

Global stability of an eco-epidemiological predator-prey model with saturation incidence

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Abstract A delayed predator-prey model with disease in the predator and stage structure for the prey is investigated. By analyzing the corresponding characteristic equations, the local stability of each of feasible equilibria is studied. The existence of Hopf bifurcations at the disease-free equilibrium and the coexistence equilibrium are addressed, respectively. By using Lyapunov functions and LaSalle invariant principle, sufficient conditions are derived for the global stability of the trivial equilibrium, the predator-extinction equilibrium and the disease-free equilibrium, respectively. Further, sufficient conditions are derived for the global attractiveness of the coexistence equilibrium of the proposed system.

Keywords Eco-epidemiological model · Stage structure · Time delay · LaSalle invariant principle · Hopf bifurcation · Stability

Mathematics Subject Classification 34K18 · 34K20 · 34K60 · 92D25

1 Introduction

Predator-prey models are important in the modelling of multi-species interactions and have received great attention among theoretical and mathematical biologists. The dynamics of the predator-prey models has been studied by many means, such

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as the adomian decomposition method (ADM), the variational iteration method (VIM) and the differential transform method (DTM) (see [1,4,6–8,13]). Since the pioneering work of Anderson and May [3] who were the first to propose an eco-epidemiological model by merging the ecological predator-prey model introduced by Lotka and Volterra, great attention has been paid to the modelling and analysis of eco-epidemiological system recently (see [9,11,12,15–18,21,22]). In [18], Xiao and Chen discussed a predator-prey model with disease in the prey. Mathematical analysis of the model equations with regard to invariance of nonnegativity, boundedness of solutions, nature of equilibria, permanence and global stability were analyzed. In [21], Zhang and Sun considered a predator-prey model with disease in the predator and Holling-II type functional response. Sufficient conditions were derived for the permanence of the eco-epidemiological system. In [22], Zhang et al. considered the following eco-epidemiological model

$$\begin{aligned}\dot{x}(t) &= rx(t) - ax^2(t) - a_{12}x(t)S(t), \\ \dot{S}(t) &= a_{21}x(t - \tau)S(t - \tau) - d_3S(t) - \beta S(t)I(t), \\ \dot{I}(t) &= \beta S(t)I(t) - d_4I(t),\end{aligned}\quad (1.1)$$

where $x(t)$, $S(t)$ and $I(t)$ represent the densities of the prey, susceptible (sound) predator and infected predator population at time t , respectively. The parameters a , a_{12} , a_{21} , d_3 , d_4 , r and β are positive constants (see [22]). In system (1.1), the authors assumed that the infectious predator would die of diseases and only the healthy predator had predation capacity, but once infected with the disease, the predator would not be able to recover. By regarding the time delay τ as the bifurcation parameter and analyzing the characteristic equation of the positive equilibrium, the local asymptotic stability of the positive equilibrium and the existence of a Hopf bifurcation of system (1.1) were investigated in [22].

The above-mentioned works all used bilinear incidence to model disease transmission. However, there are a variety of factors that emphasize the need for a modification of the bilinear incidence. For example, the underlying assumption of homogeneous mixing may not always hold. Incidence rates that increase more gradually than linearly in I and S may arise from saturation effects. It has been strongly suggested by several authors that the disease transmission process may follow saturation incidence. After studying the cholera epidemic spread in Bari in 1973, Capasso and Serio [5] introduced a saturated incidence rate with $\beta IS/(1 + \alpha I)$. This incidence rate seems more reasonable than the bilinear incidence rate βSI , because it includes the behavioral change and crowding effect of the infective individuals and prevents the unboundedness of the contact rate by choosing suitable parameters.

We note that it is assumed in system (1.1) that each individual prey admits the same risk to be attacked by predators. This assumption seems not to be realistic for many animals. In the natural world, there are many species whose individuals pass through an immature stage during which they are raised by their parents, and the rate at which they are attacked by predator can be ignored. Moreover, it can be assumed that their reproductive rate during this stage is zero. Stage-structure is a natural phenomenon and represents, for example, the division of a population into immature and mature

individuals. Stage-structured models have received great attention in recent years (see, for example, [2, 16, 19, 20]).

Based on above discussions, in this paper, we incorporate a stage structure for the prey and saturation incidence into the system (1.1). To this end, we study the following differential equations

$$\begin{aligned}\dot{x}_1(t) &= rx_2(t) - (r_1 + d_1)x_1(t), \\ \dot{x}_2(t) &= r_1x_1(t) - d_2x_2(t) - ax_2^2(t) - a_1x_2(t)S(t), \\ \dot{S}(t) &= a_2x_2(t - \tau)S(t - \tau) - d_3S(t) - bS^2(t) - \frac{\beta S(t)I(t)}{1 + \alpha I(t)}, \\ \dot{I}(t) &= \frac{\beta S(t)I(t)}{1 + \alpha I(t)} - d_4I(t),\end{aligned}\quad (1.2)$$

where $x_1(t)$ and $x_2(t)$ represent the densities of the immature and the mature prey population at time t , respectively. $\tau \geq 0$ represents the time delay due to the gestation of the susceptible predator. The parameters $a, a_1, a_2, b, d_1, d_2, d_3, d_4, r, r_1, \alpha$ and β are positive constants in which d_1 and d_2 are the death rates of the immature and the mature prey, respectively; d_3 and d_4 are the death rates of the susceptible and infected predator, respectively; a and b are the intra-specific competition rate of the mature prey and the susceptible predator, respectively; $a_1 > 0$ is the capturing rate of the susceptible predator; $a_2/a_1 > 0$ is the conversion rate of nutrients into the reproduction of the predator by consuming mature prey; the disease incidence is assumed to be the saturation incidence $\beta SI/(1 + \alpha I)$, where $\beta > 0$ is called the disease transmission coefficient.

The initial conditions for system (1.2) take the form

$$\begin{aligned}x_1(\theta) &= \varphi_1(\theta) \geq 0, \quad x_2(\theta) = \varphi_2(\theta) \geq 0, \\ S(\theta) &= \phi_1(\theta) \geq 0, \quad I(\theta) = \phi_2(\theta) \geq 0, \quad \theta \in [-\tau, 0), \\ \varphi_1(0) &> 0, \quad \varphi_2(0) > 0, \quad \phi_1(0) > 0, \quad \phi_2(0) \\ &> 0, \quad (\varphi_1(\theta), \varphi_2(\theta), \phi_1(\theta), \phi_2(\theta)) \in C([-\tau, 0], R_{+0}^4),\end{aligned}\quad (1.3)$$

where $R_{+0}^4 = \{(y_1, y_2, y_3, y_4) : y_i \geq 0, i = 1, 2, 3, 4\}$.

It is well known by the fundamental theory of functional differential equations [10] that system (1.2) has a unique solution $(x_1(t), x_2(t), S(t), I(t))$ satisfying initial conditions (1.3). It is easy to show that all solutions of system (1.2) with initial conditions (1.3) are defined on $[0, +\infty)$ and remain positive for all $t \geq 0$.

