta w_1(x)\right)$$

$$+D_x^{3\beta}\phi(x)+\frac{t^{\alpha}}{\Gamma(\alpha+1)}D_x^{3\beta}w_1(x)+D_x^{\beta}\phi(x)+\frac{t^{\alpha}}{\Gamma(\alpha+1)}D_x^{\beta}w_1(x)$$
$$+\eta\left(D_x^{2\beta}\phi(x)+\frac{t^{\alpha}}{\Gamma(\alpha+1)}D_x^{2\beta}w_1(x)+D_x^{4\beta}\phi(x)+\frac{t^{\alpha}}{\Gamma(\alpha+1)}D_x^{4\beta}w_1(x)\right)$$
$$+\mu\left(D_x^{5\beta}\phi(x)+\frac{t^{\alpha}}{\Gamma(\alpha+1)}D_x^{5\beta}w_1(x)\right). \tag{17}$$

From Eqs. (14) and (17),

$$w_1(x)=-\Big(\phi(x)D_x^{\beta}\phi(x)+D_x^{3\beta}\phi(x)+\eta\big(D_x^{2\beta}\phi(x)+D_x^{4\beta}\phi(x)\big)$$
$$+\mu D_x^{5\beta}\phi(x)+D_x^{\beta}\phi(x)\Big). \tag{18}$$

The 1st RPS approximate solution can be written in the following form:

$$u_1(x,t)=\phi(x)-\frac{t^{\alpha}}{\Gamma(\alpha+1)}\Big(\phi(x)D_x^{\beta}\phi(x)+D_x^{3\beta}\phi(x)+\eta\big(D_x^{2\beta}\phi(x)+D_x^{4\beta}\phi(x)\big)$$
$$+\mu D_x^{5\beta}\phi(x)+D_x^{\beta}\phi(x)\Big). \tag{19}$$

For $\boldsymbol{m=2}$, the 2nd residual power series solution can be written as follows:

$$u_2(x,t)=\phi(x)+w_1(x)\frac{t^{\alpha}}{\Gamma(\alpha+1)}+w_2(x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}. \tag{20}$$

Substitute the 2nd truncated series $u_2(x,t)$ into the 2-nd residual function $Res_2(x,t)$. That is,

$$Res_2(x,t)=w_1(x)+w_2(x)\frac{t^{\alpha}}{\Gamma(\alpha+1)}$$
$$+\left(\phi(x)+w_1(x)\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}w_2(x)\right)\left(D_x^{\beta}\phi(x)\right.$$
$$\left.+\frac{t^{\alpha}}{\Gamma(\alpha+1)}D_x^{\beta}w_1(x)+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}D_x^{\beta}w_2(x)\right)+D_x^{3\beta}\phi(x)$$
$$+\frac{t^{\alpha}}{\Gamma(\alpha+1)}D_x^{3\beta}w_1(x)+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}D_x^{3\beta}w_2(x)$$
$$+D_x^{\beta}\phi(x)+\frac{t^{\alpha}}{\Gamma(\alpha+1)}D_x^{\beta}w_1(x)+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}D_x^{\beta}w_2(x)$$
$$+\eta\left(D_x^{2\beta}\phi(x)+\frac{t^{\alpha}}{\Gamma(\alpha+1)}D_x^{2\beta}w_1(x)+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}D_x^{2\beta}w_2(x)\right.$$

$$+ D_x^{4\beta}\phi(x) + \frac{t^\alpha}{\Gamma(\alpha+1)} D_x^{4\beta} w_1(x) + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} D_x^{4\beta} w_2(x)\Big)$$
$$+ \mu\Big(D_x^{5\beta}\phi(x) + \frac{t^\alpha}{\Gamma(\alpha+1)} D_x^{5\beta} w_1(x) + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} D_x^{5\beta} w_2(x)\Big). \tag{21}$$

The operator D_t^α is applied on both sides of Eq. (21) as follows:

$$\begin{aligned} D_t^\alpha Res_2(x,t) = {} & w_2(x) + \phi(x) D_x^\beta w_1(x) + w_1(x) D_x^\beta \phi(x) + D_x^{3\beta} w_1(x) \\ & + \mu D_x^{5\beta} w_1(x) + \eta\Big(D_x^{2\beta} w_1(x) + D_x^{5\beta} w_1(x)\Big) + D_x^\beta w_1(x) \\ & + \frac{t^\alpha}{\Gamma(\alpha+1)}\Big(\phi(x) D_x^\beta w_2(x) + \frac{\Gamma(2\alpha+1)}{[\Gamma(\alpha+1)]^2} w_1(x) D_x^\beta w_1(x) \\ & + w_2(x) D_x^\beta \phi(x) + D_x^{3\beta} w_2(x) + D_x^\beta w_2(x) \\ & + \eta\Big(D_x^{2\beta} w_2(x) + D_x^{4\beta} w_2(x)\Big) + \mu D_x^{5\beta} w_2(x)\Big) \\ & + \frac{\Gamma(3\alpha+1)}{\Gamma(\alpha+1)[\Gamma(2\alpha+1)]^2}\Big(w_1(x) D_x^\beta w_2(x) + w_2(x) D_x^\beta w_1(x)\Big) t^{2\alpha} \\ & + \frac{\Gamma(4\alpha+1)}{\Gamma(3\alpha+1)[\Gamma(2\alpha+1)]^2}\Big(w_2(x) D_x^\beta w_2(x)\Big) t^{3\alpha}. \end{aligned} \tag{22}$$

From Eqs. (14) and (22),

$$\begin{aligned} w_2(x) = -\Big(& \phi(x) D_x^\beta w_1(x) + w_1(x) D_x^\beta \phi(x) + D_x^{3\beta} w_1(x) + \mu D_x^{5\beta} w_1(x) \\ & + \eta\Big(D_x^{2\beta} w_1(x) + D_x^{5\beta} w_1(x)\Big) + D_x^\beta w_1(x)\Big). \end{aligned} \tag{23}$$

The 2nd residual power series approximate solution can be written in the following form:

$$\begin{aligned} u_2(x,t) = {} & \phi(x) - \frac{t^\alpha}{\Gamma(\alpha+1)}\Big(\phi(x) D_x^\beta \phi(x) + D_x^{3\beta}\phi(x) + \eta\big(D_x^{2\beta}\phi(x) \\ & + D_x^{4\beta}\phi(x)\big) + \mu D_x^{5\beta}\phi(x) + D_x^\beta \phi(x)\Big) - \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\Big(\phi(x) D_x^\beta w_1(x) \\ & + w_1(x) D_x^\beta \phi(x) + D_x^{3\beta} w_1(x) + \mu D_x^{5\beta} w_1(x) \\ & + \eta\Big(D_x^{2\beta} w_1(x) + D_x^{5\beta} w_1(x)\Big) + D_x^\beta w_1(x)\Big). \end{aligned} \tag{24}$$

For $\boldsymbol{m = 3}$, substitute the 3rd truncated series,

$$u_3(x,t) = \phi(x) + w_1(x)\frac{t^\alpha}{\Gamma(\alpha+1)} + w_2(x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + w_3(x)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)},$$

of Eqs. (5) and (6) into the 3rd residual function, $Res_3(x,t)$, of Eq. (12) as

$$
\begin{aligned}
Res_3(x,t) = {} & w_1(x) + w_2(x)\frac{t^{\alpha}}{\Gamma(\alpha+1)} + w_3(x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\
& + \left(\phi(x) + w_1(x)\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}w_2(x)\right. \\
& \left. + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}w_3(x)\right)\left(D_x^{\beta}\phi(x) + \frac{t^{\alpha}}{\Gamma(\alpha+1)}D_x^{\beta}w_1(x)\right. \\
& \left. + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}D_x^{\beta}w_2(x) + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}D_x^{\beta}w_3(x)\right) \\
& + D_x^{3\beta}\phi(x) + \frac{t^{\alpha}}{\Gamma(\alpha+1)}D_x^{3\beta}w_1(x) \\
& + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}D_x^{3\beta}w_3(x) + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}D_x^{3\beta}w_2(x) \\
& + D_x^{\beta}\phi(x) + \frac{t^{\alpha}}{\Gamma(\alpha+1)}D_x^{\beta}w_1(x) \\
& + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}D_x^{\beta}w_2(x) + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}D_x^{\beta}w_3(x) \\
& + \eta\left(D_x^{2\beta}\phi(x) + \frac{t^{\alpha}}{\Gamma(\alpha+1)}D_x^{2\beta}w_1(x)\right. \\
& + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}D_x^{2\beta}w_2(x) + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}D_x^{2\beta}w_3(x) \\
& + D_x^{4\beta}\phi(x) + \frac{t^{\alpha}}{\Gamma(\alpha+1)}D_x^{4\beta}w_1(x) \\
& \left. + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}D_x^{4\beta}w_2(x) + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}D_x^{4\beta}w_3(x)\right) \\
& + \mu\left(D_x^{5\beta}\phi(x) + \frac{t^{\alpha}}{\Gamma(\alpha+1)}D_x^{5\beta}w_1(x)\right. \\
& \left. + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}D_x^{5\beta}w_2(x) + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}D_x^{5\beta}w_3(x)\right).
\end{aligned} \tag{25}
$$

