J. Appl. Math. Comput. (2016) 52:345–361
DOI 10.1007/s12190-015-0944-0

ORIGINAL RESEARCH

A new fully discrete finite difference/element approximation for fractional cable equation

Jincun Liu[1] · Hong Li[1] · Yang Liu[1]

Received: 17 July 2015 / Published online: 12 November 2015

Abstract A novel fully discrete Crank–Nicolson finite element method, which is obtained by finite difference in time and finite element in space, is presented to approximate the fractional Cable equation. Compared to the L1-formula for discretizing fractional derivatives at time t_{n+1}, the proposed approximate scheme is directly obtained at time $t_{n+\frac{1}{2}}$, in which some new coefficients $(k + \frac{1}{2})^{1-\alpha} - (k - \frac{1}{2})^{1-\alpha}$ instead of $(k + 1)^{1-\alpha} - k^{1-\alpha}$ are derived. Based on the new approximate formula, the stability and error estimate are analyzed in detail and the optimal convergence rate $O(\tau^{\min\{1+\alpha_1,1+\alpha_2\}} + h^{r+1})$ is obtained. Numerical examples in one-dimensional and two-dimensional spaces are shown to illustrate the effectiveness and feasibility of the studied algorithm.

Keywords Fractional cable equation · Novel discrete scheme · Finite difference method · Finite element method · Stability · Error estimate

Mathematics Subject Classification 65M60 · 65N15 · 65N30 · 26A33

1 Introduction

Fractional partial differential equations (PDEs), which mainly include time, space and space-time fractional PDEs, have a lot of applications (such as in chaos, mechan-

✉ Hong Li
smslh@imu.edu.cn

✉ Yang Liu
mathliuyang@aliyun.com

Jincun Liu
ljincun@163.com

[1] School of Mathematical Sciences, Inner Mongolia University, Hohhot 010021, China

ical systems, control, continuous-time random walks and so forth). Recently, some numerical solutions for Fractional PDEs have been obtained based on some different numerical methods. These numerical methods mainly cover finite element methods [6,8,15,16,18,22,27–29,38,42–44,46,47,49], finite difference methods [1,3,7,10,11,13,17,23,26,30–35,37,39,41,45,48], spectral approximations [2,24,25], DG methods [9,40] and so forth.

Here, we mainly give some introductions on finite element approximations of fractional PDEs in detail. Deng [8] discussed a finite element method for the space-time fractional Fokker–Planck equation. Zhang et al. [42] presented a Galerkin finite element scheme for symmetric space-fractional PDEs. In 2011, Li et al. [22] gave some detailed numerical analysis of finite element methods for fractional subdiffusion and superdiffusion equations. In 2012, Zhao and Li [46] gave some numerical analysis of time-space fractional telegraph equation based on finite element approximations. Wang et al. [38] studied space-fractional diffusion equations' inhomogeneous Dirichlet boundary-value problems and analyzed their finite element approximations. In 2011, Jiang and Ma [15] considered a high-order finite element method for one-dimensional time-fractional PDE. Jiang and Ma [16] presented the moving finite element method for a time fractional PDE. In [44], Zhang et al. gave some analysis and numerical results on finite element method for Grwünwald–Letnikov time-fractional PDE. In [28,29], some mixed finite element methods are studied for second and fourth order fractional PDEs. In 2010, Zheng et al. [47] gave a note on the finite element method for the advection diffusion equation with space-fractional derivative. Jin et al. [18] studied lumped mass Galerkin finite element method for the homogeneous diffusion equation with time-fractional derivative.

In this paper, our main purpose is to introduce finite element (FE) method to solve the following time-fractional Cable equation

$$\frac{\partial u(x,t)}{\partial t} = -\gamma\, {}_0D_t^{1-\alpha_1} u + {}_0D_t^{1-\alpha_2} \Delta u, \quad x \in \Omega,\ t \in [0,T], \tag{1}$$

subject to the initial condition

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{2}$$

and the boundary condition

$$u(x,t) = 0, \quad x \in \partial\Omega,\ t \in [0,T]. \tag{3}$$

Here $\Omega \subset \mathbb{R}^d (d = 1, 2)$ is a bounded space domain with the boundary $\partial\Omega$, $\Delta = \frac{\partial^2}{\partial x^2}(d = 1)$ and $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}(d = 2)$ represent the Laplacian operators, and ${}_0D_t^{\alpha}$ means to take the left Riemann–Liouville fractional derivative with respect to the time variable t by order α $(0 < \alpha < 1)$, which is defined by

$${}_0D_t^{\alpha} f(x,t) = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_0^t \frac{f(x,s)}{(t-s)^{\alpha}} \mathrm{d}s, \tag{4}$$

where $\Gamma(\cdot)$ denotes the Gamma function.

Based on the anomalous electrodiffusion in nerve cells, an important model–the fractional Cable equation has been derived and developed by many authors [5,12, 19–21]. Recently, some numerical methods for numerically solving the fractional Cable equation have been developed. Lin and Xu [24] discussed the spectral discrete method for fractional Cable equation, some detailed error analysis are made and some calculated data are provided to verify their theoretical results. Bhrawy and Zaky [4] considered an accurate spectral collocation algorithm for solving nonlinear fractional Cable equations in one- and two-dimensional cases. Zhuang et al. [49] considered a Galerkin finite element approximation for one-dimensional time fractional Cable equation and obtained a good approximation accuracy in time. Hu and Zhang [14] solved numerically the fractional Cable equation by using implicit compact difference schemes. In [32], Quintana-Murillo and Yuste gave the explicit numerical method for fractional Cable equation including two temporal Riemann–Liouville derivatives.

In this discussion, we mainly present a novel discrete scheme of fractional derivative differing from the approximate method in [24,35], and formulate then analyze a new fully discrete Crank–Nicolson type Galerkin finite element scheme, which is different from the one in [49]. In [24,35], the time fractional derivative was discretized at time $t = t_{n+1}$. Compared to the approximate formula proposed in [24,35], our approximate method is made directly at time $t_{n+\frac{1}{2}}$ and a new discrete scheme is formulated based on some different coefficients $(k+\frac{1}{2})^{1-\alpha} - (k-\frac{1}{2})^{1-\alpha}$ from the ones $(k+1)^{1-\alpha} - k^{1-\alpha}$ in [24,35]. In the full text, we give some detailed analysis on stability and error estimates for two-dimensional problem of the fractional Cable equation. Moreover, two numerical tests are made to confirm our theoretical analysis.

In this paper, the functional spaces we adopted are the standard Sobolev spaces as follow

$$
\begin{aligned}
H_0^1(\Omega) &:= \{v \in H^1(\Omega),\ v|_{\partial\Omega} = 0\},\\
H^m(\Omega) &:= \{v \in L^2(\Omega),\ D^\beta v \in L^2(\Omega), \text{for all } |\beta| \le m\},
\end{aligned}
$$

where $L^2(\Omega)$ is the space of measurable functions whose squares are Lebesgue integrable in Ω. The inner products of $L^2(\Omega)$ and $H^1(\Omega)$ are defined by

$$
(u, v) = \int_\Omega uv\mathrm{d}x, \quad (u, v)_1 = (u, v) + (\nabla u, \nabla v),
$$

and the corresponding norms are defined as

$$
\|u\|_0 = (u, u)^{\frac{1}{2}}, \quad \|u\|_1 = (u, u)_1^{\frac{1}{2}},
$$

respectively. Furthermore, $H^m(\Omega)$ is equipped with the norm

$$
\|u\|_m = \Big(\sum_{0 \le |\beta| \le m} \|D^\beta u\|_0^2 \Big)^{\frac{1}{2}}.
$$

2 The fully discrete scheme

Set $\tau = \frac{T}{N}$ and let $0 = t_0 < t_1 < \cdots < t_N = T$ be the uniform partition of the time interval $[0, T]$, where $t_n = n\tau$. In the following, we denote $u^n = u(\cdot, t_n)$ and the terms of time derivatives will be approximated at intermediate nodes $t_{n+\frac{1}{2}} = \frac{t_n+t_{n+1}}{2}$.