The organization of this paper is as follows. In the next section, we show the permanence of solutions of model (1.2) with initial conditions (1.3). In Sect. 3, by analyzing the corresponding characteristic equations, we study the local stability of each feasible boundary equilibria of system (1.2) and the existence of Hopf bifurcations of system (1.2) at the disease-free equilibrium. By means of Lyapunov functions and LaSalle invariant principle, we establish sufficient conditions for the global stability of each feasible boundary equilibria of system (1.2). In Sect. 4, by analyzing the corresponding characteristic equation, we discuss the local stability of coexistence equilibrium and the existence of Hopf bifurcations of system (1.2) at the coexistence equilibrium.

By means of Lyapunov functions and LaSalle invariant principle, we establish sufficient conditions for the global attractiveness of the coexistence equilibrium. A brief discussion is given in Sect. 5 to conclude this work.

2 Permanence

In this section, we are concerned with the permanence of system (1.2).

Lemma 2.1 *Positive solutions of system (1.2) with initial conditions (1.3) are ultimately bounded.*

Proof Let $(x_1(t), x_2(t), S(t), I(t))$ be any positive solution of system (1.2) with initial conditions (1.3). Denote $d = \min\{d_1, d_2, d_3, d_4\}$. Define

$$V(t) = x_1(t - \tau) + x_2(t - \tau) + \frac{a_1}{a_2}[S(t) + I(t)].$$

Calculating the derivative of $V(t)$ along positive solutions of system (1.2), it follows that

$$\begin{aligned} \dot{V}(t) &= -d_1x_1(t - \tau) - d_2x_2(t - \tau) - ax_2^2(t - \tau) + rx_2(t - \tau) \\ &\quad - \frac{a_1}{a_2}[d_3S(t) + d_4I(t)] - bS^2(t) \\ &\leq -dV(t) - a\left[x_2(t - \tau) - \frac{r}{2a}\right]^2 - bS^2(t) + \frac{r^2}{4a} \\ &\leq -dV(t) + \frac{r^2}{4a} \end{aligned}$$

which yields $\limsup_{t \rightarrow \infty} V(t) \leq \frac{r^2}{4ad}$.

If we choose $M_1 = r^2/(4ad)$, $M_2 = a_2r^2/(4aa_1d)$, then

$$\limsup_{t \rightarrow \infty} x_i(t) \leq M_1 (i = 1, 2), \quad \limsup_{t \rightarrow \infty} S(t) \leq M_2, \quad \limsup_{t \rightarrow \infty} I(t) \leq M_2.$$

□

Theorem 2.1 *Suppose that*

$$(H_1) \quad \beta \underline{S} > d_4,$$

where \underline{S} is defined in (2.3), then system (1.2) is permanent.

Proof Let $(x_1(t), x_2(t), S(t), I(t))$ be any positive solution of system (1.2) with initial conditions (1.3). By Lemma 2.1, it follows that $\limsup_{t \rightarrow +\infty} S(t) \leq M_2$. Hence, for $\varepsilon > 0$ being sufficiently small, there is a $T_0 > 0$ such that if $t > T_0$, $S(t) < M_2 + \varepsilon$. Accordingly, for $\varepsilon > 0$ being sufficiently small, we derive from the first and the second equations of system (1.2) that, for $t > T_0$,

$$\begin{aligned} \dot{x}_1(t) &= rx_2(t) - (r_1 + d_1)x_1(t), \\ \dot{x}_2(t) &\geq r_1x_1(t) - d_2x_2(t) - ax_2^2(t) - a_1(M_2 + \varepsilon)x_2(t), \end{aligned} \tag{2.1}$$

which yields

$$\liminf_{t \rightarrow +\infty} x_1(t) \geq \frac{r}{r_1 + d_1} \underline{x}_2 := \underline{x}_1, \quad \liminf_{t \rightarrow +\infty} x_2(t) \geq \frac{rr_1 - (r_1 + d_1)(d_2 + a_1 M_2)}{a(r_1 + d_1)} := \underline{x}_2. \tag{2.2}$$

we derive from the third equation of system (1.2), for t sufficiently large, we have

$$\dot{S}(t) \geq b \underline{x}_2 S(t - \tau) - d_3 S(t) - b S^2(t) - \frac{\beta M_2}{1 + \alpha M_2} S(t)$$

By Theorem 4.9.1 in [14], one can obtain that

$$\liminf_{t \rightarrow +\infty} S(t) \geq \frac{1}{b} [b \underline{x}_2 - d_3 - \frac{\beta M_2}{1 + \alpha M_2}] := \underline{S}. \tag{2.3}$$

we derive from the fourth equation of system (1.2), for t sufficiently large, we have

$$\dot{I}(t) \geq \frac{\beta \underline{S} I(t)}{1 + \alpha I(t)} - d_4 I(t)$$

Since (H_1) holds, then

$$\liminf_{t \rightarrow +\infty} I(t) \geq \frac{1}{\alpha d_4} [\beta \underline{S} - d_4] := \underline{I}. \tag{2.4}$$

The above calculations and Lemma 2.1 imply that system (1.2) is permanent. □

3 Boundary equilibria and their stability

In this section, we discuss the stability of the boundary equilibria and the existence of a Hopf bifurcation at the disease-free equilibrium.

System (1.2) always has a trivial equilibrium $E_0(0, 0, 0, 0)$. If $rr_1 > d_2(r_1 + d_1)$, then system (1.2) has a predator-extinction equilibrium $E_1(x_1^0, x_2^0, 0, 0)$, where

$$x_1^0 = \frac{r[rr_1 - d_2(r_1 + d_1)]}{a(r_1 + d_1)^2}, \quad x_2^0 = \frac{rr_1 - d_2(r_1 + d_1)}{a(r_1 + d_1)}.$$

If $a_2 x_2^0 > d_3$, then system (1.2) has a disease-free equilibrium $E_2(x_1^+, x_2^+, S^+, 0)$, where

$$x_1^+ = \frac{r(abx_2^0 + a_1 d_3)}{(r_1 + d_1)(ab + a_1 a_2)}, \quad x_2^+ = \frac{abx_2^0 + a_1 d_3}{ab + a_1 a_2}, \quad S^+ = \frac{a(a_2 x_2^0 - d_3)}{ab + a_1 a_2}.$$

The characteristic equation of system (1.2) at the equilibrium $E_0(0, 0, 0, 0)$ takes the form

$$(\lambda + d_3)(\lambda + d_4)[\lambda^2 + (r_1 + d_1 + d_2)\lambda + d_2(r_1 + d_1) - rr_1] = 0, \tag{3.1}$$

It is readily seen from Eq. (3.1) that if $rr_1 < d_2(r_1 + d_1)$, then E_0 is locally asymptotically stable; if $rr_1 > d_2(r_1 + d_1)$, then E_0 is unstable.

