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ORIGINAL RESEARCH

On (p, q)-analogue of modified Bernstein–Schurer operators for functions of one and two variables

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Abstract In this paper, we introduce a new kind of modified Bernstein–Schurer operators based on the concept of (p, q)-integers. We investigate statistical approximation properties, establish a local approximation theorem, give a convergence theorem for the Lipschitz continuous functions, we also obtain a Voronovskaja-type asymptotic formula. Next, we construct the bivariate operators and get some convergence properties. Finally, we give some graphs to illustrate the convergence properties of operators to some functions.

Keywords (p, q)-Integers · Bernstein–Schurer operators · A-statistical convergence · Rate of convergence · Lipschitz continuous functions

Mathematics Subject Classification 41A10 · 41A25 · 41A36

1 Introduction

Recently, Mursaleen et al. applied (p, q)-calculus in approximation theory and introduced the (p, q)-analogue of Bernstein operators in [1]. We mention some of their other works as [2–6].

In 2011, Muraru [7] introduced a generalization of the Bernstein–Schurer operators based on q-integers as follows.

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$$\widetilde{B}_{m,l}(f;q;x)=\sum_{k=0}^{m+l}\begin{bmatrix} m+l \\ k \end{bmatrix}_q x^k(1-x)_q^{m+l-k} f\left(\frac{[k]_q}{[m]_q}\right),$$

for any $m \in \mathbb{N}$ and $f \in C([0, 1+l])$, l is fixed. In 2013, Ren and Zeng [8] introduced the following modified q-Bernstein–Schurer operators which preserve linear functions.

$$\widetilde{S}_{n,l}(f;q;x)=\sum_{k=0}^{n+l}\begin{bmatrix} n+l \\ k \end{bmatrix}_q \left(\frac{[n]_q x}{[n+l]_q}\right)^k \left(1-\frac{[n]_q x}{[n+l]_q}\right)_q^{n+l-k} f\left(\frac{[k]_q}{[n]_q}\right), \quad (1)$$

for $f \in C([0, 1+l])$, $l \in \mathbb{N} \cup \{0\}$ is fixed, $n \in \mathbb{N}$, $0 < q < 1$.

In this paper, firstly, we will introduce a generalization of modified Bernstein–Schurer operators based on (p, q)-integers which will be defined in (2) and investigate some approximation properties, secondly, we will construct the bivariate type operators which will be defined in (22) and obtain some convergence properties, finally, we will give some graphics to illustrate the convergence to some functions.

Before introducing the operators, we mention certain definitions based on (p, q)-integers, details can be found in [9–13]. For any fixed real number $0 < q < p \le 1$ and each nonnegative integer k, we denote (p, q)-integers by $[k]_{p,q}$, where

$$[k]_{p,q}=\frac{p^k-q^k}{p-q}.$$

Also (p, q)-factorial and (p, q)-binomial coefficients are defined as follows:

$$[k]_{p,q}!=\begin{cases}[k]_{p,q}[k-1]_{p,q}\ldots[1]_{p,q}, & k=1,2,\ldots; \\ 1, & k=0,\end{cases}, \quad \begin{bmatrix} n \\ k \end{bmatrix}_{p,q}=\frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!},$$

for $n \ge k \ge 0$. The (p, q)-Binomial expansion is defined by

$$(x+y)_{p,q}^n=\begin{cases} 1, & \text{n}=0; \\ (x+y)(px+qy)\ldots\left(p^{n-1}x+q^{n-1}y\right), & \text{n}=1,2,\ldots \end{cases}.$$

When $p = 1$, all the definitions of (p, q)-calculus above are reduced to q-calculus.

For $f \in C(I)$, $I = [0, 1+l]$, $l \in \mathbb{N}_0$, $0 < q < p \le 1$ and $n \in \mathbb{N}$, we introduce the (p, q)-analogue of modified Bernstein–Schurer operators as follows:

$$S_{n,p,q}^l(f;x)=\sum_{k=0}^{n+l}\begin{bmatrix} n+l \\ k \end{bmatrix}_{p,q} \left(\frac{[n]_{p,q} x}{[n+l]_{p,q}}\right)^k \left(1-\frac{[n]_{p,q} x}{[n+l]_{p,q}}\right)_{p,q}^{n+l-k} f\left(\frac{[k]_{p,q}}{[n]_{p,q}}\right). \quad (2)$$

It is observed that when $p = 1$, $S_{n,p,q}^l(f;x)$ becomes to (1).

2 Auxiliary results

In order to obtain the approximation properties, we need the following lemmas:

Lemma 2.1 *For the (p, q)-analogue of modified Bernstein–Schurer operators* (2)*, we have the following equalities*

$$S_{n,p,q}^{l}(1;x) = 1, \tag{3}$$

$$S_{n,p,q}^{l}(t;x) = x, \tag{4}$$

$$S_{n,p,q}^{l}\left(t^2;x\right) = \frac{p[n+l-1]_{p,q}}{[n+l]_{p,q}}x^2 + \frac{x}{[n]_{p,q}}\left(\frac{q[n]_{p,q}x}{[n+l]_{p,q}} + 1 - \frac{[n]_{p,q}x}{[n+l]_{p,q}}\right)_{p,q}^{n+l-1}, \tag{5}$$

$$\begin{aligned} S_{n,p,q}^{l}\left(t^3;x\right) &= \frac{p^3[n+l-1]_{p,q}[n+l-2]_{p,q}x^3}{[n+l]_{p,q}^2} + \frac{\left(p^2+2pq\right)[n+l-1]_{p,q}x^2}{[n]_{p,q}[n+l]_{p,q}} \\ &\quad \times \left(\frac{q[n]_{p,q}x}{[n+l]_{p,q}} + 1 - \frac{[n]_{p,q}x}{[n+l]_{p,q}}\right)_{p,q}^{n+l-2} \\ &\quad + \frac{x}{[n]_{p,q}^2}\left(\frac{q^2[n]_{p,q}x}{[n+l]_{p,q}} + 1 - \frac{[n]_{p,q}x}{[n+l]_{p,q}}\right)_{p,q}^{n+l-1}, \end{aligned} \tag{6}$$

$$\begin{aligned} S_{n,p,q}^{l}\left(t^4;x\right) &= \frac{p^6[n+l-1]_{p,q}[n+l-2]_{p,q}[n+l-3]_{p,q}x^4}{[n+l]_{p,q}^3} \\ &\quad + \frac{\left(p^5+2p^4q+3p^3q^2\right)[n+l-1]_{p,q}[n+l-2]_{p,q}x^3}{[n]_{p,q}[n+l]_{p,q}^2} \\ &\quad \times \left(\frac{q[n]_{p,q}x}{[n+l]_{p,q}} + 1 - \frac{[n]_{p,q}x}{[n+l]_{p,q}}\right)_{p,q}^{n+l-3} \\ &\quad + \frac{\left(p^3+3p^2q+3pq^2\right)[n+l-1]_{p,q}x^2}{[n]_{p,q}^2[n+l]_{p,q}} \\ &\quad \times \left(\frac{q^2[n]_{p,q}x}{[n+l]_{p,q}} + 1 - \frac{[n]_{p,q}x}{[n+l]_{p,q}}\right)_{p,q}^{n+l-2} \\ &\quad + \frac{x}{[n]_{p,q}^3}\left(\frac{q^3[n]_{p,q}x}{[n+l]_{p,q}} + 1 - \frac{[n]_{p,q}x}{[n+l]_{p,q}}\right)_{p,q}^{n+l-1}. \end{aligned} \tag{7}$$

Proof (3) and (4) are easily obtained from (2) and the definition of (p, q)-integrals. Using (2) and $[k]_{p,q}^2 = q^{k-1}[k]_{p,q} + p[k]_{p,q}[k-1]_{p,q}$, we have

$$
\begin{aligned}
&S^l_{n,p,q}\left(t^2;x\right)\\
&=\sum_{k=0}^{n+l}\begin{bmatrix} n+l \\ k \end{bmatrix}_{p,q}\left(\frac{[n]_{p,q}x}{[n+l]_{p,q}}\right)^k\left(1-\frac{[n]_{p,q}x}{[n+l]_{p,q}}\right)^{n+l-k}_{p,q}\left(\frac{[k]_{p,q}}{[n]_{p,q}}\right)^2\\
&=\frac{x}{[n]_{p,q}}\sum_{k=0}^{n+l-1}\begin{bmatrix} n+l-1 \\ k \end{bmatrix}_{p,q}\left(\frac{[n]_{p,q}x}{[n+l]_{p,q}}\right)^k\left(1-\frac{[n]_{p,q}x}{[n+l]_{p,q}}\right)^{n+l-k-1}_{p,q} q^k\\
&\quad+\frac{p[n+l]_{p,q}[n+l-1]_{p,q}}{[n]^2_{p,q}}\sum_{k=0}^{n+l-2}\begin{bmatrix} n+l-2 \\ k \end{bmatrix}_{p,q}\left(\frac{[n]_{p,q}x}{[n+l]_{p,q}}\right)^{k+2}\\
&\quad\times\left(1-\frac{[n]_{p,q}x}{[n+l]_{p,q}}\right)^{n+l-2-k}_{p,q}\\
&=\frac{p[n+l-1]_{p,q}}{[n+l]_{p,q}}x^2+\frac{x}{[n]_{p,q}}\left(\frac{q[n]_{p,q}x}{[n+l]_{p,q}}+1-\frac{[n]_{p,q}x}{[n+l]_{p,q}}\right)^{n+l-1}_{p,q}.
\end{aligned}
$$