The first order time derivative $\frac{\partial u}{\partial t}$ at $t = t_{n+\frac{1}{2}}$ can be approximated as

$$\frac{\partial u(x, t_{n+\frac{1}{2}})}{\partial t} = \frac{u^{n+1} - u^n}{\tau} + O(\tau^2), \quad n = 0, 1, \ldots, N-1. \tag{5}$$

Recalling the relationship between the Caputo fractional derivative and Riemann–Liouville fractional derivative

$${}_0D_t^{\alpha} f(x,t) = {}_0^CD_t^{\alpha} f(x,t) + \frac{f(x,0)}{\Gamma(1-\alpha)t^{\alpha}}, \quad 0 < \alpha < 1, \tag{6}$$

as well as the definition of Caputo fractional derivative of order α

$${}_0^CD_t^{\alpha} f(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial f(x,s)}{\partial s} \frac{\mathrm{d}s}{(t-s)^{\alpha}}, \quad 0 < \alpha < 1,$$

we can approximate the Caputo fractional derivative as the following

$$\begin{aligned}
&{}_0^CD_t^{\alpha} f(x, t_{n+\frac{1}{2}}) \\
&= \frac{1}{\Gamma(1-\alpha)} \left[\sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \frac{\partial f}{\partial s} \frac{\mathrm{d}s}{\left(t_{n+\frac{1}{2}} - s\right)^{\alpha}} + \int_{t_n}^{t_{n+\frac{1}{2}}} \frac{\partial f}{\partial s} \frac{\mathrm{d}s}{\left(t_{n+\frac{1}{2}} - s\right)^{\alpha}} \right] \\
&= \frac{1}{\Gamma(1-\alpha)} \left[\sum_{j=0}^{n-1} \frac{f^{j+1} - f^j}{\tau} \int_{t_j}^{t_{j+1}} \frac{\mathrm{d}s}{\left(t_{n+\frac{1}{2}} - s\right)^{\alpha}} \right. \\
&\quad \left. + \frac{f^{n+1} - f^n}{\tau} \int_{t_n}^{t_{n+\frac{1}{2}}} \frac{\mathrm{d}s}{\left(t_{n+\frac{1}{2}} - s\right)^{\alpha}} \right] + r_t \\
&= \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \left\{ \sum_{j=0}^{n-1} (f^{j+1} - f^j) \left[\left(n + \frac{1}{2} - j\right)^{1-\alpha} - \left(n - \frac{1}{2} - j\right)^{1-\alpha} \right] \right. \\
&\quad \left. + (f^{n+1} - f^n) \frac{1}{2^{1-\alpha}} \right\} + r_t \\
&= \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^{n} c_k (f^{n+1-k} - f^{n-k}) + r_t, \qquad n = 0, 1, 2, \ldots, N-1,
\end{aligned} \tag{7}$$

where $c_k = (k+\frac{1}{2})^{1-\alpha} - (k-\frac{1}{2})^{1-\alpha}$, $(k = 1, 2, \ldots)$, $c_0 = \frac{1}{2^{1-\alpha}}$ and $r_t = O(\tau^{2-\alpha})$, which can be proved by the similar method in [24,35].

Combining (6) and (7), we have that for $n = 0, 1, 2, \ldots, N-1$

$$
\begin{aligned}
{}_0D_t^{1-\alpha_1} u\left(t_{n+\frac{1}{2}}\right) &= \frac{\tau^{\alpha_1-1}}{\Gamma(1+\alpha_1)} \sum_{k=0}^{n} c_k^1 (u^{n+1-k} - u^{n-k}) \\
&\quad + \frac{u(x,0)}{\Gamma(\alpha_1)\left[(n+\frac{1}{2})\tau\right]^{1-\alpha_1}} + O(\tau^{1+\alpha_1}), \\
{}_0D_t^{1-\alpha_2} \Delta u(t_{n+\frac{1}{2}}) &= \frac{\tau^{\alpha_2-1}}{\Gamma(1+\alpha_2)} \sum_{k=0}^{n} c_k^2 \left(\Delta u^{n+1-k} - \Delta u^{n-k}\right) \\
&\quad + \frac{\Delta u(x,0)}{\Gamma(\alpha_2)\left[(n+\frac{1}{2})\tau\right]^{1-\alpha_2}} + O(\tau^{1+\alpha_2}),
\end{aligned}
\tag{8}
$$

where

$$
\begin{aligned}
c_k^1 &= \left(k+\frac{1}{2}\right)^{\alpha_1} - \left(k-\frac{1}{2}\right)^{\alpha_1}, \quad c_0^1 = \frac{1}{2^{\alpha_1}}; \\
c_k^2 &= \left(k+\frac{1}{2}\right)^{\alpha_2} - \left(k-\frac{1}{2}\right)^{\alpha_2}, \quad c_0^2 = \frac{1}{2^{\alpha_2}}.
\end{aligned}
\tag{9}
$$

Denote $\hat{\alpha}_1 = \frac{\tau^{\alpha_1}}{\Gamma(1+\alpha_1)}, \hat{\alpha}_2 = \frac{\tau^{\alpha_2}}{\Gamma(1+\alpha_2)}, \hat{\beta}_n^1 = \frac{\tau}{\Gamma(\alpha_1)[(n+\frac{1}{2})\tau]^{1-\alpha_1}}, \hat{\beta}_n^2 = \frac{\tau}{\Gamma(\alpha_2)[(n+\frac{1}{2})\tau]^{1-\alpha_2}}$, and $R_t = O(\tau^2 + \tau^{1+\alpha_1} + \tau^{1+\alpha_2})$, then the variational problem can be stated as: Find $u^{n+1} \in H_0^1(\Omega)$ such that for all $v \in H_0^1(\Omega)$,

$$
\begin{aligned}
(u^{n+1}, v) &= (u^n, v) - \gamma\hat{\alpha}_1 \sum_{k=0}^{n} c_k^1 (u^{n+1-k} - u^{n-k}, v) \\
&\quad - \hat{\alpha}_2 \sum_{k=0}^{n} c_k^2 (\nabla u^{n+1-k} - \nabla u^{n-k}, \nabla v) - \gamma\hat{\beta}_n^1 (u^0, v) \\
&\quad - \hat{\beta}_n^2 (\nabla u^0, \nabla v) + \tau(R_t, v), \quad n = 0, 1, 2, \ldots, N-1.
\end{aligned}
\tag{10}
$$

The following properties can be proved directly.

Lemma 1 *The coefficients c_k^1 and c_k^2 defined by (9) satisfy that*

$$
\begin{aligned}
& c_k^1 > 0, \quad c_k^2 > 0, \quad j = 0, 1, 2, \ldots \\
& c_0^1 > c_1^1 > c_2^1 > \cdots > c_k^1 > 0, \ c_k^1 \to 0\,(k \to \infty); \\
& c_0^2 > c_1^2 > c_2^2 > \cdots > c_k^2 > 0, \ c_k^2 \to 0\,(k \to \infty).
\end{aligned}
\tag{11}
$$

Lemma 2 $\hat{\alpha}_1 = \frac{\tau^{\alpha_1}}{\Gamma(1+\alpha_1)}$, $\hat{\alpha}_2 = \frac{\tau^{\alpha_2}}{\Gamma(1+\alpha_2)}$, $\hat{\beta}_n^1 = \frac{\tau}{\Gamma(\alpha_1)[(n+\frac{1}{2})\tau]^{1-\alpha_1}}$, *and* $\hat{\beta}_n^2 = \frac{\tau}{\Gamma(\alpha_2)[(n+\frac{1}{2})\tau]^{1-\alpha_2}}$ *satisfy that*

$$\hat{\beta}_n^1 \le \hat{\alpha}_1 c_n^1, \quad \hat{\beta}_n^2 \le \hat{\alpha}_2 c_n^2, \quad n \ge 1. \tag{12}$$

Proof Firstly we prove that $\hat{\beta}_n^1 \le \hat{\alpha}_1 c_n^1$.