Theorem 3.1 *If $rr_1 < d_2(r_1 + d_1)$, then the trivial equilibrium $E_0(0, 0, 0, 0)$ of system (1.2) is globally asymptotically stable.*

Proof Based on above discussions, we see that if $rr_1 < d_2(r_1 + d_1)$, then E_0 is locally asymptotically stable. Hence, we only prove that all positive solutions of system (1.2) with initial conditions (1.3) converge to E_0 . Let $(x_1(t), x_2(t), S(t), I(t))$ be any positive solution of system (1.2) with initial conditions (1.3). Define

$$V_0(t) = \frac{r_1}{r_1 + d_1}x_1(t) + x_2(t) + \frac{a_1}{a_2} [S(t) + I(t)] + a_1 \int_{t-\tau}^t x_2(u)S(u)du.$$

Calculating the derivative of $V_0(t)$ along positive solutions of system (1.2), it follows that

$$\dot{V}_0(t) = -\frac{d_2(r_1 + d_1) - rr_1}{r_1 + d_1}x_2(t) - ax_2^2(t) - \frac{a_1}{a_2} [d_3S(t) + d_4I(t)] - \frac{a_1}{a_2}bS^2(t). \tag{3.2}$$

If $rr_1 < d_2(r_1 + d_1)$, it then follows from (3.2) that $\dot{V}_0(t) \leq 0$. By Theorem 5.3.1 in [10], solutions limit to Λ , the largest invariant subset of $\{\dot{V}_0(t) = 0\}$. Clearly, we see from (3.2) that $\dot{V}_0(t) = 0$ if and only if $x_2(t) = 0, S(t) = 0$ and $I(t) = 0$. Noting that Λ is invariant, for each element in Λ , we have $x_2(t) = 0$. It therefore follows from the second equation of system (1.2) that

$$0 = \dot{x}_2(t) = r_1x_1(t),$$

which yields $x_1(t) = 0$. Hence, $\dot{V}_0(t) = 0$ if and only if $(x_1(t), x_2(t), y_1(t), y_2(t)) = (0, 0, 0, 0)$. Accordingly, the global asymptotic stability of E_0 follows from LaSalle invariant principle for delay differential systems. \square

The characteristic equation of system (1.2) at the equilibrium $E_1(x_1^0, x_2^0, 0, 0)$ is of the form

$$(\lambda + d_4)[\lambda^2 + (r_1 + d_1 + d_2 + 2ax_2^0)\lambda + rr_1 - d_2(r_1 + d_1)](\lambda + d_3 - a_2x_2^0e^{-\lambda\tau}) = 0, \tag{3.3}$$

Equation (3.3) always has a negative real root: $\lambda_1 = -d_4$. If $rr_1 > d_2(r_1 + d_1)$, then the equation

$$\lambda^2 + (r_1 + d_1 + d_2 + 2ax_2^0)\lambda + rr_1 - d_2(r_1 + d_1) = 0$$

has two roots with negative real parts. All other roots of Eq. (3.3) are determined by the equation

$$\lambda + d_3 - a_2x_2^0e^{-\lambda\tau} = 0. \tag{3.4}$$

Denote $f(\lambda) = \lambda + d_3 - a_2x_2^0e^{-\lambda\tau}$. If $a_2x_2^0 > d_3$ holds, it is easy to show that, for λ real,

$$f(0) = d_3 - a_2x_2^0 < 0, \quad \lim_{\lambda \rightarrow +\infty} f(\lambda) = +\infty$$

Hence, $f(\lambda) = 0$ has a positive real root. Therefore, if $a_2x_2^0 > d_3$ holds, the equilibrium $E_1(x_1^0, x_2^0, 0, 0)$ is unstable.

If $0 < a_2x_2^0 < d_3$, we claim that E_1 is locally asymptotically stable. Otherwise, there is a root λ satisfying $Re\lambda \geq 0$. It follows from (3.4) that

$$Re\lambda = a_2x_2^0e^{-\tau Re\lambda} \cos(\tau Im\lambda) - d_3 \leq a_2x_2^0 - d_3 < 0,$$

which is a contradiction. Hence, if $0 < a_2x_2^0 < d_3$, then the equilibrium E_1 is locally asymptotically stable. □

Theorem 3.2 *If $0 < a_2x_2^0 < d_3$, then the predator-extinction equilibrium $E_1(x_1^0, x_2^0, 0, 0)$ of system (1.2) is globally asymptotically stable.*

Proof Based on above discussions, we see that if $0 < a_2x_2^0 < d_3$, then E_1 is locally asymptotically stable. Hence, we only prove that all positive solutions of system (1.2) with initial conditions (1.3) converge to E_1 . Let $(x_1(t), x_2(t), S(t), I(t))$ be any positive solution of system (1.2) with initial conditions (1.3). System (1.2) can be rewritten as

$$\begin{aligned} \dot{x}_1(t) &= \frac{r}{x_1^0}[-x_2(t)(x_1(t) - x_1^0) + x_1(t)(x_2(t) - x_2^0)] \\ \dot{x}_2(t) &= \frac{r_1}{x_2^0}[-x_1(t)(x_2(t) - x_2^0) + x_2(t)(x_1(t) - x_1^0)] \\ &\quad + x_2(t)[-a(x_2(t) - x_2^0)] - a_1x_2(t)S(t) \\ \dot{S}(t) &= a_2x_2(t - \tau)S(t - \tau) - d_3S(t) - bS^2(t) - \frac{\beta S(t)I(t)}{1 + \alpha I(t)} \\ \dot{I}(t) &= \frac{\beta S(t)I(t)}{1 + \alpha I(t)} - d_4I(t) \end{aligned} \tag{3.5}$$

Define

$$V_{11}(t) = k_1 \left(x_1 - x_1^0 - x_1^0 \ln \frac{x_1}{x_1^0} \right) + x_2 - x_2^0 - x_2^0 \ln \frac{x_2}{x_2^0} + k_2(S + I).$$

where $k_1 = r_1x_1^0/(rx_2^0)$, $k_2 = a_1/a_2$. Calculating the derivative of $V_{11}(t)$ along positive solutions of system (1.2), it follows that

$$\begin{aligned} \dot{V}_{11}(t) &= \frac{k_1(x_1(t) - x_1^0)}{x_1(t)} \dot{x}_1(t) + \frac{x_2(t) - x_2^0}{x_2(t)} \dot{x}_2(t) + k_2 (\dot{S}(t) + \dot{I}(t)) \\ &= -\frac{r_1}{x_2^0} \left(\sqrt{\frac{x_2(t)}{x_1(t)}}(x_1(t) - x_1^0) - \sqrt{\frac{x_1(t)}{x_2(t)}}(x_2(t) - x_2^0) \right)^2 \\ &\quad - a(x_2(t) - x_2^0)^2 + a_1x_2^0S(t) \\ &\quad - a_1x_2(t)S(t) + a_1x_2(t - \tau)S(t - \tau) \\ &\quad - k_2d_3S(t) - k_2bS^2(t) - k_2d_4I(t). \end{aligned} \tag{3.6}$$

