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Certain types of irregular *m*-polar fuzzy graphs

Muhammad Akram[1] · **Hafiza Raheela Younas**[1]

Abstract The notion of an *m*-polar fuzzy set is a generalization of a bipolar fuzzy set. We apply the concept of *m*-polar fuzzy sets to graphs. We introduce certain types of irregular *m*-polar fuzzy graphs and investigate some of their properties. We describe the concepts of types of irregular *m*-polar fuzzy graphs with several examples. We also present applications of *m*-polar fuzzy graphs in decision making and social network as examples.

Keywords *m*-Polar fuzzy set · *m*-Polar fuzzy graph · Irregular *m*-polar fuzzy graph

Mathematics Subject Classification 03E72 · 68R10 · 68R05

1 Introduction

In 1965, Zadeh [14] introduced the concept of a fuzzy subset of a set. Since then, the fuzzy set theory has become a robust branch of research in different fields including graph theory, decision making, social science, medical field, management, artificial intelligence, engineering. In 1994, Zhang [16] proposed the concept of bipolar fuzzy sets as a generalization of fuzzy sets. The range of the membership degree of a bipolar fuzzy set is $[-1,1]$. In 2014, Chen et al. [7] proposed the concept of *m*-polar fuzzy set as a generalization of bipolar fuzzy set. He showed that bipolar fuzzy sets and 2-polar fuzzy sets are cryptomorphic mathematical notions and that we can obtain

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concisely one from the corresponding one [7]. The idea behind this is that "multipolar information" (not just bipolar information which correspond to two-valued logic) exists because the data for the real world problems is sometimes taken from n agents ($n \geq 2$). For example, the exact degree of telecommunication safety of mankind is a point in $[0, 1]^n$ ($n = 7 \times 10^9$) because different persons have been monitored different times. There are many other examples such as truth degrees of a logic formula which are based on n logic implication operators ($n \geq 2$), similarity degrees of two logic formulas which are based on n logic implication operators ($n \geq 2$), ordering results of a magazine, ordering results of a university, and inclusion degrees (accuracy measures, rough measures, approximation qualities, fuzziness measures, decision preformation evaluations) of a rough set.

In 1973, Kaufmann [8] gave the basic concept of fuzzy graphs. In 1975, Rosenfeld [11] discussed the concept of fuzzy graphs. Rosenfeld [11] also considered the fuzzy relations between fuzzy sets and he obtained some results on fuzzy graphs. Bhattacharya [6] gave some remarks on fuzzy graphs. Mordeson and Nair [9] studied several concepts of fuzzy graphs. Nagoorgani and Latha [10] discussed irregular fuzzy graphs. Akram et al. [1–5] introduced many new concepts, including bipolar fuzzy graphs, irregular bipolar fuzzy graphs, bipolar fuzzy digraphs in decision support system, m-polar fuzzy graphs and m-polar fuzzy line graphs, certain metrices in m-polar fuzzy graphs. In this paper, we introduce certain types of irregular m-polar fuzzy graphs and investigate some of their properties. We describe the concepts of types of irregular m-polar fuzzy graphs with several examples. We also present applications of m-polar fuzzy graphs in decision making and social network as examples. We have used standard definitions and terminologies in this paper. For other notations, terminologies and applications not mentioned in the paper, the readers are referred to [9,12,13,15,17].

2 Preliminaries

We review some known definitions that are necessary for the work presented in this paper.

Definition 2.1 [11] By a *fuzzy graph* of a graph $G^* = (V, E)$, we mean a pair $G = (\mu, \nu)$, where μ is a fuzzy set on V and ν is a fuzzy set on E such that

$$\nu(xy) \leq \min\{\mu(x), \mu(y)\}$$

for all $xy \in E$. Note that $\nu(xy) = 0$ for all $xy \in V \times V - E$.

Definition 2.2 [16] Let X be a nonempty set. A *bipolar fuzzy set* B in X is an object having the form $B = \{(x, \mu_B^P(x), \mu_B^N(x)) | x \in X\}$, where $\mu_B^P : X \to [0, 1]$ and $\mu_B^N : X \to [-1, 0]$ are mappings. We use the positive membership degree $\mu_B^P(x)$ to denote the satisfaction degree of an element x to the property corresponding to a bipolar fuzzy set B, and the negative membership degree $\mu_B^N(x))$ to denote the satisfaction degree of an element x to some implicit counter-property corresponding to a bipolar fuzzy set B.

Definition 2.3 [16] Let X be a nonempty set. Then a mapping $A = (\mu_A^P, \mu_A^N) : X \times X \to [0, 1] \times [-1, 0]$ is called a *bipolar fuzzy relation* on X such that $\mu_A^P(x, y) \in [0, 1]$ and $\mu_A^N(x, y) \in [-1, 0]$.

Definition 2.4 [1] A *bipolar fuzzy graph* of a graph $G^* = (V, E)$ is a pair $G = (A, B)$, where $A = (\mu_A^P, \mu_A^N)$ is a bipolar fuzzy set on V and $B = (\mu_B^P, \mu_B^N)$ is a bipolar fuzzy relation on V such that

$$\mu_B^P(xy) \leq \min\{\mu_A^P(x), \mu_A^P(y)\} \text{ and } \mu_B^N(xy) \geq \max\{\mu_A^N(x), \mu_A^N(y)\}$$

for all $xy \in E$. Note that $0 < \mu_B^P(xy) \leq 1, -1 \leq \mu_B^N(xy) < 0$ and $\mu_B^P(xy) = 0$, $\mu_B^N(xy) = 0$ for all $xy \in V \times V - E$.

Definition 2.5 [7] An *m-polar fuzzy set* (or a $[0, 1]^m$ -set) on X is a mapping $A : X \to [0, 1]^m$. The set of all *m*-polar fuzzy sets on X is denoted by $m(X)$. Note that $\mathbf{0} = (0, 0, \ldots, 0)$ is the smallest element in $[0, 1]^m$ and $\mathbf{1} = (1, 1, \ldots, 1)$ is the largest element in $[0, 1]^m$.

3 Certain types of irregular *m*-polar fuzzy graphs

An *m*-polar fuzzy relation is a generalization of a bipolar fuzzy relation. Akram and Neha [4] defined it as follow:

Definition 3.1 Let C be an *m*-polar fuzzy subset of a non-empty set V. An *m-polar fuzzy relation* on C is an *m*-polar fuzzy subset D of $V \times V$ defined by the mapping $D : V \times V \to [0, 1]^m$ such that for all $x, y \in V$, $p_i \circ D(xy) \leq \inf\{p_i \circ C(x), p_i \circ C(y)\}$, $i = 1, 2, 3, \ldots, m$, where $p_i \circ C(x)$ denotes the ith degree of membership of the vertex x and $p_i \circ D(xy)$ denotes the *ith* degree of membership of the edge xy.

