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ORIGINAL RESEARCH

Existence and uniqueness of solutions to a fractional difference equation with p-Laplacian operator

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Abstract In this paper, we consider a discrete fractional boundary value problem with p-Laplacian operator of the form

$$\begin{cases} \Delta^{\beta}[\phi_p(\Delta^{\alpha} y)](t) + f(\alpha+\beta+t-1, y(\alpha+\beta+t-1)) = 0, t \in [0, b]_{\mathbb{N}_0}, \\ \Delta^{\alpha} y(\beta-2) = \Delta^{\alpha} y(\beta+b) = 0, \\ y(\alpha+\beta-4) = y(\alpha+\beta+b) = 0, \end{cases}$$

where $f : [\alpha+\beta-4, \alpha+\beta+b]_{\mathbb{N}_{\alpha+\beta-4}} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, and $p > 1$, $1 < \alpha, \beta \leq 2$. We study the existence and uniqueness of solution to this problem by using a variety of tools from nonlinear functional analysis including the contraction mapping theorem and Brouwer fixed point theorem.

Keywords Boundary value problem · Fractional difference equation · p-Laplacian operator · Contraction mapping theorem · Brouwer fixed point theorem

Mathematics Subject Classification 39A12 · 26A33 · 34B15

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1 Introduction

In recent years, fractional differential equations have seen tremendous growth. For some recent results and applications, we can see Wu et al. [1], which shows that a discrete fractional difference equation with Caputo derivative sense has an explicit solution in form of the discrete Mittag–Leffler function. Tarasov [2] formulated discrete models for dislocations in fractional nonlocal continua.

The non-locality of fractional differential equations can describe mathematical model better. However, it is difficult for us to calculate and analyze mathematical problems. With the development of computer, it is well known that discrete analogues of differential equations can be very useful, especially for using computer to simulate the behavior of solutions for certain dynamic equations.

A recent paper by Goodrich [3] explored a discrete fractional boundary value problem of the form

$$\begin{cases} -\Delta^\nu y(t) = f(t+\nu-1, y(t+\nu-1)), t \in [0,b]_{\mathbb{N}_0}, \\ y(\nu-2) = g(y), \\ y(\nu+b) = 0, \end{cases}$$

where $f \in [\nu-2, \nu+b-1]_{\mathbb{N}_{\nu-2}} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $g : C([\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}}, \mathbb{R})$ is given function, and $1 < \nu \leq 2$. This problem was solved by the contraction mapping theorem, Brouwer fixed point theorem, and Guo–Krasnosel'skii fixed point theorem.

Pan and Han [4] studied the existence and nonexistence of positive solutions to a discrete fractional boundary value problem with a parameter

$$\begin{cases} -\Delta^\nu y(t) = \lambda f(t+\nu-1, y(t+\nu-1)), t \in [0,b+1]_{\mathbb{N}_0}, \\ y(\nu-2) = y(\nu+b+1) = 0, \end{cases}$$

where $1 < \nu \leq 2$ is a real number, $f : [\nu-1, \nu+b]_{\mathbb{N}_{\nu-1}} \times \mathbb{R} \to (0, +\infty)$ is a continuous function, $b \geq 2$ is an integer, λ is a parameter. The eigenvalue intervals of the nonlinear fractional differential equation boundary value problem are considered by the properties of the Green function and Guo–Krasnosel'skii fixed point theorem in cones, some sufficient conditions of the nonexistence of positive solutions for the boundary value problem are established.

Differential equations with p-Laplacian operator are increasingly applied in real life, especially in physics and engineering [5]. Some theories of fractional differential equations with p-Laplacian operator are just beginning to be investigated. Lu and Han [6] investigated the existence of positive solutions for the eigenvalue problem of nonlinear fractional differential equation with generalized p-Laplacian operator

$$\begin{cases} D_{0+}^{\beta}(\phi_p(D_{0+}^{\alpha}u(t))) = \lambda f(u(t)), 0 < t < 1, \\ u(0) = u'(0) = u'(1) = 0, \phi_p(D_{0+}^{\alpha}u(0)) = (\phi_p(D_{0+}^{\alpha}u(1)))' = 0, \end{cases}$$

where $2 < \alpha \leq 3$, $1 < \beta \leq 2$, D^{α}_{0+} and D^{β}_{0+} are the standard Riemann–Liouville fractional differential, ϕ is the generalized p-Laplacian operator, $\lambda > 0$ is a parameter, and $f : (0, +\infty) \to (0, +\infty)$ is a continuous function. By using the Green function's properties and Guo–Krasnosel'skii fixed point theorem in cones, several new existence results of at least one or two positive solutions in terms of different eigenvalue interval are obtained.

Motivated by all the works above, we consider a discrete fractional boundary value problem with p-Laplacian operator

$$\Delta^{\beta}[\phi_p(\Delta^{\alpha} y)](t) + f(\alpha+\beta+t-1, y(\alpha+\beta+t-1)) = 0, t \in [0, b]_{\mathbb{N}_0}, \quad (1.1)$$

$$\Delta^{\alpha} y(\beta-2) = \Delta^{\alpha} y(\beta+b) = 0, \quad (1.2)$$

$$y(\alpha+\beta-4) = y(\alpha+\beta+b) = 0, \quad (1.3)$$

where $p > 1$, $1 < \alpha, \beta \leq 2$, Δ^{α} and Δ^{β} denote the Riemann–Liouville fractional differences of order α and β respectively, $f : [\alpha+\beta-4, \alpha+\beta+b]_{\mathbb{N}_{\alpha+\beta-4}} \times \mathbb{R} \to \mathbb{R}$ is a continuous function. ϕ_p is the p-Laplacian operator, that is, $\phi_p(u) = |u|^{p-2}u$, $p > 1$. Obviously, ϕ_p is invertible and its inverse operator is ϕ_q, where $q > 1$ is a constant with $\frac{1}{p} + \frac{1}{q} = 1$.