From the definition of c_n^1, we have

$$\begin{aligned}
&\hat{\alpha}_1 c_n^1 - \hat{\beta}_n^1 \\
&= \frac{\tau^{\alpha_1}}{\Gamma(1+\alpha_1)}\left[\left(n+\frac{1}{2}\right)^{\alpha_1} - \left(n-\frac{1}{2}\right)^{\alpha_1}\right] - \frac{\tau}{\Gamma(\alpha_1)\left[\left(n+\frac{1}{2}\right)\tau\right]^{1-\alpha_1}} \\
&= \frac{\tau^{\alpha_1}\left[\left(n+\frac{1}{2}\right)\tau\right]^{1-\alpha_1}\left[\left(n+\frac{1}{2}\right)^{\alpha_1} - \left(n-\frac{1}{2}\right)^{\alpha_1}\right] - \alpha_1\tau}{\Gamma(1+\alpha_1)\left[\left(n+\frac{1}{2}\right)\tau\right]^{1-\alpha_1}} \\
&= \tau\frac{\left(n+\frac{1}{2}\right)^{1-\alpha_1}\left[\left(n+\frac{1}{2}\right)^{\alpha_1} - \left(n-\frac{1}{2}\right)^{\alpha_1}\right] - \alpha_1}{\Gamma(1+\alpha_1)\left[\left(n+\frac{1}{2}\right)\tau\right]^{1-\alpha_1}}.
\end{aligned} \tag{13}$$

Considering the differential mean-value theorem, we know

$$\begin{aligned}
&\left(n+\frac{1}{2}\right)^{1-\alpha_1}\left[\left(n+\frac{1}{2}\right)^{\alpha_1} - \left(n-\frac{1}{2}\right)^{\alpha_1}\right] - \alpha_1 \\
&= \left(n+\frac{1}{2}\right)^{1-\alpha_1}\alpha_1\xi^{\alpha_1-1} - \alpha_1, \quad \xi \in \left(n-\frac{1}{2}, n+\frac{1}{2}\right) \\
&\ge \alpha_1 - \alpha_1 = 0.
\end{aligned} \tag{14}$$

After substituting the result of (14) to (13), we reach that $\hat{\alpha}_1 c_n^1 - \hat{\beta}_n^1 \ge 0$, which is $\hat{\beta}_n^1 \le \hat{\alpha}_1 c_n^1$.

Similarly, we can prove that $\hat{\beta}_n^2 \le \hat{\alpha}_2 c_n^2$. □

Now we consider approximating the variational problem in space. Let $\Im = \{\Im_h\}$ be a quasi-uniform partition of Ω. $\Im_h$ is a triangle of Ω, h_e is the diameter of $\Im_h$, and $h = \max\{h_e\}$. Let $S^h \subset H_0^1(\Omega)$ be a family of finite element space with the accuracy of order $r \ge 1$, i.e.,

$$S^h = \{\chi \in H_0^1(\Omega) \cap C(\Omega) : \chi|_{\Im_h} \in P_r, \ \forall \Im_h \in \Im\},$$

where P_r denotes the set of polynomials of degree up to r. The Crank–Nicolson type fully discrete finite element scheme can be stated as following: for $n \ge 1$, to find $u_h^{n+1} \in S^h$ such that $\forall v_h \in S^h$ satisfies

$$
\begin{aligned}
&\left(1+\gamma\hat{\alpha}_1 c_0^1\right)\left(u_h^{n+1}, v_h\right)+\hat{\alpha}_2 c_0^2\left(\nabla u_h^{n+1}, \nabla v_h\right) \\
&\quad=\left(u_h^n, v_h\right)+\gamma\hat{\alpha}_1 \sum_{k=0}^{n-1}\left(c_k^1-c_{k+1}^1\right)\left(u_h^{n-k}, v_h\right)+\gamma\left(\hat{\alpha}_1 c_n^1-\hat{\beta}_n^1\right)\left(u_h^0, v_h\right) \\
&\quad\quad+\hat{\alpha}_2 \sum_{k=0}^{n-1}\left(c_k^2-c_{k+1}^2\right)\left(\nabla u_h^{n-k}, \nabla v_h\right)+\left(\hat{\alpha}_2 c_n^2-\hat{\beta}_n^2\right)\left(\nabla u_h^0, \nabla v_h\right). \quad (15)
\end{aligned}
$$

When $n=0$, it equals that the two terms of sum in the above formulation don't appear, which means: to find $u_h^1 \in S^h$ such that for $\forall v_h \in S^h$ it holds

$$
\begin{aligned}
&(1+\gamma\hat{\alpha}_1 c_0^1)(u_h^1, v_h)+\hat{\alpha}_2 c_0^2(\nabla u_h^1, \nabla v_h) \\
&\quad=(u_h^0, v_h)+\gamma(\hat{\alpha}_1 c_0^1-\hat{\beta}_0^1)(u_h^0, v_h)+(\hat{\alpha}_2 c_0^2-\hat{\beta}_0^2)(\nabla u_h^0, \nabla v_h), \quad (16)
\end{aligned}
$$

here u_h^0 is the approximation to the original value $u_0(x)$.

3 Stability and error estimation for the fully discrete scheme

In this section, we will give the stability analysis and error estimation for the fully discrete scheme (15) and (16). Throughout the following C will denote positive constants, not necessarily the same at different occurrences, which is independent of the finite element mesh size h and time step size τ.

Theorem 1 *The fully discrete formulations (15) and (16) are unconditionally stable, and it holds that*

$$
\omega^n \le \|u_h^0\|^2+\frac{\gamma\tau^{\alpha_1}}{\Gamma(1+\alpha_1)}\|u_h^0\|^2+\frac{\tau^{\alpha_2}}{\Gamma(1+\alpha_2)}\|\nabla u_h^0\|^2, \quad n \ge 1, \quad (17)
$$

where $\omega^n=\|u_h^n\|^2+\gamma\hat{\alpha}_1\sum_{k=0}^{n-1} c_k^1\|u_h^{n-k}\|^2+\hat{\alpha}_2\sum_{k=0}^{n-1} c_k^2\|\nabla u_h^{n-k}\|^2.$

Proof When $n \ge 1$, taking $v_h=u_h^{n+1}$ in (15), we get

$$
\begin{aligned}
&\left(1+\gamma\hat{\alpha}_1 c_0^1\right)\|u_h^{n+1}\|^2+\hat{\alpha}_2 c_0^2\|\nabla u_h^{n+1}\|^2 \\
&\quad=\left(u_h^n, u_h^{n+1}\right)+\gamma\hat{\alpha}_1 \sum_{k=0}^{n-1}\left(c_k^1-c_{k+1}^1\right)\left(u_h^{n-k}, u_h^{n+1}\right)+\gamma\left(\hat{\alpha}_1 c_n^1-\hat{\beta}_n^1\right)\left(u_h^0, u_h^{n+1}\right) \\
&\quad\quad+\hat{\alpha}_2 \sum_{k=0}^{n-1}\left(c_k^2-c_{k+1}^2\right)\left(\nabla u_h^{n-k}, \nabla u_h^{n+1}\right)+\left(\hat{\alpha}_2 c_n^2-\hat{\beta}_n^2\right)\left(\nabla u_h^0, \nabla u_h^{n+1}\right). \quad (18)
\end{aligned}
$$

