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ORIGINAL RESEARCH

Positive definite solutions and perturbation analysis of a class of nonlinear matrix equations

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Abstract In this paper, we consider a class of nonlinear matrix equation of the type $X + \sum_{i=1}^{m} A_i^* X^{-q} A_i - \sum_{j=1}^{n} B_j^* X^{-r} B_j = Q$, where $0 < q,\ r \leq 1$ and Q is positive definite. Based on the Schauder fixed point theorem and Bhaskar–Lakshmikantham coupled fixed point theorem, we derive some sufficient conditions for the existence and uniqueness of the positive definite solution to such equations. An iterative method is provided to compute the unique positive definite solution. A perturbation estimation and the explicit expression of Rice condition number of the unique positive definite solution are also established. The theoretical results are illustrated by numerical examples.

Keywords Nonlinear matrix equation · Positive definite solution · Perturbation analysis · Coupled fixed point theorem · Condition number

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1 Introduction

In this paper, we consider the following nonlinear matrix equation

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$$X + \sum_{i=1}^{m} A_i^* X^{-q} A_i - \sum_{j=1}^{n} B_j^* X^{-r} B_j = Q \tag{1.1}$$

where $0 < q,\ r \leq 1$, and $A_i, B_j,\ i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n$ and Q are $n \times n$ complex matrices with Q Hermitian positive definite. A^* is the conjugate transpose of a matrix A. This type of nonlinear matrix equations arises in many pratical problems. When $m = n = 1$ and $q = r = 1$, the Eq. (1.1) reduces to be a special stochastic algebraic Riccati equation arising in stochastic control theory, which can be stated as follows. Some stochastic linear quadratic (LQ)control problems lead to computing the positive definite solution of the following stochastic algebraic Riccati equation [26]:

$$\begin{aligned} &C^*XC - X + S + \Pi_1(X) - (L + C^*XP + \Pi_{12}(X)) \\ &(R + P^*XP + \Pi_2(X))^+ (L + C^*XP + \Pi_{12}(X))^* = 0 \end{aligned} \tag{1.2}$$

where Z^+ stands for the Moore-Penrose inverse of a matrix Z and C, P, S, R and L are given matrices of size $n \times n$, $n \times m$, $n \times n$, $m \times m$ and $n \times m$, respectively, such that

$$T = \begin{pmatrix} S & L \\ L^* & R \end{pmatrix}.$$

is a Hermitian matrix, and the operator

$$\Pi(X) = \begin{pmatrix} \Pi_1(X) & \Pi_{12}(X) \\ \Pi_{12}(X)^* & \Pi_2(X) \end{pmatrix}$$

is positive, i.e., $X \geq 0$ implies $\Pi(X) \geq 0$. Consider the following case: C is the identity matrix, P is an $n \times n$ nonsingular matrix, L is the zero matrix and $\Pi_{12}(X) = \Pi_2(X) = 0,\ \Pi_1(X) = (R + P^*XP)^{-1}$, where $R + P^*XP$ is positive definite for all positive semidefinite matrices X. Meanwhile, the stochastic rational Riccati Equation (1.2) has the form

$$S + \left(R + P^*XP\right)^{-1} - XP\left(R + P^*XP\right)^{-1} P^*X = 0 \tag{1.3}$$

Set $Y = R + P^*XP$, then $P^{-*}(Y - R) = XP$. Thus, we have

$$S + Y^{-1} - P^{-*}(Y - R)\, Y^{-1} (Y - R)\, P^{-1} = 0$$

which impies that

$$Y + R^*Y^{-1}R - P^*Y^{-1}P = 2R + P^*SP.$$

Denote by $Q = 2R + P^*SP$, $A = R$, $B = P$, then Eq. (1.3) can be equivalently written as Eq. (1.1). Therefore, Eq. (1.1) is a special stochastic Riccati equation (1.2). When $r = 1$ and $A_i = 0$ for all $i = 1, 2, \ldots, m$, Eq.(1.1) becomes a sepcial case of nonlinear matrix equation $X = Q + B^*(\hat{X} - C)^{-1}B$ with $C = 0$ and $B = diag(B_1, \ldots, B_n)$ which plays an important role in connection with certain optimal interpolation problems, see for detail [8,19,21]. Moreover, the special case when all $A_i = 0\,(i = 1, 2, \ldots, m)$ or all $B_j = 0\,(j = 1, 2, \ldots, n)$ are also problems of practical importance and have many applications in control theory, ladder networks, dynamic programming, statistics, stochastic filtering, nano research and etc., see for instance [3,5,8–10,13,15–17,23,25] and the references therein.

In the last few years there has been a constantly increasing interest in developing the theory, numerical solutions, and perturbation analysis for the definite solutions to the nonlinear matrix equations of the form (1.1) in several special cases. For instance, [25] considered the existence and uniqueness of positive definite solutions for the matrix equation $X + \sum_{i=1}^{m} A_i^* X^{-q} A_i = Q$. Duan [6] and Lim [15] proved, respectively, that the nonlinear matrix equation $X - \sum_{i=1}^{m} A_i^* X^{-\delta_i} A_i = Q$ always has a unique positive definite solution. Applying matrix differentiation, [8] considered perturbation analysis for the unique positive definite solution of $X - \sum_{i=1}^{m} A_i^* X^{-1} A_i = Q$. Moreover, positive definite solutions of nonlinear matrix equations $X \pm A^* X^{-q} A = Q$ with $0 < q < 1$ were extensively investigated [9,10,13,16,17,23]. Similar nonlinear matrix equations such as $X^s \pm A^* X^{-t} A = Q$ [11,24], $X + A^H \bar{X}^{-1} A = I$ [14] and $X^r + \sum_{i=1}^{m} A_i^* X^{\delta_i} A_i = I$ [22] have been investigated by many authors.

Recently, Mahar Berzig [1] considered positive definite solution to the linear matrix equation $X + \sum_{i=1}^{m} A_i^* X A_i - \sum_{i=1}^{m} B_i^* X B_i = Q$. Then [2,7] considered the existence and uniqueness of the positive definite solution to the nonlinear matrix equation $X + A^* X^{-1} A - B^* X^{-1} B = I$ which is a special case of Eq. (1.1) with $m = n = 1$, $q = r = 1$ and $Q = I$. However, as far as we know, there is few literature considering the general nonlinear matrix equations $X + \sum_{i=}^{m} A_i^* X^{-q} A_i - \sum_{j=1}^{n} B_j^* X^{-r} B_j = Q$ where $0 < q, r \leq 1$, m, n are positive integers, and Q is positive definite.

Motivated by this, we consider in this work positive definite solution of the general case of Eq. (1.1) with m, n positive integer numbers and $0 < q, r \leq 1$. Based on the Lakshmikantham-Bhaskar fixed point theorem, we derive in the second section some sufficient conditions for the existence and uniqueness of positive definite solution to Eq. (1.1) and propose an iterative method to compute the unique positive definite solution. In the third section, we consider the perturbation analysis of nonlinear matrix equations of the form (1.1). An estimation bound of the unique positive definite solution which is sharp and easy to calculate is derived and an explicit expression of the Rice condition number of the unique positive definite solution is also obtained. Theoretical results are illustrated by several numerical examples in Sect. 4.

Throughout this paper, we denote by $C^{n\times n}$, $H^{n\times n}$ and $H^+(n)$ the sets of all $n \times n$ complex matrices, all $n \times n$ Hermitian matrices and all positive definite Hermitian matrices, respectively. The notation $A \geq 0 (A > 0)$ means that A is Hermitian positive semidefinite (positive definite). We denote by $\sigma_1(A)$ and $\sigma_n(A)$ the maximal and minimal singular values of A, respectively. Similarly, $\lambda_1(A)$ and $\lambda_n(A)$ stand for the maximal and the minimal eigenvalues of A, respectively. For $A, B \in H^{n\times n}$, we write $A \geq B (A > B)$ if $A - B \geq 0 (> 0)$ and let

$$[A, B] = \{X | A \leq X \leq B\}, \quad [A, B) = \{X | A \leq X < B\}.$$

The symbol $\mathrm{tr}(A)$ denotes the trace of the matrix A and $\|\cdot\|_{\mathrm{tr}}$ denotes the trace norm which is defined by

$$\|A\|_{\mathrm{tr}} = \sum_{i=1}^{n} \sigma_i(A)$$

where $\sigma_i(A), i = 1, 2, \ldots, n$ are all singular values of the matrix A. It's not difficult to verify that $\|\cdot\|_{\mathrm{tr}}$ is unitary invariant and $\|A\|_{\mathrm{tr}} = \mathrm{trace}(A)$ if A is Hermitian positive semidefinite. Unless otherwise noted, the symbol $\|\cdot\|$ stands for the spectral norm(i.e., $\|A\| = \sqrt{\rho(AA^*)} = \sigma_1(A)$). It's clear that for any positive definite matrix Q, $\|Q\| = \lambda_1(Q)$ and $\|Q^{-1}\|^{-1} = \lambda_n(Q)$.

2 Positive definite solutions

In this section, we provide several sufficient conditions for Eq. (1.1) to have positive definite solutions and also we propose an iterative method for obtaining the positive definite solution.

We start with several lemmas which we need to prove our main results:

Lemma 2.1 *[3] If $A > B > 0$ (or $A \geq B > 0$), then $A^r > B^r$ (or $A^r \geq B^r$) for all $r \in (0, 1]$, and $A^r < B^r$ (or $0 < A^r \leq B^r$) for all $r \in [-1, 0)$.*

Lemma 2.2 *[3] Let f be an operator monotone function on $(0, \infty)$ and let A, B be two positive operators bounded below by a, i.e., $A > aI$ and $B > aI$ for a positive number a. If there exists $f'(a)$, then for every unitary invariant norm $\|\cdot\|$, we have*

$$\|f(A) - f(B)\| < |f'(a)| \cdot \|A - B\|.$$

Lemma 2.3 *[5] Let $A \geq 0$ and $B \geq 0$ be $n \times n$ matrices. Then*

$$0 \leq \mathrm{tr}(AB) \leq \|A\|\mathrm{tr}(B).$$

Lemma 2.4 *[25] Suppose that $\sum_{i=1}^{m} \sigma_1^2(Q^{-q/2} A_i Q^{-1/2}) < \frac{q^q}{(q+1)^{q+1}}$. Then the nonlinear matrix equation $X + \sum_{i=1}^{m} A_i^* X^{-q} A_i = Q$ has a positive definite solution in $[\frac{q}{q+1} Q, Q]$, where $0 < q \leq 1$ and Q is positive definite.*

Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to X$ be a given mapping. We call that F has the mixed monotone property if $F(x, y)$ is increasing in x and decreasing in y, that is,

$$F(x_1, y_1) \leq F(x_2, y_2)$$

for arbitrary$x_1, x_2, y_1, y_2 \in X$ with $x_1 \leq x_2$ and $y_2 \leq y_1$.