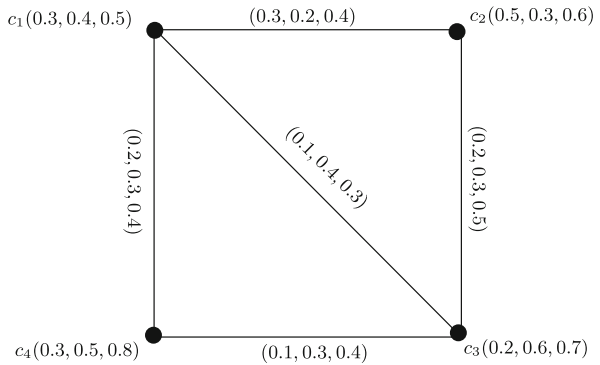Chen et al. [7] defined the concept of an *m*-polar fuzzy graph as follow.

Definition 3.2 An *m-polar fuzzy graph* of a graph $G^* = (V,E)$ is a pair $G = (C,D)$, where $C : V \to [0, 1]^m$ is an *m*-polar fuzzy set in V and $D : E \to [0, 1]^m$ is an *m*-polar fuzzy relation on V such that

$$p_i \circ D(xy) \leq \inf\{p_i \circ C(x), p_i \circ C(y)\},$$

for all $x, y \in V$.

We note that $p_i \circ D(xy) = 0$ for all $xy \in V \times V - E$ for all $i = 1, 2, 3, \ldots, m$. C is called the *m-polar fuzzy vertex set* of G and D is called the *m-polar fuzzy edge set* of G, respectively. An *m*-polar fuzzy relation D on X is called symmetric if $p_i \circ D(xy) = p_i \circ D(yx)$ for all $x, y \in V$.

Fig. 1 3-Polar fuzzy undirected graph

$c_1(0.3, 0.4, 0.5)$　　　　　　$(0.3, 0.2, 0.4)$　　　　　　$c_2(0.5, 0.3, 0.6)$

$(0.2, 0.3, 0.4)$

$(0.1, 0.4, 0.3)$

$(0.2, 0.3, 0.5)$

$c_4(0.3, 0.5, 0.8)$　　　　　　$(0.1, 0.3, 0.4)$　　　　　　$c_3(0.2, 0.6, 0.7)$

Example 3.3 Consider a crisp graph $G^* = (V, E)$ such that $V = \{c_1, c_2, c_3, c_4\}$ and $E = \{c_1c_2, c_2c_3, c_3c_4, c_4c_1, c_3c_1\}$. Let C and D be 3-polar fuzzy sets on V and E, respectively, defined by

	c_1	c_2	c_3	c_4
$p_1 \circ C$	0.3	0.5	0.2	0.3
$p_2 \circ C$	0.4	0.3	0.6	0.5
$p_3 \circ C$	0.5	0.6	0.7	0.8

	c_1c_2	c_2c_3	c_3c_4	c_4c_1	c_3c_1
$p_1 \circ D$	0.3	0.2	0.1	0.2	0.1
$p_2 \circ D$	0.2	0.3	0.3	0.3	0.4
$p_3 \circ D$	0.4	0.5	0.4	0.4	0.3

By direct calculations, it is easy to see from Fig. 1 that G is a 3-polar fuzzy graph.

Definition 3.4 An *m polar fuzzy path* P is a sequence of distinct vertices $u = v_1, v_2, \ldots, v_n = v$ such that for all j there exists at least one i, $p_i \circ B(v_j v_{j+1}) > 0$. An *m*-polar fuzzy path is called an *m-polar fuzzy cycle* if $u = v$. An *m*-polar fuzzy graph $G = (A, B)$ is said to be *connected*, if there is a path between each pair of distinct vertices.

Definition 3.5 Let $G = (C, D)$ be an *m*-polar fuzzy graph of a graph $G^* = (V, E)$. Then we define the *neighbourhood* of a vertex x in graph G as

$$N(x) = \{y \in V : p_i \circ D(xy) \leq \inf(p_i \circ C(x), p_i \circ C(y))\},$$

where $i = 1, 2, 3, \ldots, m$.

Definition 3.6 Let $G = (C, D)$ be an *m*-polar fuzzy graph of a graph $G^* = (V, E)$. Then we define the *neighbourhood degree* of a vertex x in graph G as

$$\begin{aligned} d_G(x) &= \Sigma_{y \in N(x)} p_i \circ C(y) \\ &= (\Sigma_{y \in N(x)} p_1 \circ C(y), \Sigma_{y \in N(x)} p_2 \circ C(y), \Sigma_{y \in N(x)} p_3 \circ C(y), \\ &\quad \ldots, \Sigma_{y \in N(x)} p_m \circ C(y)), \end{aligned}$$
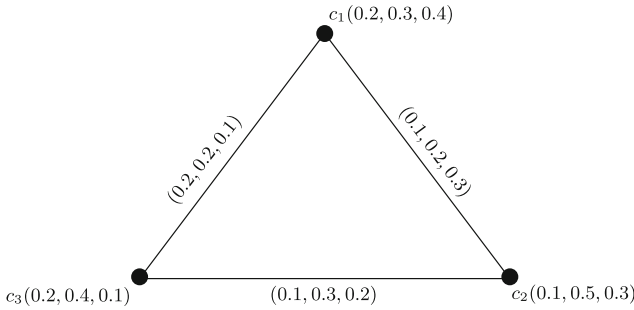
$i = 1, 2, 3, \ldots, m$.

Fig. 2 3-Polar fuzzy graph

Example 3.7 Consider a 3-polar fuzzy graph $G = (C, D)$ of a crisp graph $G^* = (V, E)$, where $V = \{c_1, c_2, c_3\}$, $E = \{c_1 c_2, c_2 c_3, c_3 c_1\}$.

By direct calculations, we have neighbourhood degree of each vertex from Fig. 2:

$$d_G(c_1) = (0.1 + 0.2, 0.4 + 0.5, 0.3 + 0.1) = (0.3, 0.9, 0.4),$$
$$d_G(c_2) = (0.2 + 0.2, 0.4 + 0.3, 0.4 + 0.1) = (0.4, 0.7, 0.5),$$
$$d_G(c_3) = (0.1 + 0.2, 0.3 + 0.5, 0.3 + 0.4) = (0.3, 0.8, 0.7).$$

Definition 3.8 Let $G = (C, D)$ be an m-polar fuzzy graph of a graph $G^* = (V, E)$. Then the graph G is called an *irregular m-polar fuzzy graph* if there exist a vertex whose adjacent vertices have distinct neighbourhood degrees. That is, $d_G(x) \neq n$ for all $x \in V$.

Example 3.9 Consider a 4-polar fuzzy graph G of a crisp graph $G^* = (V, E)$, where $V = \{c_1, c_2, c_3, c_4\}$ and $E = \{c_1 c_2, c_2 c_3, c_3 c_4, c_4 c_1\}$.