Define

$$V_1(t) = V_{11}(t) + a_1 \int_{t-\tau}^t x_2(u)S(u)du.$$

By calculation, we have that

$$\begin{aligned} \dot{V}_1(t) = & -\frac{r_1}{x_2^0} \left(\sqrt{\frac{x_2(t)}{x_1(t)}}(x_1(t) - x_1^0) - \sqrt{\frac{x_1(t)}{x_2(t)}}(x_2(t) - x_2^0) \right)^2 - a(x_2(t) - x_2^0)^2 \\ & - (k_2d_3 - a_1x_2^0)S(t) - k_2bS^2(t) - k_2d_4I(t) \end{aligned} \tag{3.7}$$

It follows from (3.7) that if $a_2x_2^0 < d_3$, i.e. $k_2d_3 > a_1x_2^0$ holds, then $\dot{V}_1(t) \leq 0$. By Theorem 5.3.1 in [10], solutions limit to Λ , the largest invariant subset of $\{\dot{V}_0(t) = 0\}$. Clearly, we see from (3.7) that $\dot{V}_1(t) = 0$ if and only if $x_1(t) = x_1^0, x_2(t) = x_2^0, S(t) = 0$ and $I(t) = 0$. Hence, the only invariant set $M = \{(x_1^0, x_2^0, 0, 0)\}$. Using LaSalle invariant principle for delay differential systems, the global asymptotic stability of E_1 follows. \square

The characteristic equation of system (1.2) at the equilibrium $E_2(x_1^+, x_2^+, S^+, 0)$ takes the form

$$(\lambda + d_4 - \beta S^+)[\lambda^3 + g_2\lambda^2 + g_1\lambda + g_0 + (f_2\lambda^2 + f_1\lambda + f_0)e^{-\lambda\tau}] = 0, \tag{3.8}$$

where

$$\begin{aligned} g_0 = & ax_2^+(r_1 + d_1)(d_3 + 2bS^+), & g_1 = & (d_3 + 2bS^+)(r_1 + d_1 + c_1) + ax_2^+(r_1 + d_1), \\ g_2 = & r_1 + d_1 + c_1 + d_3 + 2bS^+, & c_1 = & d_2 + 2ax_2^+ + a_1S^+, \\ f_0 = & a_2x_2^+[a_1S^+(r_1 + d_1) - ax_2^+(r_1 + d_1)], \\ f_1 = & a_2x_2^+[a_1S^+ - (r_1 + d_1 + c_1)], & f_2 = & -a_2x_2^+. \end{aligned}$$

Clearly, Eq. (3.8) always has a root $\lambda_1 = \beta S^+ - d_4$. All other roots of Eq. (3.8) are determined by the following equation

$$\lambda^3 + g_2\lambda^2 + g_1\lambda + g_0 + (f_2\lambda^2 + f_1\lambda + f_0)e^{-\lambda\tau} = 0. \tag{3.9}$$

When $\tau = 0$, Eq. (3.9) reduces to

$$\lambda^3 + (g_2 + f_2)\lambda^2 + (g_1 + f_1)\lambda + g_0 + f_0 = 0. \tag{3.10}$$

By calculation, we derive that

$$\begin{aligned} \Delta_1 = & g_2 + f_2 = r_1 + d_1 + c_1 + bS^+ > 0, \\ \Delta_2 = & (g_1 + f_1)(g_2 + f_2) - (g_0 + f_0) = bS^+(r_1 + d_1 + c_1)(r_1 + d_1 + c_1 + bS^+) \\ & + a_1a_2x_2^+S^+(c_1 + bS^+) + ax_2^+(r_1 + d_1)(r_1 + d_1 + c_1) > 0, \\ \Delta_3 = & (g_0 + f_0)\Delta_2 = x_2^+S^+(r_1 + d_1)(ab + a_1a_2)\Delta_2 > 0. \end{aligned}$$

Hence, if $0 < \beta S^+ < d_4$, then the equilibrium E_2 is locally asymptotically stable when $\tau = 0$.

If $i\omega (\omega > 0)$ is a solution of (3.9), separating real and imaginary parts, we have

$$\begin{aligned} f_1\omega \sin \omega\tau + (f_0 - f_2\omega^2) \cos \omega\tau &= g_2\omega^2 - g_0, \\ f_1\omega \cos \omega\tau - (f_0 - f_2\omega^2) \sin \omega\tau &= \omega^3 - g_1\omega \end{aligned} \tag{3.11}$$

Squaring and adding the two equations of (3.11), it follows that

$$\omega^6 + h_2\omega^4 + h_1\omega^2 + h_0 = 0. \tag{3.12}$$

By calculation, we derive that

$$\begin{aligned} h_2 &= g_2^2 - 2g_1 - f_2^2 = c_1^2 + (r_1 + d_1)^2 + 2rr_1 + \beta S^+(2d_3 + 3\beta S^+) > 0, \\ h_1 &= g_1^2 - 2g_0g_2 + 2f_0f_2 - f_1^2 \\ &= [ax_2^+(r_1 + d_1)]^2 + a_1S^+(d_3 + \beta S^+)^2(2d_2 + 4ax_2^+ + a_1S^+) > 0, \\ h_0 &= g_0^2 - f_0^2 = x_2^+S^+(r_1 + d_1)(ab + a_1a_2)(g_0 - f_0) \end{aligned}$$

Note that if $g_0 > f_0$, equation (3.12) has no positive real roots. Accordingly, by Theorem 3.4.1 in Kuang [14], we see that if $0 < \beta S^+ < d_4$ and $g_0 > f_0$, then E_2 is locally asymptotically stable for all $\tau \geq 0$. If $0 < \beta S^+ < d_4$ and $g_0 < f_0$, then equation (3.12) has a unique positive root ω_0 . That is, equation (3.9) has a pair of purely imaginary roots of the form $\pm i\omega_0$. Denote

$$\begin{aligned} \tau_k &= \frac{1}{\omega_0} \arccos \frac{f_1\omega_0(\omega_0^3 - g_1\omega_0) + (f_0 - f_2\omega_0^2)(g_2\omega_0^2 - g_0)}{(f_1\omega_0)^2 + (f_0 - f_2\omega_0^2)^2} \\ &\quad + \frac{2k\pi}{\omega_0}, \quad k = 0, 1, 2, \dots \end{aligned} \tag{3.13}$$

By Theorem 3.4.1 in Kuang [14], we see that if $0 < \beta S^+ < d_4$ and $g_0 < f_0$, then E_2 remains stable for $\tau < \tau_0$.