We say that (x, y) is a coupled fixed point of F if $x = F(x, y)$ and $y = F(y, x)$.

The proof of our main result in this section is based on the following two fixed point theorems.

Lemma 2.5 (Schauder fixed point theorem) *Let S be a nonempty, compact, convex subset of a normed vector space. Every continuous function $f : S \to S$ mapping S into itself has a fixed point.*

Lemma 2.6 (Bhaskar–Laksmikantam's coupled fixed point theorem *[4]*) *Let $(X, \leq)$ be a partially ordered set and d be a metric on X. Let the map $F : X \times X \to X$ be continuous and mixed monotone on X. Assume that there exists a $\delta \in [0, 1)$ with*

$$d(F(x, y), F(u, v)) \leq \frac{\delta}{2}[d(x, u) + d(y, v)]$$

for all $x \geq u$ and $y \leq v$. Suppose also that

(1) there exist $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$;
(2) every pair of elements in X has a lower bound and an upper bound, that is, for every $(x, y) \in (X \times X)$, there exist z_1 and z_2 such that $x, y \leq z_1$ and $x, y \geq z_2$.

Then there exists a unique $\bar{x} \in X$ such that $\bar{x} = F(\bar{x}, \bar{x})$. Moreover, the sequences $\{x_k\}$ and $\{y_k\}$ generated by $x_{k+1} = F(x_k, y_k)$ and $y_{k+1} = F(y_k, x_k)$ converge to $\bar{x}$, with the following estimate

$$\max\{d(x_k, \bar{x}), d(y_k, \bar{x})\} \leq \frac{\delta^k}{1-\delta}\max\{d(x_1, x_0), d(y_1, y_0)\}.$$

Theorem 2.1 *If $\sum_{i=1}^{m} \sigma_1^2(Q^{-q/2} A_i Q^{-1/2}) < \frac{q^q}{(q+1)^{q+1}}$, then Eq. (1.1) has a positive definite solution $\bar{X} \geq \frac{q}{q+1} Q$.*

Proof Suppose $\sum_{i=1}^{m} \sigma_1^2(Q^{-q/2} A_i Q^{-1/2}) < \frac{q^q}{(q+1)^{q+1}}$, then there exists a positive definite $\tilde{X} \in [\frac{q}{q+1} Q, Q]$ such that $\tilde{X} + \sum_{i=1}^{m} A_i^* \tilde{X}^{-q} A_i = Q$ according to Lemma 2.4. Then

$$\tilde{X} = Q - \sum_{i=1}^{m} A_i^* \tilde{X}^{-q} A_i \leq Q - \sum_{i=1}^{m} A_i^* \tilde{X}^{-q} A_i + \sum_{j=1}^{n} B_j^* \tilde{X}^{-r} B_j \leq Q + \sum_{j=1}^{n} B_j^* \tilde{X}^{-r} B_j.$$

Denote $D = \{X | \tilde{X} \leq X \leq Q + \sum_{j=1}^{n} B_j^* \tilde{X}^{-r} B_j\}$. Consider the map

$$G(X) = Q - \sum_{i=1}^{m} A_i^* X^{-q} A_i + \sum_{j=1}^{n} B_j^* X^{-r} B_j, \quad X \in D.$$

Clearly, D is a bounded convex set in $H^+(n)$ and G is continuous on D. Then we have for arbitrary $X \in D$,

$$G(X) = Q - \sum_{i=1}^{m} A_i^* X^{-q} A_i + \sum_{j=1}^{n} B_j^* X^{-r} B_j \geq Q - \sum_{i=1}^{m} A_i^* X^{-q} A_i$$

$$\geq Q - \sum_{i=1}^{m} A_i^* \tilde{X}^{-q} A_i = \tilde{X},$$

and

$$G(X) = Q - \sum_{i=1}^{m} A_i^* X^{-q} A_i + \sum_{j=1}^{n} B_j^* X^{-r} B_j \leq Q + \sum_{j=1}^{n} B_j^* X^{-r} B_j$$
$$\leq Q + \sum_{j=1}^{n} B_j^* \tilde{X}^{-r} B_j.$$

which implies $G(D) \subseteq D$. Applying Schauder fixed point theorem (Lemma 2.5), G has a fixed point $\bar{X}$ in D, which is a positive definite solution to Eq. (1.1) and clearly $\bar{X} \geq \tilde{X} \geq \frac{q}{q+1} Q$. □

Remark 2.1 From the proof of Theorem 2.1, if $\sum_{i=1}^{m} \sigma_1^2 (Q^{-q/2} A_i Q^{-1/2}) < \frac{q^q}{(q+1)^{q+1}}$, then Eq. (1.1) has a positive definite solution $\bar{X}$ satisfying $\frac{q}{q+1} Q \leq \tilde{X} \leq \bar{X} \leq Q + \sum_{j=1}^{n} B_j^* \tilde{X}^{-r} B_j \leq Q + \frac{(q+1)^r}{q^r} \sum_{j=1}^{n} B_j^* Q^{-r} B_j$ in which the second inequality holds from Lemma 2.1 ($\tilde{X} \geq \frac{q}{q+1} Q$ gives $\tilde{X}^{-r} \leq \frac{(q+1)^r}{q^r} Q^{-r}$.)

Theorem 2.2 *Suppose that*

$$\sum_{i=1}^{m} \|A_i^* A_i\| < \frac{1}{2} \frac{q^q}{\|(q+1) Q^{-1}\|^{q+1}}, \quad \sum_{j=1}^{n} \|B_j^* B_j\| < \frac{1}{2r} \frac{q^{r+1}}{\|(q+1) Q^{-1}\|^{r+1}}. \tag{2.1}$$

Then Eq. (1.1) has a unique positive definite solution $\bar{X} \in [\frac{q}{q+1} Q, +\infty)$ *and the sequences* $\{X_k\}$ *and* Y_k *defined by*

$$\begin{cases} X_0 = \frac{q}{q+1} Q, \quad Y_0 = \frac{2r(q+1)+q}{2r(q+1)} Q, \\ X_{k+1} = Q - \sum_{i=1}^{m} A_i^* X_k^{-q} A_i + \sum_{j=1}^{n} B_j^* Y_k^{-r} B_j, \\ Y_{k+1} = Q - \sum_{i=1}^{m} A_i^* Y_k^{-q} A_i + \sum_{j=1}^{n} B_j^* X_k^{-r} B_j, \quad k = 0, 1, 2, \ldots \end{cases} \tag{2.2}$$

converges to $\bar{X}$ *and the error estimation is given by*

$$\max\{\|X_k - X\|_{\mathrm{tr}}, \|Y_k - Y\|_{\mathrm{tr}}\} \leq \frac{\delta^k}{1-\delta} \max\{\|X_1 - X_0\|_{\mathrm{tr}}, \|Y_1 - Y_0\|_{\mathrm{tr}}\}, \tag{2.3}$$

where

$$\delta = 2 \cdot \max\left\{ \frac{\|(q+1) Q^{-1}\|^{q+1}}{q^q} \sum_{i=1}^{m} \|A_i^* A_i\|, \quad \frac{r \cdot \|(q+1) Q^{-1}\|^{r+1}}{q^{r+1}} \sum_{j=1}^{n} \|B_j^* B_j\| \right\}.$$

Proof Consider the following two-variable map

$$F(X,Y)=Q-\sum_{i=1}^{m}A_i^*X^{-q}A_i+\sum_{j=1}^{n}B_j^*Y^{-r}B_j,\quad X,Y\in\Omega,$$

where $\Omega=\{Z\in H^+(n): Z\geq \frac{q}{q+1}Q\}$. Obviously, F is continuous on $\Omega\times\Omega$.

We divide our proof into five aspects:

(1) For arbitrary $X,Y\in\Omega$, combining (2.1) with lemma 2.1, we obtain that

$$\begin{aligned}F(X,Y)&\geq Q-\sum_{i=1}^{m}A_i^*X^{-q}A_i\geq Q-\frac{(q+1)^q}{q^q}\sum_{i=1}^{m}A_i^*Q^{-q}A_i\\&\geq Q-\frac{1}{2}\frac{1}{q+1}Q\geq\frac{q}{q+1}Q,\end{aligned}$$

in which the third inequality holds since

$$\begin{aligned}\frac{(q+1)^q}{q^q}\sum_{i=1}^{m}A_i^*Q^{-q}A_i&\leq\frac{(q+1)^q}{q^q}\|Q^{-1}\|^q\sum_{i=1}^{m}A_i^*A_i\leq\frac{1}{2}\frac{1}{(q+1)\|Q^{-1}\|}I\\&=\frac{1}{2}\frac{1}{(q+1)}\lambda_n(Q)I\leq\frac{1}{2}\frac{1}{q+1}Q.\end{aligned}$$

Thus, $F(X,Y)\in\Omega$ which implies that $F:\Omega\times\Omega\to\Omega$.

(2) For arbitrary $X_1,X_2,Y_1,Y_2\in\Omega$ with $X_1\leq X_2$ and $Y_1\geq Y_2$, we have

$$\begin{aligned}F(X_1,Y_1)=Q-\sum_{i=1}^{m}A_i^*X_1^{-q}A_i+\sum_{j=1}^{n}B_j^*Y_1^{-r}B_j\leq Q\\-\sum_{i=1}^{m}A_i^*X_2^{-q}A_i+\sum_{j=1}^{n}B_j^*Y_2^{-r}B_j=F(X_2,Y_2)\end{aligned}$$

which implies that F is mixed monotone on Ω.