By direct calculations, we have

$$d_G(c_1) = (0.3 + 0.2, 0.4 + 0.1, 0.7 + 0.3, 0.5 + 0.6) = (0.5, 0.5, 1.0, 1.1),$$
$$d_G(c_2) = (0.3 + 0.2, 0.6 + 0.7, 0.1 + 0.4, 0.3 + 0.2) = (0.5, 1.3, 0.5, 0.5),$$
$$d_G(c_3) = (0.3 + 0.2, 0.4 + 0.1, 0.7 + 0.3, 0.5 + 0.6) = (0.5, 0.5, 1.0, 1.1),$$
$$d_G(c_4) = (0.3 + 0.2, 0.6 + 0.7, 0.1 + 0.4, 0.3 + 0.2) = (0.5, 1.3, 0.5, 0.5).$$

It is easy to see from Fig. 3 that G is an irregular 4-polar fuzzy graph.

Definition 3.10 Let $G = (C, D)$ be an m-polar fuzzy graph of a graph $G^* = (V, E)$. Then we define the *closed neighbourhood degree* of a vertex x in graph G as

$$d_G[x] = \Sigma_{y \in N(x)} p_i \circ C(y) + (p_1 \circ C(x), p_2 \circ C(x), p_3 \circ C(x), \ldots, p_m \circ C(x))$$
$$= d_G(x) + (p_1 \circ C(x), p_2 \circ C(x), p_3 \circ C(x), \ldots, p_m \circ C(x)),$$

$i = 1, 2, 3, \ldots, m$.

Fig. 3 Irregular 4-Polar fuzzy graph

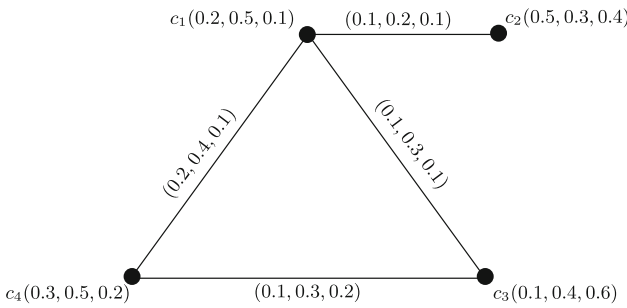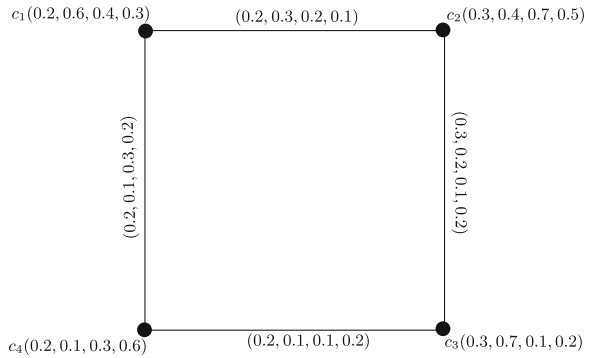$c_1(0.2, 0.6, 0.4, 0.3)$ $(0.2, 0.3, 0.2, 0.1)$ $c_2(0.3, 0.4, 0.7, 0.5)$

$(0.2, 0.1, 0.3, 0.2)$

$(0.3, 0.2, 0.1, 0.2)$

$c_4(0.2, 0.1, 0.3, 0.6)$ $(0.2, 0.1, 0.1, 0.2)$ $c_3(0.3, 0.7, 0.1, 0.2)$

$c_1(0.2, 0.5, 0.1)$ $(0.1, 0.2, 0.1)$ $c_2(0.5, 0.3, 0.4)$

$(0.2, 0.4, 0.1)$

$(0.1, 0.3, 0.1)$

$c_4(0.3, 0.5, 0.2)$ $(0.1, 0.3, 0.2)$ $c_3(0.1, 0.4, 0.6)$

Fig. 4 3-Polar fuzzy graph

Example 3.11 Consider a 3-polar fuzzy graph $G = (C, D)$ such that C and D are defined on $V = \{c_1, c_2, c_3\}$ and $E = \{c_1c_2, c_2c_3, c_3c_1\}$ by the following tables (Fig. 4).

	c_1	c_2	c_3	c_4
$p_1 \circ C$	0.2	0.5	0.1	0.3
$p_2 \circ C$	0.5	0.3	0.4	0.5
$p_3 \circ C$	0.1	0.4	0.6	0.2

	c_1c_2	c_1c_3	c_3c_4	c_4c_1
$p_1 \circ D$	0.1	0.1	0.1	0.2
$p_2 \circ D$	0.2	0.3	0.3	0.4
$p_3 \circ D$	0.1	0.1	0.2	0.1

By direct calculations, we see that:

$$dG[c_1] = (0.2 + 0.5 + 0.1 + 0.3, 0.5 + 0.3 + 0.4 + 0.5, 0.1 + 0.4 + 0.6 + 0.2)$$
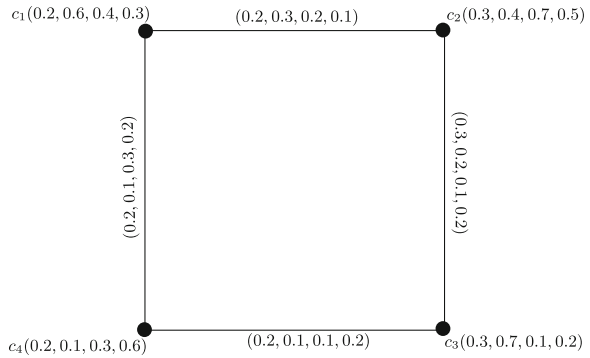$$= (1.1, 1.7, 1.3),$$
$$dG[c_2] = (0.5 + 0.2, 0.3 + 0.5, 0.4 + 0.1) = (0.7, 0.8, 0.5),$$
$$dG[c_3] = (0.3 + 0.2 + 0.1, 0.5 + 0.4 + 0.5, 0.6 + 0.1 + 0.2) = (0.6, 1.4, 0.9),$$
$$dG[c_4] = (0.3 + 0.2 + 0.1, 0.5 + 0.4 + 0.5, 0.6 + 0.1 + 0.2) = (0.6, 1.4, 0.9).$$

Definition 3.12 Let $G = (C, D)$ be an *m*-polar fuzzy graph of a graph $G^* = (V, E)$. The graph G is called a *totally irregular m-polar fuzzy graph* if there exist a vertex whose adjacent vertices have distinct closed neighbourhood degrees.

Fig. 5 Totally irregular 4-Polar fuzzy graph

$c_1(0.2, 0.6, 0.4, 0.3)$ $(0.2, 0.3, 0.2, 0.1)$ $c_2(0.3, 0.4, 0.7, 0.5)$

$(0.2, 0.1, 0.3, 0.2)$

$(0.3, 0.2, 0.1, 0.2)$

$c_4(0.2, 0.1, 0.3, 0.6)$ $(0.2, 0.1, 0.1, 0.2)$ $c_3(0.3, 0.7, 0.1, 0.2)$

Example 3.13 Consider a 4-polar fuzzy graph G as shown in Fig. 5.
By routine computations, we have

$$d_G[c_1] = (0.7, 1.1, 1.4, 1.4),$$
$$d_G[c_2] = (0.8, 1.7, 1.2, 1.0),$$
$$d_G[c_3] = (0.8, 1.2, 1.1, 1.3),$$
$$d_G[c_4] = (0.7, 1.4, 0.8, 1.1).$$

It can be easily seen from Fig. 5 that G is a totally irregular 4-polar fuzzy graph.