We now claim that

$$\left. \frac{d(Re(\lambda))}{d\tau} \right|_{\tau=\tau_0} > 0$$

This will show that there exists at least one eigenvalue with positive real part for $\tau > \tau_0$. Moreover, the conditions for the existence of a Hopf bifurcation [10] are then satisfied yielding a periodic solution. To this end, differentiating Eq. (3.9) with respect to τ , it follows that

$$\left(\frac{d\lambda}{d\tau} \right)^{-1} = \frac{3\lambda^2 + 2g_2\lambda + g_1}{-\lambda(\lambda^3 + g_2\lambda^2 + g_1\lambda + g_0)} + \frac{2f_2\lambda + f_1}{\lambda(f_2\lambda^2 + f_1\lambda + f_0)} - \frac{\tau}{\lambda}.$$

Hence, a direct calculation shows that

$$\begin{aligned} \operatorname{sign} \left\{ \frac{d(\operatorname{Re}\lambda)}{d\tau} \right\}_{\lambda=i\omega_0} &= \operatorname{sign} \left\{ \operatorname{Re} \left(\frac{d\lambda}{d\tau} \right)^{-1} \right\}_{\lambda=i\omega_0} \\ &= \operatorname{sign} \left\{ \frac{3\omega_0^4 + 2(g_2^2 - 2g_1)\omega_0^2 + g_1^2 - 2g_0g_2}{(\omega_0^3 - g_1\omega_0)^2 + (g_0 - g_2\omega_0^2)^2} + \frac{-2f_2^2\omega_0^2 + 2f_2f_0 - f_1^2}{(f_1\omega_0)^2 + (f_2\omega_0^2 - f_0)^2} \right\}. \end{aligned}$$

We derive from (3.11) that

$$(\omega_0^3 - g_1\omega_0)^2 + (g_0 - g_2\omega_0^2)^2 = (f_1\omega_0)^2 + (f_2\omega_0^2 - f_0)^2$$

Hence, it follows that

$$\operatorname{sign} \left\{ \frac{d(\operatorname{Re}\lambda)}{d\tau} \right\}_{\lambda=i\omega_0} = \operatorname{sign} \left\{ \frac{3\omega_0^4 + 2h_2\omega_0^2 + h_1}{(f_1\omega_0)^2 + (f_2\omega_0^2 - f_0)^2} \right\} > 0.$$

Therefore, the transversal condition holds and a Hopf bifurcation occurs at $\omega = \omega_0, \tau = \tau_0$.

In conclusion, we have the following results.

Theorem 3.3 For system (1.2), assume $0 < \beta S^+ < d_4$ holds, we have the following:

- (i) If $g_0 > f_0$, then the disease-free equilibrium $E_2(x_1^+, x_2^+, S^+, 0, \cdot)$ is locally asymptotically stable for all $\tau \geq 0$;
- (ii) If $g_0 < f_0$, then there exists a positive number τ_0 , such that E_2 is locally asymptotically stable if $0 < \tau < \tau_0$ and is unstable if $\tau > \tau_0$. Further, system (1.2) undergoes a Hopf bifurcation at E_2 when $\tau = \tau_0$.

Theorem 3.4 Let $0 < \beta S^+ < d_4$ hold, then the disease-free equilibrium E_2 is globally asymptotically stable provided

$$(H_2) \quad \underline{x}_2 > \frac{a_1}{a} S^+.$$

Here, \underline{x}_2 is defined in Theorem 2.1.

Proof It is easy to see that if (H_2) holds, then $ax_2^+ > a_1S^+$. It follows from (3.8) that $g_0 - f_0 = x_2^+(r_1 + d_1)[ad_3 + 2abS^+ + a_2(ax_2^+ - a_1S^+)] > 0$ holds. By Theorem 3.3, we see that if $0 < \beta S^+ < d_4$ and (H_1) hold, then the equilibrium $E_2(x_1^+, x_2^+, S^+, 0)$ is locally asymptotically stable. Hence, it suffices to show that all positive solutions of system (1.2) with initial conditions (1.3) converge to E_2 . We achieve this by constructing a global Lyapunov function. Let $(x_1(t), x_2(t), S(t), I(t))$ be any positive solution of system (1.2) with initial conditions (1.3). System (1.2) can be rewritten as

$$\begin{aligned}
 \dot{x}_1(t) &= \frac{r}{x_1^+} [-x_2(t)(x_1(t) - x_1^+) + x_1(t)(x_2(t) - x_2^+)] \\
 \dot{x}_2(t) &= \frac{r_1}{x_2^+} [-x_1(t)(x_2(t) - x_2^+) + x_2(t)(x_1(t) - x_1^+)] + x_2(t)[-a(x_2(t) - x_2^+)] \\
 &\quad + a_1 S^+ x_2(t) - a_1 x_2(t) S(t) \\
 \dot{S}(t) &= a_2 x_2(t - \tau) S(t - \tau) - d_3 S(t) - b S^2(t) - \frac{\beta S(t) I(t)}{1 + \alpha I(t)} \\
 \dot{I}(t) &= \frac{\beta S(t) I(t)}{1 + \alpha I(t)} - d_4 I(t)
 \end{aligned}
 \tag{3.14}$$

Define

$$\begin{aligned}
 V_{21}(t) &= k_3 \left(x_1 - x_1^+ - x_1^+ \ln \frac{x_1}{x_1^+} \right) + x_2 - x_2^+ - x_2^+ \ln \frac{x_2}{x_2^+} \\
 &\quad + k_4 \left(S - S^+ - S^+ \ln \frac{S}{S^+} \right) + k_4 I.
 \end{aligned}$$

where $k_3 = r_1 x_1^+ / (r x_2^+)$, $k_4 = a_1 / a_2$.

Calculating the derivative of $V_{21}(t)$ along positive solutions of system (1.2), it follows that

$$\begin{aligned}
 \dot{V}_{21}(t) &= k_3 \frac{x_1(t) - x_1^+}{x_1(t)} \dot{x}_1(t) + \frac{x_2(t) - x_2^+}{x_2(t)} \dot{x}_2(t) + k_4 \frac{S(t) - S^+}{S(t)} \dot{S}(t) + k_4 \dot{I}(t) \\
 &= -\frac{r_1}{x_2^+} \left(\sqrt{\frac{x_2(t)}{x_1(t)}} (x_1(t) - x_1^+) - \sqrt{\frac{x_1(t)}{x_2(t)}} (x_2(t) - x_2^+) \right)^2 - a(x_2(t) - x_2^+)^2 \\
 &\quad - a_1 x_2(t) S(t) + a_1 x_2(t - \tau) S(t - \tau) - k_4 \left(d_4 - \frac{\beta S^+}{1 + \alpha I(t)} \right) I(t) + a_1 x_2^+ S^+ \\
 &\quad - \frac{a_1 S^+}{S(t)} x_2(t - \tau) S(t - \tau) + a_1 S^+ (x_2(t) - x_2^+) - k_4 b (S(t) - S^+)^2
 \end{aligned}
 \tag{3.15}$$

Define

$$V_2(t) = V_{21}(t) + a_1 \int_{t-\tau}^t \left[x_2(u) S(u) - x_2^+ S^+ - x_2^+ S^+ \ln \frac{x_2(u) S(u)}{x_2^+ S^+} \right] du.$$