Thus, (5) is proved. Next, from (2) and

$$
[k]^3_{p,q}=p^3[k]_{p,q}[k-1]_{p,q}[k-2]_{p,q}+pq^{k-2}(p+2q)[k]_{p,q}[k-1]_{p,q}+q^{2k-2}[k]_{p,q},
$$

we get

$$
\begin{aligned}
&S^l_{n,p,q}\left(t^3;x\right)\\
&=\sum_{k=0}^{n+l}\begin{bmatrix} n+l \\ k \end{bmatrix}_{p,q}\left(\frac{[n]_{p,q}x}{[n+l]_{p,q}}\right)^k\left(1-\frac{[n]_{p,q}x}{[n+l]_{p,q}}\right)^{n+l-k}_{p,q}\left(\frac{[k]_{p,q}}{[n]_{p,q}}\right)^3\\
&=\frac{p^3[n+l-1]_{p,q}[n+l-2]_{p,q}x^3}{[n+l]^2_{p,q}}+\frac{\left(p^2+2pq\right)[n+l-1]_{p,q}x^2}{[n]_{p,q}[n+l]_{p,q}}\\
&\left(\frac{q[n]_{p,q}x}{[n+l]_{p,q}}+1-\frac{[n]_{p,q}x}{[n+l]_{p,q}}\right)^{n+l-2}_{p,q}+\frac{x}{[n]^2_{p,q}}\left(\frac{q^2[n]_{p,q}x}{[n+l]_{p,q}}+1-\frac{[n]_{p,q}x}{[n+l]_{p,q}}\right)^{n+l-1}_{p,q}.
\end{aligned}
$$

(9) is proved. Finally, since

$$
\begin{aligned}
[k]^4_{p,q}&=p^6[k]_{p,q}[k-1]_{p,q}[k-2]_{p,q}[k-3]_{p,q}\\
&\quad+p^3q^{k-3}\left(p^2+2pq+3q^2\right)[k]_{p,q}[k-1]_{p,q}[k-2]_{p,q}\\
&\quad+pq^{2k-4}\left(p^2+3pq+3q^2\right)[k]_{p,q}[k-1]_{p,q}+q^{3k-3}[k]_{p,q},
\end{aligned}
$$

and some simple computations, we have

$$
\begin{aligned}
&S_{n,p,q}^{l}\left(t^{4};x\right)\\
&=\sum_{k=0}^{n+l}\begin{bmatrix} n+l\\ k\end{bmatrix}_{p,q}\left(\frac{[n]_{p,q}x}{[n+l]_{p,q}}\right)^{k}\left(1-\frac{[n]_{p,q}x}{[n+l]_{p,q}}\right)_{p,q}^{n+l-k}\left(\frac{[k]_{p,q}}{[n]_{p,q}}\right)^{4}\\
&=\frac{p^{6}[n+l-1]_{p,q}[n+l-2]_{p,q}[n+l-3]_{p,q}x^{4}}{[n+l]_{p,q}^{3}}\\
&\quad+\frac{\left(p^{5}+2p^{4}q+3p^{3}q^{2}\right)[n+l-1]_{p,q}[n+l-2]_{p,q}x^{3}}{[n]_{p,q}[n+l]_{p,q}^{2}}\left(\frac{q[n]_{p,q}x}{[n+l]_{p,q}}+1\right.\\
&\quad\left.-\frac{[n]_{p,q}x}{[n+l]_{p,q}}\right)_{p,q}^{n+l-3}+\frac{\left(p^{3}+3p^{2}q+3pq^{2}\right)[n+l-1]_{p,q}x^{2}}{[n]_{p,q}^{2}[n+l]_{p,q}}\left(\frac{q^{2}[n]_{p,q}x}{[n+l]_{p,q}}\right.\\
&\quad\left.+1-\frac{[n]_{p,q}x}{[n+l]_{p,q}}\right)_{p,q}^{n+l-2}+\frac{x}{[n]_{p,q}^{3}}\left(\frac{q^{3}[n]_{p,q}x}{[n+l]_{p,q}}+1-\frac{[n]_{p,q}x}{[n+l]_{p,q}}\right)_{p,q}^{n+l-1}.
\end{aligned}
$$

Thus, (7) is proved. □

Remark 2.2 Let $\{p_n\}$ and $\{q_n\}$ denote sequences such that $0 < q_n < p_n \leq 1$. Then by Bohman and Korovkin Theorem and Lemma 2.1, for any $f \in C(I)$, if $\lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n = 1$ and $\lim_{n\to\infty}[n]_{p_n,q_n} = \infty$, operators $S_{n,p,q}^{l}(f;x)$ convergence uniformly to $f(x)$.

Lemma 2.3 *Let* $p = \{p_n\}$, $q = \{q_n\}$, $0 < q_n < p_n \leq 1$ *be sequences satisfying* $\lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n = 1$ *and* $\lim_{n\to\infty} q_n^n = a$, $a \in [0, 1)$, *then we have*

$$S_{n,p,q}^{l}(t-x;x)=0, \tag{8}$$

$$
\begin{aligned}
S_{n,p,q}^{l}\left((t-x)^{2};x\right)&=-\frac{q^{n+l-1}}{[n+l]_{p,q}}x^{2}\\
&\quad+\frac{x}{[n]_{p,q}}\left(\frac{q[n]_{p,q}x}{[n+l]_{p,q}}+1-\frac{[n]_{p,q}x}{[n+l]_{p,q}}\right)_{p,q}^{n+l-1},
\end{aligned}\tag{9}
$$

$$\lim_{n\to\infty}[n]_{p_n,q_n}S_{n,p_n,q_n}^{l}\left((t-x)^{2};x\right)=-ax^{2}+\lambda_1 x, \tag{10}$$

$$\lim_{n\to\infty}[n]_{p_n,q_n}^{2}S_{n,p_n,q_n}^{l}\left((t-x)^{4};x\right)=3a^{2}x^{4}+3\lambda_2 x^{2}. \tag{11}$$

where $\lambda_1, \lambda_2 \in (0, 1]$ *depending on the sequence* $\{q_n\}$.

Proof From (3) and (4), we get (8). Since $S_{n,p,q}^{l}\left((t-x)^{2};x\right) = S_{n,p,q}^{l}\left(t^{2};x\right) - 2xS_{n,p,q}^{l}(t;x) - x^{2}$ and $p[n+l-1]_{p,q} = [n+l]_{p,q} - q^{n+l-1}$, we obtain (9). Indeed, we can get (10) easily from (9). Finally, since

$$\begin{aligned}
&S^l_{n,p,q}\left((t-x)^4;x\right)\\
&=S^l_{n,p,q}\left(t^4;x\right)-4xS^l_{n,p,q}\left(t^3;x\right)+6x^2S^l_{n,p,q}\left(t^2;x\right)-3x^4\\
&=\Bigg(\frac{p^6[n+l-1]_{p,q}[n+l-2]_{p,q}[n+l-3]_{p,q}}{[n+l]^3_{p,q}}\\
&\quad-\frac{4p^3[n+l-1]_{p,q}[n+l-2]_{p,q}}{[n+l]^2_{p,q}}+\frac{6p[n+l-1]_{p,q}}{[n+l]_{p,q}}-3\Bigg)x^4\\
&\quad+\frac{\left(p^5+2p^4q+3p^3q^2\right)[n+l-1]_{p,q}[n+l-2]_{p,q}x^3}{[n]_{p,q}[n+l]^2_{p,q}}\\
&\quad\times\left(\frac{q[n]_{p,q}x}{[n+l]_{p,q}}+1-\frac{[n]_{p,q}x}{[n+l]_{p,q}}\right)^{n+l-3}_{p,q}+\frac{\left(p^3+3p^2q+3pq^2\right)[n+l-1]_{p,q}x^2}{[n]^2_{p,q}[n+l]_{p,q}}\\
&\quad\times\left(\frac{q^2[n]_{p,q}x}{[n+l]_{p,q}}+1-\frac{[n]_{p,q}x}{[n+l]_{p,q}}\right)^{n+l-2}_{p,q}+\frac{x}{[n]^3_{p,q}}\left(\frac{q^3[n]_{p,q}x}{[n+l]_{p,q}}+1-\frac{[n]_{p,q}x}{[n+l]_{p,q}}\right)^{n+l-1}_{p,q}\\
&\quad-\frac{4\left(p^2+2pq\right)[n+l-1]_{p,q}x^3}{[n]_{p,q}[n+l]_{p,q}}\left(\frac{q[n]_{p,q}x}{[n+l]_{p,q}}+1-\frac{[n]_{p,q}x}{[n+l]_{p,q}}\right)^{n+l-2}_{p,q}\\
&\quad-\frac{4x^2}{[n]^2_{p,q}}\left(\frac{q^2[n]_{p,q}x}{[n+l]_{p,q}}+1-\frac{[n]_{p,q}x}{[n+l]_{p,q}}\right)^{n+l-1}_{p,q}\\
&\quad+\frac{6x^3}{[n]_{p,q}}\left(\frac{q[n]_{p,q}x}{[n+l]_{p,q}}+1-\frac{[n]_{p,q}x}{[n+l]_{p,q}}\right)^{n+l-1}_{p,q}.
\end{aligned}\tag{12}$$