Definition 3.14 Let $G = (C, D)$ be a connected m-polar fuzzy graph. Then G is called a *neighbourly irregular m-polar fuzzy graph* if every two adjacent vertices of G have distinct open neighbouhood degrees.

Example 3.15 Consider a 3-polar fuzzy graph G as shown in Fig. 6.
Then,

$$d_G(c_1) = (1.2, 1.7, 0.9),$$
$$d_G(c_2) = (1.1, 1.6, 1.2),$$

$c_1(0.2, 0.6, 0.4)$ $(0.2, 0.5, 0.1)$ $c_2(0.3, 0.7, 0.1)$

$(0.1, 0.5, 0.3)$

$(0.2, 0.3, 0.2)$

$(0.2, 0.3, 0.1)$

$c_5(0.2, 0.1, 0.4)$

$(0.3, 0.4, 0.1)$

$(0.2, 0.1, 0.3)$

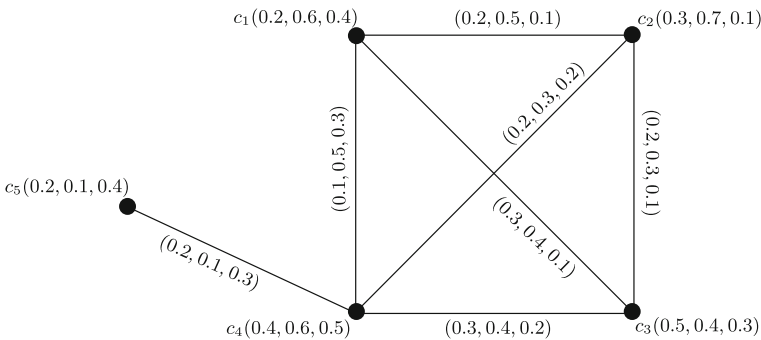$c_4(0.4, 0.6, 0.5)$ $(0.3, 0.4, 0.2)$ $c_3(0.5, 0.4, 0.3)$

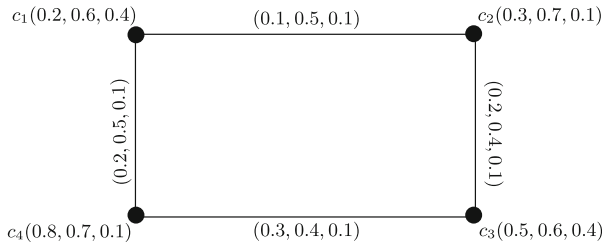**Fig. 6** Neighbourly irregular 3-Polar fuzzy graph

Fig. 7 Neighbourly totally irregular 3-Polar fuzzy graph

$$d_G(c_3) = (0.9, 1.9, 1.0),$$
$$d_G(c_4) = (1.2, 1.8, 1.2),$$
$$d_G(c_5) = (0.4, 0.6, 0.5).$$

It is clear from calculations of Fig. 6 that G is a neighbourly irregular 3-polar fuzzy graph.

Definition 3.16 Let $G = (C, D)$ be a connected m-polar fuzzy graph. Then G is called a *neighbourly totally irregular m-polar fuzzy graph* if every two adjacent vertices of G have distinct closed neighbouhood degrees.

Example 3.17 Consider a 3-polar fuzzy graph G as shown in Fig. 7.
By routine calculations, we have

$$d_G[c_1] = (1.3, 2.0, 0.6),$$
$$d_G[c_2] = (1.0, 1.9, 0.9),$$
$$d_G[c_3] = (1.6, 2.0, 0.6),$$
$$d_G[c_4] = (1.0, 1.9, 0.9).$$

It can be easily seen from Fig. 7 that G is neighbourly totally irregular 3-polar fuzzy graph.

Definition 3.18 A connected m-polar fuzzy graph $G = (C, D)$ is called a *highly irregular m-polar fuzzy graph* if every vertex of graph G is adjacent to vertices with distinct neighbourhood degrees.

Example 3.19 Consider a 3-polar fuzzy graph G as shown in Fig. 8.
By routine calculations, we have

$$d_G(c_1) = (0.7, 0.9, 1.4),$$
$$d_G(c_2) = (1.1, 1.3, 1.3),$$
$$d_G(c_3) = (0.4, 0.8, 1.0),$$
$$d_G(c_4) = (0.7, 0.9, 1.4),$$
$$d_G(c_5) = (0.2, 0.3, 0.6).$$

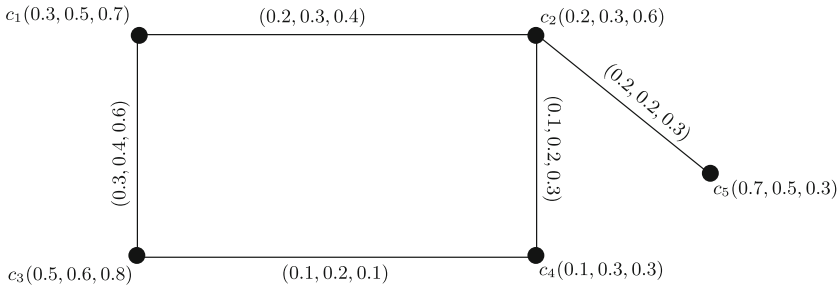It is easy to see from Fig. 8 that G is highly irregular 3-polar fuzzy graph.

Fig. 8 Highly irregular 3-Polar fuzzy graph

Remark 1 A highly irregular *m*-polar fuzzy graph may not be a neighbourly irregular *m*-polar fuzzy graph.

There is no relation between highly irregular *m*-polar fuzzy graphs and neighbourly irregular *m*-polar fuzzy graphs. We explain this concept with the following examples.

Example 3.20 Consider a 4-polar fuzzy graph G as shown in Fig. 9.