Now, solving the equation $D_t^{m-1}Res_m(x,0) = 0$, for $m = 3$ gives the required value of $w_3(x)$ as follows:

$$
\begin{aligned}
w_3(x) = {} & -\left(\phi(x)D_x^{\beta}w_2(x) + w_2(x)D_x^{\beta}\phi(x) + D_x^{3\beta}w_2(x)\right. \\
& + \mu D_x^{5\beta}w_2(x) + \eta\left(D_x^{2\beta}w_2(x) + D_x^{5\beta}w_2(x)\right) + D_x^{\beta}w_2(x) \\
& \left. + \frac{\Gamma(2\alpha+1)}{[\Gamma(\alpha+1)]^2}w_1(x)D_x^{\beta}w_1(x)\right).
\end{aligned} \tag{26}
$$

Based on the previous results for $w_0(x)$, $w_1(x)$ and $w_2(x)$, the 3rd residual power series approximate solution of Eqs. (5) and (6) may be expressed in the form of

$$\begin{aligned}
u_3(x,t) = \phi(x) - \frac{t^{\alpha}}{\Gamma(\alpha+1)}\Big(\phi(x)D_x^{\beta}\phi(x) + D_x^{3\beta}\phi(x) \\
+\eta\big(D_x^{2\beta}\phi(x) + D_x^{4\beta}\phi(x)\big) + \mu D_x^{5\beta}\phi(x) + D_x^{\beta}\phi(x)\Big) \\
-\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\Bigg(\phi(x)D_x^{\beta}w_1(x) + w_1(x)D_x^{\beta}\phi(x) + D_x^{3\beta}w_1(x) \\
+\mu D_x^{5\beta}w_1(x) + \eta\Big(D_x^{2\beta}w_1(x) + D_x^{5\beta}w_1(x)\Big) + D_x^{\beta}w_1(x)\Bigg) \\
-\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\Bigg(\phi(x)D_x^{\beta}w_2(x) + w_2(x)D_x^{\beta}\phi(x) + D_x^{3\beta}w_2(x) \\
+\mu D_x^{5\beta}w_2(x) + \eta\Big(D_x^{2\beta}w_2(x) + D_x^{5\beta}w_2(x)\Big) + D_x^{\beta}w_2(x) \\
+\frac{\Gamma(2\alpha+1)}{[\Gamma(\alpha+1)]^2}w_1(x)D_x^{\beta}w_1(x)\Bigg). \qquad (27)
\end{aligned}$$

For $\boldsymbol{m=4}$, substitute the 4th truncated series, $u_4(x,t) = \phi(x) + w_1(x)\frac{t^{\alpha}}{\Gamma(\alpha+1)} + w_2(x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + w_3(x)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + w_4(x)\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}$, of Eqs. (5) and (6) into the 4th residual function, $Res_4(x,t)$, of Eq. (12) as follows:

$$\begin{aligned}
Res_4(x,t) = \Bigg(\phi(x) + w_1(x)\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}w_2(x) \\
+\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}w_3(x) + \frac{t^{4\alpha}}{\Gamma(4\alpha+1)}w_4(x)\Bigg) \\
\times\Bigg(D_x^{\beta}\phi(x) + \frac{t^{\alpha}}{\Gamma(\alpha+1)}D_x^{\beta}w_1(x) + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}D_x^{\beta}w_2(x) \\
+\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}D_x^{\beta}w_3(x) + \frac{t^{4\alpha}}{\Gamma(4\alpha+1)}D_x^{\beta}w_4(x)\Bigg) + D_x^{3\beta}\phi(x) \\
+\frac{t^{\alpha}}{\Gamma(\alpha+1)}D_x^{3\beta}w_1(x) + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}D_x^{3\beta}w_3(x) + \frac{t^{4\alpha}}{\Gamma(4\alpha+1)}D_x^{3\beta}w_4(x) \\
+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}D_x^{3\beta}w_2(x) + D_x^{\beta}\phi(x) + \frac{t^{\alpha}}{\Gamma(\alpha+1)}D_x^{\beta}w_1(x) \\
+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}D_x^{\beta}w_2(x) + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}D_x^{\beta}w_3(x) + \frac{t^{4\alpha}}{\Gamma(4\alpha+1)}D_x^{\beta}w_4(x) \\
+\eta\Bigg(D_x^{2\beta}\phi(x) + \frac{t^{\alpha}}{\Gamma(\alpha+1)}D_x^{2\beta}w_1(x) + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}D_x^{2\beta}w_2(x) \\
+\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}D_x^{2\beta}w_3(x) + \frac{t^{4\alpha}}{\Gamma(4\alpha+1)}D_x^{2\beta}w_4(x) + D_x^{4\beta}\phi(x) \\
+\frac{t^{\alpha}}{\Gamma(\alpha+1)}D_x^{4\beta}w_1(x) + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}D_x^{4\beta}w_2(x)
\end{aligned}$$

$$+\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}D_x^{4\beta}w_3(x)+\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}D_x^{4\beta}w_4(x)\Big)$$
$$+\mu\Big(D_x^{5\beta}\phi(x)+\frac{t^{\alpha}}{\Gamma(\alpha+1)}D_x^{5\beta}w_1(x)+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}D_x^{5\beta}w_2(x)$$
$$+\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}D_x^{5\beta}w_3(x)+\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}D_x^{5\beta}w_4(x)\Big)$$
$$+w_1(x)+w_2(x)\frac{t^{\alpha}}{\Gamma(\alpha+1)}+w_3(x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+w_4(x)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}. \tag{28}$$

The operator $D_t^{3\alpha}$ is applied on both sides of Eq. (28) as follows:

$$\begin{aligned}
D_t^{3\alpha}Res_4(x,t) &= w_4(x)+\phi(x)D_x^{\beta}w_3(x)+\left[\frac{\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)\Gamma(\alpha+1)}\right]\big(w_1(x)D_x^{\beta}w_2(x)\\
&\quad+w_2(x)D_x^{\beta}w_1(x)\big)+w_3(x)D_x^{\beta}\phi(x)+D_x^{3\beta}w_3(x)+D_x^{\beta}w_3(x)\\
&\quad+\eta(D_x^{2\beta}w_3(x)+D_x^{4\beta}w_3(x))+\mu D_x^{5\beta}w_3(x)\\
&\quad+\frac{t^{\alpha}}{\Gamma(\alpha+1)}\Big(\phi(x)D_x^{\beta}w_4(x)+\frac{\Gamma(4\alpha+1)}{\Gamma(\alpha+1)\Gamma(3\alpha+1)}w_1(x)D_x^{\beta}w_3(x)\\
&\quad+\frac{\Gamma(4\alpha+1)}{[\Gamma(2\alpha+1)]^2}w_2(x)D_x^{\beta}w_2(x)\\
&\quad+\frac{\Gamma(4\alpha+1)}{\Gamma(\alpha+1)\Gamma(3\alpha+1)}w_3(x)D_x^{\beta}w_1(x)+w_4(x)D_x^{\beta}\phi(x)+D_x^{3\beta}w_4(x)\\
&\quad+D_x^{\beta}w_4(x)+\eta(D_x^{2\beta}w_4(x)+D_x^{4\beta}w_4(x))+\mu D_x^{5\beta}w_4(x)\Big)\\
&\quad+\frac{t^{2\alpha}\Gamma(5\alpha+1)}{\Gamma(2\alpha+1)}\Big(\frac{w_1(x)D_x^{\beta}w_4(x)}{\Gamma(\alpha+1)\Gamma(4\alpha+1)}+\frac{w_2(x)D_x^{\beta}w_3(x)}{\Gamma(2\alpha+1)\Gamma(3\alpha+1)}\\
&\quad+\frac{w_3(x)D_x^{\beta}w_2(x)}{\Gamma(2\alpha+1)\Gamma(3\alpha+1)}+\frac{w_4(x)D_x^{\beta}w_1(x)}{\Gamma(4\alpha+1)\Gamma(\alpha+1)}\Big)\\
&\quad+\frac{t^{3\alpha}\Gamma(6\alpha+1)}{\Gamma(3\alpha+1)}\Big(\frac{w_2(x)D_x^{\beta}w_4(x))}{\Gamma(2\alpha+1)\Gamma(4\alpha+1)}+\frac{w_3(x)D_x^{\beta}w_3(x))}{\Gamma(3\alpha+1)}\\
&\quad+\frac{w_4(x)D_x^{\beta}w_2(x)}{\Gamma(4\alpha+1)\Gamma(2\alpha+1)}\Big)+\frac{t^{4\alpha}\Gamma(7\alpha+1)}{\Gamma(4\alpha+1)}\Big(\frac{w_3(x)D_x^{\beta}w_4(x))}{\Gamma(3\alpha+1)\Gamma(4\alpha+1)}\\
&\quad+\frac{w_4(x)D_x^{\beta}w_3(x)}{\Gamma(4\alpha+1)\Gamma(3\alpha+1)}\Big)+\frac{t^{5\alpha}\Gamma(8\alpha+1)}{\Gamma(5\alpha+1)}\Big(\frac{w_4(x)D_x^{\beta}w_4(x))}{[\Gamma(4\alpha+1)]^2}\Big).
\end{aligned}\tag{29}$$