Noting (11) and (12), we can get the following by Hölder inequality and Young inequality

$$
\begin{aligned}
&\left(1+\gamma\hat{\alpha}_1 c_0^1\right)\|u_h^{n+1}\|^2+\hat{\alpha}_2 c_0^2\|\nabla u_h^{n+1}\|^2\\
&\le \frac{\|u_h^n\|^2}{2}+\frac{\|u_h^{n+1}\|^2}{2}+\gamma\hat{\alpha}_1\left[\sum_{k=0}^{n-1}\left(c_k^1-c_{k+1}^1\right)\frac{\|u_h^{n-k}\|^2}{2}+\left(c_0^1-c_n^1\right)\frac{\|u_h^{n+1}\|^2}{2}\right]\\
&\quad+\hat{\alpha}_2\left[\sum_{k=0}^{n-1}\left(c_k^2-c_{k+1}^2\right)\frac{\|\nabla u_h^{n-k}\|^2}{2}+\left(c_0^2-c_n^2\right)\frac{\|\nabla u_h^{n+1}\|^2}{2}\right]\\
&\quad+\gamma\left(\hat{\alpha}_1 c_n^1-\hat{\beta}_n^1\right)\left(\frac{\|u_h^0\|^2}{2}+\frac{\|u_h^{n+1}\|^2}{2}\right)\\
&\quad+\left(\hat{\alpha}_2 c_n^2-\hat{\beta}_n^2\right)\left(\frac{\|\nabla u_h^0\|^2}{2}+\frac{\|\nabla u_h^{n+1}\|^2}{2}\right).
\end{aligned}
\tag{19}
$$

Rearranging the terms, we obtain

$$
\begin{aligned}
&\|u_h^{n+1}\|^2+\gamma\hat{\alpha}_1\sum_{n=0}^{n} c_k^1\|u_h^{n+1-k}\|^2\\
&\quad+\hat{\alpha}_2\sum_{n=0}^{n} c_k^2\|\nabla u_h^{n+1-k}\|^2+\gamma\hat{\beta}_n^1\|u_h^{n+1}\|^2+\hat{\beta}_n^2\|\nabla u_h^{n+1}\|^2\\
&\le\|u_h^n\|^2+\gamma\hat{\alpha}_1\sum_{k=0}^{n-1} c_k^1\|u_h^{n-k}\|^2+\hat{\alpha}_2\sum_{k=0}^{n-1} c_k^2\|\nabla u_h^{n-k}\|^2.
\end{aligned}
\tag{20}
$$

Denoting $\omega^n=\|u_h^n\|^2+\gamma\hat{\alpha}_1\sum_{k=0}^{n-1} c_k^1\|u_h^{n-k}\|^2+\hat{\alpha}_2\sum_{k=0}^{n-1} c_k^2\|\nabla u_h^{n-k}\|^2$, the above formula becomes

$$
\omega^{n+1}\le\omega^n\le\omega^{n-1}\le\cdots\le\omega^1.
\tag{21}
$$

When $n=0$, taking $v_h=u_h^1$ in (15), we have

$$
\begin{aligned}
&\left(1+\gamma\hat{\alpha}_1 c_0^1\right)\|u_h^1\|^2+\hat{\alpha}_2 c_0^2\|\nabla u_h^1\|^2\\
&=\left(u_h^0,u_h^1\right)+\gamma\left(\hat{\alpha}_1 c_0^1-\hat{\beta}_0^1\right)\left(u_h^0,u_h^1\right)+\left(\hat{\alpha}_2 c_0^2-\hat{\beta}_0^2\right)\left(\nabla u_h^0,\nabla u_h^1\right)\\
&\le\frac{\|u_h^0\|^2}{2}+\frac{\|u_h^1\|^2}{2}+\gamma\frac{\left(\hat{\alpha}_1 c_0^1-\hat{\beta}_0^1\right)^2}{\hat{\alpha}_1 c_0^1}\frac{\|u_h^0\|^2}{2}+\gamma\hat{\alpha}_1 c_0^1\frac{\|u_h^1\|^2}{2}\\
&\quad+\frac{\left(\hat{\alpha}_2 c_0^2-\hat{\beta}_0^2\right)^2}{\hat{\alpha}_2 c_0^2}\frac{\|\nabla u_h^0\|^2}{2}+\hat{\alpha}_2 c_0^2\frac{\|\nabla u_h^1\|^2}{2},
\end{aligned}
\tag{22}
$$

where

$$\frac{\left(\hat{\alpha}_1 c_0^1 - \hat{\beta}_0^1\right)^2}{\hat{\alpha}_1 c_0^1} = \left(\frac{\tau^{\alpha_1}}{\Gamma(1+\alpha_1)2^{\alpha_1}} - \frac{\tau}{\Gamma(\alpha_1)(\frac{\tau}{2})^{1-\alpha_1}}\right)^2 \cdot \frac{\Gamma(1+\alpha_1)2^{\alpha_1}}{\tau^{\alpha_1}}$$
$$= \frac{\tau^{2\alpha_1}(1-2\alpha_1)^2}{(\Gamma(1+\alpha_1)2^{\alpha_1})^2} \cdot \frac{\Gamma(1+\alpha_1)2^{\alpha_1}}{\tau^{\alpha_1}}$$
$$= \frac{(1-2\alpha_1)^2}{\Gamma(1+\alpha_1)} \left(\frac{\tau}{2}\right)^{\alpha_1} \leq \frac{\tau^{\alpha_1}}{\Gamma(1+\alpha_1)},$$

and

$$\frac{(\hat{\alpha}_2 c_0^2 - \hat{\beta}_0^2)^2}{\hat{\alpha}_2 c_0^2} \leq \frac{\tau^{\alpha_2}}{\Gamma(1+\alpha_2)}.$$

Thus we obtain from (22) that

$$\omega^n \leq \omega^1 = (1+\gamma\hat{\alpha}_1 c_0^1)\|u_h^1\|^2 + \hat{\alpha}_2 c_0^2 \|\nabla u_h^1\|^2$$
$$\leq \|u_h^0\|^2 + \frac{\gamma\tau^{\alpha_1}}{\Gamma(1+\alpha_1)}\|u_h^0\|^2 + \frac{\tau^{\alpha_2}}{\Gamma(1+\alpha_2)}\|\nabla u_h^0\|^2. \tag{23}$$

This complete the proof of (17). □

Next, we define the Ritz projection operator $P^h : H_0^1(\Omega) \to S^h$ as

$$(\nabla P^h u, \nabla v) = (\nabla u, \nabla v), \quad \forall v \in S^h. \tag{24}$$

This projection operator P^h satisfies the following estimation (see [36])

$$\|P^h v - v\| + h\|\nabla(P^h v - v)\| \leq Ch^s \|v\|_s, \quad \forall v \in H^s \cap H_0^1, \; s \geq 1. \tag{25}$$