Since

$$\begin{aligned}
&p^6[n+l-1]_{p,q}[n+l-2]_{p,q}[n+l-3]_{p,q}\\
&\quad=\left([n+l]_{p,q}-q^{n+l-1}\right)\left([n+l]_{p,q}-q^{n+l-1}-pq^{n+l-2}\right)\\
&\qquad\times\left([n+l]_{p,q}-q^{n+l-1}-pq^{n+l-2}-p^2q^{n+l-3}\right)\\
&\quad=[n+l]^3_{p,q}-[n+l]^2_{p,q}q^{n+l-3}\left(p^2+2pq+3q^2\right)+[n+l]_{p,q}q^{2n+2l-5}\\
&\qquad\times\left(p^3+3q^3+4pq^2+3p^2q\right)-[2]_{p,q}[3]_{p,q}q^{3n+3l-6},
\end{aligned}$$

and $4p^3[n+l-1]_{p,q}[n+l-2]_{p,q}=4[n+l]^2_{p,q}-4q^{n+l-2}[n+l]_{p,q}(2q+p)+4[2]_{p,q}q^{2n+2l-3}$, $6p[n+l-1]_{p,q}=6[n+l]_{p,q}-6q^{n+l-1}$, by some computations, we get

$$\begin{aligned}
&\lim_{n\to\infty}[n]^2_{p,q}\Bigg(\frac{p^6[n+l-1]_{p,q}[n+l-2]_{p,q}[n+l-3]_{p,q}}{[n+l]^3_{p,q}}\\
&\qquad-\frac{4p^3[n+l-1]_{p,q}[n+l-2]_{p,q}}{[n+l]^2_{p,q}}+\frac{6p[n+l-1]_{p,q}}{[n+l]_{p,q}}-3\Bigg)x^4
\end{aligned}$$

$$= \lim_{n\to\infty} [n]^2_{p,q} \left(\frac{-q^{n+l-3}(p-q)^2}{[n+l]_{p,q}} + \frac{q^{2n+2l-5}\left(p^3-q^3+3p^2q\right)}{[n+l]^2_{p,q}} - \frac{[2]_{p,q}[3]_{p,q}q^{3n+3l-6}}{[n+l]^3_{p,q}} \right) x^4. \tag{13}$$

From conditions of Lemma 2.3, we have

$$\lim_{n\to\infty} [n]^2_{p,q} \frac{-q^{n+l-3}(p-q)^2}{[n+l]_{p,q}} = -\lim_{n\to\infty} q^{n+l-3}\left(p^n-q^n\right)(p-q) = 0,$$

thus, (13) $= 3a^2x^4$. From (12), we get

$$\lim_{n\to\infty}[n]^2_{p,q} S^l_{n,p,q}\left((t-x)^4;x\right) = 3a^2x^4 + \lim_{n\to\infty}[n]^2_{p,q} \times \left(\frac{6x^3\lambda_1}{[n]_{p,q}} + \frac{7x^2\lambda_2}{[n]^2_{p,q}} + \frac{x\lambda_3}{[n]^3_{p,q}} - \frac{12x^3\lambda_1}{[n]_{p,q}} - \frac{4x^2\lambda_2}{[n]^2_{p,q}} + \frac{6x^3\lambda_1}{[n]_{p,q}} \right)$$
$$= 3a^2x^4 + 3\lambda_2 x^2,$$

where $\lambda_1, \lambda_2, \lambda_3 \in (0, 1]$ depending on the sequence $\{q_n\}$. □

3 Statistical approximation properties

In this section, we present the statistical approximation properties of the operator $S^l_{n,p,q}(f;x)$ by using the Korovkin-type statistical approximation theorem proved in [14].

Let K be a subset of $\mathbb{N}$, the set of all natural numbers. The density of K is defined by $\delta(K) := \lim_n \frac{1}{n}\sum_{k=1}^{n}\chi_K(k)$ provided the limit exists, where χ_K is the characteristic function of K. A sequence $x := \{x_n\}$ is called statistically convergent to a number L if, for every $\varepsilon > 0$, $\delta\{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\} = 0$. Let $A := (a_{jn})$, $j, n = 1, 2, \ldots$ be an infinite summability matrix. For a given sequence $x := \{x_n\}$, the $A-$transform of x, denoted by $Ax := ((Ax)_j)$, is given by $(Ax)_j = \sum_{k=1}^{\infty} a_{jn}x_n$ provided the series converges for each j. We say that A is regular if $\lim_n (Ax)_j = L$ whenever $\lim x = L$. Assume that A is a non-negative regular summability matrix. A sequence $x = \{x_n\}$ is called A-statistically convergent to L provided that for every $\varepsilon > 0$, $\lim_j \sum_{n:|x_n-L|\geq\varepsilon} a_{jn} = 0$. We denote this limit by $st_A - \lim_n x_n = L$. For $A = C_1$, the Cesàro matrix of order one, A-statistical convergence reduces to statistical convergence. It is easy to see that every convergent sequence is statistically convergent but not conversely.

We consider sequences $p := \{p_n\}$, $q := \{q_n\}$ for $0 < q_n < p_n \leq 1$ satisfying

$$st_A - \lim_{n\to\infty} p_n = st_A - \lim_{n\to\infty} q_n = 1 \textit{ and } st_A - \lim_{n\to\infty}[n]_{p_n,q_n} = \infty. \tag{14}$$

If $e_i = t^i,\ t \in \mathbb{R}^+,\ i = 0, 1, 2, \ldots$ stands for the ith monomial, then we have

Theorem 3.1 *Let $A=(a_{nk})$ be a non-negative regular summability matrix, $p:=\{p_n\}$ and $q:=\{q_n\}$ be sequences satisfying (14), then for all $f\in C(I)$, $x\in I$, we have*

$$st_A-\lim_n \left|\left|S^l_{n,p,q}f-f\right|\right|_{C(I)}=0.$$

Proof Obviously

$$st_A-\lim_n \left|\left|S^l_{n,p_n,q_n}(e_i)-e_i\right|\right|_{C(I)}=0,\quad i=0,1. \tag{15}$$

By (5), we have

$$\left|S^l_{n,p_n,q_n}(e_2;x)-e_2(x)\right|\le \frac{1}{[n+l]_{p_n,q_n}}+\frac{1}{[n]_{p_n,q_n}}.$$

Now for a given $\varepsilon>0$, let us define the following sets:

$$U:=\left\{k:\left|\left|S^l_{n,p_k,q_k}(e_2)-e_2\right|\right|_{C(I)}\ge\varepsilon\right\},\ U_1:=\left\{k:\frac{1}{[n+l]_{p_k,q_k}}\ge\frac{\varepsilon}{2}\right\},$$
$$U_2:=\left\{k:\frac{1}{[n]_{p_k,q_k}}\ge\frac{\varepsilon}{2}\right\}.$$

Then one can see that $U\subseteq U_1\cup U_2$, so we have

$$\delta\left\{k\le n:||S^l_{n,p_k,q_k}(e_2)-e_2||_{C(I)}\right\}\le\delta\left\{k\le n:\left|\frac{1}{[n+l]_{p_k,q_k}}\right|\ge\frac{\varepsilon}{2}\right\}$$
$$+\delta\left\{k\le n:\frac{1}{[n]_{p_k,q_k}}\ge\frac{\varepsilon}{2}\right\},$$

since $st_A-\lim_n p_n=st_A-\lim_n q_n=1$ and $st_A-\lim_n[n]_{p_n,q_n}=\infty$, we have

$$st_A-\lim_n\frac{1}{[n+l]_{p_n,q_n}}=st_A-\lim_n\frac{1}{[n]_{p_n,q_n}}=0,$$

which imply that the right-hand side of the above inequality is zero, thus we have

$$st_A-\lim_n||S^l_{n,p_n,q_n}(e_2)-e_2||_{C(I)}=0. \tag{16}$$

Combining (15) and (16), Theorem 3.1 follows from the Korovkin-type statistical approximation theorem established in [14], the proof is completed. □

4 Local approximation properties

Let $f \in C(I)$, endowed with the norm $||f|| = \sup_{x\in I} |f(x)|$. The Peetre's K-functional is defined by

$$K_2(f;\delta) = \inf_{g\in C^2} \left\{||f-g|| + \delta||g''||\right\},$$

where $\delta > 0$ and $C^2 = \left\{g \in C(I) : g', g'' \in C(I)\right\}$. By [15, p. 177, Theorem2.4], there exits an absolute constant $C > 0$ such that

$$K_2(f;\delta) \leq C\omega_2(f;\sqrt{\delta}), \tag{17}$$

where

$$\omega_2(f;\delta) = \sup_{0<h\leq\delta} \sup_{x,x+h,x+2h\in I} |f(x+2h) - 2f(x+h) + f(x)|$$

is the second order modulus of smoothness of $f \in C(I)$.

Now we give a direct local approximation theorem for the operators $S^l_{n,p,q}(f, x)$.

Theorem 4.1 *For $0 < q < p \leq 1$, $x \in I$ and $f \in C(I)$, we have*

$$\left|S^l_{n,p,q}(f;x) - f(x)\right| \leq C\omega_2\left(f; \sqrt{-\frac{q^{n+l-1}x^2}{2[n+l]_{p,q}} + \frac{x}{2[n]_{p,q}}\left(\frac{q[n]_{p,q}x}{[n+l]_{p,q}} + 1 - \frac{[n]_{p,q}x}{[n+l]_{p,q}}\right)^{n+l-1}_{p,q}}\right),$$

where C is a positive constant.