By routine computations, we have

$$d_G(c_1) = (1.0, 0.7, 0.5, 1.0),$$
$$d_G(c_2) = (0.4, 0.1, 0.2, 0.5),$$
$$d_G(c_3) = (1.4, 1.4, 0.9, 1.3),$$
$$d_G(c_4) = (0.7, 0.5, 0.5, 1.1),$$
$$d_G(c_5) = (1.2, 1.0, 0.9, 1.3),$$
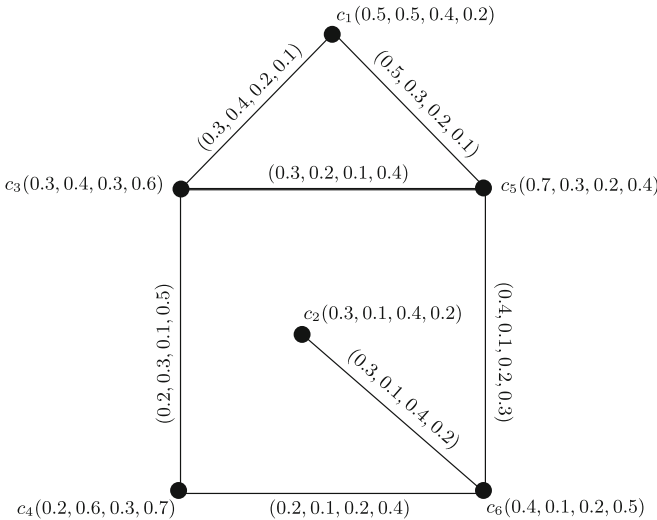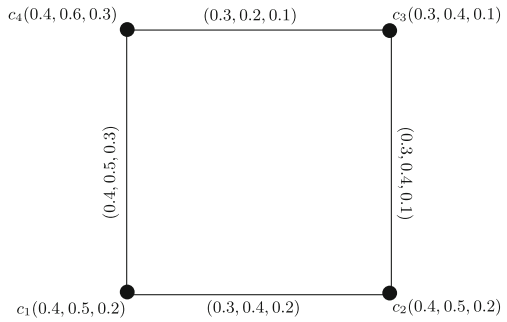$$d_G(c_6) = (1.2, 1.0, 0.9, 1.3).$$

Fig. 9 Not neighbourly irregular 4-Polar fuzzy graph

Fig. 10 Not highly irregular
3-Polar fuzzy graph

We consider a vertex $c_6 \in V$ whose adjacent vertices are c_2, c_4 and c_5 which have distinct neighbourhood degrees. But $d_G(c_6) = d_G(c_5)$. So, G is not a neighbourly irregular 4-polar fuzzy graph but it is a highly irregular 4-polar fuzzy graph.

Remark 2 A neighbourly irregular m-polar fuzzy graph may not be a highly irregular m-polar fuzzy graph.

Example 3.21 Consider a 3-polar fuzzy graph G as shown in Fig. 10.
By calculations, we have

$$d_G(c_1) = (0.8, 1.1, 0.5),$$
$$d_G(c_2) = (0.7, 0.9, 0.3),$$
$$d_G(c_3) = (0.8, 1.1, 0.5),$$
$$d_G(c_4) = (0.7, 0.9, 0.3).$$

It is clear that every two adjacent vertices of graph G have distinct neighbourhood degrees. But take a vertex c_3 whose adjacent vertices are c_2 and c_4 which have same neighbourhood degrees, that is $d_G(c_2) = d_G(c_4)$. So, G is not a highly irregular 3-polar fuzzy graph but G is a neighbourly irregular 3-polar fuzzy graph.

Theorem 3.22 *Let $G = (C, D)$ be an m-polar fuzzy graph. Then G is highly irregular m-polar fuzzy graph and neighbourly irregular m-polar fuzzy graph if and only if the neighbourhood degrees of all the vertices of G are distinct.*

Proof Let G be an m-polar fuzzy graph with r vertices $c_1, c_2, \ldots, c_r$. Suppose that G is highly irregular m-polar fuzzy graph and neighbourly irregular m-polar fuzzy graph.

We claim that the neighbourhood degrees of all the vertices of G are distinct. Let $d_G(c_1) = (l_1, l_2, \ldots, l_m)$, $d_G(c_2) = (t_1, t_2, \ldots, t_m)$, $d_G(c_3) = (q_1, q_2, \ldots, q_m), \ldots,$ $d_G(c_r) = (s_1, s_2, \ldots, s_m)$. Suppose that the adjacent vertices of c_1 are $c_2, c_3, \ldots, c_r$ with neighbourhood degrees $(t_1, t_2, \ldots, t_m)$, $(q_1, q_2, \cdots, q_m), \ldots, (s_1, s_2, \ldots, s_m)$ respectively. Then we have, $t_1 \neq q_1 \neq \cdots \neq s_1, t_2 \neq q_2 \neq \cdots \neq s_2, \ldots,$ $t_m \neq q_m \neq \cdots \neq s_m$, since G is highly irregular. Also $l_1 \neq t_1 \neq q_1 \neq \cdots \neq s_1,$ $l_2 \neq t_2 \neq q_2 \neq \cdots \neq s_2, \ldots, l_m \neq t_m \neq q_m \neq \cdots \neq s_m$, since G is neighbourly irregular. Hence, the neighbourhood degrees of all the vertices of G are distinct.

Conversely, suppose that neighbourhood degrees of all the vertices of G are distinct. We claim that G is highly irregular and neighbourly irregular m-polar fuzzy graph. Let $d_G(c_1) = (l_1, l_2, \ldots, l_m)$, $d_G(c_2) = (t_1, t_2, \ldots, t_m)$, $d_G(c_3) = (q_1, q_2, \ldots, q_m), \ldots, d_G(c_r) = (s_1, s_2, \ldots, s_m)$. Then by our supposition $l_1 \neq t_1 \neq q_1 \neq \cdots \neq s_1$, $l_2 \neq t_2 \neq q_2 \neq \cdots \neq s_2, \ldots$, $l_m \neq t_m \neq q_m \neq \cdots \neq s_m$, which implies that every two adjacent vertices have distinct neighbourhood degrees and every vertex is adjacent to vertices with distinct neighbourhood degrees. Hence, G is neighbourly irregular m-polar fuzzy graph and also highly irregular m-polar fuzzy graph. □

Proposition 3.23 *A neighbourly irregular m-polar fuzzy graph may not be a neighbourly totally irregular m-polar fuzzy graph.*

Example 3.24 Consider a 4-polar fuzzy graph G as shown in Fig. 11.
By calculations, we have

$$d_G(c_1) = (0.8, 1.1, 0.5, 1.2),$$
$$d_G(c_2) = (0.7, 0.9, 0.3, 1.3),$$
$$d_G(c_3) = (0.8, 1.1, 0.5, 1.2),$$
$$d_G(c_4) = (0.7, 0.9, 0.3, 1.3).$$

and

$$d_G[c_1] = (1.2, 1.6, 0.7, 1.9),$$
$$d_G[c_2] = (1.1, 1.4, 0.5, 2.0),$$
$$d_G[c_3] = (1.1, 1.5, 0.6, 1.8),$$
$$d_G[c_4] = (1.1, 1.5, 0.6. 1.8).$$