By calculation, we have that

$$\begin{aligned}
 \dot{V}_2(t) &= -\frac{r_1}{x_2^+} \left(\sqrt{\frac{x_2(t)}{x_1(t)}} (x_1(t) - x_1^+) - \sqrt{\frac{x_1(t)}{x_2(t)}} (x_2(t) - x_2^+) \right)^2 - k_4 b (S(t) - S^+)^2 \\
 &\quad - a_1 x_2^+ S^+ \left[\frac{x_2(t - \tau) S(t - \tau)}{x_2^+ S(t)} - 1 - \ln \frac{x_2(t - \tau) S(t - \tau)}{x_2^+ S(t)} \right]
 \end{aligned}$$

$$\begin{aligned}
 & -a_1x_2^+S^+ \left[\frac{x_2^+}{x_2(t)} - 1 - \ln \frac{x_2^+}{x_2(t)} \right] - k_4(d_4 - \frac{\beta S^+}{1 + \alpha I(t)})I(t) \\
 & -(x_2(t) - x_2^+)^2 \left[a - \frac{a_1S^+}{x_2(t)} \right]
 \end{aligned} \tag{3.16}$$

It follows from (3.16) that if $0 < \beta S^+ < d_4$ and (H_1) hold true, then $\dot{V}_2(t) \leq 0$ with equality if and only if $x_1(t) = x_1^+, x_2(t) = x_2^+, S(t) = S^+, I(t) = 0$ Using LaSalle invariant principle for delay differential systems, the global asymptotic stability of the equilibrium E_2 of system (1.2) follows. \square

4 Coexistence equilibrium and its stability

In this section, we discuss the stability of the coexistence equilibrium.

It is easy to show that if $\beta S^+ > d_4$, system (1.2) has a unique coexistence equilibrium $E^*(x_1^*, x_2^*, S^*, I^*)$, where

$$\begin{aligned}
 x_1^* &= \frac{r}{r_1 + d_1}x_2^*, & x_2^* &= \frac{rr_1 - d_2(r_1 + d_1)}{a(r_1 + d_1)} - \frac{a_1}{a}S^*, \\
 S^* &= \frac{1}{2}\Delta + \sqrt{\frac{1}{4}\Delta^2 + \frac{ad_4}{\alpha(ab + a_1a_2)}}, & I^* &= \frac{\beta S^* - d_4}{\alpha d_4}, \\
 \Delta &= S^+ - \frac{a\beta}{\alpha(ab + a_1a_2)}.
 \end{aligned}$$

The characteristic equation of system (1.2) at the equilibrium E^* is of the form

$$\lambda^4 + p_3\lambda^3 + p_2\lambda^2 + p_1\lambda + p_0 + (q_3\lambda^3 + q_2\lambda^2 + q_1\lambda + q_0)e^{-\lambda\tau} = 0, \tag{4.1}$$

where

$$\begin{aligned}
 p_3 &= r_1 + d_1 + c_2 + c_3 + \frac{d_4\alpha I^*}{1 + \alpha I^*}, \\
 p_2 &= \frac{d_4\alpha I^*}{1 + \alpha I^*}(r_1 + d_1 + c_2 + c_3) + c_3(r_1 + d_1 + c_2) + \frac{d_4\beta I^*}{(1 + \alpha I^*)^2} + ax_2^*(r_1 + d_1), \\
 p_1 &= (r_1 + d_1 + c_2) \left(\frac{c_3d_4\alpha I^*}{1 + \alpha I^*} + \frac{d_4\beta I^*}{(1 + \alpha I^*)^2} \right) + ax_2^*(r_1 + d_1) \left(c_3 + \frac{d_4\alpha I^*}{1 + \alpha I^*} \right), \\
 p_0 &= ax_2^*(r_1 + d_1) \left(\frac{c_3d_4\alpha I^*}{1 + \alpha I^*} + \frac{d_4\beta I^*}{(1 + \alpha I^*)^2} \right), \\
 q_3 &= -a_2x_2^*, & q_2 &= -a_2x_2^* \left(r_1 + d_1 + d_2 + 2ax_2^* + \frac{d_4\alpha I^*}{1 + \alpha I^*} \right), \\
 q_1 &= -a_2x_2^* \left[(r_1 + d_1 + c_2) \frac{d_4\alpha I^*}{1 + \alpha I^*} + ax_2^*(r_1 + d_1) - a_1S^*(r_1 + d_1) + \frac{d_4\alpha I^*}{1 + \alpha I^*} \right], \\
 q_0 &= -a_2x_2^* \frac{d_4\alpha I^*}{1 + \alpha I^*} [ax_2^*(r_1 + d_1) - a_1S^*(r_1 + d_1)], \\
 c_2 &= d_2 + 2ax_2^* + a_1S^*, & c_3 &= a_2x_2^* + bS^*.
 \end{aligned}$$

When $\tau = 0$, Eq. (4.1) becomes

$$\lambda^4 + (p_3 + q_3)\lambda^3 + (p_2 + q_2)\lambda^2 + (p_1 + q_1)\lambda + p_0 + q_0 = 0. \tag{4.2}$$

It is easy to show that

$$\begin{aligned} \Delta_1 &= p_3 + q_3 = r_1 + d_1 + c_2 + bS^* + \frac{d_4\alpha I^*}{1 + \alpha I^*} > 0, \\ \Delta_2 &= (p_3 + q_3)(p_2 + q_2) - (p_1 + q_1) \\ &= ax_2^*(r_1 + d_1)(r_1 + d_1 + c_2) + a_1a_2x_2^*S^*(c_2 + bS^*) \\ &\quad + bS^*(r_1 + d_1 + c_2 + bS^*) \left(r_1 + d_1 + c_2 + \frac{d_4\alpha I^*}{1 + \alpha I^*} \right) \\ &\quad + \frac{d_4\beta I^*}{(1 + \alpha I^*)^2} \left(bS^* + \frac{d_4\alpha I^*}{1 + \alpha I^*} \right) \\ &\quad + \frac{d_4\alpha I^*}{1 + \alpha I^*}(r_1 + d_1 + c_2) \left(r_1 + d_1 + c_2 + bS^* + \frac{d_4\alpha I^*}{1 + \alpha I^*} \right) > 0, \\ \Delta_3 &= (p_1 + q_1)\Delta_2 - (p_0 + q_0)(p_3 + q_3)^2 \\ \Delta_4 &= (p_0 + q_0)\Delta_3 \end{aligned}$$

Note $p_0 + q_0 > 0$, hence, if $\Delta_3 > 0$, we have $\Delta_4 > 0$. By the Routh-Hurwitz criterion, we know that if $\beta S^+ > d_4$ and $\Delta_3 > 0$ hold, the coexistence equilibrium E^* of system (1.2) is locally asymptotically stable when $\tau = 0$.