(3) For arbitrary $X,Y,U,V\in\Omega$ with $X\geq U\geq\frac{q}{q+1}Q\geq\frac{q}{q+1}\lambda_n(Q)I$ and $\frac{q}{q+1}\lambda_n(Q)I\leq\frac{q}{q+1}Q\leq Y\leq V$, we have by Lemma 2.3 that

$$\begin{aligned}\|F(X,Y)-F(U,V)\|_{\mathrm{tr}}&=\|\sum_{i=1}^{m}A_i^*\left(U^{-q}-X^{-q}\right)A_i\\&\quad+\sum_{j=1}^{n}B_j^*\left(Y^{-r}-V^{-r}\right)B_j\|_{\mathrm{tr}}\\&\leq\|\sum_{i=1}^{m}A_i^*\left(U^{-q}-X^{-q}\right)A_i\|_{\mathrm{tr}}\end{aligned}$$

$$
\begin{aligned}
&+\|\sum_{j=1}^{n} B_j^* \left(Y^{-r} - V^{-r}\right) B_j\|_{\mathrm{tr}} \\
&= \sum_{i=1}^{m} \operatorname{tr}\left(A_i^* \left(U^{-q} - X^{-q}\right) A_i\right) \\
&\quad + \sum_{j=1}^{n} \operatorname{tr}\left(B_j^* \left(Y^{-r} - V^{-r}\right) B_j\right) \\
&\leq \sum_{i=1}^{m} \|A_i^* A_i\| \cdot \|U^{-q} - X^{-q}\|_{\mathrm{tr}} \\
&\quad + \sum_{j=1}^{n} \|B_j^* B_j\| \cdot \|Y^{-r} - V^{-r}\|_{\mathrm{tr}}.
\end{aligned}
$$

Using Lemma 2.2, we obtain that

$$
\begin{aligned}
\|U^{-q} - X^{-q}\|_{\mathrm{tr}} &\leq q \cdot \frac{(q+1)^{q+1}}{q^{q+1}} \cdot \frac{1}{\lambda_n^{q+1}(Q)} \|U - X\|_{\mathrm{tr}} \\
&= \frac{(q+1)^{q+1}}{q^{q}} \cdot \frac{1}{\lambda_n^{q+1}(Q)} \|U - X\|_{\mathrm{tr}},
\end{aligned}
$$

Similarly,

$$
\|Y^{-r} - V^{-r}\|_{\mathrm{tr}} \leq r \cdot \frac{(q+1)^{r+1}}{q^{r+1}} \cdot \frac{1}{\lambda_n^{r+1}(Q)} \|Y - V\|_{\mathrm{tr}},
$$

It is clear that $\|Q^{-1}\| = \sigma_1(Q^{-1}) = \lambda_1(Q^{-1}) = \frac{1}{\lambda_n(Q)}$ since Q is positive definite. Then we have

$$
\begin{aligned}
\|F(X,Y) - F(U,V)\|_{\mathrm{tr}} &\leq \tfrac{(q+1)^{q+1}}{q^{q}} \cdot \tfrac{1}{\lambda_n^{q+1}(Q)} \cdot \textstyle\sum_{i=1}^{m} \|A_i^* A_i\| \cdot \|U - X\|_{\mathrm{tr}} \\
&\quad + r \cdot \tfrac{(q+1)^{r+1}}{q^{r+1}} \cdot \tfrac{1}{\lambda_n^{r+1}(Q)} \cdot \textstyle\sum_{j=1}^{n} \|B_j^* B_j\| \cdot \|Y - V\|_{\mathrm{tr}} \\
&= \tfrac{\|(q+1)Q^{-1}\|^{q+1}}{q^{q}} \textstyle\sum_{i=1}^{m} \|A_i^* A_i\| \cdot \|U - X\|_{\mathrm{tr}} \\
&\quad + r \cdot \tfrac{\|(q+1)Q^{-1}\|^{r+1}}{q^{r+1}} \textstyle\sum_{j=1}^{n} \|B_j^* B_j\| \cdot \|Y - V\|_{\mathrm{tr}} \\
&\leq \delta \cdot \tfrac{1}{2} \cdot (\|U - X\|_{\mathrm{tr}} + \|Y - V\|_{\mathrm{tr}})
\end{aligned}
$$

where

$$0 < \delta = 2 \cdot \max\{\frac{\|(q+1)Q^{-1}\|^{q+1}}{q^q} \sum_{i=1}^{m} \|A_i^* A_i\|, \quad \frac{r \cdot \|(q+1)Q^{-1}\|^{r+1}}{q^{r+1}} \sum_{j=1}^{n} \|B_j^* B_j\|\} < 1.$$

(4) Let $X_0 = \frac{q}{q+1} Q, \ Y_0 = \frac{2r(q+1)+q}{2r(q+1)} Q$. Then we have by Lemma 2.1 that

$$\begin{aligned}
F(X_0, \ Y_0) &= Q - \sum_{i=1}^{m} A_i^* X_0^{-q} A_i + \sum_{j=1}^{n} B_j^* Y_0^{-r} B_j \\
&= Q - \frac{(q+1)^q}{q^q} \sum_{i=1}^{m} A_i^* Q^{-q} A_i + \sum_{j=1}^{n} B_j^* Y_0^{-r} B_j \\
&\geq Q - \frac{(q+1)^q}{q^q} \|Q^{-q}\| \sum_{i=1}^{m} \|A_i^* A_i\| \cdot I \\
&\geq Q - \frac{(q+1)^q}{q^q} \|Q^{-1}\|^q \cdot \frac{1}{2} \frac{q^q}{\|(q+1)Q^{-1}\|^{q+1}} I \\
&= Q - \frac{1}{2(q+1)} \lambda_n(Q) I \\
&\geq Q - \frac{1}{q+1} \lambda_n(Q) I \\
&\geq Q - \frac{1}{q+1} Q \\
&= \frac{q}{q+1} Q = X_0,
\end{aligned}$$

and similarly,

$$\begin{aligned}
F(Y_0, X_0) &= Q - \sum_{i=1}^{m} A_i^* Y_0^{-q} A_i + \sum_{j=1}^{n} B_j^* X_0^{-r} B_j \\
&\leq Q + \frac{(q+1)^r}{q^r} \sum_{j=1}^{n} B_j^* Q^{-r} B_j \\
&\leq Q + \frac{(q+1)^r}{q^r} \cdot \|Q^{-1}\|^r \sum_{j=1}^{n} \|B_j^* B_j\| \cdot I
\end{aligned}$$

$$\leq Q + \frac{(q+1)^r}{q^r}\|Q^{-1}\|^r \cdot \frac{q^{r+1}}{2r(q+1)^{r+1}\|Q^{-1}\|^{r+1}} I$$

$$= Q + \frac{1}{2r}\frac{q}{q+1}\lambda_n(Q)I$$

$$\leq \frac{2r(q+1)+q}{2r(q+1)} Q$$

$$= Y_0.$$

(5) For arbitrary pair $X, Y \in H^+(n)$, it's well known that

$$\min\{\lambda_n(X),\ \ \lambda_n(Y)\}I \leq X, Y \leq \max\{\lambda_1(X),\ \lambda_1(Y)\}I$$

which means that each pair has both lower bound and upper bound.

□

Combining the above 5 aspects with Lemma 2.6, there exists a unique $\bar{X} \in \Omega$ such that $\bar{X} = F(\bar{X},\ \bar{X})$. Moreover, the sequences $\{X_k\}$ and Y_k generated by (2.2) converge to $\bar{X}$, with the estimation (2.3).

Remark 2.2 In case $m = n = 1$, $q = r = 1$ and $Q = I$ in Theorem 2.2, we obtain Theorem 2.3 in [7].

3 Perturbation analysis

Consider the perturbed matrix equation

$$\tilde{X} + \sum_{i=1}^{m} \tilde{A}_i^* \tilde{X}^{-q} \tilde{A}_i - \sum_{j=1}^{n} \tilde{B}_j^* \tilde{X}^{-r} \tilde{B}_j = \tilde{Q} \tag{3.1}$$

where $\tilde{A}_i, i = 1, 2, \ldots, m;\ \tilde{B}_j, j = 1, \ldots, n$ and $\tilde{Q}$ are the slightly perturbed matrices of the matrices $A_i, i = 1, \ldots, m;\ B_j, j = 1, \ldots, n$ and Q, respectively. In this section, we show that if $\|\tilde{A}_i - A_i\|, i = 1, \ldots, m;\ \|\tilde{B}_j - B_j\|, j = 1, \ldots, n.$ and $\|\tilde{Q} - Q\|$ are sufficiently small, then the unique positive definite solution $\tilde{X}$ to the perturbed matrix equation (3.1) exists. We derive a perturbation estimate for the unique positive definite solution X and give an explicit expression of the Rice condition number of X.

Denote $\Delta Q = \tilde{Q} - Q,\ \Delta A_i = \tilde{A}_i - A_i, i = 1, 2, \ldots, m;\ \Delta B_j = \tilde{B}_j - B_j, j = 1, 2, \ldots, n,\ \Delta X = \tilde{X} - X.$

Theorem 3.1 *Let*

(i) $\theta_1 := \dfrac{q^q}{(q+1)^{q+1}} - 2\displaystyle\sum_{i=1}^{m}\|A_i\|^2\|Q^{-1}\|^{q+1} > 0,\ \ \theta_2 := \dfrac{q^{r+1}}{r(q+1)^{r+1}}$

$$-2\sum_{j=1}^{n}\|B_j\|^2\|Q^{-1}\|^{r+1} > 0, \tag{3.2}$$

$$\text{(ii)}\ \ \|\Delta Q\| \le \frac{1}{\|Q^{-1}\|}\cdot \min\left\{1-\sqrt[q+1]{1-\theta_1},\ 1-\sqrt[r+1]{1-\theta_2}\right\}, \tag{3.3}$$

$$\text{(iii)}\ \ \sum_{i=1}^{m}\left(\|\tilde{A}_i\|^2-\|A_i\|^2\right) < \frac{(q+1)^{q+1}-q^q}{2(q+1)^{q+1}\|Q^{-1}\|^{q+1}}\theta_1,\ \sum_{j=1}^{n}\left(\|\tilde{B}_j\|^2-\|B_j\|^2\right) < \frac{(q+1)^{r+1}-q^{r+1}/r}{2(q+1)^{r+1}\|Q^{-1}\|^{r+1}}\theta_2. \tag{3.4}$$