Our work presented in this article has the following features which are worth emphasizing.

(i) As far as we know, there is less literature available concerned with four-point boundary value problems of fractional difference equation which Δ^{α} and Δ^{β} are the standard Riemann–Liouville fractional differences.
(ii) We consider the boundary value problem with p-Laplacian which arises in the modeling of different physical and natural phenomena.
(iii) When $p = 2$, the fractional difference equation (1.1) reduce to $\Delta^{\beta}(\Delta^{\alpha} y)(t) + f(\alpha+\beta+t-1, y(\alpha+\beta+t-1)) = 0$ which involves mixed fractional difference equations.
(iv) When $p = 2$ and $\beta = 0$, the fractional difference equation (1.1) reduce to $\Delta^{\alpha} y(t) + f(\alpha+t-1, y(\alpha+t-1)) = 0$ which is the form studied in [3] and [7].

The plan of the paper is as follows. In Sect. 2, we give some definitions and lemmas which are needed in this paper. In Sect. 3, we study the existence and uniqueness of solution to problem (1.1)–(1.3) by using Banach contraction mapping theorem and Brouwer fixed point theorem. In Sect. 4, we give examples to illustrate the theorems.

2 Preliminaries

For the convenience of the reader, we give some necessary basic definitions and lemmas in discrete fractional calculus theory.

Definition 2.1 ([3]) We define $t^{\underline{\nu}} := \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)}$ for any t and ν, for which the right-hand side is defined. We also appeal to the convention that if $t+1-\nu$ is a pole of the Gamma function and $t+1$ is not a pole, then $t^{\underline{\nu}} = 0$.

Definition 2.2 ([3]) The νth fractional sum of a function f, for $\nu > 0$, is defined by

$$\Delta^{-\nu} f(t; a) := \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{\underline{\nu-1}} f(s),$$

for $t \in \{a+\nu, a+\nu+1, \cdots\} := \mathbb{N}_{a+\nu}$. We also define the νth fractional difference for $\nu > 0$ by $\Delta^{\nu} f(t) := \Delta^N \Delta^{\nu-N} f(t)$, where $t \in \mathbb{N}_{a+\nu}$ and $\nu \in \mathbb{N}$ is chosen so that $0 \le N-1 < \nu \le N$.

Lemma 2.1 ([3]) *Let $0 \le N-1 < \nu \le N$, where $N \in \mathbb{N}$ and $N-1 \ge 0$. Then*

$$\Delta^{-\nu}\Delta^{\nu} y(t) = y(t) + C_1 t^{\underline{\nu-1}} + C_2 t^{\underline{\nu-2}} + \cdots + C_N t^{\underline{\nu-N}},$$

for some $C_i \in \mathbb{R}$, with $1 \le i \le N$.

Lemma 2.2 ([3]) *Let t and ν be any numbers for which $t^{\underline{\nu}}$ and $t^{\underline{\nu-1}}$ are defined. Then*

$$\Delta t^{\underline{\nu}} = \nu t^{\underline{\nu-1}}.$$

Lemma 2.3 ([3]) *For t and s, for which both $(t-s-1)^{\underline{\nu}}$ and $(t-s-2)^{\underline{\nu}}$ are defined, we find that*

$$\Delta_s[(t-s-1)^{\underline{\nu}}] = -\nu(t-s-2)^{\underline{\nu-1}}.$$

Lemma 2.4 ([7]) *et $0 \le N-1 < \nu \le N$, where positive integer N greater than or equal to ν and $\nu > 0$. Defined*

$$\Delta^{\nu} t^{\underline{\nu}} = \Delta^{N-(N-\nu)} t^{\underline{\nu}} = \Delta^N \Delta^{-(N-\nu)} t^{\underline{\nu}}.$$

Replace ν by $\nu - m$ to obtain

$$\Delta^{\nu} t^{\underline{\nu-m}} = \frac{\Gamma(\nu-m+1)}{\Gamma(N-m+1)} \Delta^N t^{N-m},$$

when $\Delta^N t^{N-m} = 0, m = 1 \ldots N$, we have $\Delta^{\nu} t^{\underline{\nu-m}} = 0, m = 1 \ldots N$.

We now state and prove an important lemma. This lemma will give a representation for the solution to (1.1)–(1.3), provided that the solution exists. This representation will be crucial in Sect. 3 of this paper when we prove our existence and uniqueness theorems.

Lemma 2.5 *Let $f : [\alpha+\beta-4, \alpha+\beta+b]_{\mathbb{N}_{\alpha+\beta-4}} \times \mathbb{R} \to \mathbb{R}$ be given. A function y is a solution of the (1.1)–(1.3), if and only if it has the form*

$$y(t) = \sum_{s=\beta-2}^{\beta+b} H(t,s)h(s), t \in [\alpha+\beta-4, \alpha+\beta+b]_{\mathbb{N}_{\alpha+\beta-4}}, \tag{2.1}$$

where $H(t,s)$, $h(s)$ and $G(s,l)$ are given by

$$H(t,s)=\frac{1}{\Gamma(\alpha)}\begin{cases}\frac{(t-\beta+2)^{\underline{\alpha-1}}(\alpha+\beta+b-s-1)^{\underline{\alpha-1}}}{(\alpha+b+2)^{\underline{\alpha-1}}}-(t-s-1)^{\underline{\alpha-1}}, s<t-\alpha+1\leq\beta+b,\\ \frac{(t-\beta+2)^{\underline{\alpha-1}}(\alpha+\beta+b-s-1)^{\underline{\alpha-1}}}{(\alpha+b+2)^{\underline{\alpha-1}}}, t-\alpha+1\leq s\leq\beta+b,\end{cases} \tag{2.2}$$