Proof Let $g \in C^2$. By Taylor's expansion

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)du,$$

and Lemma 2.1, we get

$$S^l_{n,p,q}(g;x) = g(x) + S^l_{n,p,q}\left(\int_x^t (t-u)g''(u)du; x\right).$$

Hence, by (9), we have

$$\begin{aligned}
&\left|S_{n,p,q}^{l}(g;x)-g(x)\right| \\
&\quad\leq \left|S_{n,p,q}^{l}\left(\int_{x}^{t}(t-u)g''(u)du;x\right)\right| \\
&\quad\leq S_{n,p,q}^{l}\left(\left|\int_{x}^{t}(t-u)|g''(u)|du\right|;x\right) \\
&\quad\leq S_{n,p,q}^{l}\left((t-x)^2;x\right)||g''|| \\
&\quad= \left[-\frac{q^{n+l-1}}{[n+l]_{p,q}}x^2+\frac{x}{[n]_{p,q}}\left(\frac{q[n]_{p,q}x}{[n+l]_{p,q}}+1-\frac{[n]_{p,q}x}{[n+l]_{p,q}}\right)_{p,q}^{n+l-1}\right]||g''||.
\end{aligned} \tag{18}$$

On the other hand, by (3), we have

$$\begin{aligned}
&\left|S_{n,p,q}^{l}(f;x)\right| \\
&\quad\leq \sum_{k=0}^{n+l}\begin{bmatrix}n+l\\k\end{bmatrix}_{p,q}\left(\frac{[n]_{p,q}x}{[n+l]_{p,q}}\right)^{k}\left(1-\frac{[n]_{p,q}x}{[n+l]_{p,q}}\right)_{p,q}^{n+l-k}\left|f\left(\frac{[k]_{p,q}}{[n]_{p,q}}\right)\right| \\
&\quad\leq ||f||.
\end{aligned} \tag{19}$$

Now (18) and (19) imply

$$\begin{aligned}
&\left|S_{n,p,q}^{l}(f;x)-f(x)\right| \\
&\quad\leq \left|S_{n,p,q}^{l}(f-g;x)-(f-g)(x)\right|+\left|S_{n,p,q}^{l}(g;x)-g(x)\right| \\
&\quad\leq 2||f-g||+\left[-\frac{q^{n+l-1}}{[n+l]_{p,q}}x^2+\frac{x}{[n]_{p,q}}\right. \\
&\qquad\left.\times\left(\frac{q[n]_{p,q}x}{[n+l]_{p,q}}+1-\frac{[n]_{p,q}x}{[n+l]_{p,q}}\right)_{p,q}^{n+l-1}\right]||g''||.
\end{aligned}$$

Hence taking infimum on the right hand side over all $g\in C^2$, we get

$$\begin{aligned}
&\left|S_{n,p,q}^{l}(f;x)-f(x)\right| \\
&\quad\leq 2K_2\left(f;-\frac{q^{n+l-1}x^2}{2[n+l]_{p,q}}+\frac{x}{2[n]_{p,q}}\left(\frac{q[n]_{p,q}x}{[n+l]_{p,q}}+1-\frac{[n]_{p,q}x}{[n+l]_{p,q}}\right)_{p,q}^{n+l-1}\right).
\end{aligned}$$

By (17), for every $0 < q < p \leq 1$, we have

$$\begin{aligned}&\left|S^{l}_{n,p,q}(f;x) - f(x)\right|\\&\leq C\omega_2\left(f;\sqrt{-\frac{q^{n+l-1}x^2}{2[n+l]_{p,q}} + \frac{x}{2[n]_{p,q}}\left(\frac{q[n]_{p,q}x}{[n+l]_{p,q}} + 1 - \frac{[n]_{p,q}x}{[n+l]_{p,q}}\right)^{n+l-1}_{p,q}}\right),\end{aligned}$$

where C is a positive constant. This completes the proof of Theorem 4.1. □

Remark 4.2 For any fixed $x \in I$, $l \in \mathbb{N}_0$ and $n \in \mathbb{N}$, let $p := \{p_n\}$ and $q := \{q_n\}$ are sequences satisfying $0 < p_n < q_n \leq 1$, $\lim_n p_n = \lim_n q_n = 1$ and $\lim_n [n]_{p_n,q_n} = \infty$, we have

$$\lim_{n\to\infty}\left[-\frac{q^{n+l-1}x^2}{[n+l]_{p,q}} + \frac{x}{[n]_{p,q}}\left(\frac{q[n]_{p,q}x}{[n+l]_{p,q}} + 1 - \frac{[n]_{p,q}x}{[n+l]_{p,q}}\right)^{n+l-1}_{p,q}\right] = 0.$$

These gives us a rate of pointwise convergence of the operators $S^{l}_{n,p_n,q_n}(f;x)$ to $f(x)$.

Next we study the rate of convergence of the operators $S^{l}_{n,p,q}(f;x)$ with the help of functions of Lipschitz class $Lip_M(\alpha)$, where $M > 0$ and $0 < \alpha \leq 1$. A function f belongs to $Lip_M(\alpha)$ if

$$|f(y) - f(x)| \leq M|y - x|^{\alpha} \quad (y, x \in \mathbb{R}). \tag{20}$$

We have the following theorem.

Theorem 4.3 *Let $p := \{p_n\}$ and $q := \{q_n\}$ are sequences satisfying $0 < q_n < p_n \leq 1$, $\lim_n p_n = \lim_n q_n = 1$, $\lim_n [n]_{p_n,q_n} = \infty$ and $f \in Lip_M(\alpha)$, $0 < \alpha \leq 1$. Then we have*

$$\begin{aligned}&\left|S^{l}_{n,p,q}(f;x) - f(x)\right|\\&\leq M\left[-\frac{q^{n+l-1}x^2}{[n+l]_{p,q}} + \frac{x}{[n]_{p,q}}\left(\frac{q[n]_{p,q}x}{[n+l]_{p,q}} + 1 - \frac{[n]_{p,q}x}{[n+l]_{p,q}}\right)^{n+l-1}_{p,q}\right]^{\frac{\alpha}{2}}.\end{aligned}$$

Proof Since $S^{l}_{n,p,q}(f;x)$ are linear positive operators and $f \in Lip_M(\alpha)$ $(0 < \alpha \leq 1)$, we have

$$\begin{aligned}
&\left|S_{n,p,q}^{l}(f;x)-f(x)\right| \\
&\quad\leq S_{n,p,q}^{l}\left(|f(t)-f(x)|;x\right) \\
&\quad=\sum_{k=0}^{n+l}\begin{bmatrix} n+l \\ k \end{bmatrix}_{p,q}\left(\frac{[n]_{p,q}x}{[n+l]_{p,q}}\right)^{k}\left(1-\frac{[n]_{p,q}x}{[n+l]_{p,q}}\right)_{p,q}^{n+l-k}\left|f\left(\frac{[k]_{p,q}}{[n]_{p,q}}\right)-f(x)\right| \\
&\quad\leq M\sum_{k=0}^{n+l}\begin{bmatrix} n+l \\ k \end{bmatrix}_{p,q}\left(\frac{[n]_{p,q}x}{[n+l]_{p,q}}\right)^{k}\left(1-\frac{[n]_{p,q}x}{[n+l]_{p,q}}\right)_{p,q}^{n+l-k}\left|\frac{[k]_{p,q}}{[n]_{p,q}}-x\right|^{\alpha} \\
&\quad\leq M\sum_{k=0}^{n+l}\left(\begin{bmatrix} n+l \\ k \end{bmatrix}_{p,q}\left(\frac{[n]_{p,q}x}{[n+l]_{p,q}}\right)^{k}\left(1-\frac{[n]_{p,q}x}{[n+l]_{p,q}}\right)_{p,q}^{n+l-k}\left(\frac{[k]_{p,q}}{[n]_{p,q}}-x\right)^{2}\right)^{\frac{\alpha}{2}} \\
&\qquad\times\left(\begin{bmatrix} n+l \\ k \end{bmatrix}_{p,q}\left(\frac{[n]_{p,q}x}{[n+l]_{p,q}}\right)^{k}\left(1-\frac{[n]_{p,q}x}{[n+l]_{p,q}}\right)_{p,q}^{n+l-k}\right)^{\frac{2-\alpha}{2}}.
\end{aligned}$$

Applying Hölder's inequality for sums, we obtain

$$\begin{aligned}
&\left|S_{n,p,q}^{l}(f;x)-f(x)\right| \\
&\quad\leq M\left(\sum_{k=0}^{n+l}\begin{bmatrix} n+l \\ k \end{bmatrix}_{p,q}\left(\frac{[n]_{p,q}x}{[n+l]_{p,q}}\right)^{k}\left(1-\frac{[n]_{p,q}x}{[n+l]_{p,q}}\right)_{p,q}^{n+l-k}\left(\frac{[k]_{p,q}}{[n]_{p,q}}-x\right)^{2}\right)^{\frac{\alpha}{2}} \\
&\qquad\times\left(\sum_{k=0}^{n+l}\begin{bmatrix} n+l \\ k \end{bmatrix}_{p,q}\left(\frac{[n]_{p,q}x}{[n+l]_{p,q}}\right)^{k}\left(1-\frac{[n]_{p,q}x}{[n+l]_{p,q}}\right)_{p,q}^{n+l-k}\right)^{\frac{2-\alpha}{2}} \\
&\quad=M\left(S_{n,p,q}^{l}\left((t-x)^{2};x\right)\right)^{\frac{\alpha}{2}} \\
&\quad=M\left[-\frac{q^{n+l-1}x^{2}}{[n+l]_{p,q}}+\frac{x}{[n]_{p,q}}\left(\frac{q[n]_{p,q}x}{[n+l]_{p,q}}+1-\frac{[n]_{p,q}x}{[n+l]_{p,q}}\right)_{p,q}^{n+l-1}\right]^{\frac{\alpha}{2}}.
\end{aligned}$$

Thus, Theorem 4.3 is proved. □

Now, we give a Voronovskaja-type asymptotic formula for $S_{n,p,q}^{l}(f;x)$.

