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ORIGINAL RESEARCH

Asymptotic behavior for a weak viscoelastic wave equations with a dynamic boundary and time varying delay term

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Abstract In this paper, we consider the weak viscoelastic wave equations with dynamic boundary conditions related to the Kelvin Voigt damping and delay term acting on the boundary in a bounded domain. Under appropriate conditions on μ_1 and μ_2, we prove the asymptotic behavior by making use an appropriate Lyapunov functional.

Keywords Asymptotic behavior · Source term · Kelvin Voigt damping

Mathematics Subject Classification 35L60 · b5K55 · 26A33 · 35B44 · 35B33

1 Introduction

In this paper, we consider the following wave equation with dynamic boundary conditions and time varying delay term:

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$$\begin{cases} u_{tt} - \Delta u - \delta \Delta u_t - \alpha(t) \int_0^t g(t-s)\Delta u(s) ds = |u|^{p-2} u, & in \quad \Omega \times (0, +\infty), \\ u = 0, & on \quad \Gamma_0 \times (0, +\infty), \\ u_{tt} = -a \Big[\frac{\partial u}{\partial \upsilon}(x,t) + \delta \frac{\partial u_t}{\partial \upsilon}(x,t) + \alpha(t) \int_0^t g(t-s)\Delta u(s) \frac{\partial u}{\partial \upsilon}(x,s) ds & \\ \qquad + \mu_1 u_t(x,t) + \mu_2 u_t(x, t-\tau(t)) \Big], & on \quad \Gamma_1 \times (0, +\infty), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & x \in \Omega, \\ u_t(x, t-\tau(t)) = f_0(x, t-\tau(t)), & on \quad \Gamma_1 \times (0, +\infty) \end{cases} \tag{1}$$

where $u = u(x,t)$, $t \geq 0$, $x \in \Omega$, Δ denotes the Laplacian operator with respect to the x variable, Ω is a regular and bounded domain of R^N, $(N \geq 1), \partial\Omega = \Gamma_1 \cup \Gamma_0$, $\Gamma_1 \cap \Gamma_0 = \emptyset$ and $\frac{\partial}{\partial \upsilon}$ denotes the unit outer normal derivative, μ_1 and μ_2 are positive constants. Moreover, $\tau(t) > 0$ represents the time varying delay term and u_0, u_1, f_0 are given functions belonging to suitable spaces that will be precised later. This type of problems arises (for example) in modeling of longitudinal vibrations in a homogeneous bar on which there are viscous effects. The term Δu_t, indicates that the stress is proportional not only to the strain, but also to the strain rate see [5]. This type of problem without delay (i.e $\mu_i = 0$), has been considered by many authors during the past decades and many results have been obtained (see [2,4,6,7,13,32–36]). The main difficulty of the problem considered is related to the non ordinary boundary conditions defined on Γ_1. Very little attention has been paid to this type of boundary conditions. We mention a few particular results in the one dimensional without delay term for a linear damping ($m = 1$) and $g = 0$ see [9–23,32]. From the mathematical point of view, these problems do not neglect acceleration terms on the boundary. Such types of boundary conditions are usually called dynamic boundary conditions. They are not only important from the theoretical point of view but also arise in several physical applications. For instance in one space dimension, problem (1) can modelize the dynamic evolution of a viscoelastic rod that is fixed at one end and has a tip mass attached to its free end. The dynamic boundary conditions represent the Newton's law for the attached mass, (see [1,4,6]) for more details. Which arise when we consider the transverse motion of a flexible membrane whose boundary may be affected by the vibrations only in a region. Also some of them as in problem (1) appear when we assume that is an exterior domain of R^3 in which homogeneous fluid is at rest except for sound waves. Each point of the boundary is subjected to small normal displacements into the obstacle (see [2] for more details). Among the early results dealing with the dynamic boundary conditions are those of Grobbelaar-Van Dalsen [7,8] in which the author has made contributions to this field and in Gerbi and Said-Houari [13] the authors have studied the following problem:

$$\begin{cases} u_{tt} - \Delta u + \delta \Delta u_t = |u|^{p-1}u, & in \quad \Omega \times (0, +\infty), \\ u = 0, & on \quad \Gamma_0 \times (0, +\infty), \\ u_{tt} = -a\left[\frac{\partial u}{\partial \upsilon}(x,t) + \delta \frac{\partial u_t}{\partial \upsilon}(x,t) + \alpha |u_t|^{m-1} u_t(x,t)\right], & on \quad \Gamma_1 \times (0, +\infty), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & x \in \Omega, \end{cases}$$

and they have obtained several results concerning local existence which extended to the global existence by using stable sets, the authors have obtained also the energy decay and the blow up of the solutions for initial energy positive.