Substituting $\lambda = i\nu (\nu > 0)$ into (4.1) and separating the real and imaginary parts, one obtains that

$$\begin{aligned} (q_3\nu^3 - q_1\nu) \sin \nu\tau + (q_2\nu^2 - q_0) \cos \nu\tau &= \nu^4 - p_2\nu^2 + p_0, \\ (q_3\nu^3 - q_1\nu) \cos \nu\tau - (q_2\nu^2 - q_0) \sin \nu\tau &= p_1\nu - p_3\nu^3. \end{aligned} \tag{4.3}$$

Squaring and adding the two equations of (4.3), it follows that

$$\nu^8 + \hat{h}_3\nu^6 + \hat{h}_2\nu^4 + \hat{h}_1\nu^2 + \hat{h}_0 = 0, \tag{4.4}$$

where

$$\begin{aligned} \hat{h}_3 &= p_3^2 - 2p_2 - q_3^2, & \hat{h}_2 &= p_2^2 + 2p_0 - 2p_1p_3 - q_2^2 + 2q_1q_3, \\ \hat{h}_1 &= p_1^2 - 2p_0p_2 - q_1^2 + 2q_0q_2, & \hat{h}_0 &= p_0^2 - q_0^2. \end{aligned}$$

Letting $z = \nu^2$, Eq. (4.4) can be written as

$$\hat{h}(z) := z^4 + \hat{h}_3z^3 + \hat{h}_2z^2 + \hat{h}_1z + \hat{h}_0 = 0. \tag{4.5}$$

If $\hat{h}_i > 0 (i = 0, 1, 2, 3)$, then $\hat{h}(z)$ has always no positive roots. Hence, under these conditions, Eq. (4.4) has no purely imaginary roots for any $\tau > 0$ and accordingly, the equilibrium E^* is locally asymptotically stable for all $\tau \geq 0$.

If $\hat{h}_i > 0 (i = 1, 2, 3)$ and $\hat{h}_0 < 0$, then Eq. (4.5) has one positive root z^* . Accordingly, Eq. (4.4) has one positive roots $v_0 = \sqrt{z^*}$. From (4.3) one can get the corresponding $\tau^{(j)} > 0$ such that (4.1) has a pair of purely imaginary roots $\pm i v_0$ given by

$$\tau^{(j)} = \frac{1}{v_0} \arccos \frac{(q_2 v_0^2 - q_0)(v_0^4 - p_2 v_0^2 + p_0) + (q_3 v_0^3 - q_1 v_0)(p_1 v_0 - p_3 v_0^3)}{(q_2 v_0^2 - q_0)^2 + (q_3 v_0^3 - q_1 v_0)^2} + \frac{2j\pi}{v_0},$$

$j = 0, 1, 2, \dots$

Differentiating the two sides of (4.1) with respect to τ , it follows that

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{4\lambda^3 + 3p_3\lambda^2 + 2p_2\lambda + p_1}{-\lambda(\lambda^4 + p_3\lambda^3 + p_2\lambda^2 + p_1\lambda + p_0)} + \frac{3q_3\lambda^2 + 2q_2\lambda + q_1}{\lambda(q_3\lambda^3 + q_2\lambda^2 + q_1\lambda + q_0)} - \frac{\tau}{\lambda}.$$

After some algebra, one obtains that

$$\begin{aligned} \text{sign} \left\{ \frac{d\text{Re}\lambda}{d\tau} \right\}_{\tau=\tau^{(j)}} &= \text{sign} \left\{ \text{Re} \left(\frac{d\lambda}{d\tau} \right)^{-1} \right\}_{\tau=\tau^{(j)}} \\ &= \text{sign} \left\{ \frac{(p_1 - 3p_3v_0^2)(p_3v_0^2 - p_1) + 2(p_2 - 2v_0^2)(v_0^4 - p_2v_0^2 + p_0)}{v_0^2(p_1 - p_3v_0^2)^2 + (v_0^4 - p_2v_0^2 + p_0)^2} \right. \\ &\quad \left. + \frac{(q_1 - 3q_3v_0^2)(q_3v_0^2 - q_1) + 2q_2v_0(q_0 - q_2v_0^2)}{(q_0 - q_2v_0^2)^2 + (q_1v_0 - q_3v_0^3)^2} \right\} \end{aligned}$$

We derive from (4.3) that

$v_0^2(p_1 - p_3v_0^2)^2 + (v_0^4 - p_2v_0^2 + p_0)^2 = (q_0 - q_2v_0^2)^2 + (q_1v_0 - q_3v_0^3)^2$. Hence, it follows that

$$\text{sign} \left\{ \frac{d\text{Re}\lambda}{d\tau} \right\}_{\tau=\tau^{(j)}} = \text{sign} \left\{ \frac{4v_0^6 + 3\hat{h}_3v_0^4 + 2\hat{h}_2v_0^2 + \hat{h}_1}{(q_2v_0^2 - q_0)^2 + (q_1v_0 - q_3v_0^3)^2} \right\} > 0$$

From what has been discussed above, we have the following results.

Theorem 4.1 Assume that $\beta S^+ > d_4$ and $\Delta_3 > 0$ hold, we have

- (i) If $\hat{h}_i > 0 (i = 0, 1, 2, 3)$, then the coexistence equilibrium E^* is locally asymptotically stable for all $\tau \geq 0$.
- (ii) If $\hat{h}_i > 0 (i = 1, 2, 3)$ and $\hat{h}_0 < 0$, then there exists a positive number $\tau^{(0)}$, such that E^* is locally asymptotically stable if $0 < \tau < \tau^{(0)}$ and is unstable if $\tau > \tau^{(0)}$. Further, system (1.2) undergoes a Hopf bifurcation at E_2 when $\tau = \tau^{(0)}$.

Now, we are concerned with the global attractiveness of the coexistence equilibrium E^* .

Theorem 4.2 Assume that $\beta S^+ > d_4$, then the coexistence equilibrium $E^*(x_1^*, x_2^*, S^*, I^*)$ of system (1.2) is globally attractive provided

(H3) $\underline{x}_2 > \frac{a_1}{a} S^*$, $\underline{I} > [\alpha\beta(I^*)^2 - 4bS^*(1 + \alpha I^*)]/[4\alpha bS^*(1 + \alpha I^*)]$ Here, \underline{x}_2 and \underline{I} are defined in Theorem 2.1.