Then nonlinear matrix equations $X+\sum_{i=1}^{m}A_i^*X^{-q}A_i-\sum_{j=1}^{n}B_j^*X^{-r}B_j=Q$ *and* $\tilde{X}+\sum_{i=1}^{m}\tilde{A}_i^*\tilde{X}^{-q}\tilde{A}_i-\sum_{j=1}^{n}\tilde{B}_j^*\tilde{X}^{-r}\tilde{B}_j=\tilde{Q}$ *have unique positive definite solutions* X *and* $\tilde{X}$*, respectively. Moreover, we have*

$$\begin{aligned}\|\Delta X\| \le \tfrac{1}{\xi}\cdot\Big[&\|\Delta Q\|+2\textstyle\sum_{i=1}^{m}\left(\|X^{-q}A_i\|\cdot\|\Delta A_i\|+\|X^{-q}\|\cdot\|\Delta A_i\|^2\right)\\ &+2\textstyle\sum_{j=1}^{n}\left(\|X^{-r}B_j\|\cdot\|\Delta B_j\|+\|X^{-r}\|\cdot\|\Delta B_j\|^2\right)\Big],\end{aligned} \tag{3.5}$$

where

$$\xi=1-qb^{-(q+1)}\sum_{i=1}^{m}\|\tilde{A}_i\|^2-rb^{-(r+1)}\sum_{j=1}^{n}\|\tilde{B}_j\|^2,\ \ b=\frac{q}{q+1}\min\left\{\lambda_n(Q),\lambda_n(\tilde{Q})\right\}.$$

Proof Since $\theta_1>0,\ \theta_2>0$, we know from Theorem 2.2 that Eq. (1.1) has unique positive definite solution $X\ge\frac{q}{q+1}Q$. Notice that $\theta_1<1$ and

$$\|\tilde{Q}^{-1}\|\le\|Q^{-1}\|+\|Q^{-1}\|\cdot\|\Delta Q\|\cdot\|\tilde{Q}^{-1}\|\le\|Q^{-1}\|+\left(1-\sqrt[q+1]{1-\theta_1}\right)\|\tilde{Q}^{-1}\|,$$

which gives

$$\|\tilde{Q}^{-1}\|^{q+1}\le\frac{\|Q^{-1}\|^{q+1}}{1-\theta_1}.$$

Similarly, $\|\tilde{Q}^{-1}\|^{r+1}\le\frac{\|Q^{-1}\|^{r+1}}{1-\theta_2}$.

Consequently, we have

$$\begin{aligned}\sum_{i=1}^{m}\|\tilde{A}_i\|^2\|\tilde{Q}^{-1}\|^{q+1} &< \frac{\|Q^{-1}\|^{q+1}}{1-\theta_1}\left[\sum_{i=1}^{m}\|A_i\|^2+\frac{(q+1)^{q+1}-q^q}{2(q+1)^{q+1}\|Q^{-1}\|^{q+1}}\theta_1\right]\\ &= \frac{2(q+1)^{q+1}\sum_{i=1}^{m}\|A_i\|^2\|Q^{-1}\|^{q+1}+[(q+1)^{q+1}-q^q]\theta_1}{2(1-\theta_1)(q+1)^{q+1}}\\ &= \frac{q^q}{2(q+1)^{q+1}}\cdot\frac{2\sum_{i=1}^{m}\|A_i\|^2\frac{(q+1)^{q+1}}{q^q}\|Q^{-1}\|^{q+1}+[\frac{(q+1)^{q+1}}{q^q}-1]\theta_1}{1-\theta_1}\\ &= \frac{q^q}{2(q+1)^{q+1}},\end{aligned} \tag{3.6}$$

and similarly

$$\begin{aligned}\sum_{j=1}^{n}\|\tilde{B}_j\|^2\|\tilde{Q}^{-1}\|^{r+1} &< \frac{\|Q^{-1}\|^{r+1}}{1-\theta_2}\left[\sum_{j=1}^{n}\|B_j\|^2+\frac{(q+1)^{r+1}-q^{r+1}/r}{2(q+1)^{r+1}\|Q^{-1}\|^{r+1}}\theta_2\right]\\ &= \frac{2(q+1)^{r+1}\sum_{j=1}^{n}\|B_j\|^2\|Q^{-1}\|^{r+1}+[(q+1)^{r+1}-q^{r+1}/r]\theta_2}{2(1-\theta_2)(q+1)^{r+1}}\\ &= \frac{q^{r+1}}{2r(q+1)^{r+1}}\cdot\frac{2\sum_{j=1}^{n}\|B_j\|^2\cdot\frac{r(q+1)^{r+1}}{q^{r+1}}\|Q^{-1}\|^{r+1}+[\frac{r(q+1)^{r+1}}{q^{r+1}}-1]\theta_2}{1-\theta_2}\\ &= \frac{q^{r+1}}{2r(q+1)^{r+1}},\end{aligned} \tag{3.7}$$

Applying Theorem 2.2, we obtain that the perturbed matrix equation (3.1) has unique positive definite solution $\tilde{X} > \frac{q}{q+1}\tilde{Q}$.

In the following, we show the estimate (3.5):

Since $X > \frac{q}{q+1}\lambda_n(Q)I$ and $\tilde{X} > \frac{q}{q+1}\lambda_n(\tilde{Q})I$. Let $b = \frac{q}{q+1}\min\{\lambda_n(Q), \lambda_n(\tilde{Q})\}$. Then $X, \tilde{X} > bI$ and consequently,

$$\|X^{-q}-\tilde{X}^{-q}\| \le qb^{-(q+1)}\|\Delta X\|, \quad \text{and } \|X^{-r}-\tilde{X}^{-r}\| \le rb^{-(r+1)}\|\Delta X\|$$

from Lemma 2.2.

Since $X+\sum_{i=1}^{m}A_i^*X^{-q}A_i-\sum_{j=1}^{n}B_j^*X^{-r}B_j = Q$ and $\tilde{X}+\sum_{i=1}^{m}\tilde{A}_i^*\tilde{X}^{-q}\tilde{A}_i - \sum_{j=1}^{n}\tilde{B}_j^*\tilde{X}^{-r}\tilde{B}_j = \tilde{Q}$, then

$$\tilde{X}-X+\sum_{i=1}^{m}\tilde{A}_i^*\tilde{X}^{-q}\tilde{A}_i-\sum_{i=1}^{m}A_i^*X^{-q}A_i-\sum_{j=1}^{n}\tilde{B}_j^*\tilde{X}^{-r}\tilde{B}_j+\sum_{j=1}^{n}B_j^*X^{-r}B_j=\tilde{Q}-Q,$$

i.e.,

$$\begin{aligned}\Delta X &= \Delta Q + \sum_{i=1}^{m} \tilde{A}_i^* \left(X^{-q} - \tilde{X}^{-q}\right) \tilde{A}_i + \sum_{i=1}^{m} \left[A_i^* X^{-q} A_i - \tilde{A}_i^* X^{-q} \tilde{A}_i\right] \\ &\quad - \sum_{j=1}^{n} \tilde{B}_j^* \left(X^{-r} - \tilde{X}^{-r}\right) \tilde{B}_j + \sum_{j=1}^{n} \left[-B_j^* X^{-r} B_j + \tilde{B}_j^* X^{-r} \tilde{B}_j\right] \\ &= \Delta Q + \sum_{i=1}^{m} \tilde{A}_i^* \left(X^{-q} - \tilde{X}^{-q}\right) \tilde{A}_i \\ &\quad - \sum_{i=1}^{m} \left[\Delta A_i^* X^{-q} A_i + \Delta A_i^* X^{-q} \Delta A_i + A_i^* X^{-q} \Delta A_i\right] \\ &\quad + \sum_{j=1}^{n} \tilde{B}_j^* \left(\tilde{X}^{-r} - X^{-r}\right) \tilde{B}_j \\ &\quad + \sum_{j=1}^{n} \left[\Delta B_j^* X^{-r} B_j + \Delta B_j^* X^{-r} \Delta B_j + B_j^* X^{-r} \Delta B_j\right].\end{aligned}$$

Hence

$$\begin{aligned}\|\Delta X\| &\le \|\Delta Q\| + \|X^{-q} - \tilde{X}^{-q}\| \sum_{i=1}^{m} \|\tilde{A}_i\|^2 + \sum_{i=1}^{m} \left(\|\Delta A_i^* X^{-q} \Delta A_i\| + 2\|\Delta A_i^* X^{-q} A_i\|\right) \\ &\quad + \|X^{-r} - \tilde{X}^{-r}\| \sum_{j=1}^{n} \|\tilde{B}_j\|^2 + \sum_{j=1}^{n} \left(\|\Delta B_j^* X^{-r} \Delta B_j\| + 2\|\Delta B_j^* X^{-r} B_j\|\right) \\ &\le \|\Delta Q\| + q b^{-(q+1)} \|\Delta X\| \sum_{i=1}^{m} \|\tilde{A}_i\|^2 \\ &\quad + \sum_{i=1}^{m} \left(2\|X^{-q} A_i\| \cdot \|\Delta A_i\| + \|X^{-q}\| \cdot \|\Delta A_i\|^2\right) \\ &\quad + r b^{-(r+1)} \|\Delta X\| \sum_{j=1}^{n} \|\tilde{B}_j\|^2 + \sum_{j=1}^{n} \left(2\|X^{-r} B_j\| \cdot \|\Delta B_j\| + \|X^{-r}\| \cdot \|\Delta B_j\|^2\right).\end{aligned}$$