From Eqs. (14) and (29),

$$\begin{aligned}
w_4(x) &= -\Big(\phi(x)D_x^{\beta}w_3(x)+\left[\frac{\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)\Gamma(\alpha+1)}\right]\big(w_1(x)D_x^{\beta}w_2(x)\\
&\quad+w_2(x)D_x^{\beta}w_1(x)\big)+w_3(x)D_x^{\beta}\phi(x)+D_x^{3\beta}w_3(x)+D_x^{\beta}w_3(x)\\
&\quad+\eta(D_x^{2\beta}w_3(x)+D_x^{4\beta}w_3(x))+\mu D_x^{5\beta}w_3(x)\Big).
\end{aligned}\tag{30}$$

Based on the previous results for $w_0(x)$, $w_1(x)$, $w_2(x)$ and $w_3(x)$, the 4th residual power series approximate solution of Eq. (5) and (6) may be expressed in the form of

$$\begin{aligned}
u_4(x,t) &= \phi(x) - \frac{t^{\alpha}}{\Gamma(\alpha+1)}\Big(\phi(x)D_x^{\beta}\phi(x) + D_x^{3\beta}\phi(x) + \eta\big(D_x^{2\beta}\phi(x) + D_x^{4\beta}\phi(x)\big) \\
&\quad + \mu D_x^{5\beta}\phi(x) + D_x^{\beta}\phi(x)\Big) - \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\Big(\phi(x)D_x^{\beta}w_1(x) + w_1(x)D_x^{\beta}\phi(x) \\
&\quad + D_x^{3\beta}w_1(x) + \mu D_x^{5\beta}w_1(x) + \eta\Big(D_x^{2\beta}w_1(x) + D_x^{5\beta}w_1(x)\Big) + D_x^{\beta}w_1(x)\Big) \\
&\quad - \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\Big(\phi(x)D_x^{\beta}w_2(x) + w_2(x)D_x^{\beta}\phi(x) + D_x^{3\beta}w_2(x) + \mu D_x^{5\beta}w_2(x) \\
&\quad + \eta\Big(D_x^{2\beta}w_2(x) + D_x^{5\beta}w_2(x)\Big) + D_x^{\beta}w_2(x) + \frac{\Gamma(2\alpha+1)}{[\Gamma(\alpha+1)]^2}w_1(x)D_x^{\beta}w_1(x)\Big) \\
&\quad - \frac{t^{4\alpha}}{\Gamma(4\alpha+1)}\Big(\phi(x)D_x^{\beta}w_3(x) + \left[\frac{\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)\Gamma(\alpha+1)}\right]\big(w_1(x)D_x^{\beta}w_2(x) \\
&\quad + w_2(x)D_x^{\beta}w_1(x)\big) + w_3(x)D_x^{\beta}\phi(x) + D_x^{3\beta}w_3(x) + D_x^{\beta}w_3(x) \\
&\quad + \eta(D_x^{2\beta}w_3(x) + D_x^{4\beta}w_3(x)) + \mu D_x^{5\beta}w_3(x)\Big).
\end{aligned} \tag{31}$$

4 Convergence analysis

Lemma 4.1 [1] *If $f(x)$ is continuous function and $\alpha,\ \beta > 0$, then the following result hold:*

$$I_a^{\alpha}I_a^{\beta}f(x) = I_a^{\beta}I_a^{\alpha}f(x) = I_a^{\alpha+\beta}f(x). \tag{32}$$

Theorem 4.1 [18] *For any fractional power series (FPS) of the form $\sum_{k=0}^{\infty}c_k x^{k\alpha}$, $x \geq 0$,*

(a) *If FPS converges for $x = x_1$, then FPS converges absolutely for all real x satisfying $|x| < |x_1|$,*
(b) *If FPS diverges for $x = x_1$, then FPS diverges for all real x satisfying $|x| > |x_1|$.*

Theorem 4.2 *For $0 \leq m-1 < \alpha \leq m$, suppose that $D_t^{r+k\alpha}$, $D_t^{r+\alpha(k+1)} \in C[R, t_0] \times [R, t_0+R])$, then*

$$\begin{aligned}
&\left(I_t^{r+k\alpha}D_t^{r+k\alpha}u\right)(x,t) - \left(I_t^{r+(k+1)\alpha}D_t^{r+(k+1)\alpha}u\right)(x,t) \\
&= \frac{(t-t_0)^{r+k\alpha}}{\Gamma(r+k\alpha+1)}D_t^{r+k\alpha}u(x,t_0),
\end{aligned}$$

where $D_t^{r+k\alpha} = (\underbrace{D_t\cdot D_t\cdot D_t\cdots D_t}_{r-times})(\underbrace{D_t^{\alpha}\cdot D_t^{\alpha}\cdot D_t^{\alpha}\cdots D_t^{\alpha}}_{k-times})$.

Proof From Eq. (32),

$$(I_t^{r+k\alpha}D_t^{r+k\alpha}u)(x,t) - (I_t^{r+(k+1)\alpha}D_t^{r+(k+1)\alpha}u)(x,t)$$

$$= I_t^{r+k\alpha}((D_t^{r+k\alpha}u)(x,t) - (I_t^{\alpha}D_t^{r+(k+1)\alpha}u)(x,t))$$
$$= I_t^{r+k\alpha}((D_t^{r+k\alpha}u)(x,t) - (I_t^{\alpha}D_t^{\alpha})(D_t^{r+k\alpha}u)(x,t))$$
$$= I_t^{r+k\alpha}((D_t^{r+k\alpha}u)(x,t_0)) \;\; (using\ Theorem\ 2.1.\ (b))$$
$$= \frac{(t-t_0)^{r+k\alpha}}{\Gamma(r+k\alpha+1)} D_t^{r+k\alpha}u(x,t_0). \tag{33}$$

Theorem 4.3 *Suppose that* $u(x,t) \in C[R,t_0] \times [R,t_0+R])$, $D_t^{k\alpha}u(x,t) \in C[R,t_0] \times [R,t_0+R])$ *where* $k = 0, 1, \ldots, N+1$ *and* $j = 0, 1, \ldots, m-1$. *Also,* $D_t^{k\alpha}u(x,t)$ *can be differentiated* $m-1$ *times w.r.t "*t'' *on* (t_0, t_0+R). *Then*

$$u(x,t) \cong \sum_{j=0}^{m-1}\sum_{i=0}^{N} W_{j+i\alpha}(x)(t-t_0)^{j+i\alpha}, \tag{34}$$

where $W_{j+i\alpha}(x) = \frac{D_t^{j+i\alpha}u(x,t_0)}{\Gamma(j+i\alpha+1)}$. *Moreover,* $\exists$ *a value* $\varepsilon,\ 0 \le \varepsilon \le t$, *the error term has the following form,*

$$\|R_N(x,t)\| = \sup_{t\in[0,T]} \left| \sum_{j=0}^{m-1} \left[\frac{D^{j+(N+1)\alpha}u(x,\varepsilon)}{\Gamma((N+1)\alpha+j+1)} t^{(j+(N+1)\alpha)} \right] \right|. \tag{35}$$