Theorem 4.4 *Let* $p=\{p_n\}$, $q=\{q_n\}$, $0<q_n<p_n\leq 1$ *be sequences satisfying* $\lim_{n\to\infty}p_n=\lim_{n\to\infty}q_n=1$, $\lim_n[n]_{p_n,q_n}=\infty$ *and* $\lim_{n\to\infty}q_n^n=a$, $a\in[0,1)$, *then we have*

$$\lim_{n\to\infty}[n]_{p,q}\left(S_{n,p,q}^{l}(f;x)-f(x)\right)=\frac{f''(x)}{2}\left(-ax^{2}+\lambda_1 x\right),$$

where $\lambda_1\in(0,1]$ *depending on the sequence* $\{q_n\}$.

Proof Let $x\in[0,1]$ be fixed. By the Taylor formula, we may write

$$f(t)=f(x)+f'(x)(t-x)+\frac{1}{2}f''(x)(t-x)^{2}+r(t;x)(t-x)^{2}, \tag{21}$$

where $r(t; x)$ is the Peano form of the remainder, $r(t; x) \in C(I)$, using L'Hopital's rule, we have

$$\lim_{t \to x} r(t; x) = \lim_{t \to \infty x} \frac{f(t) - f(x) - f'(x)(t-x) - \frac{1}{2} f''(x)(t-x)^2}{(t-x)^2}$$
$$= \lim_{t \to x} \frac{f'(t) - f'(x) - f''(x)(t-x)}{2(t-x)} = \lim_{t \to x} \frac{f''(t) - f''(x)}{2} = 0.$$

Since (8), applying $S^l_{n,p,q}(f; x)$ to (21), we obtain

$$[n]_{p,q} \left(S^l_{n,p,q}(f; x) - f(x) \right)$$
$$= \frac{1}{2} f''(x) S^l_{n,p,q} \left((t-x)^2; x \right) + [n]_{p,q} S^l_{n,p,q} \left(r(t; x)(t-x)^2; x \right).$$

By the Cauchy–Schwarz inequality, we have

$$S^l_{n,p,q} \left(r(t; x)(t-x)^2; x \right) \leq \sqrt{S^l_{n,p,q} \left(r^2(t; x); x \right)} \sqrt{S^l_{n,p,q} \left((t-x)^4; x \right)}.$$

Since $r^2(x; x) = 0$, then it is obtained easily that $\lim_{n \to \infty} [n]_{p,q} S^l_{n,p,q} \big(r(t; x)(t-x)^2; x \big) = 0$ by (11). Thus, from (10), we have

$$\lim_{n \to \infty} [n]_{p,q} \left(S^l_{n,p,q}(f; x) - f(x) \right) = \lim_{n \to \infty} \frac{1}{2} [n]_{p,q} f''(x) S^l_{n,p,q} \left((t-x)^2; x \right)$$
$$= \frac{f''(x)}{2} \left(-ax^2 + \lambda_1 x \right).$$

Theorem 4.4 is proved. □

5 Construction of bivariate operators and some approximation properties

In this section, we construct a bivariate (p, q)-analogue of modified Bernstein–Schurer operators and get some approximation properties.

For $f \in C(I_1 \times I_2)$, $I_1 \times I_2 = [0, 1 + l_1] \times [0, 1 + l_2]$, $l_1, l_2 \in \mathbb{N}_0$, $x \in I_1$, $y \in I_2$, $0 < q_{n_1}, q_{n_2} < p_{n_1}, p_{n_2} \leq 1$ and $n_1, n_2 \in \mathbb{N}$, the bivariate (p, q)-analogue of modified Bernstein–Schurer operators are defined as follows

$$S_{p_{n_1},p_{n_2},q_{n_1},q_{n_2}}^{n_1,n_2,l_1,l_2}(f;x,y)=\sum_{k_1=0}^{n_1+l_1}\sum_{k_2=0}^{n_2+l_2}\begin{bmatrix}n_1+l_1\\k_1\end{bmatrix}_{p_{n_1},q_{n_1}}\begin{bmatrix}n_2+l_2\\k_2\end{bmatrix}_{p_{n_2},q_{n_2}}$$
$$\left(\frac{[n_1]_{p_{n_1},q_{n_1}}x}{[n_1+l_1]_{p_{n_1},q_{n_1}}^{k_1}}\right)^{k_1}\left(1-\frac{[n_1]_{p_{n_1},q_{n_1}}x}{[n_1+l_1]_{p_{n_1},q_{n_1}}}\right)_{p_{n_1},q_{n_1}}^{n_1+l_1-k_1}\left(\frac{[n_2]_{p_{n_2},q_{n_2}}y}{[n_2+l_2]_{p_{n_2},q_{n_2}}}\right)^{k_2}$$
$$\left(1-\frac{[n_2]_{p_{n_2},q_{n_2}}y}{[n_2+l_2]_{p_{n_2},q_{n_2}}}\right)_{p_{n_2},q_{n_2}}^{n_2+l_2-k_2}f\left(\frac{[k_1]_{p_{n_1},q_{n_1}}}{[n_1]_{p_{n_1},q_{n_1}}},\frac{[k_2]_{p_{n_2},q_{n_2}}}{[n_2]_{p_{n_2},q_{n_2}}}\right). \tag{22}$$

Lemma 5.1 *Let* $e_{i,j}(x,y)=x^iy^j$, $i,j\in\mathbb{N}$, $(x,y)\in(I_1\times I_2)$ *be the two-dimensional test functions, the bivariate* (p,q)*-analogue of modified Bernstein–Schurer operators defined in* (22) *satisfy the following equalities*

$$S_{p_{n_1},p_{n_2},q_{n_1},q_{n_2}}^{n_1,n_2,l_1,l_2}(e_{0,0};x,y)=1; \tag{23}$$
$$S_{p_{n_1},p_{n_2},q_{n_1},q_{n_2}}^{n_1,n_2,l_1,l_2}(e_{1,0};x,y)=x; \tag{24}$$
$$S_{p_{n_1},p_{n_2},q_{n_1},q_{n_2}}^{n_1,n_2,l_1,l_2}(e_{0,1};x,y)=y; \tag{25}$$
$$S_{p_{n_1},p_{n_2},q_{n_1},q_{n_2}}^{n_1,n_2,l_1,l_2}(e_{1,1};x,y)=xy; \tag{26}$$
$$S_{p_{n_1},p_{n_2},q_{n_1},q_{n_2}}^{n_1,n_2,l_1,l_2}(e_{2,0};x,y)=\frac{p_{n_1}[n_1+l_1-1]_{p_{n_1},q_{n_1}}x^2}{[n_1+l_1]_{p_{n_1},q_{n_1}}}+\frac{x}{[n_1]_{p_{n_1},q_{n_1}}}$$
$$\times\left(\frac{q_1[n_1]_{p_{n_1},q_{n_1}}x}{[n_1+l_1]_{p_{n_1},q_{n_1}}}+1-\frac{[n_1]_{p_{n_1},q_{n_1}}x}{[n_1+l_1]_{p_{n_1},q_{n_1}}}\right)_{p_{n_1},q_{n_1}}^{n_1+l_1-1}; \tag{27}$$
$$S_{p_{n_1},p_{n_2},q_{n_1},q_{n_2}}^{n_1,n_2,l_1,l_2}(e_{0,2};x,y)=\frac{p_{n_2}[n_2+l_2-1]_{p_{n_2},q_{n_2}}y^2}{[n_2+l_2]_{p_{n_2},q_{n_2}}}+\frac{y}{[n_2]_{p_{n_2},q_{n_2}}}$$
$$\times\left(\frac{q_2[n_2]_{p_{n_2},q_{n_2}}y}{[n_2+l_2]_{p_{n_2},q_{n_2}}}+1-\frac{[n_2]_{p_{n_2},q_{n_2}}x}{[n_2+l_2]_{p_{n_2},q_{n_2}}}\right)_{p_{n_2},q_{n_2}}^{n_2+l_2-1}. \tag{28}$$

Remark 5.2 Let $\{p_{n_1}\}$, $\{p_{n_2}\}$, $\{q_{n_1}\}$ and $\{q_{n_2}\}$ are sequences such that $0<q_{n_1},q_{n_2}<p_{n_1},p_{n_2}\le 1$. Then by [16] and Lemma 5.1, for any $f\in C(I_1\times I_2)$, if $\lim_{n_1\to\infty}p_{n_1}=\lim_{n_1\to\infty}q_{n_1}=\lim_{n_2\to\infty}p_{n_2}=\lim_{n_2\to\infty}q_{n_2}=1$ and $\lim_{n_1\to\infty}[n_1]_{p_{n_1},q_{n_1}}=\lim_{n_2\to\infty}[n_2]_{p_{n_2},q_{n_2}}=\infty$, operators $S_{p_{n_1},p_{n_2},q_{n_1},q_{n_2}}^{n_1,n_2,l_1,l_2}(f;x,y)$ convergence uniformly to $f(x,y)$.