Lemma 3 *Suppose*

$$R_s^n = \frac{u(t_{n+1}) - u(t_n)}{\tau} - \frac{P^h u(t_{n+1}) - P^h u(t_n)}{\tau}, n \geq 0,$$

then it satisfies the following estimate

$$|R_s^n| \leq C\left(\tau^2 + h^{r+1}\right), \quad n \geq 0. \tag{26}$$

Proof For $n \geq 0$, by (5) and (25), we get

$$
\begin{aligned}
|R_s^n| &= \left| \frac{u(t_{n+1}) - u(t_n)}{\tau} - \frac{P^h u(t_{n+1}) - P^h u(t_n)}{\tau} \right| \\
&\leq \left| \frac{u(t_{n+1}) - u(t_n)}{\tau} - \frac{\partial u\left(t_{n+\frac{1}{2}}\right)}{\partial t} \right| + \left| \frac{\partial u\left(t_{n+\frac{1}{2}}\right)}{\partial t} - P^h \frac{\partial u\left(t_{n+\frac{1}{2}}\right)}{\partial t} \right| \\
&\quad + \left| P^h \frac{\partial u\left(t_{n+\frac{1}{2}}\right)}{\partial t} - \frac{P^h u(t_{n+1}) - P^h u(t_n)}{\tau} \right| \\
&\leq C(\tau^2 + h^{r+1}).
\end{aligned} \tag{27}
$$

□

Lemma 4 *Suppose*

$$
Q_s^n = \frac{\hat{\alpha}_1}{\tau} \sum_{k=0}^{n} c_k \left(u^{n+1-k} - u^{n-k} \right) - \frac{\hat{\alpha}_1}{\tau} \sum_{k=0}^{n} c_k \left(P^h u^{n+1-k} - P^h u^{n-k} \right), \ n \geq 0,
$$

then it satisfies the following estimate

$$
|Q_s^n| \leq C \left(\tau^{1+\alpha_1} + h^{r+1} \right), \ n \geq 0. \tag{28}
$$

Proof For $n \geq 0$, by (7) and (25), we obtain

$$
\begin{aligned}
|Q_s^n| &\leq \left| \frac{\hat{\alpha}_1}{\tau} \sum_{k=0}^{n} c_k \left(u^{n+1-k} - u^{n-k} \right) - {}_0D_t{}^{1-\alpha_1} u\left(t_{n+\frac{1}{2}}\right) \right| \\
&\quad + \left| {}_0D_t{}^{1-\alpha_1} u\left(t_{n+\frac{1}{2}}\right) - P^h {}_0D_t{}^{1-\alpha_1} u\left(t_{n+\frac{1}{2}}\right) \right| \\
&\quad + \left| P^h {}_0D_t{}^{1-\alpha_1} u\left(t_{n+\frac{1}{2}}\right) - \frac{\hat{\alpha}_1}{\tau} \sum_{k=0}^{n} c_k \left(P^h u^{n+1-k} - P^h u^{n-k} \right) \right| \\
&\leq C \left(\tau^{1+\alpha_1} + h^{r+1} \right).
\end{aligned} \tag{29}
$$

□

Theorem 2 *Assume $u^n = u(x, y, t_n)$ is the exact solution of (1), (2) and (3), u_h^n is the FE approximate solution of the fully discrete scheme (15) and (16). Then u_h^n satisfies the following error estimation*

$$
\max_{1 \leq n \leq N} \|u^n - u_h^n\| \leq C \sqrt{\frac{\Gamma(\alpha_1)}{\gamma}} T^{1-\frac{\alpha_1}{2}} (\tau^{\min\{1+\alpha_1, 1+\alpha_2\}} + h^{r+1}). \tag{30}
$$

Proof Splitting $e^n = u(t_n) - u_h^n = (u(t_n) - P^h u(t_n)) + (P^h u(t_n) - u_h^n) = \rho^n + \theta^n$, we only need to estimate $\|\theta^n\|$ by virtue of (25).

Noting that u^n satisfies (10), it holds that

$$\begin{aligned}&(1+\gamma\hat{\alpha}_1 c_0^1)(u^1, v) + \hat{\alpha}_2 c_0^2(\nabla u^1, \nabla v)\\&\quad = (u^0, v) + \gamma(\hat{\alpha}_1 c_0^1 - \hat{\beta}_0^1)(u^0, v) + (\hat{\alpha}_2 c_0^2 - \hat{\beta}_0^2)(\nabla u^0, \nabla v) + \tau(R_t, v). \qquad (31)\end{aligned}$$

Therefore, subtracting (16) from (31), then setting $v_h = \theta^1$, we get

$$\begin{aligned}&(1+\gamma\hat{\alpha}_1 c_0^1)\|\theta^1\|^2 + \hat{\alpha}_2 c_0^2\|\nabla\theta^1\|^2\\&\quad = (\theta^0, \theta^1) + \gamma(\hat{\alpha}_1 c_0^1 - \hat{\beta}_0^1)(\theta^0, \theta^1) + (\hat{\alpha}_2 c_0^2 - \hat{\beta}_0^2)(\nabla\theta^0, \nabla\theta^1)\\&\qquad + \tau(R_t + R_s^n + Q_s^n, \theta^1),\\&\quad \le \frac{\|\theta^1\|^2}{2} + \frac{\|\theta^0\|^2}{2} + \gamma(\hat{\alpha}_1 c_0^1 - \hat{\beta}_0^1)\left(\frac{\|\theta^0\|^2}{2} + \frac{\|\theta^1\|^2}{2}\right)\\&\qquad + (\hat{\alpha}_2 c_0^2 - \hat{\beta}_0^2)\left(\frac{\|\nabla\theta^0\|^2}{2} + \frac{\|\nabla\theta^1\|^2}{2}\right)\\&\qquad + \frac{\tau^2}{\gamma\hat{\beta}_0^1}\frac{\|R_t + R_s^n + Q_s^n\|^2}{2} + \gamma\hat{\beta}_0^1\frac{\|\theta^1\|^2}{2}. \qquad (32)\end{aligned}$$

Just take $u_h^0 = P^h u_0$ for simplicity, then $\theta^0 = 0$ and consequently,

$$\begin{aligned}(1+\gamma\hat{\alpha}_1 c_0^1)\|\theta^1\|^2 + \hat{\alpha}_2 c_0^2\|\nabla\theta^1\|^2 &\le \frac{\tau^2}{\gamma\hat{\beta}_0^1}\|R_t + R_s^n + Q_s^n\|^2\\&\le C\frac{\Gamma(\alpha_1)}{\gamma 2^{1-\alpha_1}}\tau^{2-\alpha_1}\left(\tau^{\min\{1+\alpha_1, 1+\alpha_2\}} + h^{r+1}\right)^2.\end{aligned} \qquad (33)$$

When $n \ge 1$, subtracting (10) by (15) and taking $v_h = \theta^{n+1}$, we get

$$\begin{aligned}&(1+\gamma\hat{\alpha}_1 c_0^1)\|\theta^{n+1}\|^2 + \hat{\alpha}_2 c_0^2\|\nabla\theta^{n+1}\|^2\\&\quad = (\theta^n, \theta^{n+1}) + \gamma\hat{\alpha}_1\sum_{k=0}^{n-1}(c_k^1 - c_{k+1}^1)(\theta^{n-k}, \theta^{n+1})\\&\qquad + \gamma(\hat{\alpha}_1 c_n^1 - \hat{\beta}_n^1)(\theta^0, \theta^{n+1}) + \hat{\alpha}_2\sum_{k=0}^{n-1}(c_k^2 - c_{k+1}^2)(\nabla\theta^{n-k}, \nabla\theta^{n+1})\\&\qquad + (\hat{\alpha}_2 c_n^2 - \hat{\beta}_n^2)(\nabla\theta^0, \nabla\theta^{n+1}) + \tau(R_t + R_s^n + Q_s^n, \theta^{n+1}). \qquad (34)\end{aligned}$$