Proof From Eq. (33),

$$\sum_{j=0}^{m-1}\sum_{i=0}^{N}\left(\left(I_t^{j+i\alpha}D_t^{j+i\alpha}u\right)(x,t) - \left(I_t^{j+(i+1)\alpha}D_t^{j+(i+1)\alpha}u\right)(x,t)\right)$$
$$= \sum_{j=0}^{m-1}\sum_{i=0}^{N} \frac{(t-t_0)^{j+i\alpha}}{\Gamma(j+i\alpha+1)} D_t^{j+i\alpha}u(x,t_0)$$
$$= \sum_{j=0}^{m-1}\sum_{i=0}^{N} W_{j+i\alpha}(x)(t-t_0)^{j+i\alpha}. \tag{36}$$

That is,

$$u(x,t) - \sum_{j=0}^{m-1}\left[I_t^{j+(N+1)\alpha}D_t^{j+(N+1)\alpha}u)(x,t)\right] = \sum_{j=0}^{m-1}\sum_{i=0}^{N} W_{j+i\alpha}(x)(t-t_0)^{j+i\alpha}. \tag{37}$$

Consider the second term of the above equation as follows:

$$
\begin{aligned}
&\sum_{j=0}^{m-1}\left[I_t^{j+(N+1)\alpha}D_t^{j+(N+1)\alpha}u)(x,t)\right]\\
&\quad=\sum_{j=0}^{m-1}\left[\frac{1}{\Gamma((N+1)\alpha+j)}\int_0^t\frac{D^{j+(N+1)\alpha}u(x,\varepsilon)}{(t-\tau)^{1-(j+(N+1)\alpha)}}d\tau\right]\\
&\quad=\sum_{j=0}^{m-1}\left[\frac{D^{j+(N+1)\alpha}u(x,\varepsilon)}{\Gamma((N+1)\alpha+j)}\int_0^t\frac{d\tau}{(t-\tau)^{1-(j+(N+1)\alpha)}}\right]\\
&\quad=\sum_{j=0}^{m-1}\left[\frac{D^{j+(N+1)\alpha}u(x,\varepsilon)}{\Gamma((N+1)\alpha+j+1)}t^{(j+(N+1)\alpha)}\right].\\
&\qquad\text{(By applying the Integral Mean Value Theorem)} \qquad (38)
\end{aligned}
$$

From Eqs. (37) and (38),

$$
u(x,t)-\sum_{j=0}^{m-1}\sum_{i=0}^{N}W_{j+i\alpha}(x)(t-t_0)^{j+i\alpha}=\sum_{j=0}^{m-1}\left[\frac{D^{j+(N+1)\alpha}u(x,\varepsilon)}{\Gamma((N+1)\alpha+j+1)}t^{(j+(N+1)\alpha)}\right].
$$

The error term can be calculated as follows

$$
\begin{aligned}
\|R_N(x,t)\|&=\left\|u(x,t)-\sum_{j=0}^{m-1}\sum_{i=0}^{N}W_{j+i\alpha}(x)(t-t_0)^{j+i\alpha}\right\|\\
&=\left\|\sum_{j=0}^{m-1}\left[\frac{D^{j+(N+1)\alpha}u(x,\varepsilon)}{\Gamma((N+1)\alpha+j+1)}t^{(j+(N+1)\alpha)}\right]\right\|. \qquad (39)
\end{aligned}
$$

This implies

$$
\|R_N(x,t)\|=\sup_{t\in[0,T]}\left|\sum_{j=0}^{m-1}\left[\frac{D^{j+(N+1)\alpha}u(x,\varepsilon)}{\Gamma((N+1)\alpha+j+1)}t^{(j+(N+1)\alpha)}\right]\right|.
$$

As $N\to\infty$, $\|R_N(x,t)\|\to 0$. Hence $u(x,t)$ can be approximated as

$$
u(x,t)\cong\sum_{j=0}^{m-1}\sum_{i=0}^{N}W_{j+i\alpha}(x)(t-t_0)^{j+i\alpha},
$$

with error term appear in Eq. (35). □

Remark For fixed $m=1$, the Eq. (34) can be reduced to generalized Taylor's series formula [27]. Consequently the error term "R_n'' also satisfy the generalized Taylor's series.

5 Numerical results

To show the the potentiality and the superiority of RPSM to solve the nonlinear fractional Benney-Lin equation, three applications are considered. It should be noted that for the development of the solutions, all calculations have been carried out using the software Maple 16.

Application 4.1 Consider the following nonlinear time fractional Benney-Lin equation:

$$D_t^{\alpha} u + uu_x + u_{xxx} + \eta(u_{xx} + u_{xxxx}) + \mu u_{xxxxx} + u_x = 0, \;\; 0 < \alpha < 1, \; \eta > 0, \; \mu \in \mathbb{R}. \tag{40}$$

Using RPSM, the numerical values of the probability density function $u(x, t)$ are determined for various time fractional Brownian motions $\alpha = 0.25$, $\alpha = 0.5$, $\alpha = 0.75$ and also for the standard motion $\alpha = 1$ and their graphical results are shown in Figs. 1, 3, 5, 7, 9, 11, 13 and 15 for the three-dimensional study and Figs. 2, 4, 6, 8, 10, 12, 14 and 16 for the two-dimensional study, for eight different case studies, respectively.

$$\textbf{Case study 1: } \phi = \alpha - 2\kappa^2 tan(\kappa x) \tag{41}$$
$$\textbf{Case study 2: } \phi = \alpha - 2\kappa^2 tan^2(\kappa x) \tag{42}$$
$$\textbf{Case study 3: } \phi = \alpha - 2\kappa^2 tanh(\kappa x) \tag{43}$$
$$\textbf{Case study 4: } \phi = \alpha - 2\kappa^2 tanh^2(\kappa x) \tag{44}$$
$$\textbf{Case study 5: } \phi = \alpha - 2\kappa^2 sec(\kappa x) \tag{45}$$

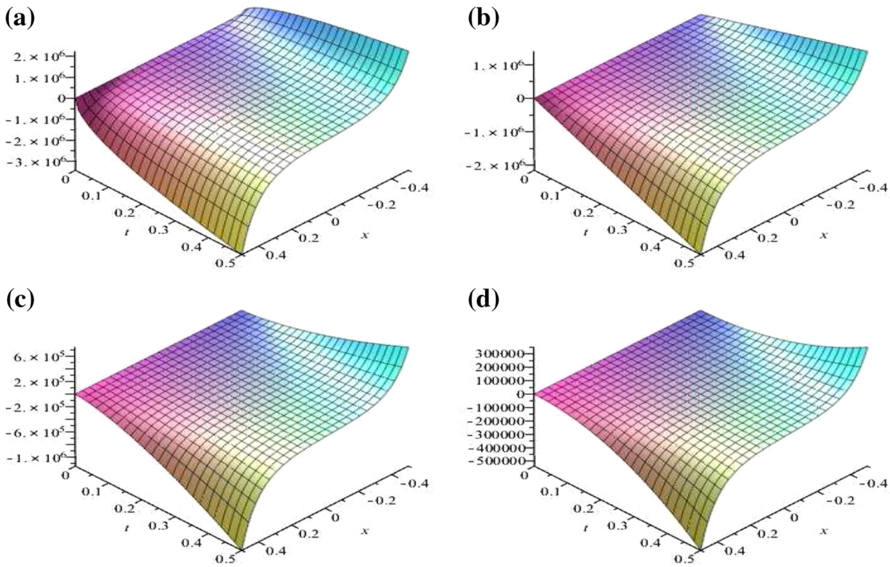

Fig. 1 Plot of the results by RPSM for the time-fractional Benney-Lin equation with the first initial condition Eq. (41) for **a** $\alpha = 0.25$, **b** $\alpha = 0.50$, **c** $\alpha = 0.75$ and **d** $\alpha = 1$ at $\eta = \mu = \kappa = 1$

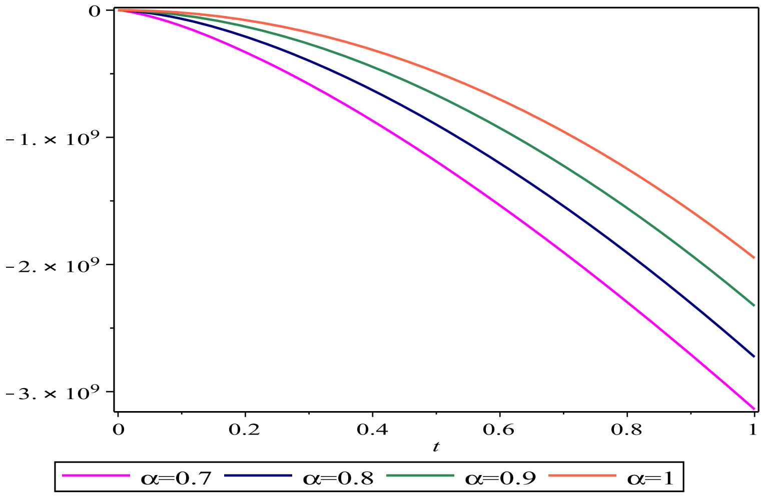

Fig. 2 Plot of approximate solutions versus time t for different value of α at $x = 1$ with the first initial condition Eq. (41)