Denote $\xi = 1 - q b^{-(q+1)} \sum_{i=1}^{m} \|\tilde{A}_i\|^2 - r b^{-(r+1)} \sum_{j=1}^{n} \|\tilde{B}_j\|^2$. Notice from the proof of (3.6) and (3.7) that

$$\begin{aligned}\|Q^{-1}\|^{q+1} \cdot \sum_{i=1}^{m} \|\tilde{A}_i\|^2 &< \frac{\|Q^{-1}\|^{q+1}}{1 - \theta_1} \left[\sum_{i=1}^{m} \|A_i\|^2 + \frac{(q+1)^{q+1} - q^q}{2(q+1)^{q+1} \|Q^{-1}\|^{q+1}} \theta_1\right] \\ &= \frac{q^q}{2(q+1)^{q+1}}\end{aligned}$$

and

$$\|Q^{-1}\|^{r+1} \cdot \sum_{j=1}^{n} \|\tilde{B}_j\|^2 < \frac{\|Q^{-1}\|^{r+1}}{1-\theta_2} \left[\sum_{j=1}^{n} \|B_j\|^2 + \frac{(q+1)^{r+1} - q^{r+1}/r}{2(q+1)^{r+1}\|Q^{-1}\|^{r+1}} \theta_2 \right]$$
$$= \frac{q^{r+1}}{2r(q+1)^{r+1}}.$$

which implies that if $b = \frac{q}{q+1}\lambda_n(Q) = \frac{q}{q+1}\frac{1}{\|Q^{-1}\|}$, then

$$\begin{aligned}
\xi &= 1 - qb^{-(q+1)} \textstyle\sum_{i=1}^{m} \|\tilde{A}_i\|^2 - rb^{-(r+1)} \sum_{j=1}^{n} \|\tilde{B}_j\|^2 \\
&= 1 - q\frac{(q+1)^{q+1}}{q^{q+1}} \|Q^{-1}\|^{q+1} \textstyle\sum_{i=1}^{m} \|\tilde{A}_i\|^2 - r\frac{(q+1)^{r+1}}{q^{r+1}} \|Q^{-1}\|^{r+1} \sum_{j=1}^{n} \|\tilde{B}_j\|^2 \\
&> 1 - q\frac{(q+1)^{q+1}}{q^{q+1}} \frac{q^q}{2(q+1)^{q+1}} - r\frac{(q+1)^{r+1}}{q^{r+1}} \cdot \frac{1}{2r} \cdot \frac{q^{r+1}}{(q+1)^{r+1}} = 0,
\end{aligned}$$

and if $b = \frac{q}{q+1}\lambda_n(\tilde{Q}) = \frac{q}{q+1}\frac{1}{\|\tilde{Q}^{-1}\|}$, then

$$\begin{aligned}
\xi &= 1 - qb^{-(q+1)} \textstyle\sum_{i=1}^{m} \|\tilde{A}_i\|^2 - rb^{-(r+1)} \sum_{j=1}^{n} \|\tilde{B}_j\|^2 \\
&= 1 - q\frac{(q+1)^{q+1}}{q^{q+1}} \|\tilde{Q}^{-1}\|^{q+1} \textstyle\sum_{i=1}^{m} \|\tilde{A}_i\|^2 - r\frac{(q+1)^{r+1}}{q^{r+1}} \|\tilde{Q}^{-1}\|^{r+1} \sum_{j=1}^{n} \|\tilde{B}_j\|^2 \\
&> 1 - q\frac{(q+1)^{q+1}}{q^{q+1}} \frac{q^q}{2(q+1)^{q+1}} - r\frac{(q+1)^{r+1}}{q^{r+1}} \cdot \frac{1}{2r} \cdot \frac{q^{r+1}}{(q+1)^{r+1}} \\
&= 0.
\end{aligned}$$

Thus we obtain that

$$\|\Delta X\| \leq \frac{1}{\xi} \cdot \left[\|\Delta Q\| + \sum_{i=1}^{m} \left(2\|X^{-q} A_i\| \cdot \|\Delta A_i\| + \|X^{-q}\| \cdot \|\Delta A_i\|^2 \right) \right.$$
$$\left. + \sum_{j=1}^{n} \left(2\|X^{-r} B_j\| \cdot \|\Delta B_j\| + \|X^{-r}\| \cdot \|\Delta B_j\|^2 \right) \right].$$

□

By the theory of condition number developed by Rice [20], we give in the following an explicit expression of the condition number of the unique positive definite solution X in case $0 < q, r < 1$.

3.1 The complex case

Lemma 3.1 *[13] Let X be any $n \times n$ positive definite matrix,* $0 < q < 1$. *Then*

$$\text{(i)}\ \ X^{-q} = \frac{\sin \text{q}\pi}{\pi} \int_0^{\infty} (\lambda I + X)^{-1} \lambda^{-q} d\lambda,$$
$$\text{(ii)}\ \ X^{-q} = \frac{\sin q\pi}{q\pi} \int_0^{\infty} (\lambda I + X)^{-1} X (\lambda I + X)^{-1} \lambda^{-q} d\lambda.$$

From Theorem 3.1, we see that if $\|\Delta A_i\|$, $i = 1, 2, \ldots, m$, $\|\Delta B_j\|$, $j = 1, 2, \ldots, n$ and $\|\Delta Q\|$ are sufficiently small, then the perturbed matrix equation (3.1) has a unique positive definite solution $\tilde{X}$. Subtracting (1.1) from (3.1) gives rise to

$$\Delta X + \sum_{i=1}^{m} \left(\tilde{A}_i^* \tilde{X}^{-q} \tilde{A}_i - A_i^* X^{-q} A_i \right) + \sum_{j=1}^{n} \left(-\tilde{B}_j^* \tilde{X}^{-r} \tilde{B}_j + B_j^* X^{-r} B_j \right) = \Delta Q,$$

i.e.,

$$\begin{aligned} \Delta X + \sum_{i=1}^{m} \Big[A_i^* \left(\tilde{X}^{-q} - X^{-q} \right) A_i + \tilde{A}_i^* \left(\tilde{X}^{-q} - X^{-q} \right) \Delta A_i + \Delta A_i^* \left(\tilde{X}^{-q} - X^{-q} \right) A_i \Big] \\ + \sum_{j=1}^{n} \Big[-B_j^* \left(\tilde{X}^{-r} - X^{-r} \right) B_j - \tilde{B}_j^* \left(\tilde{X}^{-r} - X^{-r} \right) \Delta B_j - \Delta B_j^* \left(\tilde{X}^{-r} - X^{-r} \right) B_j \Big] \\ = \Delta Q - \sum_{i=1}^{m} \left(\Delta A_i^* X^{-q} A_i + \Delta A_i^* X^{-q} \Delta A_i + A_i^* X^{-q} \Delta A_i \right) \\ + \sum_{j=1}^{n} \left(\Delta B_j^* X^{-r} B_j + \Delta B_j^* X^{-r} \Delta B_j + B_j^* X^{-r} \Delta B_j \right), \end{aligned} \tag{3.8}$$

Notice that

$$\begin{aligned} (\lambda I + X + \Delta X)^{-1} - (\lambda I + X)^{-1} &= - (\lambda I + X + \Delta X)^{-1} \Delta X (\lambda I + X)^{-1} \\ &= -(\lambda I + X)^{-1} \Delta X \left(\lambda I + X + \Delta X\right)^{-1}. \end{aligned}$$

Using Lemma 3.1, we have

$$\begin{aligned} \tilde{X}^{-q} - X^{-q} &= \tfrac{\sin q\pi}{\pi} \textstyle\int_0^{\infty} [(\lambda I + X + \Delta X)^{-1} - (\lambda I + X)^{-1}] \lambda^{-q} d\lambda \\ &= \tfrac{\sin q\pi}{\pi} \textstyle\int_0^{\infty} -(\lambda I + X)^{-1} \Delta X (\lambda I + X + \Delta X)^{-1} \lambda^{-q} d\lambda \\ &= \tfrac{\sin q\pi}{\pi} \textstyle\int_0^{\infty} -(\lambda I + X)^{-1} \Delta X (\lambda I + X)^{-1} \lambda^{-q} d\lambda \\ &\quad + \tfrac{\sin q\pi}{\pi} \textstyle\int_0^{\infty} (\lambda I + X)^{-1} \Delta X (\lambda I + X + \Delta X)^{-1} \Delta X (\lambda I + X)^{-1} \lambda^{-q} d\lambda \end{aligned},$$

$$\begin{aligned}\tilde{X}^{-r} - X^{-r} &= \tfrac{\sin r\pi}{\pi}\int_0^\infty [(\lambda I + X + \Delta X)^{-1} - (\lambda I + X)^{-1}]\lambda^{-r} d\lambda \\ &= \tfrac{\sin r\pi}{\pi}\int_0^\infty -(\lambda I + X)^{-1}\Delta X(\lambda I + X + \Delta X)^{-1}\lambda^{-r} d\lambda \\ &= \tfrac{\sin r\pi}{\pi}\int_0^\infty -(\lambda I + X)^{-1}\Delta X(\lambda I + X)^{-1}\lambda^{-r} d\lambda \\ &\quad + \tfrac{\sin r\pi}{\pi}\int_0^\infty (\lambda I + X)^{-1}\Delta X(\lambda I + X + \Delta X)^{-1}\Delta X(\lambda I + X)^{-1}\lambda^{-r} d\lambda\end{aligned} \tag{3.9}$$

Combining (3.9) with (3.8), we obtain that

$$\begin{aligned}&\Delta X - \tfrac{\sin q\pi}{\pi}\textstyle\sum_{i=1}^m \int_0^\infty A_i^*(\lambda I + X)^{-1}\Delta X(\lambda I + X)^{-1}\lambda^{-q} A_i d\lambda \\ &\quad + \tfrac{\sin r\pi}{\pi}\textstyle\sum_{j=1}^n \int_0^\infty B_j^*(\lambda I + X)^{-1}\Delta X(\lambda I + X)^{-1}\lambda^{-r} B_j d\lambda \\ &= E + h(\Delta X),\end{aligned} \tag{3.10}$$

where

$$\begin{aligned}E &= \Delta Q - \sum_{i=1}^m \left(\Delta A_i^* X^{-q} A_i + \Delta A_i^* X^{-q}\Delta A_i + A_i^* X^{-q}\Delta A_i\right) \\ &\quad + \sum_{j=1}^n \left(\Delta B_j^* X^{-r} B_j + \Delta B_j^* X^{-r}\Delta B_j + B_j^* X^{-r}\Delta B_j\right), \\ h(\Delta X) &= -\frac{\sin q\pi}{\pi}\sum_{i=1}^m A_i^* \int_0^\infty (\lambda I + X)^{-1}\Delta X(\lambda I + X + \Delta X)^{-1} \\ &\quad \times \Delta X(\lambda I + X)^{-1}\lambda^{-q} d\lambda A_i \\ &\quad + \frac{\sin q\pi}{\pi}\sum_{i=1}^m \tilde{A}_i^* \int_0^\infty (\lambda I + X)^{-1}\Delta X(\lambda I + X + \Delta X)^{-1}\lambda^{-q} d\lambda\Delta A_i \\ &\quad + \frac{\sin q\pi}{\pi}\sum_{i=1}^m \Delta A_i^* \int_0^\infty (\lambda I + X)^{-1}\Delta X(\lambda I + X + \Delta X)^{-1}\lambda^{-q} d\lambda A_i \\ &\quad + \frac{\sin r\pi}{\pi}\sum_{j=1}^n B_j^* \int_0^\infty (\lambda I + X)^{-1}\Delta X(\lambda I + X + \Delta X)^{-1} \\ &\quad \times \Delta X(\lambda I + X)^{-1}\lambda^{-r} d\lambda B_j \\ &\quad - \frac{\sin r\pi}{\pi}\sum_{j=1}^n \tilde{B}_j^* \int_0^\infty (\lambda I + X)^{-1}\Delta X(\lambda I + X + \Delta X)^{-1}\lambda^{-r} d\lambda\Delta B_j \\ &\quad - \frac{\sin r\pi}{\pi}\sum_{j=1}^n \Delta B_j^* \int_0^\infty (\lambda I + X)^{-1}\Delta X(\lambda I + X + \Delta X)^{-1}\lambda^{-r} d\lambda B_j.\end{aligned}$$