In the above deduction, (24) is used. Using Hölder inequality, Young inequality, (12) and (13), then we have

$$
\begin{aligned}
&(1+\gamma\hat{\alpha}_1 c_0^1)\|\theta^{n+1}\|^2+\hat{\alpha}_2 c_0^2\|\nabla\theta^{n+1}\|^2\\
&\quad\le\|\theta^n\|^2+\gamma\hat{\alpha}_1\sum_{k=0}^{n-1}(c_k^1-c_{k+1}^1)\|\theta^{n-k}\|^2+\gamma(\hat{\alpha}_1 c_n^1-\hat{\beta}_n^1)\|\theta^0\|^2\\
&\qquad-\gamma\hat{\beta}_n^1\|\theta^{n+1}\|^2-\hat{\beta}_n^2\|\nabla\theta^{n+1}\|^2+\hat{\alpha}_2\sum_{k=0}^{n-1}(c_k^2-c_{k+1}^2)\|\nabla\theta^{n-k}\|^2\\
&\qquad+(\hat{\alpha}_2 c_n^2-\hat{\beta}_n^2)\|\nabla\theta^0\|^2+2\tau|(R_t+R_s^n+Q_s^n,\theta^{n+1})|. \qquad (35)
\end{aligned}
$$

Since we have taken $u_h^0=P^h u_0, \theta^0=0$ and $\nabla\theta^0=0$ are used in the above derivation process. After transposition, it holds

$$
\begin{aligned}
&\|\theta^{n+1}\|^2+\gamma\hat{\alpha}_1\sum_{n=0}^{n+1}c_k^1\|\theta^{n+1-k}\|^2+\hat{\alpha}_2\sum_{n=0}^{n+1}c_k^2\|\nabla\theta^{n+1-k}\|^2\\
&\quad\le\|\theta^n\|^2+\gamma\hat{\alpha}_1\sum_{k=0}^{n}c_k^1\|\theta^{n-k}\|^2+\hat{\alpha}_2\sum_{k=0}^{n}c_k^2\|\nabla\theta^{n-k}\|^2\\
&\qquad-\gamma\hat{\beta}_n^1\|\theta^{n+1}\|^2+2\tau|(R_t+R_s^n+Q_s^n,\theta^{n+1})|\\
&\quad\le\|\theta^n\|^2+\gamma\hat{\alpha}_1\sum_{k=0}^{n}c_k^1\|\theta^{n-k}\|^2+\hat{\alpha}_2\sum_{k=0}^{n}c_k^2\|\nabla\theta^{n-k}\|^2\\
&\qquad+\frac{1}{\gamma\hat{\beta}_n^1}\tau^2\|R_t+R_s^n+Q_s^n\|^2. \qquad (36)
\end{aligned}
$$

Denotes $\eta^n=\|\theta^n\|^2+\gamma\hat{\alpha}_1\sum_{k=0}^{n}c_k\|\theta^{n-k}\|^2+\hat{\alpha}_2\sum_{k=0}^{n}d_k\|\nabla\theta^{n-k}\|^2$, and the above formula becomes

$$
\begin{aligned}
\eta^{n+1}&\le\eta^n+\frac{C}{\gamma\hat{\beta}_n^1}\tau^2\left(\tau^{\min\{1+\alpha_1,1+\alpha_2\}}+h^{r+1}\right)^2\le\cdots\\
&\le\eta^1+\frac{C}{\gamma}\left(\frac{1}{\hat{\beta}_n^1}+\frac{1}{\hat{\beta_{n-1}^1}}+\cdots+\frac{1}{\hat{\beta}_1^1}\right)\tau^2\left(\tau^{\min\{1+\alpha_1,1+\alpha_2\}}+h^{r+1}\right)^2\\
&\le\eta^1+\frac{C\Gamma(\alpha_1)}{\gamma}\left\{\left(n+\frac{1}{2}\right)^{1-\alpha_1}+\left(n-\frac{1}{2}\right)^{1-\alpha_1}+\cdots\right.\\
&\qquad\left.+\left(\frac{3}{2}\right)^{1-\alpha_1}\right\}\tau^{2-\alpha_1}\left(\tau^{\min\{1+\alpha_1,1+\alpha_2\}}+h^{r+1}\right)^2\\
&\le\eta^1+\frac{C\Gamma(\alpha_1)}{\gamma}n\left(n+\frac{1}{2}\right)^{1-\alpha_1}\tau^{2-\alpha_1}\left(\tau^{\min\{1+\alpha_1,1+\alpha_2\}}+h^{r+1}\right)^2\\
&\le\eta^1+\frac{C\Gamma(\alpha_1)}{\gamma}T^{2-\alpha_1}\left(\tau^{\min\{1+\alpha_1,1+\alpha_2\}}+h^{r+1}\right)^2. \qquad (37)
\end{aligned}
$$

Substitute (33) to the above inequality, we obtain

$$\begin{aligned}\eta^{n+1} &\leq C\frac{\Gamma(\alpha_1)}{\gamma 2^{1-\alpha_1}}\tau^{2-\alpha_1}\left(\tau^{\min\{1+\alpha_1,1+\alpha_2\}}+h^{r+1}\right)^2\\ &\quad+\frac{C\Gamma(\alpha_1)}{\gamma}T^{2-\alpha_1}\left(\tau^{\min\{1+\alpha_1,1+\alpha_2\}}+h^{r+1}\right)^2\\ &\leq C\frac{\Gamma(\alpha_1)T^{2-\alpha_1}}{\gamma}\left(\tau^{\min\{1+\alpha_1,1+\alpha_2\}}+h^{r+1}\right)^2, \end{aligned} \tag{38}$$

so

$$\begin{aligned}\|e^{n+1}\| &= \|\theta^{n+1}+\rho^{n+1}\| \leq \|\theta^{n+1}\|+\|\rho^n\|\\ &\leq C\sqrt{\frac{\Gamma(\alpha_1)}{\gamma}}T^{1-\frac{\alpha_1}{2}}\left(\tau^{\min\{1+\alpha_1,1+\alpha_2\}}+h^{r+1}\right). \end{aligned} \tag{39}$$

Thus (30) is proved. □

4 Numerical examples

In this section, we show two numerical examples to verify and illustrate the theoretical results of finite element approximation for the fractional Cable equation. Now we compute some numerical data based on one-dimensional case and two-dimensional case, respectively.

4.1 One-dimensional example

Based on the fully discrete scheme (15) and (16), we first check the convergence rate by the example of one-dimensional space. Therefore, we consider the Cable equation with a forcing term f:

$$u_t = -\gamma\, {}_0D_t^{1-\alpha_1}u + {}_0D_t^{1-\alpha_2}u_{xx} + f(x,t), \quad x\in[0,1],\ t\in[0,T]; \tag{40}$$

Considering the following exact solution

$$u(x,t) = t^2\sin(2\pi x),$$

the associated forcing term is

$$f(x,t) = 2\left(t+\frac{\gamma}{\Gamma(2+\alpha_1)}t^{1+\alpha_1}+\frac{4\pi^2}{\Gamma(2+\alpha_2)}t^{1+\alpha_2}\right)\sin(2\pi x).$$

Let S^h be the space of continuous piecewise linear functions on $\Im$. We first make the step of space $h = 1/1000$ to be small enough to test the convergence order in time. The value of γ is taken as $\gamma = 1$ and $\gamma = 1000$, respectively. From the results

Table 1 The convergence results in time when $\gamma = 1,\ h = 1/1000$ at $T = 1$

τ	$\frac{\alpha_1=0.9}{\alpha_2=0.1}$	Rate	$\frac{\alpha_1=0.5}{\alpha_2=0.5}$	Rate	$\frac{\alpha_1=0.2}{\alpha_2=0.6}$	Rate
1/8	4.8713e−003	–	6.4240e−003	–	5.5563e−003	–
1/16	2.1492e−003	1.1805	2.1061e−003	1.6089	1.7146e−003	1.6962
1/32	9.6707e−004	1.1521	7.0170e−004	1.5857	5.3830e−004	1.6714
1/64	4.4111e−004	1.1325	2.3699e−004	1.5660	1.7185e−004	1.6472
1/128	2.0298e−004	1.1198	8.0895e−005	1.5507	5.5735e−005	1.6245
TCRs	–	1.1000	–	1.5000	–	1.2000