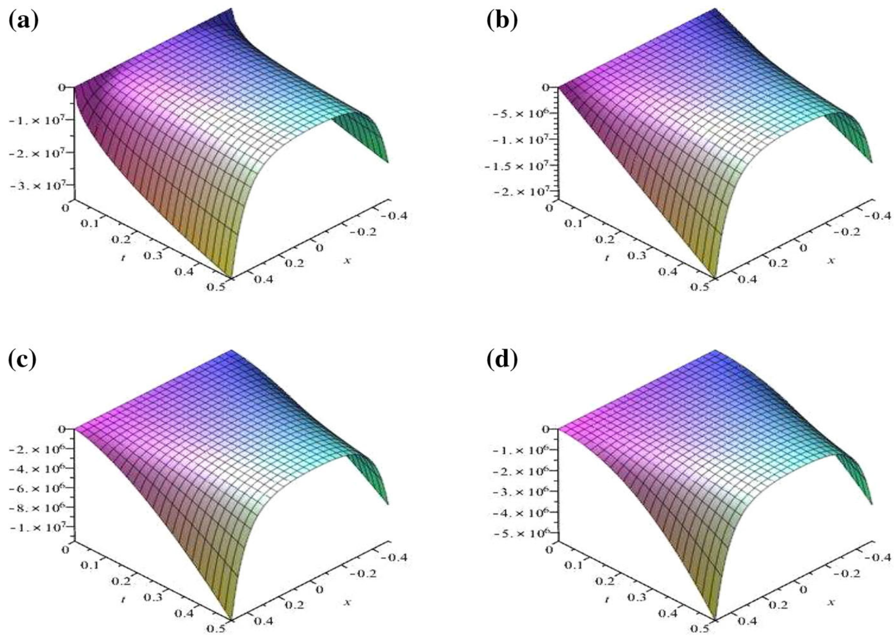

Fig. 3 Plot of the results by RPSM for the time-fractional Benney-Lin equation with the second initial condition Eq. (42) for **a** $\alpha = 0.25$, **b** $\alpha = 0.50$, **c** $\alpha = 0.75$ and **d** $\alpha = 1$ at $\eta = \mu = \kappa = 1$

$$\textbf{Case study 6: } \phi = \alpha - 2\kappa^2 sec^2(\kappa x) \tag{46}$$

$$\textbf{Case study 7: } \phi = \alpha - 2\kappa^2 sech(\kappa x) \tag{47}$$

$$\textbf{Case study 8: } \phi = \alpha - 2\kappa^2 sech^2(\kappa x) \tag{48}$$

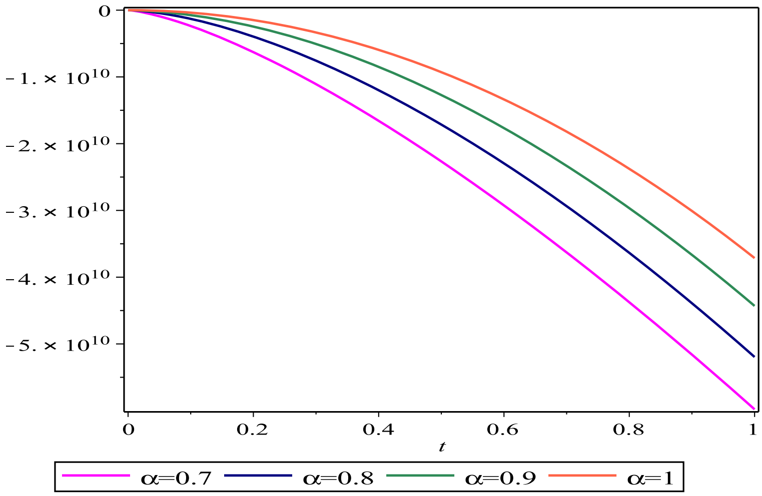

Fig. 4 Plot of approximate solutions versus time t for different value of α at $x = 1$ with the second initial condition Eq. (42)

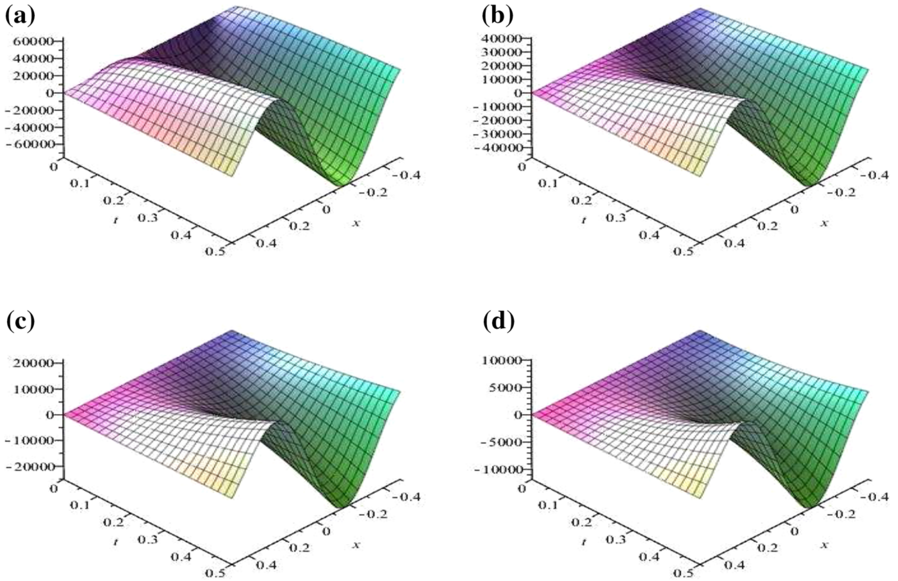

Fig. 5 Plot of the results by RPSM for the time-fractional Benney-Lin equation with the third initial condition Eq. (43) for **a** $\alpha = 0.25$, **b** $\alpha = 0.5$, **c** $\alpha = 0.75$ and **d** $\alpha = 1$ at $\eta = \mu = \kappa = 1$

The new iterative approach gives the solution in the form of a rapidly convergent series with computable components. It can be observed from Figs. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16 that each of the subfigures demonstrated the nearly similar and coinciding behavior of the RPS approximate solution and for the standard case $\alpha = 1$, nearly matching and excellent agreement of the subfigures (c) and (d). Figures 2, 4, 6,

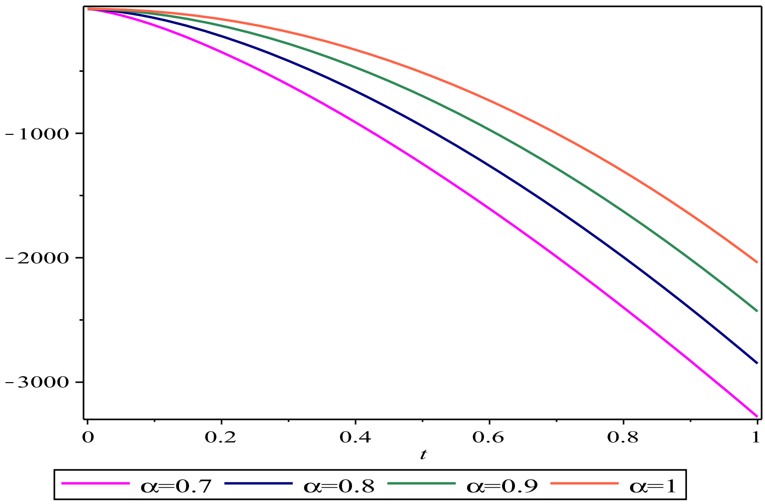

Fig. 6 Plot of approximate solutions versus time t for different value of α at $x = 1$ with the third initial condition Eq. (43)