$$h(s)=-\phi_q\left[\sum_{l=0}^{b}G(s,l)f(l+\alpha+\beta-1,y(l+\alpha+\beta-1))\right], s\in[\beta-2,\beta+b]_{\mathbb{N}_{\beta-2}}, \tag{2.3}$$

$$G(s,l)=\frac{1}{\Gamma(\beta)}\begin{cases}\frac{s^{\underline{\beta-1}}(\beta+b-l-1)^{\underline{\beta-1}}}{(\beta+b)^{\underline{\beta-1}}}-(s-l-1)^{\underline{\beta-1}}, 0\leq l<s-\beta+1\leq b,\\ \frac{s^{\underline{\beta-1}}(\beta+b-l-1)^{\underline{\beta-1}}}{(\beta+b)^{\underline{\beta-1}}}, 0\leq s-\beta+1\leq l\leq b.\end{cases}$$

Proof If $y(t)$ is a solution to (1.1)–(1.3), by using Lemma 2.1, we find that

$$\begin{aligned}[\phi_p(\Delta^\alpha y)](t)&=-\Delta^{-\beta}f(t+\alpha+\beta-1,y(t+\alpha+\beta-1))+C_1t^{\underline{\beta-1}}+C_2t^{\underline{\beta-2}}\\&=-\frac{1}{\Gamma(\beta)}\sum_{l=0}^{t-\beta}(t-l-1)^{\underline{\beta-1}}f(l+\alpha+\beta-1,y(l+\alpha+\beta-1))\\&\quad+C_1t^{\underline{\beta-1}}+C_2t^{\underline{\beta-2}},\end{aligned}$$

where $t\in[\beta-2,\beta+b]_{\mathbb{N}_{\beta-2}}$. Consequently, boundary condition (1.2) implies that $C_2=0$ and

$$C_1=\frac{1}{(\beta+b)^{\underline{\beta-1}}\Gamma(\beta)}\sum_{l=0}^{b}(\beta+b-l-1)^{\underline{\beta-1}}f(l+\alpha+\beta-1,y(l+\alpha+\beta-1)),$$

then we deduce that

$$\begin{aligned}[\phi_p(\Delta^\alpha y)](t)&=-\frac{1}{\Gamma(\beta)}\sum_{l=0}^{t-\beta}(t-l-1)^{\underline{\beta-1}}f(l+\alpha+\beta-1,y(l+\alpha+\beta-1))\\&\quad+\frac{1}{(\beta+b)^{\underline{\beta-1}}\Gamma(\beta)}\sum_{l=0}^{b}(\beta+b-l-1)^{\underline{\beta-1}}f(l+\alpha+\beta-1,\\&\quad y(l+\alpha+\beta-1))t^{\underline{\beta-1}},\end{aligned}$$

for $t\in[\beta-2,\beta+b]_{\mathbb{N}_{\beta-2}}$. Hence, we have

$$[\phi_p(\Delta^\alpha y)](t)=\sum_{l=0}^{b}G(t,l)f(l+\alpha+\beta-1,y(l+\alpha+\beta-1)), t\in[\beta-2,\beta+b]_{\mathbb{N}_{\beta-2}}. \tag{2.4}$$

Then taking p-Laplacian inverse of operators on both sides of (2.4), we find that

$$(\Delta^{\alpha} y)(t) = \phi_q\bigg[\sum_{l=0}^{b} G(t,l) f(l+\alpha+\beta-1, y(l+\alpha+\beta-1))\bigg],$$
$$t \in [\beta-2, \beta+b]_{\mathbb{N}_{\beta-2}}.$$

Next let

$$-h(t) = \phi_q\bigg[\sum_{l=0}^{b} G(t,l) f(l+\alpha+\beta-1, y(l+\alpha+\beta-1))\bigg],$$
$$t \in [\beta-2, \beta+b]_{\mathbb{N}_{\beta-2}}.$$

By using Lemma 2.1, we deduce that

$$y(t) = -\Delta^{-\alpha} h(t) + D_1 (t-\beta+2)^{\underline{\alpha-1}} + D_2 (t-\beta+2)^{\underline{\alpha-2}},$$
$$t \in [\alpha+\beta-4, \alpha+\beta+b]_{\mathbb{N}_{\alpha+\beta-4}}.$$

Consequently, boundary condition (1.3) implies that $D_2 = 0$ and

$$D_1 = \frac{1}{(\alpha+b+2)^{\underline{\alpha-1}}\Gamma(\alpha)} \sum_{s=\beta-2}^{\beta+b} (\alpha+\beta+b-s-1)^{\underline{\alpha-1}} h(s),$$

thus we get function (2.1) that

$$y(t) = \sum_{s=\beta-2}^{\beta+b} H(t,s) h(s),\ t \in [\alpha+\beta-4, \alpha+\beta+b]_{\mathbb{N}_{\alpha+\beta-4}}.$$

On the other hand, the function $y(t)$ is satisfied to (2.1), then $y(\alpha+\beta-4) = 0$, if $t = \alpha+\beta-4$; and $y(\alpha+\beta+b) = 0$, if $t = \alpha+\beta+b$. It is to say function (2.1) meet the boundary condition (1.3).