Lemma 5.3 *Let* $\{p_{n_1}\}$, $\{p_{n_2}\}$, $\{q_{n_1}\}$, $\{q_{n_2}\}$, $0<q_{n_1},q_{n_2}<p_{n_1},p_{n_2}\le 1$ *be sequences satisfying* $\lim_{n_1\to\infty}p_{n_1}=\lim_{n_1\to\infty}q_{n_1}=\lim_{n_2\to\infty}p_{n_2}=\lim_{n_2\to\infty}q_{n_2}=1$ *and* $\lim_{n_1\to\infty}[n_1]_{p_{n_1},q_{n_1}}=\lim_{n_2\to\infty}[n_2]_{p_{n_2},q_{n_2}}=\infty$. *The following equalities hold*

$$S_{p_{n_1},p_{n_2},q_{n_1},q_{n_2}}^{n_1,n_2,l_1,l_2}(t-x;x,y)=0; \tag{29}$$

$$S^{n_1,n_2,l_1,l_2}_{p_{n_1},p_{n_2},q_{n_1},q_{n_2}}(s-y;x,y)=0; \tag{30}$$

$$S^{n_1,n_2,l_1,l_2}_{p_{n_1},p_{n_2},q_{n_1},q_{n_2}}\left((t-x)^2;x,y\right)=-\frac{q_{n_1}^{n_1+l_1-1}x^2}{[n_1+l_1]_{p_{n_1},q_{n_1}}}+\frac{x}{[n_1]_{p_{n_1},q_{n_1}}}$$
$$\times\left(\frac{q_{n_1}[n_1]_{p_{n_1},q_{n_1}}x}{[n_1+l_1]_{p_{n_1},q_{n_1}}}+1-\frac{[n_1]_{p_{n_1},q_{n_1}}x}{[n_1+l_1]_{p_{n_1},q_{n_1}}}\right)^{n_1+l_1-1}_{p_{n_1},q_{n_1}}:=\delta_{n_1}(x); \tag{31}$$

$$S^{n_1,n_2,l_1,l_2}_{p_{n_1},p_{n_2},q_{n_1},q_{n_2}}\left((s-y)^2;x,y\right)=-\frac{q_{n_2}^{n_2+l_2-1}y^2}{[n_2+l_2]_{p_{n_2},q_{n_2}}}+\frac{y}{[n_2]_{p_{n_2},q_{n_2}}}$$
$$\times\left(\frac{q_{n_2}[n_2]_{p_{n_2},q_{n_2}}y}{[n_2+l_2]_{p_{n_2},q_{n_2}}}+1-\frac{[n_2]_{p_{n_2},q_{n_2}}y}{[n_2+l_2]_{p_{n_2},q_{n_2}}}\right)^{n_2+l_2-1}_{p_{n_2},q_{n_2}}:=\delta_{n_2}(y). \tag{32}$$

For $f\in C(I_1\times I_2)$, the complete modulus of continuity for the bivariate case is defined as

$$\begin{aligned}&\omega(f;\delta_1,\delta_2)\\&\quad=\sup\left\{|f(t,s)-f(x,y)|:(t,s),(x,y)\in(I_1\times I_2),|t-x|\leq\delta_1,|s-y|\leq\delta_2\right\},\end{aligned}$$

where $\delta_1,\delta_2>0$. Furthermore, $\omega(f;\delta_1,\delta_2)$ satisfies the following properties:

$$(i)\ \omega(f;\delta_1,\delta_2)\to0,\ if\ \delta_1,\delta_2\to0;$$
$$(ii)\ |f(t,s)-f(x,y)|\leq\omega(f;\delta_1,\delta_2)\left(1+\frac{|t-x|}{\delta_1}\right)\left(1+\frac{|s-y|}{\delta_2}\right).$$

The partial moduli of continuity with respect to x and y is defined as

$$\omega^{(1)}(f;\delta)=\sup\left\{|f(x_1,y)-f(x_2,y)|:y\in[0,1]\ and\ |x_1-x_2|\leq\delta\right\},$$
$$\omega^{(2)}(f;\delta)=\sup\left\{|f(x,y_1)-f(x,y_2)|:x\in[0,1]\ and\ |y_1-y_2|\leq\delta\right\}.$$

Details of the modulus of continuity for bivariate case can be found in [17].

Now, we give the estimate of the rate of convergence of bivariate (p, q)-analogue of modified Bernstein–Schurer operators defined in (22).

Theorem 5.4 *For $f\in C(I_1\times I_2)$, under the conditions of Lemma 5.3, we have*

$$\left|S^{n_1,n_2,l_1,l_2}_{p_{n_1},p_{n_2},q_{n_1},q_{n_2}}(f;x,y)-f(x,y)\right|\leq4\omega\left(f;\sqrt{\delta_{n_1}(x)},\sqrt{\delta_{n_2}(y)}\right),$$

where $\delta_{n_1}(x)$ and $\delta_{n_2}(y)$ are defined in (31) and (32).

Proof From Lemma 5.1, using the property (ii) above and Cauchy–Schwarz inequality, we easily get

$$\begin{aligned}
&\left|S^{n_1,n_2,l_1,l_2}_{p_{n_1},p_{n_2},q_{n_1},q_{n_2}}(f;x,y)-f(x,y)\right| \\
&\le S^{n_1,n_2,l_1,l_2}_{p_{n_1},p_{n_2},q_{n_1},q_{n_2}}(|f(t,s)-f(x,y)|;x,y) \\
&\le \omega\left(f;\sqrt{\delta_{n_1}(x)},\sqrt{\delta_{n_2}(y)}\right)\left(1+\sqrt{\frac{S^{n_1,n_2,l_1,l_2}_{p_{n_1},p_{n_2},q_{n_1},q_{n_2}}\left((t-x)^2;x,y\right)}{\delta_{n_1}(x)}}\right) \\
&\quad\times\left(1+\sqrt{\frac{S^{n_1,n_2,l_1,l_2}_{p_{n_1},p_{n_2},q_{n_1},q_{n_2}}\left((s-y)^2;x,y\right)}{\delta_{n_2}(y)}}\right).
\end{aligned}$$

Theorem 5.4 is proved. □

Theorem 5.5 *For $f\in C(I_1\times I_2)$, under the conditions of Lemma 5.3, we have*

$$\left|S^{n_1,n_2,l_1,l_2}_{p_{n_1},p_{n_2},q_{n_1},q_{n_2}}(f;x,y)-f(x,y)\right|\le 2\left[f;\omega^{(1)}\left(f;\sqrt{\delta_{n_1}(x)}\right)+\omega^{(2)}\left(f;\sqrt{\delta_{n_2}(y)}\right)\right],$$

where $\delta_{n_1}(x)$ and $\delta_{n_2}(y)$ are defined in (31) *and* (32).

Proof Using the definition of partial moduli of continuity above and Cauchy–Schwarz inequality, we have

$$\begin{aligned}
&\left|S^{n_1,n_2,l_1,l_2}_{p_{n_1},p_{n_2},q_{n_1},q_{n_2}}(f;x,y)-f(x,y)\right| \\
&\le S^{n_1,n_2,l_1,l_2}_{p_{n_1},p_{n_2},q_{n_1},q_{n_2}}(|f(t,s)-f(x,y)|;x,y) \\
&\le S^{n_1,n_2,l_1,l_2}_{p_{n_1},p_{n_2},q_{n_1},q_{n_2}}(|f(t,s)-f(t,y)|;x,y) \\
&\quad+S^{n_1,n_2,l_1,l_2}_{p_{n_1},p_{n_2},q_{n_1},q_{n_2}}(|f(t,y)-f(x,y)|;x,y) \\
&\le S^{n_1,n_2,l_1,l_2}_{p_{n_1},p_{n_2},q_{n_1},q_{n_2}}\left(\omega^{(2)}(f;|s-y|);x,y\right) \\
&\quad+S^{n_1,n_2,l_1,l_2}_{p_{n_1},p_{n_2},q_{n_1},q_{n_2}}\left(\omega^{(1)}(f;|t-x|);x,y\right) \\
&\le \omega^{(2)}\left(f;\sqrt{\delta_{n_2}(y)}\right)\left(1+\sqrt{\frac{S^{n_1,n_2,l_1,l_2}_{p_{n_1},p_{n_2},q_{n_1},q_{n_2}}\left((s-y)^2;x,y\right)}{\delta_{n_2}(y)}}\right) \\
&\quad+\omega^{(1)}\left(f;\sqrt{\delta_{n_1}(y)}\right)\left(1+\sqrt{\frac{S^{n_1,n_2,l_1,l_2}_{p_{n_1},p_{n_2},q_{n_1},q_{n_2}}\left((t-x)^2;x,y\right)}{\delta_{n_1}(x)}}\right).
\end{aligned}$$

Theorem 5.5 is proved. □

Finally, we study the rate of convergence of $S^{n_1,n_2,l_1,l_2}_{p_{n_1},p_{n_2},q_{n_1},q_{n_2}}(f;x,y)$ by means of functions of Lipschitz class $Lip_M(\alpha_1,\alpha_2)$ if

$$|f(t,s)-f(x,y)|\le M|t-x|^{\alpha_1}|s-y|^{\alpha_2},\quad (t,s),(x,y)\in(I_1\times I_2).$$