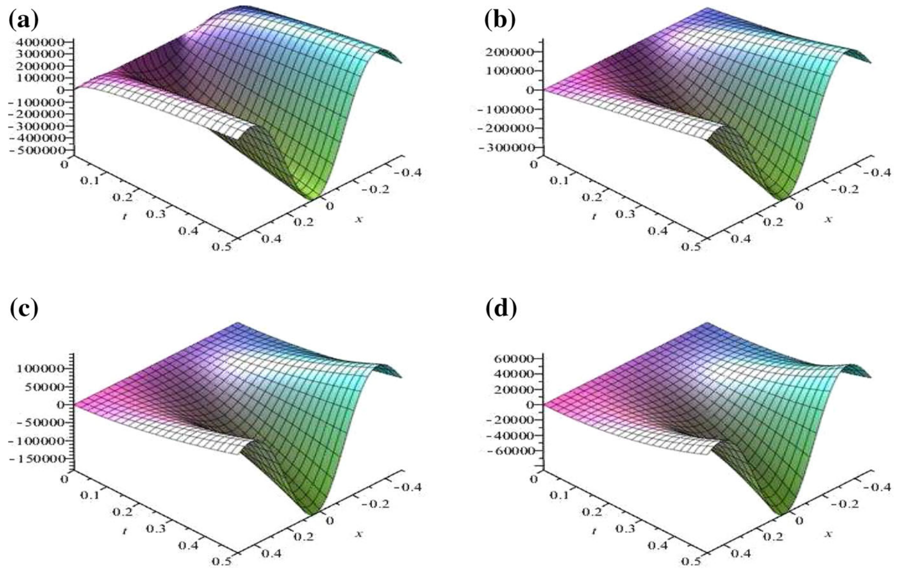

Fig. 7 Plot of the results by RPSM for the time-fractional Benney-Lin equation with the fourth initial condition Eq. (44) for **a** $\alpha = 0.25$, **b** $\alpha = 0.5$, **c** $\alpha = 0.75$ and **d** $\alpha = 1$ at $\eta = \mu = \kappa = 1$

8, 10, 12, 14 and 16 show the behavior of the approximate analytical solutions obtained by RPSM for different fractional Brownian motion $\alpha = 0.7$, $\alpha = 0.8$, $\alpha = 0.9$, and standard Benney-Lin's equation, i.e., $\alpha = 1$. Thus RPSM gives a good approximated solution using few terms only and now it is clear that the adding new terms of the residual power series approximations can make the overall error smaller.

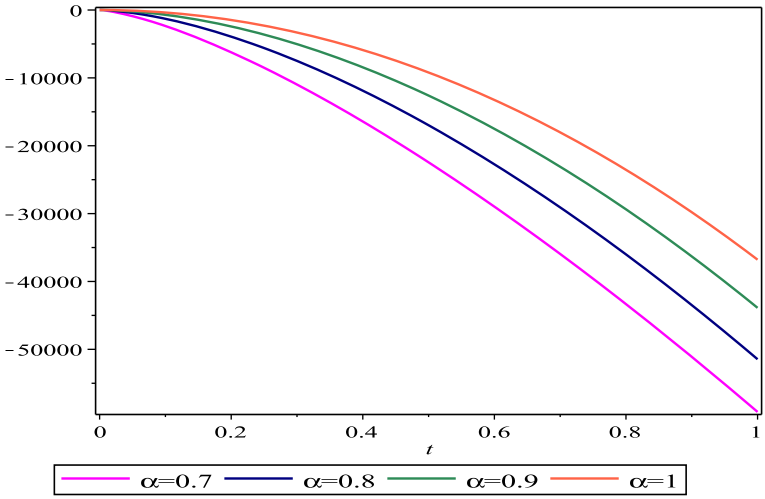

Fig. 8 Plot of approximate solutions versus time t for different value of α at $x = 1$ with the fourth initial condition Eq. (44)

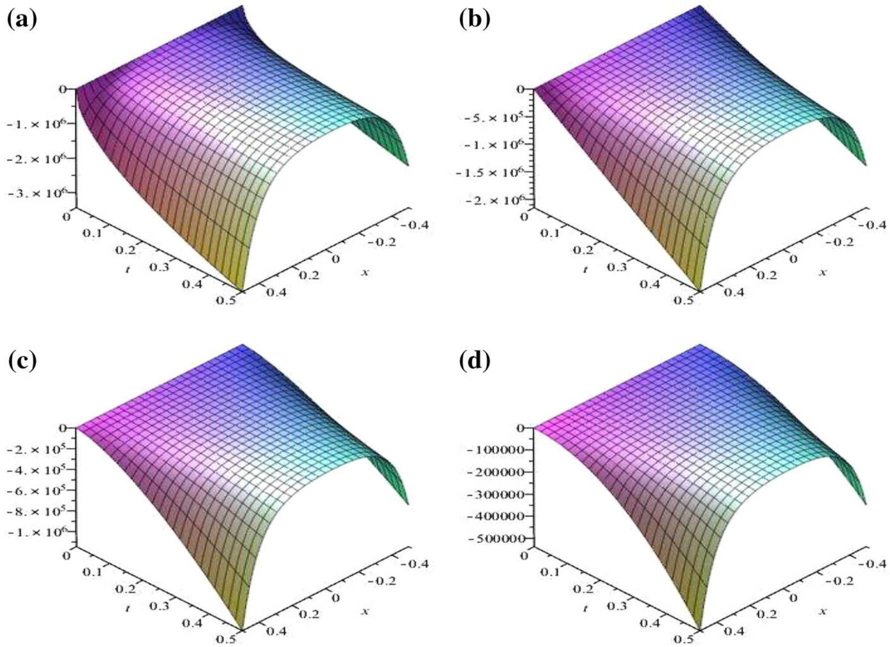

Fig. 9 Plot of the results by RPSM for the time-fractional Benney-Lin equation with the fifth initial condition Eq. (45) for **a** $\alpha = 0.25$, **b** $\alpha = 0.5$, **c** $\alpha = 0.75$ and **d** $\alpha = 1$ at $\eta = \mu = \kappa = 1$

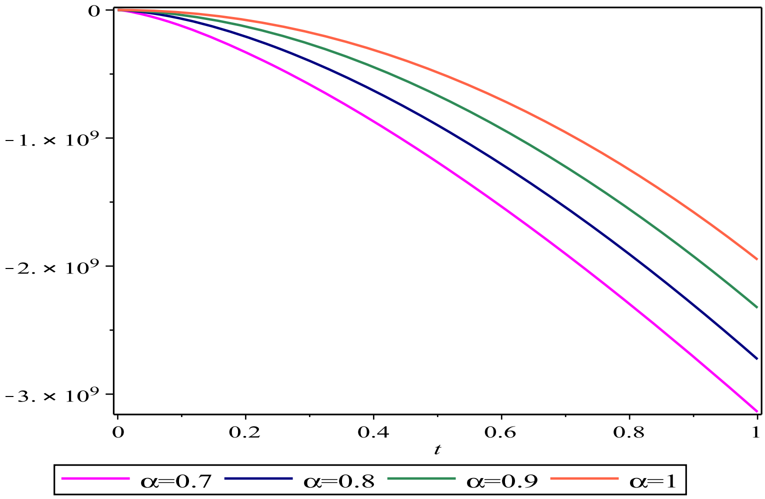

Fig. 10 Plot of approximate solutions versus time t for different value of α at $x = 1$ with the fifth initial condition Eq. (45)

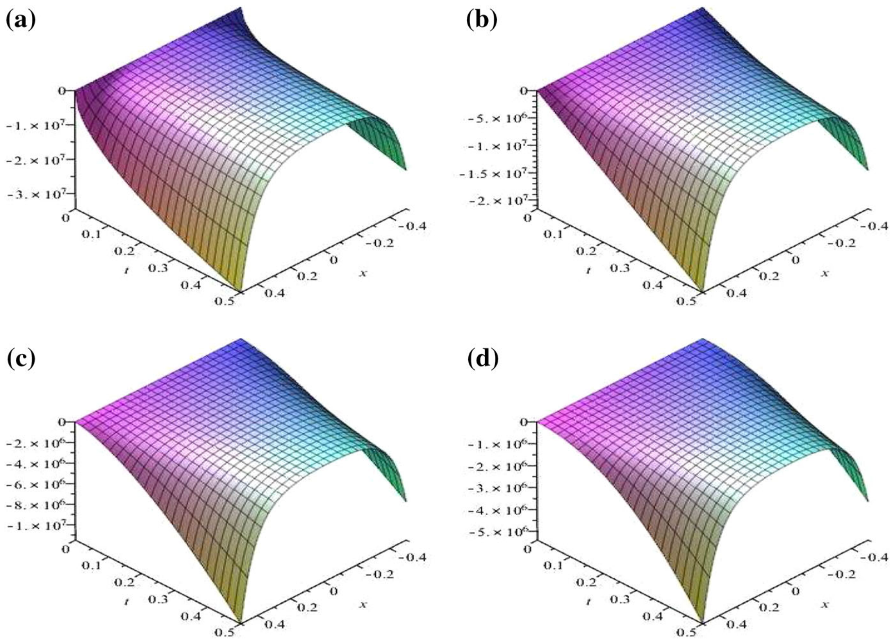

Fig. 11 Plot of the results by RPSM for the time-fractional Benney-Lin equation with the sixth initial condition Eq. (46) for **a** $\alpha = 0.25$, **b** $\alpha = 0.5$, **c** $\alpha = 0.75$ and **d** $\alpha = 1$ at $\eta = \mu = \kappa = 1$