What's more, function $y(t)$ defined by (2.1) can transform to

$$y(t) = -\Delta^{-\alpha} h(t) + D_1 (t-\beta+2)^{\underline{\alpha-1}},\ t \in [\alpha+\beta-4, \alpha+\beta+b]_{\mathbb{N}_{\alpha+\beta-4}}.$$

Then we find that

$$\Delta^{\alpha} y(t) = -\Delta^{\alpha}\Delta^{-\alpha} h(t) + D_1 \Delta^{\alpha} (t-\beta+2)^{\underline{\alpha-1}},\ t \in [\beta-2, \beta+b]_{\mathbb{N}_{\beta-2}}.$$

By using Lemma 2.1, we know that

$$\Delta^{\alpha} (t-\beta+2)^{\underline{\alpha-1}} = \frac{\Gamma(\alpha)}{\Gamma(2)} \Delta^2 (t-\beta+2) = 0. \tag{2.5}$$

By Eq. (2.5), function $\Delta^{\alpha} y(t)$ has the form

$$\Delta^{\alpha} y(t) = -h(t) = \phi_q\left[\sum_{s=0}^{b} G(t,s) f(s+\alpha+\beta-1, y(s+\alpha+\beta-1))\right],$$
$$t \in [\beta-2, \beta+b]_{\mathbb{N}_{\beta-2}}. \tag{2.6}$$

Then $\Delta^{\alpha} y(\beta-2) = 0$, if $t = \beta-2$; and then $\Delta^{\alpha} y(\beta+b) = 0$, if $t = \beta+b$. It is to say function (2.1) meet the boundary condition (1.2). Taking p-Laplacian operators on both sides of (2.6), we find that

$$[\phi_p(\Delta^{\alpha} y)](t) = \sum_{s=0}^{b} G(t,s) f(s+\alpha+\beta-1, y(s+\alpha+\beta-1)),$$
$$t \in [\beta-2, \beta+b]_{\mathbb{N}_{\beta-2}},$$

or the form

$$\begin{aligned}[\phi_p(\Delta^{\alpha} y)](t) &= -\Delta^{-\beta} f(t+\alpha+\beta-1, y(t+\alpha+\beta-1)) \\ &\quad + \frac{1}{(\beta+2)^{\underline{\beta-1}}\Gamma(\beta)} \sum_{s=0}^{b} (\beta+b-s-1)^{\underline{\beta-1}} f(s) t^{\underline{\beta-1}}, \\ & t \in [\beta-2, \beta+b]_{\mathbb{N}_{\beta-2}}.\end{aligned} \tag{2.7}$$

By Eq. (2.7), function $\Delta^{\beta}[\phi_p(\Delta^{\alpha} y)](t)$ has the form

$$\begin{aligned}\Delta^{\beta}[\phi_p(\Delta^{\alpha} y)](t) &= -\Delta^{\beta}\Delta^{-\beta} f(t+\alpha+\beta-1, y(t+\alpha+\beta-1)) \\ &\quad + \frac{1}{(\beta+2)^{\underline{\beta-1}}\Gamma(\beta)} \sum_{s=0}^{b} (\beta+b-s-1)^{\underline{\beta-1}} f(s) \Delta^{\beta} t^{\underline{\beta-1}}, \\ & t \in [0, b]_{\mathbb{N}_0}.\end{aligned}$$

By using Lemma 2.4, we know that

$$\Delta^{\beta} t^{\underline{\beta-1}} = \frac{\Gamma(\beta)}{\Gamma(2)} \Delta^2 t = 0. \tag{2.8}$$

By Eq. (2.8), we find that

$$\Delta^{\beta}[\phi_p(\Delta^{\alpha} y)](t) = -f(t+\alpha+\beta-1, y(t+\alpha+\beta-1)), t \in [0, b]_{\mathbb{N}_0},$$

which shows that if (1.1)–(1.3) has a solution, then it can be represented by (2.1) and that every function of the form (2.1) is a solution of (1.1)–(1.3), which completes the proof. □

3 Existence of solutions

In this section, we will show the existence of solutions for boundary value problem (1.1)–(1.3).

Lemma 3.1 *The Green's function $G(s,l)$ and function $H(t,s)$ satisfy the following conditions:*

(i) $G(s,l) \geq 0$, *for* $(s,l) \in [\beta-2, \beta+b]_{\mathbb{N}_{\beta-2}} \times [0,b]_{\mathbb{N}_0}$, $H(t,s) \geq 0$, *for* $(t,s) \in [\alpha+\beta-4, \alpha+\beta+b]_{\mathbb{N}_{\alpha+\beta-4}} \times [\beta-2, \beta+b]_{\mathbb{N}_{\beta-2}}$;

(ii) $\max_{s\in[\beta-2,\beta+b]_{\mathbb{N}_{\beta-2}}} G(s,l) = G(l+\beta+1, l)$, *for* $l \in [0,b]_{\mathbb{N}_0}$, $\max_{t\in[\alpha+\beta-4,\alpha+\beta+b]_{\mathbb{N}_{\alpha+\beta-4}}} H(t,s) = H(s+\alpha+1, s)$, *for* $s \in [\beta-2, \beta+b]_{\mathbb{N}_{\beta-2}}$;

(iii) *there exists a number* $\gamma \in (0,1)$ *such that*

$$\min_{s\in\left[\frac{b+\nu}{4}, \frac{3(b+\nu)}{4}\right]} G(s,l) \geq \gamma \max_{s\in[\nu-2,\nu+b+1]_{\mathbb{N}_{\nu-2}}} G(s,l)$$
$$= \gamma G(l+\nu-1, l), \quad \text{for } l \in [0,b]_{\mathbb{N}_0}.$$

Remark 3.1 We omit the proof here, which is similar to Theorem 3.2 in [7].