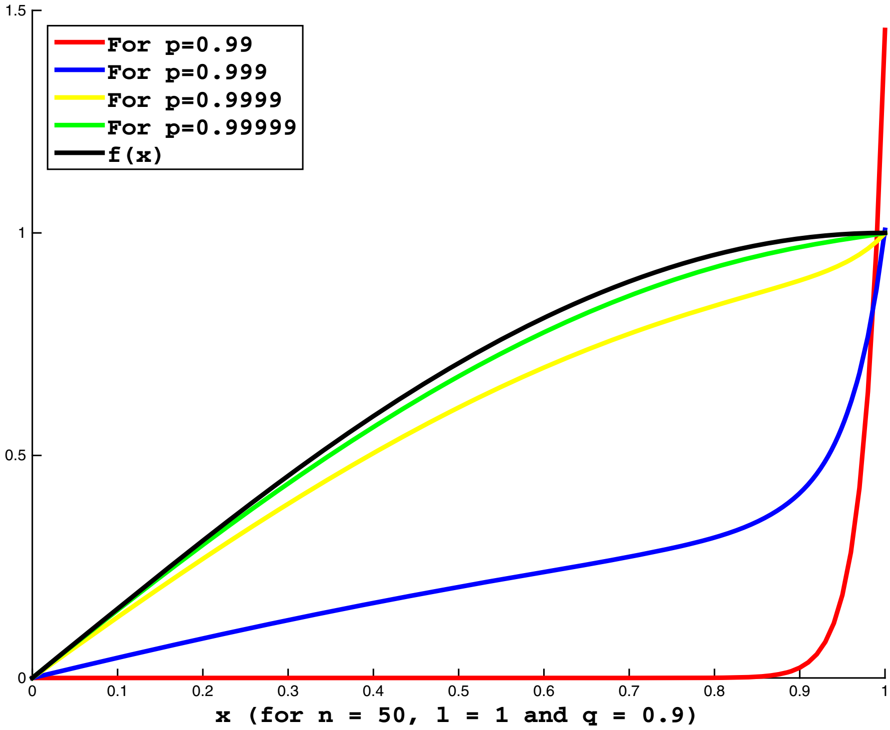

Fig. 1 Convergence of $S^{l}_{n,p,q}(f; x)$ for $n = 50$, $l = 1$, $q = 0.9$ and different values of p

Theorem 5.6 *Let $f \in Lip_M(\alpha_1, \alpha_2)$, under the conditions of Lemma 5.3, we have*

$$\left| S^{n_1,n_2,l_1,l_2}_{p_{n_1},p_{n_2},q_{n_1},q_{n_2}}(f; x, y) - f(x, y) \right| \leq M \delta^{\alpha_1/2}_{n_1}(x) \delta^{\alpha_2/2}_{n_2}(y),$$

where $\delta_{n_1}(x)$ and $\delta_{n_2}(y)$ are defined in (31) and (32).

Proof Since $f \in Lip_M(\alpha_1, \alpha_2)$, we get

$$\begin{aligned} &\left| S^{n_1,n_2,l_1,l_2}_{p_{n_1},p_{n_2},q_{n_1},q_{n_2}}(f; x, y) - f(x, y) \right| \\ &\leq S^{n_1,n_2,l_1,l_2}_{p_{n_1},p_{n_2},q_{n_1},q_{n_2}}(|f(t, s) - f(x, y)|; x, y) \\ &\leq M S^{n_1,n_2,l_1,l_2}_{p_{n_1},p_{n_2},q_{n_1},q_{n_2}}(|t - x|^{\alpha_1}; x, y) S^{n_1,n_2,l_1,l_2}_{p_{n_1},p_{n_2},q_{n_1},q_{n_2}}(|s - y|^{\alpha_2}; x, y), \end{aligned}$$

using the Hölder's inequality for last formula, respectively, we obtain

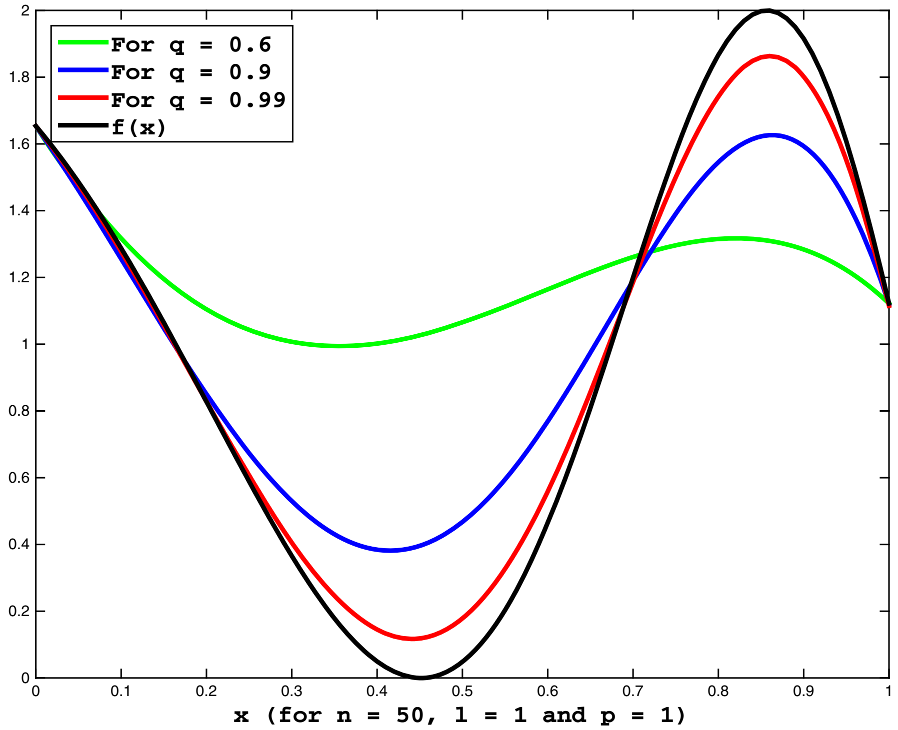

Fig. 2 Convergence of $S^{l}_{n,p,q}(f;x)$ for $n=50$, $l=1$, $p=1$ and different values of q

$$\begin{aligned}
&\left|S^{n_1,n_2,l_1,l_2}_{p_{n_1},p_{n_2},q_{n_1},q_{n_2}}(f;x,y)-f(x,y)\right|\\
&\quad\le M\left[S^{n_1,n_2,l_1,l_2}_{p_{n_1},p_{n_2},q_{n_1},q_{n_2}}\left((t-x)^2;x,y\right)\right]^{\frac{\alpha_1}{2}}\left[S^{n_1,n_2,l_1,l_2}_{p_{n_1},p_{n_2},q_{n_1},q_{n_2}}(1;x,y)\right]^{\frac{2-\alpha_1}{2}}\\
&\qquad\times\left[S^{n_1,n_2,l_1,l_2}_{p_{n_1},p_{n_2},q_{n_1},q_{n_2}}\left((s-y)^2;x,y\right)\right]^{\frac{\alpha_2}{2}}\left[S^{n_1,n_2,l_1,l_2}_{p_{n_1},p_{n_2},q_{n_1},q_{n_2}}(1;x,y)\right]^{\frac{2-\alpha_2}{2}}\\
&\quad=M\delta^{\alpha_1/2}_{n_1}(x)\delta^{\alpha_2/2}_{n_2}(y),
\end{aligned}$$

where $\delta_{n_1}(x)$ and $\delta_{n_2}(y)$ are defined in (31) and (32). Theorem 5.6 is proved. □

6 Graphical analysis

In this section, we give several graphs to show the convergence of $S^{l}_{n,p,q}(f;x)$ to $f(x)$ and $S^{n_1,n_2,l_1,l_2}_{p_{n_1},p_{n_2},q_{n_1},q_{n_2}}(f;x,y)$ to $f(x,y)$ with different values of parameters.

Let $f(x)=\sin(\pi x/2)$, for $n=50$, $l=1$ and $q=0.9$, the graphs of $S^{l}_{n,p,q}(f;x)$ with different values of p are shown in Fig. 1. Moreover, let $f(x)=1-\cos(4e^{x})$, for $n=50$, $l=1$ and $p=1$, the graphs of $S^{l}_{n,p,q}(f;x)$ with different values of q are shown in Fig. 2.