It is clear that $d_G[c_3] = d_G[c_4]$. Hence, G is not a neighbourly totally irregular 4-polar fuzzy graph but a neighbourly irregular 4-polar fuzzy graph.
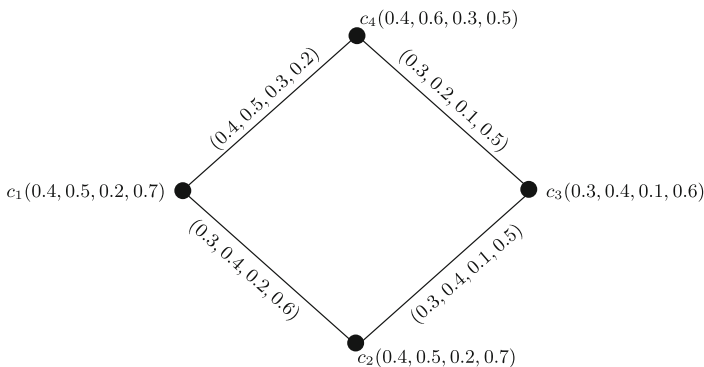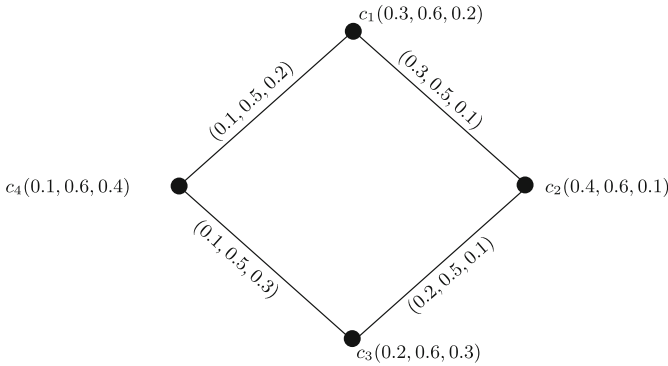

Fig. 11 Simple 4-Polar fuzzy graph

Fig. 12 3-Polar fuzzy graph

Remark 3 A neighbourly totally irregular m-polar fuzzy graph may not be a neighbourly irregular m-polar fuzzy graph.

Example 3.25 Consider a 3-polar fuzzy graph G as shown in Fig. 12.
By calculations of Fig. 12, we have

$$dG[c_1] = (0.8, 1.8, 0.7),$$
$$dG[c_2] = (0.9, 1.8, 0.6),$$
$$dG[c_3] = (0.7, 1.8, 0.8),$$
$$dG[c_4] = (0.6, 1.8, 0.9).$$

But on the other hand

$$dG(c_1) = dG(c_2) = dG(c_3) = dG(c_4)$$

Hence G is not a neighbourly irregular 3-polar fuzzy graph but G is a neighbourly totally irregular 3-polar fuzzy graph.

Proposition 3.26 *Let G be an m-polar fuzzy graph. If G is neighbourly irregular m-polar fuzzy graph and $(p_1 \circ C(x), p_2 \circ C(x), \ldots, p_m \circ C(x)) = (c_1, c_2, \ldots, c_m)$ is a constant function, then G is a neighbourly totally irregular m-polar fuzzy graph.*

Proof Suppose that G is a neighbourly irregular m-polar fuzzy graph, i.e. every two adjacent vertices have distinct neighbourhood degrees. Let $v_j, v_k \in V$, where v_j and v_k are adjacent vertices with distinct neighbourhood degrees, $(l_1, l_2, \ldots, l_m)$ and $(t_1, t_2, \ldots, t_m)$ respectively. That is $dG(v_j) = (l_1, l_2, \ldots, l_m)$ and $dG(v_k) = (t_1, t_2, \ldots, t_m)$, where $l_1 \neq t_1, l_2 \neq t_2, \ldots, l_m \neq t_m$.
Let us assume that,

$$(p_1 \circ C(x)(v_j), p_2 \circ C(x)(v_j), \ldots, p_m \circ C(x)(v_j)) = (p_1 \circ C(x)(v_k),$$
$$p_2 \circ C(x)(v_k), \ldots, p_m \circ C(x)(v_k)) = (c_1, c_2, \ldots, c_m),$$

where $c_1, c_2, \ldots, c_m$ are constants and $c_1, c_2, \ldots, c_m \in [0, 1]$.
Therefore,

$$
\begin{aligned}
d_G[v_j] &= d_G(v_j) + (p_1 \circ C(x)(v_j), p_2 \circ C(x)(v_j), \ldots, p_m \circ C(x)(v_j)) \\
&= (l_1, l_2, \ldots, l_m) + (c_1, c_2, \ldots, c_m) \\
d_G[v_k] &= d_G(v_k) + (p_1 \circ C(x)(v_k), p_2 \circ C(x)(v_k), \ldots, p_m \circ C(x)(v_k)) \\
&= (t_1, t_2, \ldots, t_m) + (c_1, c_2, \ldots, c_m)
\end{aligned}
$$

We claim that, $d_G[v_j] \neq d_G[v_k]$
Suppose on contrary that, $d_G[v_j] = d_G[v_k]$,i.e.,

$$
\begin{aligned}
(l_1, l_2, \ldots, l_m) + (c_1, c_2, \ldots, c_m) &= (t_1, t_2, \ldots, t_m) + (c_1, c_2, \ldots, c_m) \\
\Rightarrow (l_1, l_2, \ldots, l_m) &= (t_1, t_2, \ldots, t_m) \text{, i.e.,}
\end{aligned}
$$

$l_1 = t_1, l_2 = t_2, \ldots, l_m = t_m$, which is a contradiction to the fact that $l_1 \neq t_1, l_2 \neq t_2, \ldots, l_m \neq t_m$

Therefore, $d_G[v_j] \neq d_G[v_k]$. Hence, G is a neighbourly totally irregular m-polar fuzzy graph. □

Theorem 3.27 *Let G be an m-polar fuzzy graph. If G is a neighbourly totally irregular and $(p_1 \circ C(x), p_2 \circ C(x), \ldots, p_m \circ C(x)) = (c_1, c_2, \ldots, c_m)$ is a constant function, then G is neighbourly irregular m-polar fuzzy graph.*

Proof Suppose that G is a neighbourly totally irregular m-polar fuzzy graph, i.e, every two adjacent vertices have distinct closed neighbourhood degrees. Let $v_j, v_k \in V$ and $d_G[v_j] = (l_1, l_2, \ldots, l_m), d_G[v_k] = (t_1, t_2, \ldots, t_m)$, where $l_1 \neq t_1, l_2 \neq t_2, \ldots, l_m \neq t_m$

Assume that,

$$
\begin{aligned}
&(p_1 \circ C(x)(v_j), p_2 \circ C(x)(v_j), \ldots, p_m \circ C(x)(v_j)) = (p_1 \circ C(x)(v_k), \\
&\quad p_2 \circ C(x)(v_k), \ldots, p_m \circ C(x)(v_k)) \\
&= (c_1, c_2, \ldots, c_m),
\end{aligned}
$$

where $c_1, c_2, \ldots, c_m$ are constants and $c_1, c_2, \ldots, c_m$
$\in [0, 1]$ and $d_G[v_j] \neq d_G[v_k]$.