Lemma 3.2 *Let*

$$\sum_{i=1}^{m} \|A_i^* A_i\| < \frac{q^q}{2\|(q+1)Q^{-1}\|^{q+1}}, \quad \sum_{j=1}^{n} \|B_j^* B_j\| < \frac{q^{r+1}}{2r\|(q+1)Q^{-1}\|^{r+1}}.$$

Then the linear operator $\mathbf{L} : H^{n\times n} \to H^{n\times n}$ *defined by*

$$\begin{aligned} \mathbf{L}W = W - \tfrac{\sin q\pi}{\pi} \textstyle\sum_{i=1}^{m} \int_0^\infty A_i^* (\lambda I + X)^{-1} W(\lambda I + X)^{-1} A_i \lambda^{-q} d\lambda \\ + \tfrac{\sin r\pi}{\pi} \textstyle\sum_{j=1}^{n} \int_0^\infty B_j^* (\lambda I + X)^{-1} W (\lambda I + X)^{-1} B_j \lambda^{-r} d\lambda \end{aligned} \tag{3.11}$$

is invertible.

Proof It suffices to show that for any matrix $V \in H^{n\times n}$, the following equation

$$\mathbf{L}W = V \tag{3.12}$$

has a unique solution. Define the operator $\mathbf{G} : H^{n\times n} \to H^{n\times n}$ by

$$\begin{aligned} \mathbf{G}Z = \frac{\sin q\pi}{\pi} \sum_{i=1}^{m} \int_0^\infty X^{-1/2} A_i^* (\lambda I + X)^{-1} X^{1/2} Z X^{1/2} (\lambda I + X)^{-1} A_i X^{-1/2} \lambda^{-q} d\lambda \\ - \frac{\sin r\pi}{\pi} \sum_{j=1}^{n} \int_0^\infty X^{-1/2} B_j^* (\lambda I + X)^{-1} X^{1/2} Z X^{1/2} (\lambda I + X)^{-1} B_j X^{-1/2} \lambda^{-r} d\lambda. \end{aligned}$$

Let $Y = X^{-1/2} W X^{-1/2}$. Thus (3.12) is equivalent to

$$Y - \mathbf{G}Y = X^{-1/2} V X^{-1/2}. \tag{3.13}$$

Notice that $\|X^{-1}\| < \frac{q+1}{q}\|Q^{-1}\|$. According to Lemma 3.1 (ii), we have

$$\begin{aligned} \|\mathbf{G}Y\| &\le \Big\|\frac{\sin q\pi}{\pi} \sum_{i=1}^{m} \int_0^\infty X^{-1/2} A_i^* (\lambda I + X)^{-1} X^{1/2} Y X^{1/2} (\lambda I + X)^{-1} A_i X^{-1/2} \lambda^{-q} d\lambda\Big\| \\ &\quad + \Big\|\frac{\sin r\pi}{\pi} \sum_{j=1}^{n} \int_0^\infty X^{-1/2} B_j^* (\lambda I + X)^{-1} X^{1/2} Y X^{1/2} (\lambda I + X)^{-1} B_j X^{-1/2} \lambda^{-r} d\lambda\Big\| \\ &\le \Big\|\frac{\sin q\pi}{\pi} \sum_{i=1}^{m} \int_0^\infty X^{-1/2} A_i^* (\lambda I + X)^{-1} X (\lambda I + X)^{-1} A_i X^{-1/2} \lambda^{-q} d\lambda\Big\| \cdot \|Y\| \\ &\quad + \Big\|\frac{\sin r\pi}{\pi} \sum_{j=1}^{n} \int_0^\infty X^{-1/2} B_j^* (\lambda I + X)^{-1} X (\lambda I + X)^{-1} B_j X^{-1/2} \lambda^{-r} d\lambda\Big\| \cdot \|Y\| \\ &= q\|Y\| \cdot \sum_{i=1}^{m} \|X^{-1/2} A_i^* X^{-q} A_i X^{-1/2}\| + r\|Y\| \cdot \sum_{j=1}^{n} \|X^{-1/2} B_j^* X^{-r} B_j X^{-1/2}\| \end{aligned}$$

$$
\begin{aligned}
&\leq q\|Y\| \cdot \sum_{i=1}^{m} \|X^{-1/2}A_i^* X^{-q/2}\|^2 + r\|Y\| \cdot \sum_{j=1}^{n} \|X^{-1/2}B_j^* X^{-r/2}\|^2 \\
&\leq q\|Y\| \cdot \sum_{i=1}^{m} \|A_i\|^2 \|X^{-1}\|^{q+1} + r\|Y\| \cdot \sum_{j=1}^{n} \|B_j\|^2 \|X^{-1}\|^{r+1} \\
&\leq q\|Y\| \cdot \frac{(q+1)^{q+1}}{q^{q+1}} \cdot \sum_{i=1}^{m} \|A_i\|^2 \|Q^{-1}\|^{q+1} \\
&\quad + r\|Y\| \cdot \frac{(q+1)^{r+1}}{q^{r+1}} \cdot \sum_{j=1}^{n} \|B_j\|^2 \|Q^{-1}\|^{r+1} \\
&< \|Y\|.
\end{aligned}
$$

Then $\|\mathbf{G}\| < 1$ which implies that $\mathbf{I} - \mathbf{G}$ is invertible. Therefore, for any matrix $V \in H^{n\times n}$, Eq. (3.13) has a unique solution Y. Thus equation (3.12) has a unique solution W for any $V \in H^{n\times n}$ which implies that the operator $\mathbf{L}$ is invertible. The proof is then completed. □

Let $C_i = X^{-q}A_i, i = 1, 2, \ldots, m;\ D_j = X^{-r}B_j, j = 1, 2, \ldots, n$. We can rewrite (3.10) as

$$
\begin{aligned}
\Delta X = &\mathbf{L}^{-1}\left[\Delta Q - \textstyle\sum_{i=1}^{m}\left(C_i^*\Delta A_i + \Delta A_i^* C_i\right) + \sum_{j=1}^{n}\left(D_j^*\Delta B_j + \Delta B_j^* D_j\right)\right] \\
&- \mathbf{L}^{-1}\left(\textstyle\sum_{i=1}^{m}\Delta A_i^* X^{-q}\Delta A_i + \sum_{j=1}^{n}\Delta B_j^* X^{-r}\Delta B_j\right) \\
&+ \mathbf{L}^{-1}(h(\Delta X)).
\end{aligned}
$$

Then we have

$$
\begin{aligned}
\Delta X = &\mathbf{L}^{-1}\left(\Delta Q - \textstyle\sum_{i=1}^{m}\left(C_i^*\Delta A_i + \Delta A_i^* C_i\right) + \sum_{j=1}^{n}\left(D_j^*\Delta B_j + \Delta B_j^* D_j\right)\right) \\
&+ O\left(\|(\Delta A_1, \cdots, \Delta A_m, \Delta B_1, \cdots, \Delta B_n, \Delta Q)\|_F^2\right),
\end{aligned}
\tag{3.14}
$$

$(\Delta A_1, \cdots, \Delta A_m, \Delta B_1, \cdots, \Delta B_n, \Delta Q) \to 0.$

By Rice's condition number theory [20], we define the condition number of the unique positive definite solution X of Eq. (1.1) as follows:

$$
C(X) = \lim_{\delta\to 0} \sup_{\left\|\left(\frac{\Delta A_1}{\mu_1}, \cdots, \frac{\Delta A_m}{\mu_m}, \frac{\Delta B_1}{\eta_1}, \ldots, \frac{\Delta B_n}{\eta_n}, \frac{\Delta Q}{\rho}\right)\right\|_F \leq \delta} \frac{\|\Delta X\|_F}{\xi\delta}, \tag{3.15}
$$

where $\xi, \mu_1, \cdots, \mu_m, \eta_1, \cdots, \eta_n, \rho$ are positive parameters. Taking $\xi = \mu_1 = \cdots = \mu_m = \eta_1 = \cdots = \eta_n = \rho = 1$ in (3.15) gives the absolute condition number $C_{\text{abs}}(X)$ and taking $\xi = \|X\|_F, \mu_i = \|A_i\|_F, i = 1, \ldots, m, \eta_j = \|B_j\|_F, j = 1, \ldots, n, \rho = \|Q\|_F$ gives the relative condition number $C_{\text{rel}}(X)$.