Table 2 The convergence results in time when $\gamma = 1000,\ h = 1/1000$ at $T = 1$

τ	$\frac{\alpha_1=0.9}{\alpha_2=0.1}$	Rate	$\frac{\alpha_1=0.5}{\alpha_2=0.5}$	Rate	$\frac{\alpha_1=0.2}{\alpha_2=0.6}$	Rate
1/8	4.0640e−003	–	6.5472e−003	–	7.1902e−003	–
1/16	9.0639e−004	2.1647	2.1457e−003	1.6094	2.9373e−003	1.2915
1/32	2.4907e−004	1.8636	7.1371e−004	1.5881	1.2229e−003	1.2642
1/64	7.0563e−005	1.8196	2.3972e−004	1.5740	5.1586e−004	1.2452
1/128	2.0828e−005	1.7604	8.0462e−005	1.5750	2.1909e−004	1.2354
TCRs	–	1.1000	–	1.5000	–	1.2000

Table 3 The convergence results in space when $\gamma = 1000,\ \tau = 1/1000$ at $T = 1$

h	$\frac{\alpha_1=0.9}{\alpha_2=0.1}$	Rate	$\frac{\alpha_1=0.5}{\alpha_2=0.5}$	Rate	$\frac{\alpha_1=0.2}{\alpha_2=0.6}$	Rate
1/8	3.4089e−002	–	3.5260e−002	–	3.5715e−002	–
1/16	8.4840e−003	2.0065	8.7609e−003	2.0089	8.8579e−003	2.0115
1/32	2.1170e−003	2.0027	2.1834e−003	2.0045	2.1963e−003	2.0119
1/64	5.2819e−004	2.0029	5.4291e−004	2.0078	5.3503e−004	2.0374
TCRs	–	2.0000	–	2.0000	–	2.0000

listed in Tables 1 and 2, we can see that the value of γ affects the convergence order in time, which also verifies that the error results is related to the constant γ. Taking $\gamma = 1000$, $\tau = 1/1000$ for the purpose of testing the convergence order in space, we find from Table 3 that the convergence rates, which bring into correspondence with the theoretical convergence results (TCRs), almost reach to the second order for different values of α_1 and α_2.

4.2 Two-dimensional example

For validating the correctness of convergence results for the case in two-dimensional space, we need to consider the following two-dimensional problem with an exact solution

Table 4 The convergence results at $T = 1$

$\tau = h$	$\frac{\alpha_1=0.9}{\alpha_2=0.1}$	Rate	$\frac{\alpha_1=0.5}{\alpha_2=0.5}$	Rate	$\frac{\alpha_1=0.2}{\alpha_2=0.6}$	Rate
1/8	2.0988e−002	–	2.1333e−002	–	2.0543e−002	–
1/16	5.8736e−003	1.8373	5.6993e−003	1.9042	5.3969e−003	1.9284
1/32	1.7458e−003	1.7503	1.5408e−003	1.8871	1.4284e−003	1.9177
1/64	5.6682e−004	1.6229	4.2554e−004	1.8563	3.8430e−004	1.8941
TCRs	–	1.1000	–	1.5000	–	1.2000

$$\begin{cases} \frac{\partial u}{\partial t} = -{}_0D_t^{1-\alpha_1} u + {}_0D_t^{1-\alpha_2} \Delta u + f(x, y, t), & (x, y) \in \Omega,\ t \in [0, T]; \\ u(x, y, t) = 0, & (x, y) \in \partial\Omega,\ t \in [0, T]; \\ u(x, y, 0) = 0, & (x, y) \in \Omega, \end{cases} \tag{41}$$

where the source term is chosen as

$$f(x, y, t) = 2\left(t + \frac{1}{\Gamma(2+\alpha_1)} t^{1+\alpha_1} + \frac{2\pi^2}{\Gamma(2+\alpha_2)} t^{1+\alpha_2}\right) \sin(\pi x)\sin(\pi y),$$

which results in an exact solution of (41)

$$u(x, y, t) = t^2 \sin(\pi x)\sin(\pi y).$$

Now we take the isosceles right triangle mesh for the spatial domain $\Omega = [0, 1] \times [0, 1]$ and choose the continuous piecewise linear space to get the convergence results. In Table 4, we give some numerical results of errors and orders of convergence with different space-time mesh $\tau = h = 1/8, 1/16/1/32, 1/64$ and changed values of parameters α_1, α_2. From the results in Table 4, we easily find that on every case the rate of convergence, which are inclined to the spatial convergence order, is higher than the theoretical convergence results (TCRs) $\min\{1 + \alpha_1, 1 + \alpha_2\}$. This is due to the dominant position of space in our numerical example of two-dimensional space.

In view of the discussion on the numerical results for two examples in one and two dimensional spaces, we claim that the theoretical results derived in this paper are correct.

Acknowledgments Authors would like to thank three anonymous reviewers for their valuable comments and suggestions. This work is supported by National Natural Science Fund (11361035, 11361034, 11301258), Natural Science Fund of Inner Mongolia Autonomous Region (2012M-S0108, 2012MS0106).

References

1. Atangana, A., Baleanu, D.: Numerical solution of a kind of fractional parabolic equations via two difference schemes. Abstr. Appl. Anal. 2013 Article ID 828764, 8 pages (2013)
2. Băleanu, D., Bhrawy, A.H., Taha, T.M.: Two efficient generalized Laguerre spectral algorithms for fractional initial value problems. Abstr. Appl. Anal. 2013 Article ID 546502, 10 pages (2013)
3. Băleanu, D., Diethelm, K., Scalas, E., Trujillo, J.J.: Fractional Calculus Models and Numerical Methods. Series on Complexity. Nonlinearity and Chaos. World Scientific, Boston (2012)