Lemma 3.2 *([8])*

(1) If $1 < p < 2$, $uv > 0$ *and* $|u|, |v| \geq m > 0$, *then*

$$|\phi_p(v) - \phi_p(u)| \leq (p-1)m^{p-2}|v-u|.$$

(2) If $p \geq 2$ *and* $|u|, |v| \leq M$, *then*

$$|\phi_p(v) - \phi_p(u)| \leq (p-1)M^{p-2}|v-u|.$$

Define Banach spaces

$$B = \{y : [\beta-2, \beta+b]_{\mathbb{N}_{\beta-2}} \to \mathbb{R}\}$$

and

$$C = \{y : [\alpha+\beta-4, \alpha+\beta+b]_{\mathbb{N}_{\alpha+\beta-4}} \to \mathbb{R}\}$$

with maximum norm.

We now show that problem (1.1)–(1.3) has at least one solution under certain conditions. From Lemma 2.5, we observe that problem (1.1)–(1.3) may be recast as an equivalent summation equation. In order to get the main results, we introduce a operator $T : B \to B$ by

$$Ty(s) := \sum_{l=0}^{b} G(s,l) f(l+\alpha+\beta-1, y(l+\alpha+\beta-1)), s \in [\beta-2, \beta+b]_{\mathbb{N}_{\beta-2}}. \quad (3.1)$$

From Lemma 2.5, we know that y is a solution of (1.1)–(1.3) if and only if y is a fixed point of the operator

$$S : C \to C$$

which is defined by

$$Sy(t) = -\sum_{s=\beta-2}^{\beta+b} H(t,s)\phi_q(Ty)(s),\ t \in [\alpha+\beta-4, \alpha+\beta+b]_{\mathbb{N}_{\alpha+\beta-4}}, \tag{3.2}$$

where $H(t,s)$ and $Ty(s)$ are defined as (2.2) and (3.1).

We shall appeal to the contraction mapping theorem to get a unique solution of boundary value problem (1.1)–(1.3) when $p \geq 2$.

Theorem 3.1 *Suppose that $f(t,y)$ is Lipschitz in y, that is, there exists constant $L > 0$ such that $|f(t,y_1) - f(t,y_2)| \leq L|y_1 - y_2|$ whenever $y_1, y_2 \in \mathbb{R}$, $t \in [\alpha+\beta-4, \alpha+\beta+b]_{\mathbb{N}_{\alpha+\beta-4}}$, and there exists a function $A(t)$ such that $|f(t,y)| \leq A(t)$, for any $y \in C$, $t \in [\beta-2, \beta+b]_{\mathbb{N}_{\beta-2}}$. If $p \geq 2$ and $M < 1$, then the boundary value problem (1.1)–(1.3) has a unique solution.*

Proof By Lemma 3.1, for all $y \in B$, we can get that

$$\begin{aligned}
\|Ty\| &= \max_{s\in[\beta-2,\beta+b]_{\mathbb{N}_{\beta-2}}} |Ty(s)| \\
&= \max_{s\in[\beta-2,\beta+b]_{\mathbb{N}_{\beta-2}}} \sum_{l=0}^{b} G(s,l)|f(l+\alpha+\beta-1, y(l+\alpha+\beta-1))| \\
&\leq \sum_{l=0}^{b} G(l+\beta-1, l)A(l+\alpha+\beta-1).
\end{aligned}$$

Let $M_1 = \sum_{l=0}^{b} G(l+\beta-1, l)A(l+\alpha+\beta-1)$. What's more, for any $y_1, y_2 \in B$,

$$\begin{aligned}
\|Ty_1 - Ty_2\| &\leq L\|y_1 - y_2\| \max_{s\in[\beta-2,\beta+b]_{\mathbb{N}_{\beta-2}}} \left[\frac{1}{\Gamma(\beta)}\sum_{l=0}^{s-\beta}(s-l-1)^{\underline{\beta-1}}\right] \\
&+ L\|y_1 - y_2\| \max_{s\in[\beta-2,\beta+b]_{\mathbb{N}_{\beta-2}}} \left[\frac{(s)^{\underline{\beta-1}}}{\Gamma(\beta)(\beta+b)^{\underline{\beta-1}}}\sum_{l=0}^{b}(\beta+b-l-1)^{\underline{\beta-1}}\right].
\end{aligned} \tag{3.3}$$

We analyze the right-hand side of (3.3), by an application of Lemma 2.3, that

$$L\|y_1 - y_2\|\left[\frac{1}{\Gamma(\beta)}\sum_{l=0}^{s-\beta}(s-l-1)^{\underline{\beta-1}}\right] = \frac{L\|y_1 - y_2\|}{\Gamma(\beta)}\left[\frac{-1}{\beta}(s-l)^{\underline{\beta}}\right]_{l=0}^{s-\beta+1}$$
$$= L\|y_1 - y_2\|\left[\frac{\Gamma(s+1)}{\Gamma(s-\beta+1)\Gamma(\beta+1)}\right]$$
$$\leq L\|y_1 - y_2\|\left[\frac{\Gamma(\beta+b+1)}{\Gamma(b+1)\Gamma(\beta+1)}\right]$$
$$= L\prod_{j=1}^{b}\left(\frac{\beta+j}{j}\right)\|y_1 - y_2\|, \tag{3.4}$$

for $s \in [\beta-2, \beta+b]_{\mathbb{N}_{\beta-2}}$. Similarly, we have

$$L\|y_1-y_2\|\left[\frac{(s)^{\underline{\beta-1}}}{\Gamma(\beta)(\beta+b)^{\underline{\beta-1}}}\sum_{l=0}^{b}(\beta+b-l-1)^{\underline{\beta-1}}\right] \leq L\prod_{j=1}^{b}\left(\frac{\beta+j}{j}\right)\|y_1-y_2\|, \tag{3.5}$$

for $s \in [\beta-2, \beta+b]_{\mathbb{N}_{\beta-2}}$. So, putting (3.3)–(3.5) together, we conclude that

$$\|Ty_1 - Ty_2\| \leq 2L\prod_{j=1}^{b}\left(\frac{\beta+j}{j}\right)\|y_1 - y_2\|, \tag{3.6}$$

and let $M_2 = 2L\prod_{j=1}^{b}\left(\frac{\beta+j}{j}\right)$.