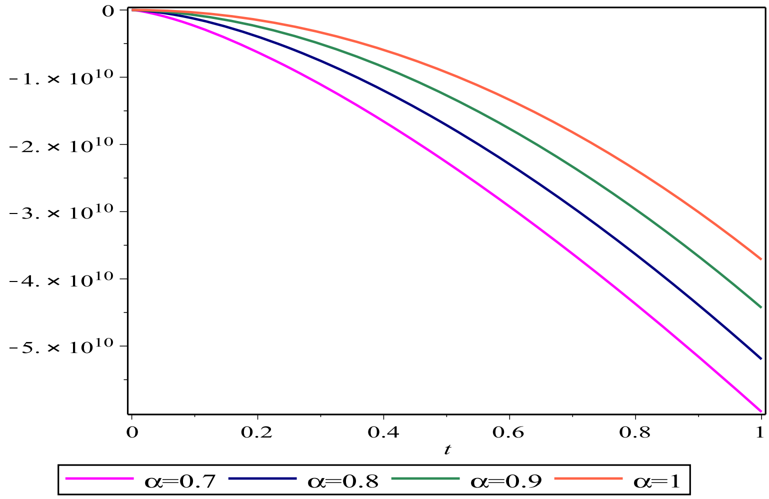

Fig. 12 Plot of approximate solutions versus time t for different value of α at $x = 1$ with the sixth initial condition Eq. (46)

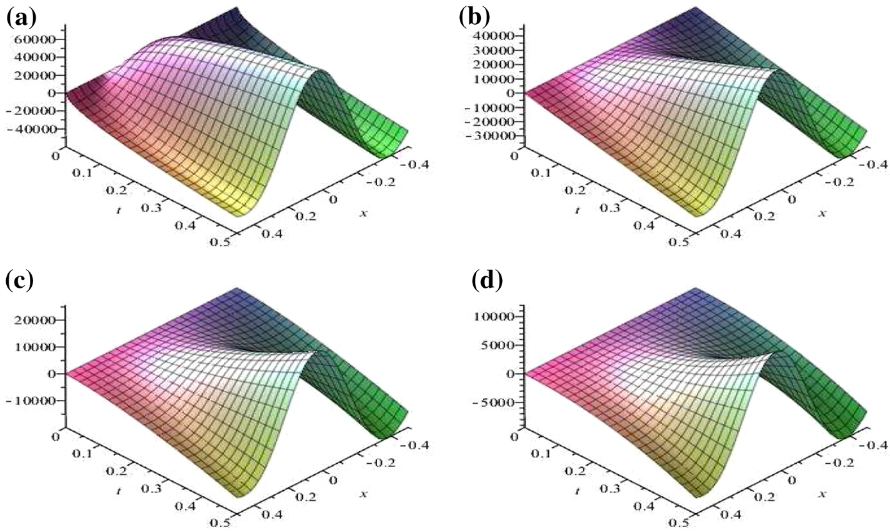

Fig. 13 Plot of the results by RPSM for the time-fractional Benney-Lin equation with the seventh initial condition Eq. (47) for **a** $\alpha = 0.25$, **b** $\alpha = 0.5$, **c** $\alpha = 0.75$ and **d** $\alpha = 1$ at $\eta = \mu = \kappa = 1$

Application 4.2 Consider the following nonlinear space fractional Benney-Lin equation:

$$u_t + uD_x^{\beta}u + D_x^{3\beta}u + \eta(D_x^{2\beta}u + D_x^{4\beta}u) + \mu D_x^{5\beta}u + D_x^{\beta}u = 0, \quad 0 < \beta < 1, \ \eta > 0, \ \mu \in \mathbb{R}, \tag{49}$$

subject to the initial condition

$$u(x, 0) = x^{12}. \tag{50}$$

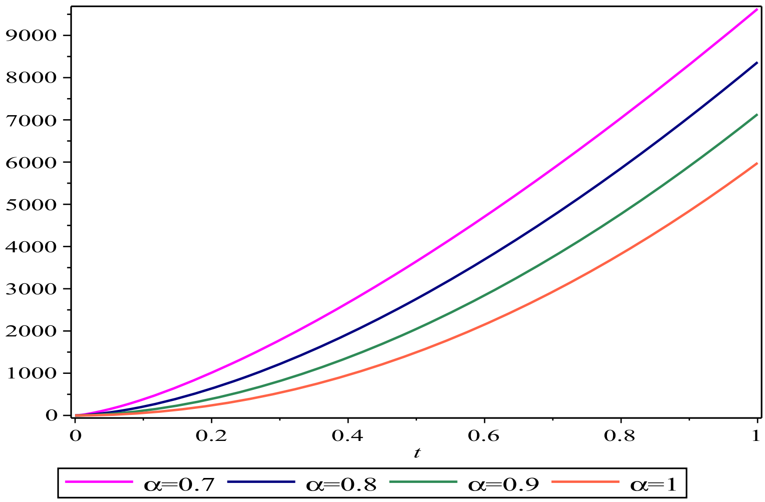

Fig. 14 Plot of approximate solutions versus time t for different value of α at $x = 1$ with the seventh initial condition Eq. (47)

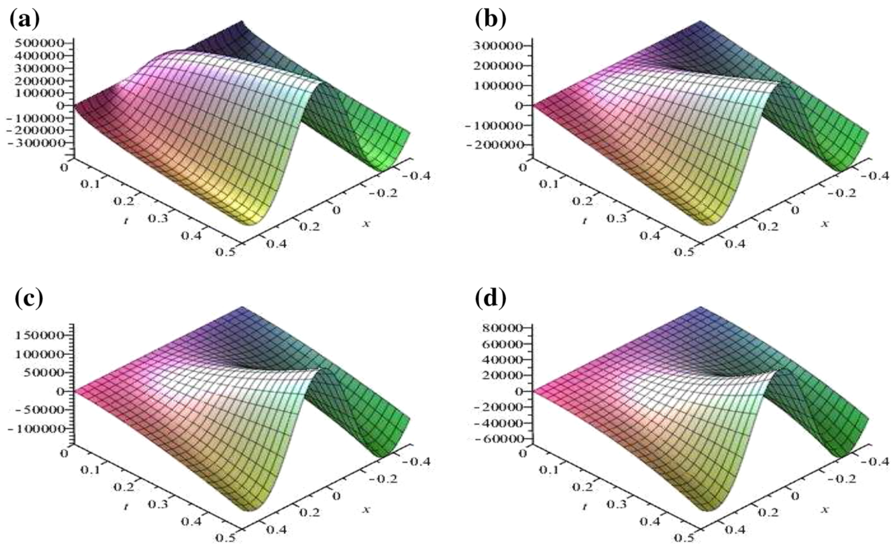

Fig. 15 Plot of the results by RPSM for the time-fractional Benney-Lin equation with the eighth initial condition Eq. (48) for **a** $\alpha = 0.25$, **b** $\alpha = 0.5$, **c** $\alpha = 0.75$ and **d** $\alpha = 1$ at $\eta = \mu = \kappa = 1$

By drawing the 3-dimensional space figure of the 3rd RPS approximate solution of Eq. (49), the geometric behavior of the approximate solution $u_3(x, t)$ with various values of β can be studied. It can be noted from the subfigures of Fig. 17 that the surface graph solutions decreases gradually when the values of x and t increase gradually on the specific domain. Also each surface nearly agrees well in its behavior. Also the