4. Bhrawy, A.H., Zaky, M.A.: Numerical simulation for two-dimensional variable-order fractional nonlinear cable equation. Nonlinear Dynam. **80**(1–2), 101–116 (2015)
5. Bisquert, J.: Fractional diffusion in the multipletrapping regime and revision of the equivalence with the continuous-time random walk. Phys. Rev. Lett. **91**(1), 010602(4) (2003)
6. Bu, W.P., Tang, Y.F., Yang, J.Y.: Galerkin finite element method for two-dimensional Riesz space fractional diffusion equations. J. Comput. Phys. **276**, 26–38 (2014)
7. Deng, W.H., Du, S.D., Wu, Y.J.: High order finite difference WENO schemes for fractional differential equations. Appl. Math. Lett. **26**(3), 362–366 (2013)
8. Deng, W.H.: Finite element method for the space and time fractional Fokker-Planck equation. SIAM J. Numer. Anal. **47**(1), 204–226 (2008)
9. Deng, W.H., Hesthaven, J.S.: Local discontinuous Galerkin methods for fractional ordinary differential equations. BIT 1–19 (2014). doi:10.1007/s10543-014-0531-z
10. Ding, H.F., Li, C.P.: High-order compact difference schemes for the modified anomalous subdiffusion equation. arXiv preprint arXiv:1408.5591 (2014)
11. Ding, H.F., Li, C.P.: High-order numerical methods for Riesz space fractional turbulent diffusion equation. arXiv preprint arXiv:1409.7464 (2014)
12. Henry, B., Langlands, T.A.M.: Fractional cable models for spiny neuronal dendrites. Phys. Rev. Lett. **100**, 128103 (2008)
13. Huang, J.F., Tang, Y.F., Vzquez, L., Yang, J.Y.: Two finite difference schemes for time fractional diffusion-wave equation. Numer. Algorithms **64**(4), 707–720 (2013)
14. Hu, X., Zhang, L.: Implicit compact difference schemes for the fractional cable equation. Appl. Math. Model. **36**(9), 4027–4043 (2012)
15. Jiang, Y.J., Ma, J.T.: High-order finite element methods for time-fractional partial differential equations. J. Comput. Appl. Math. **235**(11), 3285–3290 (2011)
16. Jiang, Y.J., Ma, J.T.: Moving finite element methods for time fractional partial differential equations. Sci. China Math. **56**(6), 1287–1300 (2013)
17. Ji, C.C., Sun, Z.Z., Hao, Z.P.: Numerical algorithms with high spatial accuracy for the fourth-order fractional sub-diffusion equations with the first Dirichlet boundary conditions. J. Sci. Comput. 1–27 (2015). doi:10.1007/s10915-015-0059-7
18. Jin, B.T., Lazarov, R., Zhou, Z.: Error estimates for a semidiscrete finite element method for fractional order parabolic equations. SIAM J. Numer. Anal. **51**(1), 445–466 (2013)
19. Langlands, T.A.M., Henry, B., Wearne, S.: Fractional cable equation models for anomalous electrodiffusion in nerve cells: infinite domain solutions. J. Math. Biol. **59**(6), 761–808 (2009)
20. Langlands, T.A.M., Henry, B., Wearne, S.: Solution of a fractional cable equation: Finite case. Preprint, Submitted to Elsevier Science http://www.maths.unsw.edu.au/applied/filed/2005/amr05-33 (2005)
21. Langlands, T.A.M., Henry, B., Wearne, S.: Solution of a fractional cable equation: Infinite case. (2005)
22. Li, C.P., Zhao, Z.G., Chen, Y.Q.: Numerical approximation of nonlinear fractional differential equations with subdiffusion and superdiffusion. Comput. Math. Appl. **62**(3), 855–875 (2011)
23. Lin, R., Liu, F., Anh, V., Turner, I.: Stability and convergence of a new explicit finite-difference approximation for the variable-order nonlinear fractional diffusion equation. Appl. Math. Comput. **212**(2), 435–445 (2009)
24. Lin, Y.M., Li, X.J., Xu, C.J.: Finite difference/spectral approximations for the fractional cable equation. Math. Comput. **80**(275), 1369–1396 (2011)
25. Lin, Y.M., Xu, C.J.: Finite difference/spectral approximations for the time-fractional diffusion equation. J. Comput. Phys. **225**(2), 1533–1552 (2007)
26. Liu, F., Zhuang, P., Anh, V., Turner, I., Burrage, K.: Stability and convergence of the difference methods for the space-time fractional advection-diffusion equation. Appl. Math. Comput. **191**(1), 12–20 (2007)
27. Liu, Y., Du, Y.W., Li, H., Wang, J.F.: An H^1-Galerkin mixed finite element method for time fractional reaction-diffusion equation. J. Appl. Math. Comput. **47**(1–2), 103–117 (2015)
28. Liu, Y., Fang, Z.C., Li, H., He, S.: A mixed finite element method for a time-fractional fourth-order partial differential equation. Appl. Math. Comput. **243**, 703–717 (2014)
29. Liu, Y., Li, H., Gao, W., He, S., Fang, Z.C.: A new mixed element method for a class of time-fractional partial differential equations. The Scientific World Journal 2014 Article ID 141467 (2014)
30. Meerschaert, M.M., Tadjeran, C.: Finite difference approximations for fractional advection-dispersion flow equations. J. Comput. Appl. Math. **172**(1), 65–77 (2004)
31. Quintana-Murillo, J., Yuste, S.B.: A finite difference method with non-uniform timesteps for fractional diffusion and diffusion-wave equations. Eur. Phys. J. Spec. Top. **222**(8), 1987–1998 (2013)

32. Quintana-Murillo, J., Yuste, S.B.: An explicit numerical method for the fractional cable equation. Int. J. Differ. Equ. 2011 Article ID 231920 (2011)
33. Shen, S., Liu, F., Anh, V., Turner, I., Chen, J.: A characteristic difference method for the variable-order fractional advection-diffusion equation. J. Appl. Math. Comput. **42**(1–2), 371–386 (2013)
34. Sousa, E.: A second order explicit finite difference method for the fractional advection diffusion equation. Comput. Math. Appl. **64**(10), 3141–3152 (2012)
35. Sun, Z.Z., Wu, X.N.: A fully discrete difference scheme for a diffusion-wave system. Appl. Numer. Math. **56**(2), 193–209 (2006)
36. Thomée, V.: Galerkin Finite Element Methods for Parabolic Problems. Springer, Berlin (1997)
37. Wang, H., Wang, K., Sircar, T.: A direct O($Nlog^2N$) finite difference method for fractional diffusion equations. J. Comput. Phys. **229**(21), 8095–8104 (2010)
38. Wang, H., Yang, D.P., Zhu, S.F.: Inhomogeneous Dirichlet boundary-value problems of space-fractional diffusion equations and their finite element approximations. SIAM J. Numer. Anal. **52**(3), 1292–1310 (2014)
39. Wang, Z.B., Vong, S.W.: Compact difference schemes for the modified anomalous fractional sub-diffusion equation and the fractional diffusion-wave equation. J. Comput. Phys. **277**, 1–15 (2014)
40. Wei, L.L., He, Y.N., Zhang, X.D., Wang, S.L.: Analysis of an implicit fully discrete local discontinuous Galerkin method for the time-fractional Schrödinger equation. Finite Elem. Anal. Des. **59**, 28–34 (2012)
41. Zeng, F.H., Liu, F.W., Li, C.P., Burrage, K., Turner, I., Anh, V.: A Crank–Nicolson ADI spectral method for a two-dimensional Riesz space fractional nonlinear reaction-diffusion equation. SIAM J. Numer. Anal. **52**(6), 2599–2622 (2014)
42. Zhang, H., Liu, F., Anh, V.: Galerkin finite element approximation of symmetric space-fractional partial differential equations. Appl. Math. Comput. **217**(6), 2534–2545 (2010)
43. Zhang, N., Deng, W.H., Wu, Y.J.: Finite difference/element method for a two-dimensional modified fractional diffusion equation. Adv. Appl. Math. Mech. **4**(4), 496–518 (2012)
44. Zhang, X.D., Liu, J., Wei, L.L., Ma, C.X.: Finite element method for Grwünwald-Letnikov time-fractional partial differential equation. Appl. Anal. **92**(10), 2103–2114 (2013)
45. Zhang, Y.N., Sun, Z.Z., Liao, H.L.: Finite difference methods for the time fractional diffusion equation on non-uniform meshes. J. Comput. Phys. **265**, 195–210 (2014)
46. Zhao, Z.G., Li, C.P.: Fractional difference/finite element approximations for the time-space fractional telegraph equation. Appl. Math. Comput. **219**(6), 2975–2988 (2012)
47. Zheng, Y.Y., Li, C.P., Zhao, Z.G.: A note on the finite element method for the space-fractional advection diffusion equation. Comput. Math. Appl. **59**(5), 1718–1726 (2010)
48. Zhuang, P., Liu, F., Anh, V., Turner, I.: Numerical methods for the variable-order fractional advection diffusion equation with a nonlinear source term. SIAM J. Numer. Anal. **47**(3), 1760–1781 (2009)
49. Zhuang, P., Liu, F., Anh, V., Turner, I.: The Galerkin finite element approximation of the fractional cable equation. In: The Proceedings of the fifth symposium on fractional Differentiation and Its Applications, Hohai University, Nanjing 1–8 (2012)