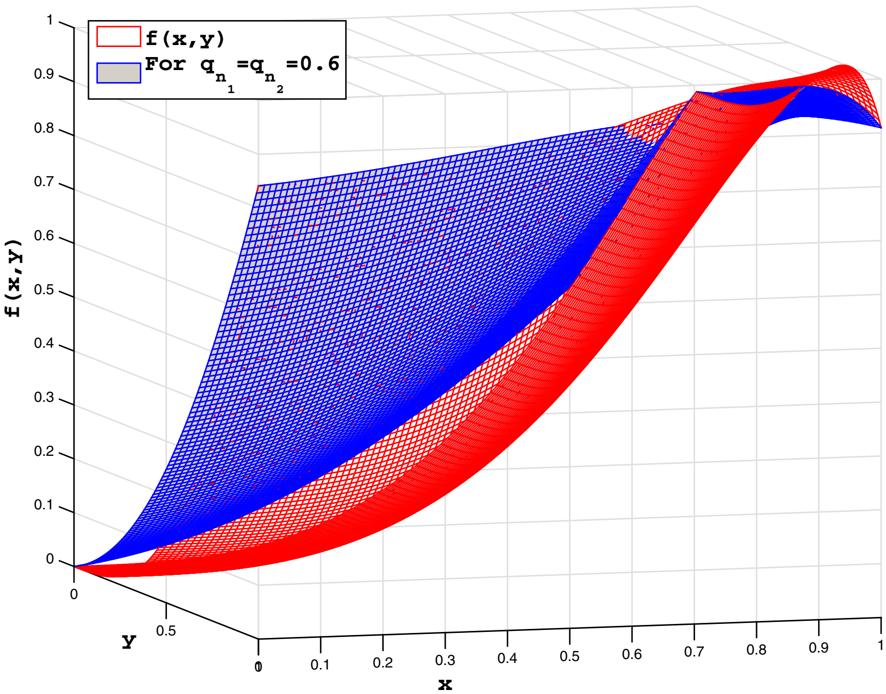

Fig. 3 $S^{n_1,n_2,l_1,l_2}_{p_{n_1},p_{n_2},q_{n_1},q_{n_2}}(f; x, y)$ for $n_1 = n_2 = 50$, $l_1 = l_2 = 1$, $p_{n_1} = p_{n_2} = 1$ and $q_{n_1} = q_{n_2} = 0.6$

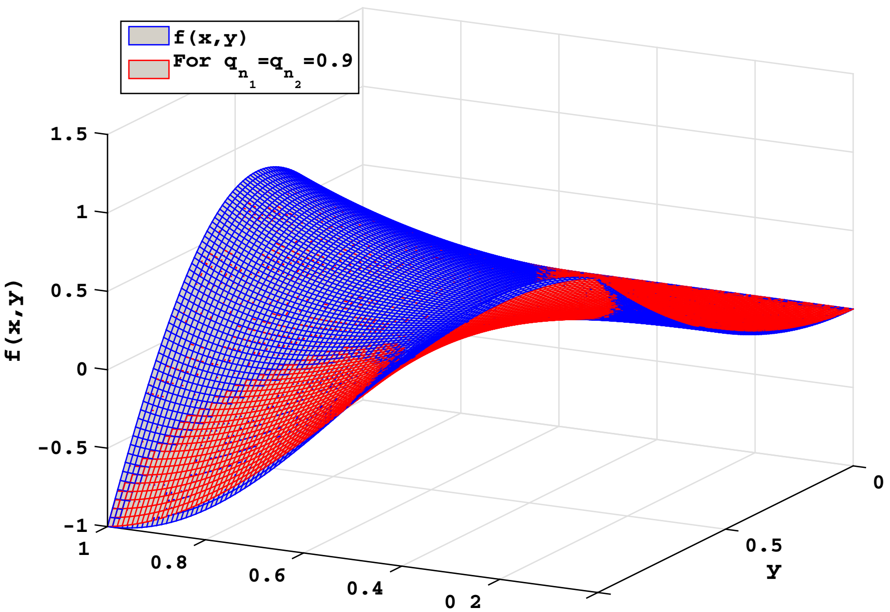

Fig. 4 $S^{n_1,n_2,l_1,l_2}_{p_{n_1},p_{n_2},q_{n_1},q_{n_2}}(f; x, y)$ for $n_1 = n_2 = 50$, $l_1 = l_2 = 1$, $p_{n_1} = p_{n_2} = 1$ and $q_{n_1} = q_{n_2} = 0.9$

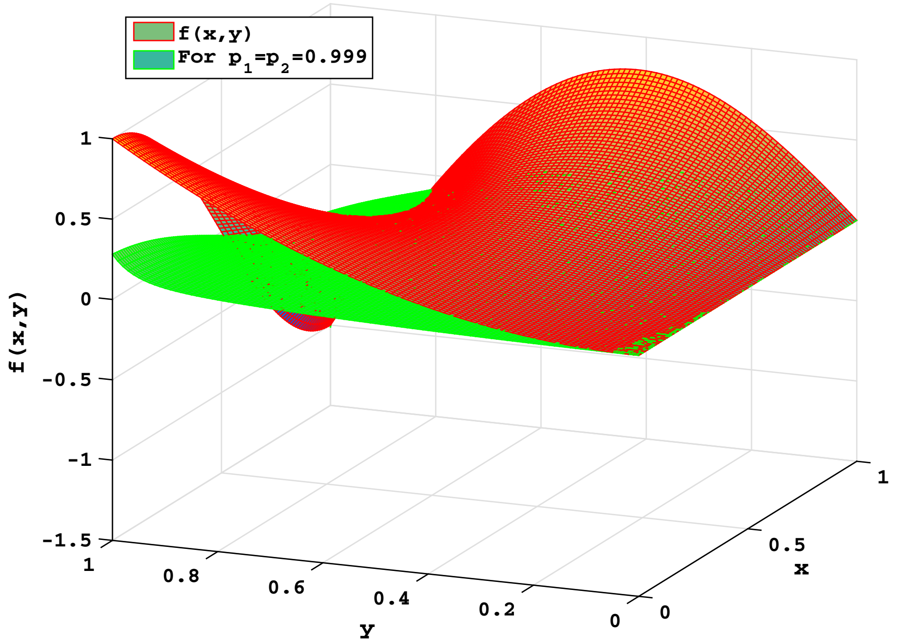

Fig. 5 $S^{n_1,n_2,l_1,l_2}_{p_{n_1},p_{n_2},q_{n_1},q_{n_2}}(f; x, y)$ for $n_1 = n_2 = 50$, $l_1 = l_2 = 1$, $p_{n_1} = p_{n_2} = 0.999$ and $q_{n_1} = q_{n_2} = 0.9$

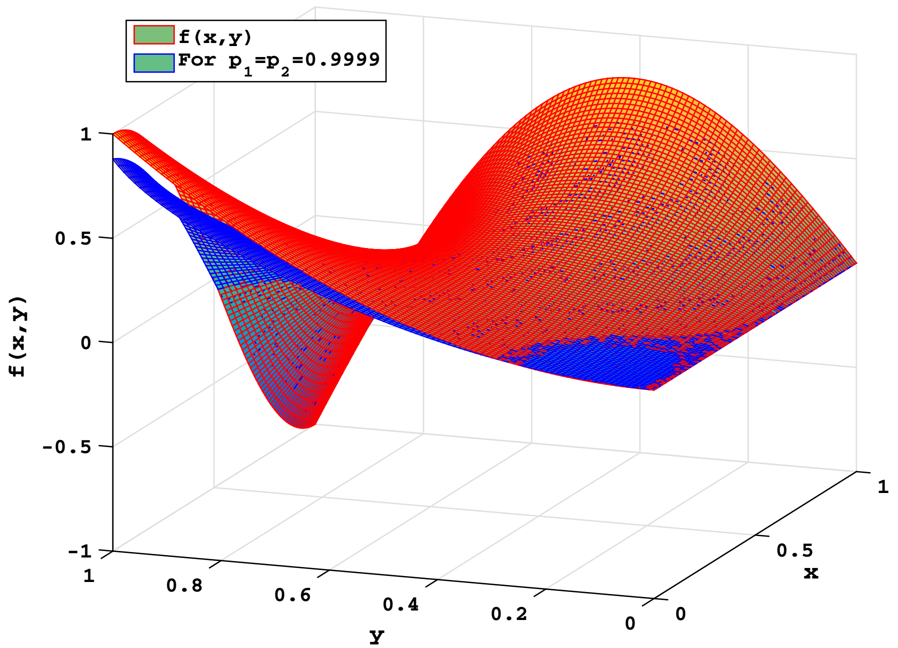

Fig. 6 $S^{n_1,n_2,l_1,l_2}_{p_{n_1},p_{n_2},q_{n_1},q_{n_2}}(f; x, y)$ for $n_1 = n_2 = 50$, $l_1 = l_2 = 1$, $p_{n_1} = p_{n_2} = 0.9999$ and $q_{n_1} = q_{n_2} = 0.9$

Let $f(x, y) = \sin(x^3 + y^3)$, Fig. 3 shows the graphs of $S^{n_1,n_2,l_1,l_2}_{p_{n_1},p_{n_2},q_{n_1},q_{n_2}}(f; x, y)$ (blue) and $f(x, y)$ (red) for $n_1 = n_2 = 50$, $l_1 = l_2 = 1$, $p_{n_1} = p_{n_2} = 1$ and $q_{n_1} = q_{n_2} = 0.6$. Let $f(x, y) = x^2 \sin(\pi y) + \cos(\pi x)y^2$, in Fig. 4, the values of q_{n_1}, q_{n_2} are replaced by 0.9, the graphs of $S^{n_1,n_2,l_1,l_2}_{p_{n_1},p_{n_2},q_{n_1},q_{n_2}}(f; x, y)$ (red) and $f(x, y)$ (blue) are shown.

Let $f(x, y) = x^2 \sin(\pi y) + \cos(\pi x)y^2$, Fig. 5 shows the graphs of $S^{n_1,n_2,l_1,l_2}_{p_{n_1},p_{n_2},q_{n_1},q_{n_2}}$ $(f; x, y)$ (green) and $f(x, y)$ (red) for $n_1 = n_2 = 50, l_1 = l_2 = 1, p_{n_1} = p_{n_2} = 0.999$ and $q_{n_1} = q_{n_2} = 0.9$. Finally, the values of p_{n_1}, p_{n_2} are substituted for 0.9999, the graphs of $S^{n_1,n_2,l_1,l_2}_{p_{n_1},p_{n_2},q_{n_1},q_{n_2}}(f; x, y)$ (blue) and $f(x, y)$ (red) are shown in Fig. 6.

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