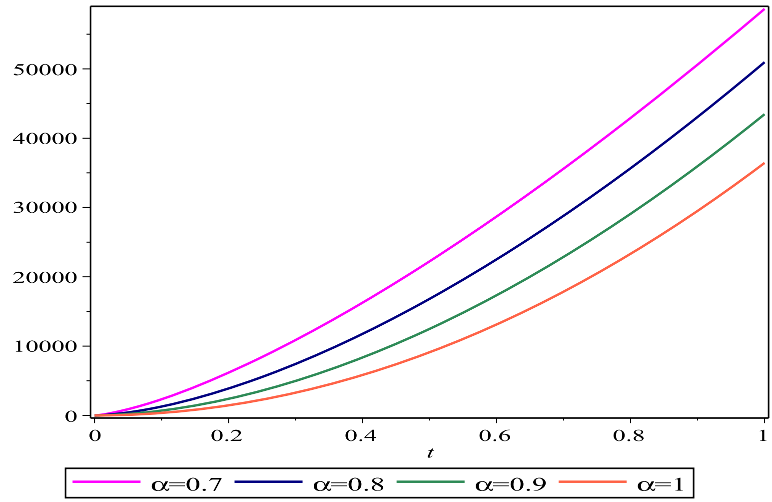

Fig. 16 Plot of approximate solutions versus time t for different value of α at $x = 1$ with the eighth initial condition Eq. (48)

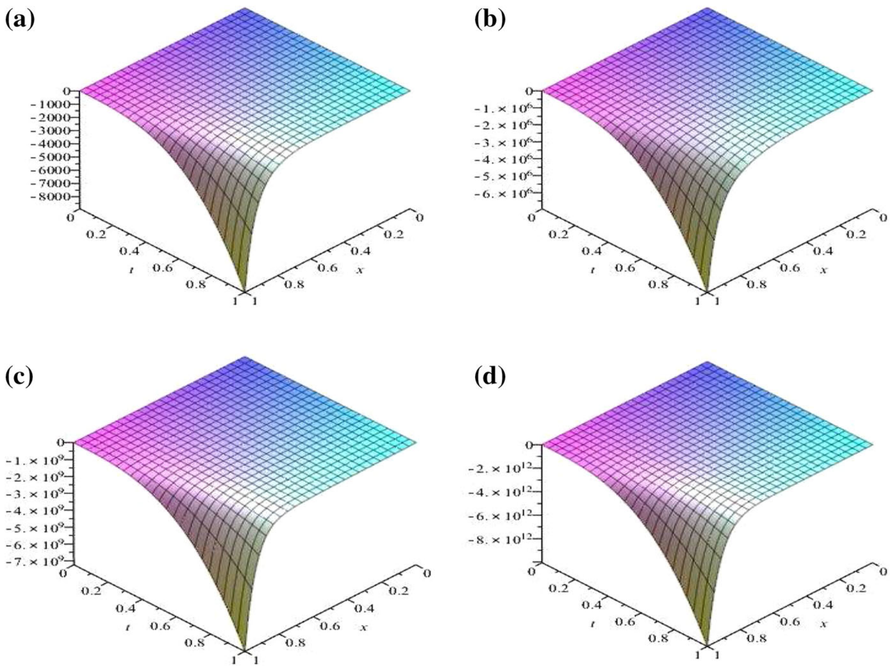

Fig. 17 Plot of the results by RPSM for the space-fractional Benney-Lin equation for **a** $\beta = 0.25$, **b** $\beta = 0.5$, **c** $\beta = 0.75$ and **d** $\beta = 1$ at $\eta = \mu = \kappa = 1$

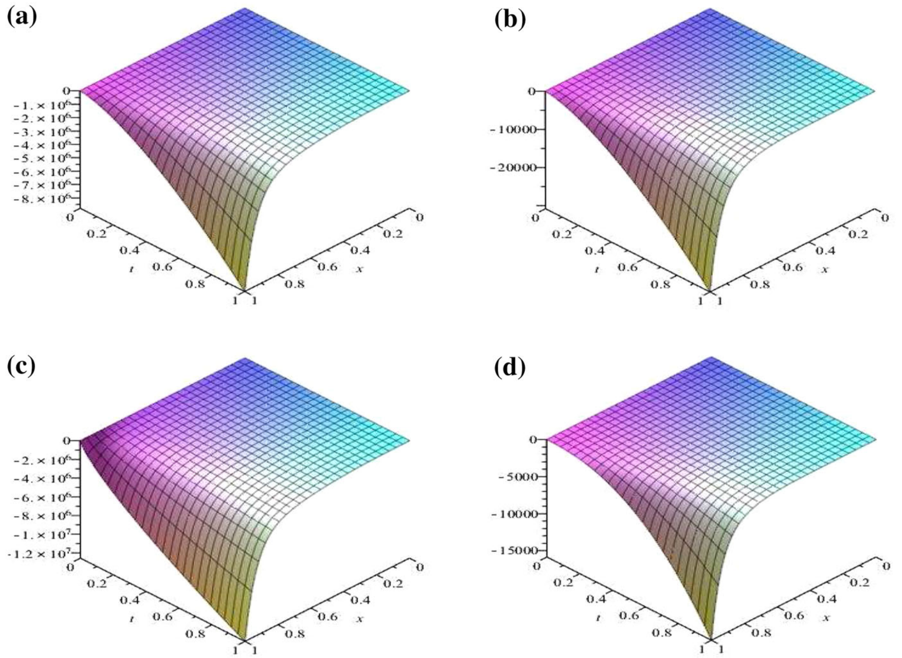

Fig. 18 Plot of the results by RPSM for the time-space-fractional Benney-Lin equation for **a** $\alpha = 0.5$, $\beta = 0.25$, **b** $\alpha = 0.5$, $\beta = 0.5$ **c** $\alpha = 0.25$, $\beta = 0.5$ and **d** $\alpha = 0.75$, $\beta = 0.25$ at $\eta = \mu = \kappa = 1$

approximate solutions are continuously depend on the space fractional derivative by viewing the comparison results of the subfigures (a) up to (d).

Application 5.3 Consider the following nonlinear time space-fractional Benney-Lin equation:

$$D_t^{\alpha} u + u D_x^{\beta} u + D_x^{3\beta} u + \eta (D_x^{2\beta} u + D_x^{4\beta} u) + \mu D_x^{5\beta} u + D_x^{\beta} u = 0, \quad 0 < \alpha, \beta < 1, \ \eta > 0, \ \mu \in \mathbb{R}, \tag{51}$$

subject to the initial condition

$$u(x, 0) = x^{10}. \tag{52}$$

The main advantage of RPSM is that it provides continuous approximate solution which is continuously depending on time space-fractional derivative. The mathematical behavior of the approximate solutions of Eqs. (51) and (52) are shown geometrically in Fig. 18. It can be noted that the efficiency of this method is dramatically enhanced using further terms of $u(x, t)$ from its mth truncated series of Eq. (8). Figure 19 shows the behavior of the solutions obtained for different values of $\alpha = 0.7$, $\alpha = 0.8$, $\alpha = 0.9$, and standard Benney-Lin's equation, i.e., $\alpha = 1$ with $\beta = 0.75$. The solutions obtained by RPSM increase very rapidly with the increases in t.

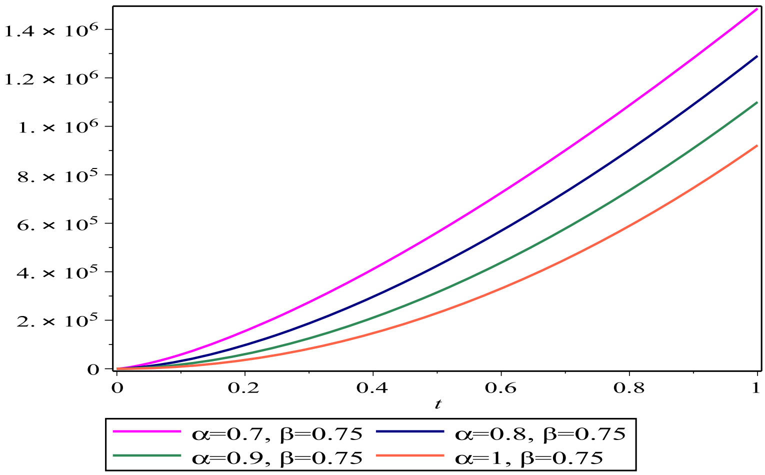

Fig. 19 Plot of approximate solutions versus time t for different value of α with $\beta = 0.75$ at $x = 1$

6 Conclusions

The explicit and approximate solutions of time-space-fractional Benney-Lin equation subject to the given initial conditions are constructed with high accuracy using the RPSM, which is the modern analytical iterative technique. The convergence analysis is also presented to show the effectiveness and leverage of the suggested method. The graphical results demonstrated that the nearly similar and coinciding behavior of the RPS approximate solution for $\alpha = 0.25$, $\alpha = 0.5$, $\alpha = 0.75$ and for the standard case $\alpha = 1$, in terms of the accuracy. Consequently, the presented method is a reliable iterative technique to handle linear and nonlinear fractional time-space-fractional partial differential equations and the RPSM provides remarkable advantages in terms of its straightforward applicability, its accuracy and its computational effectiveness.

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