Next, we will show that S defined as (3.2) is a contraction map. To this end, we note that for given $y_1, y_2 \in C$,

$$|Sy_1(t) - Sy_2(t)| \leq (q-1)\sum_{s=\beta-2}^{\beta+b} H(s+\alpha-1, s)M_1^{(q-2)}|Ty_1(t) - Ty_2(t)|$$
$$\leq (q-1)\sum_{s=\beta-2}^{\beta+b} H(s+\alpha-1, s)M_1^{(q-2)}\|Ty_1 - Ty_2\|$$
$$\leq (q-1)\sum_{s=\beta-2}^{\beta+b} H(s+\alpha-1, s)M_1^{(q-2)}M_2\|y_1 - y_2\|, \tag{3.7}$$

for $t \in [\alpha+\beta-4, \alpha+\beta+b]_{\mathbb{N}_{\alpha+\beta-4}}$. For convenience, let us put

$$M = (q-1)\sum_{s=\beta-2}^{\beta+b} H(s+\alpha-1, s)M_1^{(q-2)}M_2. \tag{3.8}$$

By inequality (3.7) and Eq. (3.8), we conclude that

$$||Sy_1(t) - Sy_2(t)|| \leq M\|y_1 - y_2\|, \tag{3.9}$$

whence by $M < 1$, we find that (1.1)–(1.3) has a unique solution. This completes the proof. □

We shall next appeal to the contraction mapping theorem to get a unique solution of boundary value problem (1.1)–(1.3) when $1 < p < 2$.

Theorem 3.2 *Suppose that $f(t, y)$ is Lipschitz in y, that is, there exists $L > 0$ such that $|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2|$ whenever $y_1, y_2 \in \mathbb{R}$, $t \in [\alpha + \beta - 4, \alpha + \beta + b]_{\mathbb{N}_{\alpha+\beta-4}}$, and there exist a nonnegative function $B(t)$ satisfying $|f(t, y)| \geq B(t)$, for any $y \in C$, $t \in [\beta - 2, \beta + b]_{\mathbb{N}_{\beta-2}}$. If $1 < p < 2$ and $K < 1$, the boundary value problem (1.1)–(1.3) has a unique solution.*

Proof By Lemma 3.1, for all $y \in B$, we can get that

$$\begin{aligned}\|Ty\| &= \max_{s\in[\beta-2,\beta+b]_{\mathbb{N}_{\beta-2}}} |Ty(s)| \\ &\geq \max_{s\in[\beta-2,\beta+b]_{\mathbb{N}_{\beta-2}}} \sum_{l=0}^{b} G(s, l)B(l + \alpha + \beta - 1) \\ &= \sum_{l=0}^{b} G(l + \beta - 1, l)B(l + \alpha + \beta - 1).\end{aligned}$$

Set $K_1 = \sum_{l=0}^{b} G(l + \beta - 1, l)B(l + \alpha + \beta - 1)$. We will show that S defined as (3.2) is a contraction map. By inequality (3.6), we note that for any $y_1, y_2 \in C$, we can get that

$$\begin{aligned}|Sy_1(t) - Sy_2(t)| &\leq (q-1) \sum_{s=\beta-2}^{\beta+b} H(s + \alpha - 1, s)K_1^{(q-2)}\|Ty_1 - Ty_2\| \\ &\leq (q-1) \sum_{s=\beta-2}^{\beta+b} H(s + \alpha - 1, s)K_1^{(q-2)} M_2\|y_1 - y_2\|,\end{aligned} \tag{3.10}$$

for $t \in [\alpha + \beta - 4, \alpha + \beta + b]_{\mathbb{N}_{\alpha+\beta-4}}$, here $M_2 = 2L \prod_{j=1}^{b} \binom{\beta+j}{j}$. For convenience, we denote

$$K = (q-1) \sum_{s=\beta-2}^{\beta+b} H(s + \alpha - 1, s)K_1^{(q-2)} M_2. \tag{3.11}$$

By inequality (3.10) and Eq. (3.11), we conclude that

$$||Sy_1(t) - Sy_2(t)|| \leq K\|y_1 - y_2\|, \tag{3.12}$$

hence (1.1)–(1.3) has a unique solution. This completes the proof. □

By weakening the condition imposed on $f(t, y)$, we can still deduce the existence of solutions to (1.1)–(1.3). We shall appeal to Brouwer fixed point theorem to accomplish this.

Theorem 3.3 *Suppose that there exists a constant $G > 0$ such that $f(t, y)$ satisfies the inequality*

$$\max_{t\in[\alpha+\beta-4,\alpha+\beta+b]_{\mathbb{N}_{\alpha+\beta-4}},y\in[-G,G]} |f(t, y)| \leq \phi_p\left[\frac{G}{\frac{4\Gamma(\beta+b+1)\Gamma(\alpha+b+3)}{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(b+1)\Gamma(b+3)}}\right]. \tag{3.13}$$

Then (1.1)–(1.3) has at least one solution, say y_0, satisfying $|y_0(t)| \leq G$, for all $t \in [\alpha + \beta - 4, \alpha + \beta + 4]_{\mathbb{N}_{\alpha+\beta-4}}$.