Proof Let $(x_1(t), x_2(t), S(t), I(t))$ be any positive solution of system (1.2) with initial conditions (1.3). System (1.2) can be rewritten as

$$\begin{aligned}
 \dot{x}_1(t) &= \frac{r}{x_1^*}[-x_2(t)(x_1(t) - x_1^*) + x_1(t)(x_2(t) - x_2^*)] \\
 \dot{x}_2(t) &= \frac{r_1}{x_2^*}[-x_1(t)(x_2(t) - x_2^*) + x_2(t)(x_1(t) - x_1^*)] \\
 &\quad + x_2(t)[-a(x_2(t) - x_2^*)] + a_1 S^* x_2(t) - a_1 x_2(t) S(t) \\
 \dot{S}(t) &= a_2 x_2(t - \tau) S(t - \tau) - d_3 S(t) - b S^2(t) - \frac{\beta S(t) I(t)}{1 + \alpha I(t)} \\
 \dot{I}(t) &= \frac{\beta S(t) I(t)}{1 + \alpha I(t)} - d_4 I(t)
 \end{aligned}
 \tag{4.6}$$

Define

$$\begin{aligned}
 V_{31}(t) &= k_5 \left(x_1 - x_1^* - x_1^* \ln \frac{x_1}{x_1^*} \right) + x_2 - x_2^* - x_2^* \ln \frac{x_2}{x_2^*} + k_6 \left(S - S^* - S^* \ln \frac{S}{S^*} \right) \\
 &\quad + k_6 \left(I - I^* - I^* \ln \frac{I}{I^*} \right).
 \end{aligned}$$

where $k_5 = r_1 x_1^*/(r x_2^*)$, $k_6 = a_1/a_2$.

Calculating the derivative of $V_{31}(t)$ along positive solutions of system (1.2), it follows that

$$\begin{aligned}
 \dot{V}_{31}(t) &= k_5 \frac{x_1(t) - x_1^*}{x_1(t)} \dot{x}_1(t) + \frac{x_2(t) - x_2^*}{x_2(t)} \dot{x}_2(t) + k_6 \frac{S(t) - S^*}{S(t)} \dot{S}(t) + k_6 \frac{I(t) - I^*}{I(t)} \dot{I}(t) \\
 &= -\frac{r_1}{x_2^*} \left(\sqrt{\frac{x_2(t)}{x_1(t)}} (x_1(t) - x_1^*) - \sqrt{\frac{x_1(t)}{x_2(t)}} (x_2(t) - x_2^*) \right)^2 - a(x_2(t) - x_2^*)^2 \\
 &\quad + a_1 S^* (x_2(t) - x_2^*) - a_1 (x_2(t) - x_2^*) S(t) + a_1 x_2(t - \tau) S(t - \tau) \frac{S(t) - S^*}{S(t)} \\
 &\quad - k_6 d_3 (S(t) - S^*) - k_6 b S(t) (S(t) - S^*) - \frac{k_6 \beta I(t) (S(t) - S^*)}{1 + \alpha I(t)} \\
 &\quad + \frac{k_6 \beta S(t) (I(t) - I^*)}{1 + \alpha I(t)} - k_6 d_4 (I(t) - I^*)
 \end{aligned}
 \tag{4.7}$$

Define

$$V_3(t) = V_{31}(t) + a_1 \int_{t-\tau}^t \left[x_2(u) S(u) - x_2^* S^* - x_2^* S^* \ln \frac{x_2(u) S(u)}{x_2^* S^*} \right] du.$$

By calculation, we have that

$$\dot{V}_3(t) = -\frac{r_1}{x_2^*} \left(\sqrt{\frac{x_2(t)}{x_1(t)}} (x_1(t) - x_1^*) - \sqrt{\frac{x_1(t)}{x_2(t)}} (x_2(t) - x_2^*) \right)^2$$

$$\begin{aligned}
& -a_1 x_2^* S^* \left[\frac{x_2(t-\tau)S(t-\tau)}{x_2^* S(t)} - 1 - \ln \frac{x_2(t-\tau)S(t-\tau)}{x_2^* S(t)} \right] \\
& -a_1 x_2^* S^* \left[\frac{x_2^*}{x_2(t)} - 1 - \ln \frac{x_2^*}{x_2(t)} \right] - \frac{k_6 \alpha \beta S^*}{(1 + \alpha I^*)(1 + \alpha I(t))} \\
& \left[(I(t) - I^*) - \frac{I^*(S(t) - S^*)}{2S^*} \right]^2 - (x_2(t) - x_2^*)^2 \left(a - \frac{a_1 S^*}{x_2(t)} \right) \\
& -k_6 (S(t) - S^*)^2 \left[b - \frac{\alpha \beta (I^*)^2}{4S^*(1 + \alpha I^*)} \cdot \frac{1}{1 + \alpha I(t)} \right] \tag{4.8}
\end{aligned}$$

Note that the function $g(x) = x - 1 - \ln x$ is always non-negative for any $x > 0$, and $g(x) = 0$ if and only if $x = 1$. Hence, if $x_2(t) > a_1 S^*/a$ and $I(t) > [\alpha \beta (I^*)^2 - 4bS^*(1 + \alpha I^*)]/[4\alpha \beta S^*(1 + \alpha I^*)]$ for $t \geq T$, we have $-(x_2(t) - x_2^*)^2 \left[a - \frac{a_1 S^*}{x_2(t)} \right] \leq 0$ and $-k_6 (S(t) - S^*)^2 \left[b - \frac{\alpha \beta (I^*)^2}{4S^*(1 + \alpha I^*)} \cdot \frac{1}{1 + \alpha I(t)} \right] \leq 0$ with equality if and only if $x_2(t) = x_2^*$ and $S(t) = S^*$. This, together with (4.8), implies that if (H_3) holds, then $\dot{V}_3(t) \leq 0$ with equality if and only if $x_1(t) = x_1^*$, $x_2(t) = x_2^*$, $S(t) = S^*$ and $I(t) = I^*$. Therefore, the global attractiveness of E^* follows from LaSalle invariant principle for delay differential systems. This completes the proof. \square

5 Conclusion

In this paper, we have incorporated a stage structure for the prey and time delay due to the gestation of the predator into an eco-epidemiological model. By analyzing the corresponding characteristic equations, the local stability of each of feasible equilibria of system (1.2) has been established, respectively. It has been shown that, under some conditions, the time delay may destabilize both the disease-free equilibrium and the coexistence equilibrium of the eco-epidemiological system and cause the population to fluctuate. By means of Lyapunov functions and LaSalle invariant principle, sufficient conditions were obtained for the global asymptotic stability of each of feasible equilibria of system (1.2), respectively. By Theorem 3.1, we see that if $rr_1 < d_2(r_1 + d_1)$, then both the prey population and the predator population go to extinction. By Theorem 3.2, we see that if $0 < a_2 x_2^0 < d_3$ holds, the prey population persists and the predator population goes to extinction. By Theorem 3.4, we see that if $0 < \beta S^+ < d_4$ and $\underline{x}_2 > a_1 S^+/a$ hold, that is the disease transmission coefficient β is sufficiently small and the prey population is always abundant enough, the disease among the predator population dies out and in this case, the prey and the sound predator coexist. By Theorem 4.2, we see that if $\beta S^+ > d_4$ and (H_3) hold, that is the prey population is always abundant enough and the disease transmission coefficient β is sufficiently large, the coefficient equilibrium is a global attractor of the system (1.2). In this case, the disease spreading in the predator becomes endemic and the prey, sound predator and the infected predator coexist.

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