Substituting (3.14) into (3.15), we get

$$
\begin{aligned}
C(X) &= \frac{1}{\xi} \max_{\substack{\left(\frac{\Delta A_1}{\mu_1},\cdots,\frac{\Delta A_m}{\mu_m},\frac{\Delta B_1}{\eta_1},\dots,\frac{\Delta B_n}{\eta_n},\frac{\Delta Q}{\rho}\right)\neq 0\\ \Delta A_i,\Delta B_j\in C^{n\times n},\Delta Q\in H^{n\times n}}} \\
&\quad\times \frac{\|\mathbf{L}^{-1}\left(\Delta Q-\sum_{i=1}^m\left(C_i^*\Delta A_i+\Delta A_i^*C_i\right)+\sum_{j=1}^n\left(D_j^*\Delta B_j+\Delta B_j^*D_j\right)\right)\|_F}{\|\left(\frac{\Delta A_1}{\mu_1},\cdots,\frac{\Delta A_m}{\mu_m},\frac{\Delta B_1}{\eta_1},\dots,\frac{\Delta B_n}{\eta_n},\frac{\Delta Q}{\rho}\right)\|_F}\\
&= \frac{1}{\xi} \max_{\substack{(K_1,\cdots,K_m,F_1,\dots,F_n,H)\neq 0\\ K_i,F_j\in C^{n\times n},H\in H^{n\times n}}} \\
&\quad\times \frac{\|\mathbf{L}^{-1}\left(\rho H-\sum_{i=1}^m\mu_i\left(C_i^*K_i+K_i^*C_i\right)+\sum_{j=1}^n\eta_j\left(D_j^*F_j+F_j^*D_j\right)\right)\|_F}{\|(K_1,\cdots,K_m,F_1,\dots,F_n,H)\|_F}\\
&= \frac{1}{\xi} \max_{\substack{(K_1,\cdots,K_m,F_1,\dots,F_n,H)\neq 0\\ K_i,F_j\in C^{n\times n},H\in H^{n\times n}}} \\
&\quad\times \frac{\|\mathbf{L}^{-1}\left(\rho H-\sum_{i=1}^m\mu_i\left(C_i^*K_i+K_i^*C_i\right)+\sum_{j=1}^n\eta_j\left(D_j^*F_j+F_j^*D_j\right)\right)\|_F}{\|(-K_1,\cdots,-K_m,F_1,\dots,F_n,H)\|_F}\\
&= \frac{1}{\xi} \max_{\substack{(E_1,\cdots,E_m,F_1,\dots,F_n,H)\neq 0\\ E_i,F_j\in C^{n\times n},H\in H^{n\times n}}} \\
&\quad\times \frac{\|\mathbf{L}^{-1}\left(\rho H+\sum_{i=1}^m\mu_i\left(C_i^*E_i+E_i^*C_i\right)+\sum_{j=1}^n\eta_j\left(D_j^*F_i+F_j^*D_j\right)\right)\|_F}{\|(E_1,\cdots,E_m,F_1,\dots,F_n,H)\|_F}.
\end{aligned}
$$

Let L be the matrix of the operator $\mathbf{L}$. Then it is not difficult to see that

$$
\begin{aligned}
L = I\otimes I &- \frac{\sin q\pi}{\pi}\sum_{i=1}^m\int_0^\infty\left[(\lambda I+X)^{-1}A_i\right]^T\otimes\left[(\lambda I+X)^{-1}A_i\right]^*\lambda^{-q}d\lambda\\
&+\frac{\sin r\pi}{\pi}\sum_{j=1}^n\int_0^\infty\left[(\lambda I+X)^{-1}B_j\right]^T\otimes\left[(\lambda I+X)^{-1}B_j\right]^*\lambda^{-r}d\lambda.
\end{aligned}
$$

Denote by $\beta=\mathrm{vec}(H)=a+kb$, $w_i=\mathrm{vec}(E_i)=u^{(i)}+kv^{(i)}$, $i=1,\dots,m$; $\varepsilon_j=\mathrm{vec}(F_j)=p^{(j)}+kq^{(j)}$, $j=1,2,\dots,n$, where k is the imaginary unit satisfying $k^2=-1$. Let

$$
g_1^{(i)}=\begin{pmatrix}u^{(i)}\\ v^{(i)}\end{pmatrix},\ g_2^{(j)}=\begin{pmatrix}p^{(j)}\\ q^{(j)}\end{pmatrix},\ g_1=\begin{pmatrix}g_1^{(1)}\\ \vdots\\ g_1^{(m)}\end{pmatrix},\ g_2=\begin{pmatrix}g_2^{(1)}\\ \vdots\\ g_2^{(n)}\end{pmatrix},\ g=\begin{pmatrix}a\\ b\\ g_1\\ g_2\end{pmatrix};
$$

$$
L^{-1}\left(I\otimes C_i^*\right)=L^{-1}\left(I\otimes\left(X^{-q}A_i\right)^*\right)=U_1^{(i)}+k\Omega_1^{(i)},
$$

$$L^{-1}\left(C_i^T \otimes I\right)\Pi = L^{-1}\left(\left(X^{-q}A_i\right)^T \otimes I\right)\Pi = U_2^{(i)} + k\Omega_2^{(i)},$$
$$L^{-1}\left(I \otimes D_i^*\right) = L^{-1}\left(I \otimes \left(X^{-r}B_i\right)^*\right) = V_1^{(j)} + k\Phi_1^{(j)},$$
$$L^{-1}\left(D_i^T \otimes I\right)\Pi = L^{-1}\left(\left(X^{-r}B_i\right)^T \otimes I\right)\Pi = V_2^{(j)} + k\Phi_2^{(j)},$$

where $U_1^{(i)}, U_2^{(i)}, V_1^{(j)}, V_2^{(j)}, \Omega_1^{(i)}, \Omega_2^{(i)}, \Phi_1^{(j)}, \Phi_2^{(j)} \in R^{n^2\times n^2}, i = 1, \ldots, m, j = 1, \ldots, n$, and Π is the vec-permutation matrix, such that $\mathrm{vec}(K^T) = \Pi \mathrm{vec} K$. For $i = 1, 2, \ldots, m$; $j = 1, \ldots, n$.

Denote

$$L^{-1} = S + k\Sigma, S, \Sigma \in R^{n^2\times n^2}, \quad S_c = \begin{bmatrix} S & -\Sigma \\ \Sigma & S \end{bmatrix},$$
$$U_c^{(i)} = \begin{bmatrix} U_1^{(i)} + U_2^{(i)} & \Omega_2^{(i)} - \Omega_1^{(i)} \\ \Omega_1^{(i)} + \Omega_2^{(i)} & U_1^{(i)} - U_2^{(i)} \end{bmatrix}, \quad V_c^{(j)} = \begin{bmatrix} V_1^{(j)} + V_2^{(j)} & \Phi_2^{(j)} - \Phi_1^{(j)} \\ \Phi_1^{(j)} + \Phi_2^{(j)} & V_1^{(j)} - V_2^{(j)} \end{bmatrix}.$$

Then we obtain that

$$\begin{aligned}
C(X) &= \frac{1}{\xi} \max_{\substack{(E_1,\cdots,E_m,F_1,\ldots,F_n,H)\neq 0 \\ E_i,F_j\in C^{n\times n},H\in H^{n\times n}}} \\
&\quad\times \frac{\|\mathbf{L}^{-1}\left(\rho H + \sum_{i=1}^m \mu_i\left(C_i^*E_i + E_i^*C_i\right) + \sum_{j=1}^n \eta_j\left(D_j^*F_j + F_j^*D_j\right)\right)\|_F}{\|\left(E_1,\cdots,E_m,F_1,\ldots,F_n,H\right)\|_F} \\
&= \frac{1}{\xi} \max_{\substack{(E_1,\cdots,E_m,F_1,\ldots,F_n,H)\neq 0 \\ E_i,F_j\in C^{n\times n},H\in H^{n\times n}}} \\
&\quad\times \frac{\|\rho L^{-1}\mathrm{vec}(H) + \sum_{i=1}^m \mu_i L^{-1}\mathrm{vec}\left(C_i^*E_i + E_i^*C_i\right) + \sum_{j=1}^n \eta_j L^{-1}\mathrm{vec}\left(D_j^*F_j + F_j^*D_j\right)\|}{\|\mathrm{vec}\left(E_1,\cdots,E_m,F_1,\ldots,F_n,H\right)\|} \\
&= \frac{1}{\xi} \max_{\substack{(E_1,\cdots,E_m,F_1,\ldots,F_n,H)\neq 0 \\ E_i,F_j\in C^{n\times n},H\in H^{n\times n}}} \\
&\quad\times \frac{\|\rho S_c\begin{pmatrix} a \\ b \end{pmatrix} + \sum_{i=1}^m \mu_i U_c^{(i)}\begin{pmatrix} u^{(i)} \\ v^{(i)} \end{pmatrix} + \sum_{j=1}^n \eta_j V_c^{(j)}\begin{pmatrix} p^{(j)} \\ q^{(j)} \end{pmatrix}\|}{\|\mathrm{vec}\left(E_1,\cdots,E_m,F_1,\ldots,F_n,H\right)\|} \\
&= \frac{1}{\xi} \max_{g\neq 0} \frac{\|(\rho S_c, \mu_1 U_c^{(1)}, \cdots, \mu_m U_c^{(m)}, \eta_1 V_c^{(1)}, \ldots, \eta_n V_c^{(n)})g\|}{\|g\|} \\
&= \frac{1}{\xi}\|\left(\rho S_c, \mu_1 U_c^{(1)}, \cdots, \mu_m U_c^{(m)}, \eta_1 V_c^{(1)}, \ldots, \eta_n V_c^{(n)}\right)\|.
\end{aligned}$$

Theorem 3.2 *Let*

$$\sum_{i=1}^m \|A_i^*A_i\| < \frac{1}{2}\frac{q^q}{\|(q+1)Q^{-1}\|^{q+1}}, \quad \sum_{j=1}^n \|B_j^*B_j\| < \frac{1}{2r}\frac{q^{r+1}}{\|(q+1)Q^{-1}\|^{r+1}}.$$

Then the condition number $C(X)$ defined by (3.15) has the following explicit expression

$$C(X) = \frac{1}{\xi}\left\| \left(\rho S_c, \mu_1 U_c^{(1)}, \cdots, \mu_m U_c^{(m)}, \eta_1 V_c^{(1)}, \ldots, \eta_n V_c^{(n)}\right) \right\|, \tag{3.16}$$

where $S_c, U_c^{(i)}, V_c^{(j)}, i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$ are defined as above.

Remark 3.1 From (3.16), we have the relative condition number

$$C_{\text{rel}}(X) = \frac{\left\| \left(\|Q\|_F S_c, \|A_1\|_F U_c^{(1)}, \cdots, \|A_m\|_F U_c^{(m)}, \|B_1\|_F V_c^{(1)}, \cdots, \|B_n\|_F V_c^{(n)}\right) \right\|}{\|X\|_F}. \tag{3.17}$$

3.2 The real case

In this section we consider the real case, i.e., all the coefficient matrices $A_i, i = 1, 2, \ldots, m; B_j, j = 1, 2, \ldots, n$, and Q of Eq. (1.1) are real. In such a case the corresponding unique positive definite solution X is also real. Similar to Theorem 3.2, we obtain the following theorem.