Proof Denote $D = \{y \in C : \|y\| \leq G\}$. Let S be the operator defined in (3.2). It is easy to prove that S is a continuous operator. By using Brouwer fixed point theorem, we will to show that $S : D \to D$, that is, whenever $\|y\| \leq G$, it follows that $\|Sy\| \leq G$. Once this is established, we deduce the conclusion. To this end, assume that inequality (3.13) hold for $f(t, y)$. For notational convenience in what follows, let us put

$$\Psi := \phi_p\left[\frac{G}{\frac{4\Gamma(\beta+b+1)\Gamma(\alpha+b+3)}{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(b+1)\Gamma(b+3)}}\right], \tag{3.14}$$

which is a positive constant. For any a $y \in D$, from (2.3), we observe that

$$\begin{aligned}
\|h\| \leq &\max_{s\in[\beta-2,\beta+b]_{\mathbb{N}_{\beta-2}}} \phi_q\Bigg[\frac{1}{\Gamma(\beta)}\sum_{l=0}^{s-\beta}(s-l-1)^{\underline{\beta-1}}|f(l+\alpha+\beta-1, y(l+\alpha+\beta-1))| \\
&+\frac{(s)^{\underline{\beta-1}}}{\Gamma(\beta)(\beta+b)^{\underline{\beta-1}}}\sum_{l=0}^{b}(\beta+b-l-1)^{\underline{\beta-1}}|f(l+\alpha+\beta-1, y(l+\alpha+\beta-1))|\Bigg] \\
\leq &\ \Psi \max_{s\in[\beta-2,\beta+b]_{\mathbb{N}_{\beta-2}}} \phi_q \\
&\times\left[\frac{1}{\Gamma(\beta)}\sum_{l=0}^{s-\beta}(s-l-1)^{\underline{\beta-1}}+\frac{(s)^{\underline{\beta-1}}}{\Gamma(\beta)(\beta+b)^{\underline{\beta-1}}}\sum_{l=0}^{b}(\beta+b-l-1)^{\underline{\beta-1}}\right].
\end{aligned} \tag{3.15}$$

Note that

$$\begin{aligned}
&\frac{1}{\Gamma(\beta)}\sum_{l=0}^{s-\beta}(s-l-1)^{\underline{\beta-1}}+\frac{(s)^{\underline{\beta-1}}}{\Gamma(\beta)(\beta+b+)^{\underline{\beta-1}}}\sum_{l=0}^{b}(\beta+b-l-1)^{\underline{\beta-1}} \\
&\leq \frac{1}{\Gamma(\beta)}\sum_{l=0}^{s-\beta}(s-l-1)^{\underline{\beta-1}}+\frac{1}{\Gamma(\beta)}\sum_{l=0}^{b}(\beta+b-l-1)^{\underline{\beta-1}}
\end{aligned}$$

$$\leq \frac{1}{\Gamma(\beta)}\sum_{l=0}^{b}(\beta+b-l-1)^{\underline{\beta-1}} + \frac{1}{\Gamma(\beta)}\sum_{l=0}^{b}(\beta+b-l-1)^{\underline{\beta-1}}$$
$$= \frac{2}{\Gamma(\beta)}\sum_{l=0}^{b}(\beta+b-l-1)^{\underline{\beta-1}}, \tag{3.16}$$

here we use the fact that $t^{\underline{\beta-1}}$ is increasing in t since $\beta-1>0$. Furthermore,

$$\sum_{l=0}^{b}(\beta+b-l-1)^{\underline{\beta-1}} = \left[-\frac{1}{\beta}(\beta+b-l)^{\underline{\beta}}\right]_{l=0}^{b+1} = \frac{\Gamma(\beta+b+1)}{\beta\Gamma(b+1)}. \tag{3.17}$$

If we now put (3.15)–(3.17) together, by the definition of Ψ given in (3.14), then we find that

$$\|h\| \leq \frac{G\Gamma(\alpha+1)\Gamma(b+3)}{2\Gamma(\alpha+b+3)}. \tag{3.18}$$

By inequality (3.18), we conclude that

$$\begin{aligned}\|Sy\| \leq{}& \max_{t\in[\alpha+\beta-4,\alpha+\beta+b]_{\mathbb{N}_{\alpha+\beta-4}}} \frac{1}{\Gamma(\alpha)}\sum_{s=\beta-2}^{t-\alpha}(t-s-1)^{\underline{\alpha-1}}|h(s)| \\ &+ \max_{t\in[\alpha+\beta-4,\alpha+\beta+b]_{\mathbb{N}_{\alpha+\beta-4}}} \frac{(t-\beta+2)^{\underline{\alpha-1}}}{\Gamma(\alpha)(\alpha+b+2)^{\underline{\alpha-1}}}\sum_{s=\beta-2}^{\beta+b}(\alpha+\beta+b-s-1)^{\underline{\alpha-1}}|h(s)| \\ \leq{}& \frac{G}{\frac{2\Gamma(\alpha+b+3)}{\Gamma(\alpha+1)\Gamma(b+3)}} \max_{t\in[\alpha+\beta-4,\alpha+\beta+b]_{\mathbb{N}_{\alpha+\beta-4}}} \left[\frac{1}{\Gamma(\alpha)}\sum_{s=\beta-2}^{t-\alpha}(t-s-1)^{\underline{\alpha-1}}\right. \\ &\left.+ \frac{(t-\beta+2)^{\underline{\alpha-1}}}{\Gamma(\alpha)(\alpha+b+2)^{\underline{\alpha-1}}}\sum_{s=\beta-2}^{\beta+b}(\alpha+\beta+b-s-1)^{\underline{\alpha-1}}\right]. \end{aligned} \tag{3.19}$$