We claim that $d_G(v_j) \neq d_G(v_k)$
By our supposition,

$$
\begin{aligned}
d_G[v_j] &\neq d_G[v_k] \\
&\Rightarrow (l_1, l_2, \ldots, l_m) + (c_1, c_2, \ldots, c_m) \neq (t_1, t_2, \ldots, t_m) + (c_1, c_2, \ldots, c_m) \\
&\Rightarrow l_1 \neq t_1, l_2 \neq t_2, \cdots, l_m \neq t_m
\end{aligned}
$$

i.e. the neighbourhood degrees of adjacent vertices of G are distinct. Hence, every two adjacent vertices have distinct neighbourhood degrees in G. Hence, G is neighbourly irregular m-polar fuzzy graph. □

Remark 4 If G is neighbourly irregular m-polar fuzzy graph, then m-polar fuzzy subgraph $H = (C', D')$ of G may not be neighbourly irregular m-polar fuzzy graph.

Example 3.28 Consider a 3-polar fuzzy graph G as shown in the following Figure

3-polar fuzzy graph

3-polar fuzzy subgraph
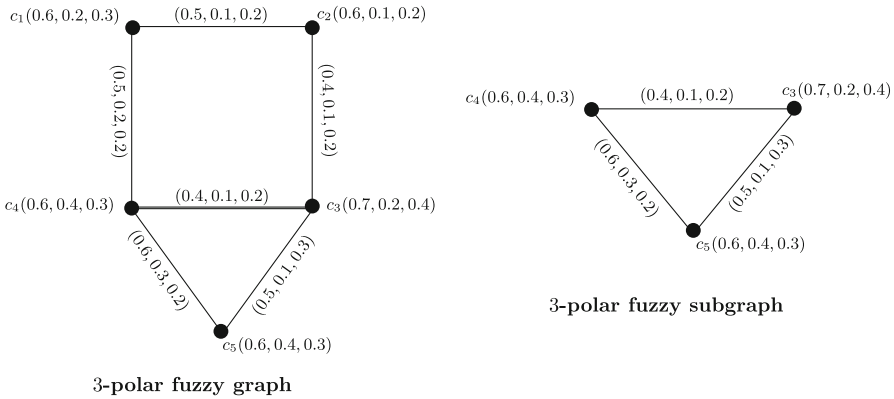
(1) For G: by calculations, we have

$$d_G(c_1) = (1.2, 0.5, 0.5),$$
$$d_G(c_2) = (1.3, 0.4, 0.7),$$
$$d_G(c_3) = (1.8, 0.9, 0.8),$$
$$d_G(c_4) = (1.9, 0.8, 1.0),$$
$$d_G(c_5) = (1.3, 0.6, 0.7).$$
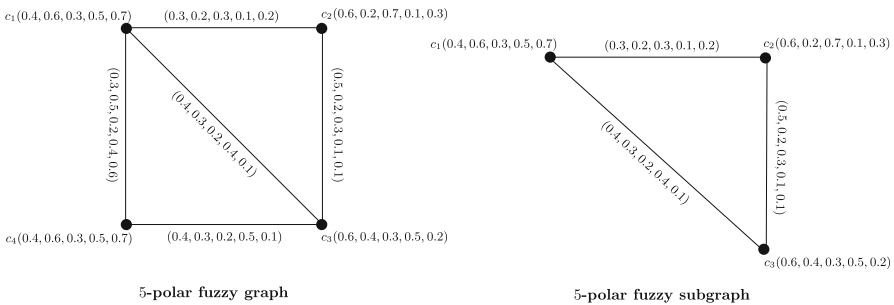
(2) For H: by calculations, we have

$$d_G(c_3) = (1.2, 0.8, 0.6),$$
$$d_G(c_4) = (1.3, 0.6, 0.7),$$
$$d_G(c_5) = (1.3, 0.6, 0.7).$$

It is clear that c_4 and c_5 are adjacent vertices with same neighbourhood degrees in H. Hence H is not a neighbourly irregular 3-polar fuzzy graph but G is neighbourly irregular 3-polar fuzzy graph.

Remark 5 If G is totally irregular m-polar fuzzy graph, then m-polar fuzzy subgraph $H = (C', D')$ of G needed not be totally irregular m-polar fuzzy graph.

Example 3.29 Consider a 5-polar fuzzy graph G as shown in the following Figure

$c_1(0.4, 0.6, 0.3, 0.5, 0.7)$ $(0.3, 0.2, 0.3, 0.1, 0.2)$ $c_2(0.6, 0.2, 0.7, 0.1, 0.3)$

$(0.3, 0.5, 0.2, 0.4, 0.6)$

$(0.4, 0.3, 0.2, 0.4, 0.1)$

$(0.5, 0.2, 0.3, 0.1, 0.1)$

$c_4(0.4, 0.6, 0.3, 0.5, 0.7)$ $(0.4, 0.3, 0.2, 0.5, 0.1)$ $c_3(0.6, 0.4, 0.3, 0.5, 0.2)$

$c_1(0.4, 0.6, 0.3, 0.5, 0.7)$ $(0.3, 0.2, 0.3, 0.1, 0.2)$ $c_2(0.6, 0.2, 0.7, 0.1, 0.3)$

$(0.4, 0.3, 0.2, 0.4, 0.1)$

$(0.5, 0.2, 0.3, 0.1, 0.1)$

$c_3(0.6, 0.4, 0.3, 0.5, 0.2)$

5-polar fuzzy graph   5-polar fuzzy subgraph

(1) For G: By calculations, we have

$$d_G[c_1] = (2.0, 1.8, 1.6, 1.6, 1.9),$$
$$d_G[c_2] = (1.6, 1.2, 1.3, 1.1, 1.2),$$
$$d_G[c_3] = (2.0, 1.8, , 1.6, 1.6, 1.9),$$
$$d_G[c_4] = (1.4, 1.6, 0.9, 1.5, 1.6).$$

Here is a vertex c_3 which is adjacent to the vertices c_1, c_2 and c_4, where $d_G[c_1] \neq d_G[c_2] \neq d_G[c_4]$.

(2) For H: By calculations, we have

$$d_G[c_1] = (1.6, 1.2, 1.3, 1.1, 1.2),$$
$$d_G[c_2] = (1.6, 1.2, 1.3, 1.1, 1.2),$$
$$d_G[c_3] = (1.6, 1.2, 1.3, 1.1, 1.2).$$

Here is a vertex c_1 whose adjacent vertices are c_2 and c_3 with same closed neighbourhood degrees. A vertex c_2 whose adjacent vertices are c_1 and c_3 with same closed neighbourhood degrees and a vertex c_3 whose adjacent vertices are c_1 and c_2 with same closed neighbourhood degrees. Hence, G is totally irregular 5-polar fuzzy graph but H is not a totally irregular 5-polar fuzzy graph.

4 Applications of m-polar fuzzy graphs

4.1 m-Polar fuzzy graphs in decision making

m-polar fuzzy sets play an important role in decision making problems, when it is necessary to gather information. This happens when a group of friends decides which place to visit, when a company decides which product design to manufacture, when a democratic company elects its leader, when a company selects its employe, in cooperative games, in weighted games etc.