Theorem 3.3 *Let $A_i, i = 1, 2, \ldots, m; B_j, j = 1, 2, \ldots, n$, and Q be real. Suppose that*

$$\sum_{i=1}^{m} \|A_i^* A_i\| < \frac{1}{2}\frac{q^q}{\|(q+1)Q^{-1}\|^{q+1}}, \quad \sum_{j=1}^{n} \|B_j^* B_j\| < \frac{1}{2r}\frac{q^{r+1}}{\|(q+1)Q^{-1}\|^{r+1}}.$$

Then the condition number $C(X)$ defined by (3.15) has the explicit expression

$$C(X) = \frac{1}{\xi}\left\| \left(\rho S_r, \mu_1 U_r^{(1)}, \cdots, \mu_m U_r^{(m)}, \eta_1 V_r^{(1)}, \cdots, \eta_n V_r^{(n)}\right) \right\|, \tag{3.18}$$

where

$$\begin{aligned} S_r = \Bigg[& I \otimes I - \frac{\sin q\pi}{\pi} \sum_{i=1}^{m} \int_0^\infty \left((\lambda I + X)^{-1} A_i\right)^T \otimes \left((\lambda I + X)^{-1} A_i\right)^T \lambda^{-q} d\lambda \\ & + \frac{\sin r\pi}{\pi} \sum_{j=1}^{n} \int_0^\infty \left((\lambda I + X)^{-1} B_j\right)^T \otimes \left((\lambda I + X)^{-1} B_j\right)^T \lambda^{-r} d\lambda \Bigg]^{-1}, \\ U_r^{(i)} &= S_r \left[I \otimes \left(A_i^T X^{-q}\right) + \left(\left(A_i^T X^{-q}\right) \otimes I\right) \Pi \right], \quad i = 1, 2, \ldots, m; \\ V_r^{(j)} &= S_r \left[I \otimes \left(B_j^T X^{-r}\right) + \left(\left(B_j^T X^{-r}\right) \otimes I\right) \Pi \right], \quad j = 1, 2, \ldots, n. \end{aligned}$$

Remark 3.2 In the real case the relative condition number is given by

$$C_{\text{rel}}(X) = \frac{\| \left(\|Q\|_F S_r, \|A_1\|_F U_r^{(1)} \cdots \|A_m\|_F U_r^{(m)}, \|B_1\|_F V_r^{(1)} \cdots \|B_n\|_F V_r^{(n)} \right) \|}{\|X\|_F}. \tag{3.19}$$

4 Numerical experiments

In this section, several numerical examples are given to illustrate the theoretical results. All the tests are carried out using MATLAB 7.1 with machine precision around 10^{-16}. The practical stopping criterion used is the residual $\|X + \sum_{i=1}^{m} A_i^* X^{-q} A_i - \sum_{j=1}^{n} B_j^* X^{-r} B_j - Q\| < 10^{-15}$.

Example 4.1 Consider Eq. (1.1) with the case $q = 0.5$, $r = 0.7$, $m = n = 2$, and the matrices A_1, A_2, B_1, B_2 and Q as follows:

$$A_1 = \begin{pmatrix} 0.1 & 0.05 & 0.05 \\ 0.05 & 0.1 & 0.05 \\ 0.05 & 0.05 & 0.1 \end{pmatrix}, \quad A2 = \begin{pmatrix} 0.5 & -0.02 & -0.02 \\ -0.02 & 0.5 & -0.02 \\ -0.02 & -0.02 & 0.5 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} -0.04 & 0.01 & -0.02 \\ 0.05 & 0.07 & -0.013 \\ 0.011 & 0.09 & 0.06 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0.01 & 0.001 & 0.01 \\ 0.001 & 0.01 & 0.001 \\ 0.01 & 0.001 & 0.01 \end{pmatrix},$$

$$Q = \begin{pmatrix} 2 & 0.2 & 0.2 \\ 0.2 & 2 & 0.2 \\ 0.2 & 0.2 & 2 \end{pmatrix}.$$

By computation, $\|A_1^* A_1\| + \|A_2^* A_2\| - \frac{1}{2} \cdot \frac{q^q}{\|(q+1)Q^{-1}\|^{q+1}} = -0.1544$, $\|B_1^* B_1\| + \|B_2^* B_2\| - \frac{1}{2r} \cdot \frac{q^{r+1}}{\|(q+1)Q^{-1}\|^{r+1}} = -0.2831$. According to Theorem 2.2, using iteration (2.2) and iterating 15 steps, then we get the unique positive definite solution to Eq. (1.1):

$$\bar{X} \approx X_{15} = \begin{pmatrix} 1.8026 & 0.2190 & 0.2168 \\ 0.2190 & 1.8077 & 0.2191 \\ 0.2168 & 0.2191 & 1.8025 \end{pmatrix}$$

with the residual $\|X_{15} + A_1^* X_{15}^{-q} A_1 + A_2^* X_{15}^{-q} A_2 - B_1^* X_{15}^{-r} B_1 - B_2^* X_{15}^{-r} B_2 - Q\| = 4.4524e - 016$ or

$$\bar{X} \approx Y_{15} = \begin{pmatrix} 1.8026 & 0.2190 & 0.2168 \\ 0.2190 & 1.8077 & 0.2191 \\ 0.2168 & 0.2191 & 1.8025 \end{pmatrix}$$

with the residual $\|Y_{15} + A_1^* Y_{15}^{-q} A_1 + A_2^* Y_{15}^{-q} A_2 - B_1^* Y_{15}^{-r} B_1 - B_2^* Y_{15}^{-r} B_2 - Q\| = 4.4593e - 016$.

Example 4.2 Let $m = n = 1$ and

$$A = \frac{\sqrt{3}}{45}\begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 1 \\ -1 & -1 & 1 & 0 & 1 \\ -1 & -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 \end{pmatrix}, \ B = \frac{A + A^*}{2},$$

$q = 0.5, r = 0.3,\ X = \mathrm{diag}(0.725, 2, 3, 2, 1),\ Q = X + A^* X^{-q} A - B^* X^{-r} B.$

Consider the perturbed matrix equation

$$\tilde{X} + \tilde{A}_j^* \tilde{X}^{-q} \tilde{A}_j - \tilde{B}_j^* \tilde{X}^{-r} \tilde{B}_j = \tilde{Q}_j,$$

where $\epsilon_j = 0.1^{2j},\ \tilde{A}_j = A + \epsilon_j (I + E),\ \tilde{B}_j = B + \epsilon_j (I + 2E)\ \tilde{X}_j = X + \epsilon_j (I - E),\ \tilde{Q}_j = \tilde{X}_j + \tilde{A}_j^* \tilde{X}_j^{-q} \tilde{A}_j - \tilde{B}_j^* \tilde{X}_j^{-q} \tilde{B}_j$, with $e = (1, 1, 1, 1, 1)$ and $E = e'e$.

Now we compute the perturbation bounds for Eq. (1.1).

By computation, $\theta_1 = \frac{q^q}{(q+1)^{q+1}} - 2\|A\|^2 \|Q^{-1}\|^{q+1} = 0.3378 > 0$, $\theta_2 = \frac{q^{r+1}}{r(q+1)^{r+1}} - 2\|B\|^2 \|Q^{-1}\|^{r+1} = 0.7891 > 0$, and $\lambda_n (X - \frac{q}{q+1} Q) = 0.4821 > 0$ which implies that X is the unique positive definite solution of Eq. (1.1) by Theorem 2.2. Obviously, $\tilde{X}_j$ are positive definite solutions of the perturbed matrix equations $\tilde{X} + \tilde{A}_j^* \tilde{X}^{-q} \tilde{A}_j - \tilde{B}_j^* \tilde{X}^{-q} \tilde{B}_j = \tilde{Q}_j$. Moreover, it is not difficult to verify that for each $j = 1, 2, 3, 4, 5$, the corresponding equations $\tilde{X} + \tilde{A}_j^* \tilde{X}^{-q} \tilde{A}_j - \tilde{B}_j^* \tilde{X}^{-q} \tilde{B}_j = \tilde{Q}_j$ and $\tilde{X}_j$ satisfy the assumption $\theta_{j1} = \frac{q^q}{(q+1)^{q+1}} - 2\|\tilde{A}_j\|^2 \|\tilde{Q}_j^{-1}\|^{q+1} > 0, \theta_{j2} = \frac{q^{r+1}}{r(q+1)^{r+1}} - 2\|\tilde{B}_j\|^2 \|\tilde{Q}_j^{-1}\|^{r+1} > 0$, and the conditions $\lambda_n (\tilde{X}_j - \frac{q}{q+1} \tilde{Q}_j) > 0$,respectively. Thus by Theorem 2.2, $\tilde{X}_j$ $(j = 1, 2, \ldots, 5)$ are the unique positive definite solutions of the corresponding perturbed matrix equations, respectively. We denote $\Delta X^{(j)} = \tilde{X}^j - X$. All the conditions of Theorem 3.1 are satisfied for $j = 1, 2, 3, 4, 5$. The results are given in the following table.

	j = 1	j = 2	j = 3	j = 4	j = 5
True error $\|\Delta X^{(j)}\|$	0.0400	4.0000e−004	4.0000e−006	4.0000e−008	4.0000e−010
Our result (3.5)	0.0985	6.9607e−004	6.9346e−006	6.9343e−008	6.9343e−010

Example 4.3 Consider Eq. (1.1) with $q = r = 0.5, m = 2, n = 1$ and

$$A_1 = \begin{pmatrix} 0 & a_1 \\ 0.02 & 0 \end{pmatrix}, \ A_2 = \begin{pmatrix} 0 & a_2 \\ 0 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}, \ Q = \begin{pmatrix} 1.1 & 0 \\ 0 & 1.2 \end{pmatrix},$$

where $a_1 = 0.25 + 10^{-k}$, $a_2 = 0.15 + 10^{-k}$ and $b = 0.35 + 10^{-k}$. Denote $\theta_1 = \|A_1^* A_1\| + \|A_2^* A_2\| - \frac{1}{2} \frac{q^q}{\|(q+1)Q^{-1}\|^{q+1}}$, $\theta_2 = \|B^* B\| - \frac{1}{2r} \frac{q^{r+1}}{\|(q+1)Q^{-1}\|^{r+1}}$, For k from

1 to 6, we compute θ_1 and θ_2 to see that the conditions of Theorem 3.3 are always satisfied. Results for $C_{\text{rel}}(X)$ by (3.19) with different vales of k are listed below where $C_{\text{rel}}(X)$ is the relative condition number of the unique positive definite solution of Eq.(1.1).

k	1	2	3	4	5	6
θ_1	−0.037027	−0.128827	−0.136225	−0.136947	−0.137019	−0.137026
θ_2	−0.019527	−0.092427	−0.098826	−0.099457	−0.099520	−0.099526
$C_{\text{rel}}(X_L)$	1.014454	1.000561	0.999674	0.999590	0.999582	0.999581

The numerical results listed in the second line show that the unique positive definite solution X is well-conditioned in such cases.

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