Since

$$\frac{1}{\Gamma(\alpha)}\sum_{s=\beta-2}^{t-\alpha}(t-s-1)^{\underline{\alpha-1}} + \frac{(t-\beta+2)^{\underline{\alpha-1}}}{\Gamma(\alpha)(\alpha+b+2)^{\underline{\alpha-1}}}\sum_{s=\beta-2}^{\beta+b}(\alpha+\beta+b-s-1)^{\underline{\alpha-1}}$$
$$\leq \frac{2}{\Gamma(\alpha)}\sum_{s=\beta-2}^{\beta+b}(\alpha+\beta+b-s-1)^{\underline{\alpha-1}}, \tag{3.20}$$

where to get inequality (3.20) we have used the fact that $t^{\underline{\alpha-1}}$ is increasing in t since $\alpha-1>0$, furthermore,

$$\sum_{s=\beta-2}^{\beta+b}(\alpha+\beta+b-s-1)^{\underline{\alpha-1}} = \left[-\frac{1}{\alpha}(\alpha+\beta+b-s)^{\underline{\alpha}}\right]_{s=\beta-2}^{\beta+b+1} = \frac{\Gamma(\alpha+b+3)}{\alpha\Gamma(b+3)}. \tag{3.21}$$

If we now put (3.19)–(3.21) together, then we find that

$$\|Sy\| \leq \frac{G}{\frac{2\Gamma(\alpha+b+3)}{\Gamma(\alpha+1)\Gamma(b+3)}}\left[\frac{2\Gamma(\alpha+b+3)}{\Gamma(\alpha+1)\Gamma(b+3)}\right] = G. \tag{3.22}$$

Thus, from (3.22) we deduce that $S : D \to D$, as desired. Consequently, it follow at once by Brouwer fixed point theorem that there exists a fixed point of the map S, say $Sy_0 = y_0$ with $y_0 \in C$. But this function y_0 is a solution of (1.1)–(1.3). Moreover, y_0 satisfies the bound $|y_0(t)| \leq G$, for each $t \in [\alpha+\beta-4, \alpha+\beta+b]_{\mathbb{N}_{\alpha+\beta-4}}$. This completes the proof. □

4 Examples

In this section, we will present some examples to illustrate main results.

Example 4.1 Consider boundary value problem of discrete fractional equation

$$\Delta^{1.2}[\phi_p(\Delta^{1.8}y)](t) + t^2 + \frac{1}{700}\sin y(t) = 0, t \in [0,2]_{\mathbb{N}_0}, \tag{4.1}$$

$$\Delta^{1.8}y(-0.8) = \Delta^{1.8}y(3.2) = 0, \tag{4.2}$$

$$y(-1) = y(5) = 0, \tag{4.3}$$

where $f(t,y) := t^2 + \frac{1}{700}\sin y(t)$, for $t \in [0,2]_{\mathbb{N}_0}$, $y \in \mathbb{R}$, is Lipschitz with Lipschitz constants $L = \frac{1}{700}$. When $p = 3$, for this choice of L, inequality (3.9) is satisfied with $M \approx 0.89 < 1$. Therefore, we deduce from Theorem 3.1 that problem (4.1)–(4.3) has a unique solution.

Example 4.2 Consider boundary value problem of discrete fractional equation

$$\Delta^{1.2}[\phi_p(\Delta^{1.8}y)](t) + \frac{1}{300} + \frac{1}{300}\sin y(t) = 0, t \in [0,2]_{\mathbb{N}_0}, \tag{4.4}$$

$$\Delta^{1.8}y(-0.8) = \Delta^{1.8}y(3.2) = 0, \tag{4.5}$$

$$y(-1) = y(5) = 0, \tag{4.6}$$

where $f(t,y) := \frac{1}{300} + \frac{1}{300}\sin y(t)$, for $t \in [0,2]_{\mathbb{N}_0}$, $y \in \mathbb{R}$, is Lipschitz with Lipschitz constants $L = \frac{1}{300}$. When $p = \frac{3}{2}$, for this choice of L, inequality (3.12) is satisfied with $K \approx 0.835 < 1$. Therefore, we deduce from Theorem 3.2 that problem (4.4)–(4.6) has a unique solution.

Example 4.3 We suppose that $f(t,y) := t^2 + \sin y(t)$, for $t \in [0,2]_{\mathbb{N}_0}$, $y \in \mathbb{R}$. Consider boundary value problem of discrete fractional equation

$$\Delta^{1.2}[\phi_p(\Delta^{1.8}y)](t) + t^2 + \sin y(t) = 0, t \in [0,2]_{\mathbb{N}_0}, \tag{4.7}$$

$$\Delta^{1.8}y(-0.8) = \Delta^{1.8}y(3.2) = 0, \tag{4.8}$$

$$y(-1) = y(5) = 0, \tag{4.9}$$

and the Banach space C in this case is $C = \{y : [-1, 5]_{\mathbb{N}} \to \mathbb{R}\}$.

We claim that (4.7)–(4.9) has at least one solution.Suppose that $G = 400$ and $p = 2$. To check the hypotheses of Theorem 3.3 hold, we note that

$$\phi_p\left[\frac{G}{\frac{4\Gamma(\beta+b+1)\Gamma(\alpha+b+3)}{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(b+1)\Gamma(b+3)}}\right] = \phi_p\left[\frac{G}{\frac{4\Gamma(1.2+2+1)\Gamma(1.8+2+3)}{\Gamma(2.8)\Gamma(2.2)\Gamma(2+1)\Gamma(2+3)}}\right] \approx 26.3.$$

Now, it is clear that $|f(t, y)| \leq 5 < 26.3$, whenever $|y(t)| \leq 400$. By Theorem 3.3 we deduce that this solution, say $y_0(t)$, satisfies $|y_0(t)| \leq G$ for $t \in [-1, 5]_{\mathbb{N}}$.

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