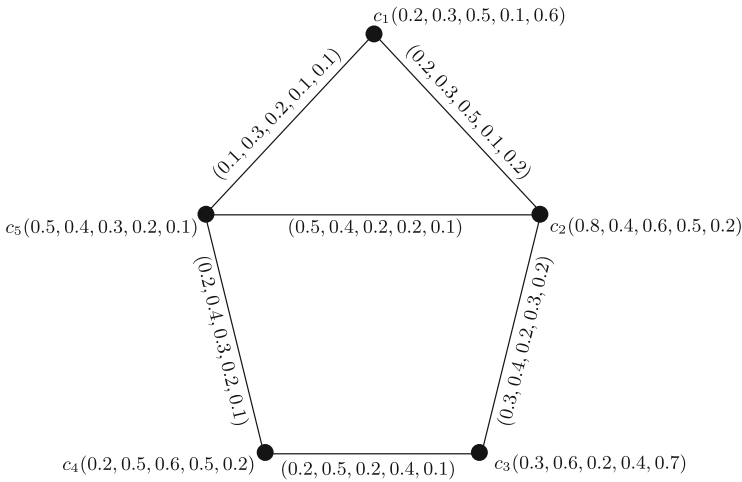We discuss here an example of a company who want to select a manager.

Fig. 13 5-Polar fuzzy graph for decision making

Let $A = \{$Ali, Usman, Umar, Amir, Musa$\}$ be the set of candidates who have applied for the post of manager and $B = \{v_1, v_2, v_3, v_4, v_5\}$ be the set of members of HR department of a company. The members have to decide a candidate on the basis of their qualities that are {intelligence, communication, leadership, adaptability, relationshipbuilding}. For each candidate a member from set B can send a value in $[0, 1]$ to $a \in A$, such as, $C(c_1) = (0.2, 0.3, 0.5, 0.1, 0.6)$, $C(c_2) = (0.8, 0.4, 0.6, 0.5, 0.2)$, $C(c_3) = (0.3, 0.6, 0.2, 0.4, 0.7)$, $C(c_4) = (0.2, 0.5, 0.6, 0.5, 0.2)$, $C(c_5) = (0.5, 0.4, 0.3, 0.2, 0.1)$. Then we obtain a 5-polar fuzzy graph, in which each member gives his preference degree to $a \in A$ on the basis of his qualities. So, $C(c_1), C(c_2), C(c_3), C(c_4)$ and $C(c_5)$ show the degree of intelligence, communication, leadership, adaptability and relationship building of each candidate given by the set of members of HR department and edges represent the common qualities of two candidates (Fig. 13).

The membership degree of edge $c_1 c_2$ shows that the members c_1 and c_2 decide that Ali is 20% eligible, Usman is 30% eligible, Umar is 50% eligible, Amir is 10% eligible and Musa is 20% eligible. So, according to $c_1 c_2$ Umar is an eligible candidate for the post of manager. Similarly, $c_2 c_3$ selects Usman, $c_3 c_4$ selects Usman, $c_4 c_5$ also selects Usman, $c_5 c_2$ selects Ali and $c_5 c_1$ selects Usman. Hence, according to all members Usman has more degree of preference as compared to others.

4.2 *m*-Polar fuzzy graphs in social network

Graph models find broad applications in many disciplines of mathematics, social sciences, natural sciences and computer sciences. In studies of group behavior, it is inspected that many people can influence thinking of others. A digraph can be used to model such behavior and this graph is called an influence graph.
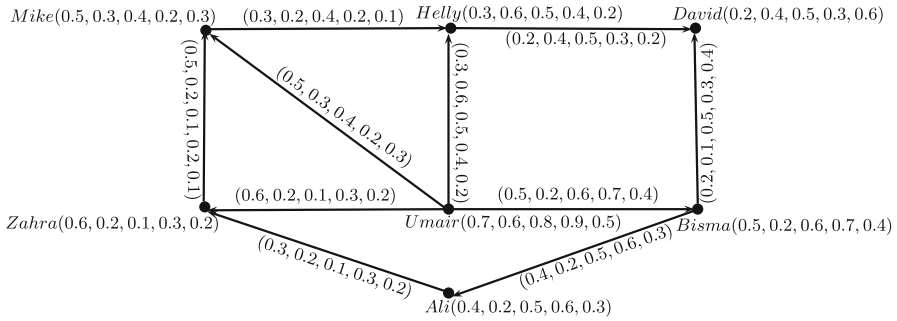
Fig. 14 5-Polar fuzzy digraph

Definition 4.1 An *m-polar fuzzy digraph* of a digraph $G^* = (V, E)$ is a pair $G = (C, D)$, where $C : V \to [0, 1]^m$ is an m-polar fuzzy set in V and $D : E \to [0, 1]^m$ is an m-polar fuzzy relation on V such that $p_i \circ D(xy) \leq \inf\{p_i \circ C(x), p_i \circ C(y)\}$ for all $x, y \in V$ and $i = 1, 2, 3, \cdots, m$. We note that D need not to be symmetric, i.e. $p_i \circ D(xy) \neq p_i \circ D(yx)$.

We discuss the influence of a person in a social group on skype.

Let $G = \{$Mike, Helly, David, Bisma, Ali, Umair, Zahra$\}$ be the set of seven persons in a social group. The influence degree depends on logical persuading, exchanging, alliance building, legitimizing and appealing to values. Then, we get a 5-polar fuzzy influence graph $D = (G, I)$, in which, vertices representing the persons of a social group and edges representing the influence of a person on other (Fig. 14).

Here, G is the set of vertices defined as
{(Mike,0.5,0.3,0.4,0.2,0.3), (Helly,0.3,0.6,0.5,0.4,0.2), (David,0.2,0.4,0.5,0.3,0.6), (Bisma,0.5,0.2,0.6,0.7,0.4), (Ali,0.4,0.2,0.5,0.6,0.3), (Umair,0.7,0.6,0.8,0.9,0.5), (Zahra,0.6,0.2,0.1,0.3,0.2)}.

I is the set of edges. The membership degree of edges can be calculated as

$$p_i \circ I(xy) \leq \inf\{p_i \circ G(x), p_i \circ G(y)\}$$

A graph shows that Umair influence Mike, Helly, Bisma and Zahra. We can see that Umair's 50 % impact on Mike is due to logical persuading, 30 % is due to exchanging, 40 % is due to alliance building, 20 % is due to legitimizing and 30 % impact is due to appealing to values. Similarly, we can see for Bisma and Zahra. Mike influence Helly, Helly influence David, Bisma influence David and Ali, Ali influence Zahra, Zahra influence Mike. So, observations show that Umair is a most influential person in the group.

5 Conclusion and future work

Fuzzy graph theory is highly utilized in research fields of computer science along with the theory of control, mining of data, expert systems and theory of database. An m-polar fuzzy model is a generalization of the fuzzy model. In this paper, we have

discussed certain types of irregular m-polar fuzzy graphs. We have also presented applications of m-polar fuzzy graphs. We are extending our research work to (1) m-polar fuzzy planar graphs, (2) m-polar fuzzy competition graphs, (3) m-polar fuzzy hypergraphs.

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