In absence of delay ($\mu_2 = 0$), the problem of existence and energy decay have been extensively studied by several authors (see [2,4,6,7,13,32–36]) and many energy estimates have been derived for arbitrary growing feedbacks (polynomial, exponential or logarithmic decay). Very recently the authors in [31] studied the following problem:

$$\begin{cases} u_{tt} - \Delta u + b(x) + f(u) = 0, & in \quad \Omega \times (0, +\infty), \\ u(x,t) = 0, & on \quad \Gamma_0 \times (0, +\infty), \\ \frac{\partial u}{\partial \nu} + g(u_t(x,t)) = 0, & on \quad \Gamma_1 \times (0, +\infty), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), \ x \in \Omega, \end{cases}$$

they proved the existence, uniqueness and uniform stability of strong and weak solutions of the nonlinear wave equation in bounded domains with nonlinear damped boundary conditions with restrictions on function $f(u)$; $g(u_t)$ and $b(x)$. They proved the existence by means of the Galerkin method and obtain the asymptotic behavior by using perturbed energy method and combining some ideas of Kmornik and Zuazua (see [37]).

It is widely known that delay effects, which arise in many practical problems, source of some instabilities, in this way Datko and Nicaise [10,12,21] showed that a small delay in a boundary control turns a well-behave hyperbolic system into a wild one which in turn, becomes a source of instability, where they proved that the energy is exponentially stable under the condition

$$\mu_2 < \mu_1. \tag{2}$$

Recently, inspired by the works of Al and Nicaise [11], Sthéphane Gherbi and Said-Houari [15] considered the following problem in bounded domain:

$$\begin{cases} u_{tt} - \Delta u - \alpha \Delta u_t = 0, & in \quad \Omega \times (0, +\infty), \\ u = 0, & on \quad \Gamma_0 \times (0, +\infty), \\ u_{tt} = -a\Big[\frac{\partial u}{\partial \upsilon}(x,t) + \alpha \frac{\partial u_t}{\partial \upsilon}(x,t) & \\ \qquad + \mu_1 u_t(x,t) + \mu_2 u_t(x, t-\tau)\Big], & on \quad \Gamma_1 \times (0, +\infty), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & x \in \Omega, \\ u_t(x, t-\tau) = f_0(x, t-\tau), & on \quad \Gamma_1 \times (0, +\infty), \end{cases}$$

and obtained several results concerning global existence and exponential decay rates for various signs of μ_1,μ_2.

The case of time varying delay in the wave equation has been studied recently by Nicaise, Valein and Fridman [11] in one-space dimension and in the linear case in problem (1) and proved an exponential stability result under the condition

$$\mu_2 < \sqrt{1-d}\mu_1,$$

where the constant d satisfies

$$\tau'(t) \leq d < 1, \quad \forall t > 0.$$

Nicaise et al. [12] extended the above result to higher-space dimension and established an exponential decay.

Very recently, Zhang et al. [30], have studied a more general model than the above one, namely

$$\begin{cases} u_{tt} - \Delta u + \int_0^t h(t-s)ds + au_t(x, t-\tau(t)) = 0, \; in & \Omega \times (0, +\infty), \\ u(x,t) = 0, & on \quad \Gamma_0 \times (0, +\infty), \\ \frac{\partial u}{\partial \nu} + g(u_t(x,t)) = 0, & on \quad \Gamma_1 \times (0, +\infty), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & x \in \Omega, \\ u_t(x, t-\tau(t)) = f_0(x, t-\tau(t)), & on \quad \Gamma_1 \times (0, +\infty). \end{cases}$$

Since it contains nonlinear term in the boundary. They investigated a nonlinear viscoelastic equation with interior time-varying delay and nonlinear dissipative boundary feedback. Under suitable assumptions on the relaxation function and time-varying delay effect together with nonlinear dissipative boundary feedback, they proved the global existence of weak solutions and asymptotic behavior of the energy by using the Faedo-Galerkin method and the perturbed energy method, respectively. This result improves earlier ones in the literature, such as Kirane and Said-Houari [38].

Motivated by the previous works, it is interesting to investigate the rate of decay of solutions by using an appropriate Lyapunov functional precisely, we show that the decay rate of energy function is exponential depending on both functions $\sigma(t)$ and $\alpha(t)$ that will be precised later.

The plan of this paper is organized as follows. In Sect. 2, we provide assumptions that will be needed for our work. In Sect. 3, we prove stability result that is given in Theorem 2.

2 Preliminary results

In this section, we present some material for the proof of our result. For the relaxation function g, α and σ we assume
(A_0) $g, \alpha : R_+ \to R_+$ are nonincreasing differentiable functions satisfying

$$g(0) > 0,\ l_0 = \int_0^\infty g(s)ds < \infty,\ \alpha(t) > 0,$$
$$1 - \alpha(t)\int_0^t g(s)ds = l > 0 \quad \text{for} \quad t > 0, \tag{3}$$

there exists a nonincreasing differentiable function $\sigma : R^+ \to R^+$ satisfying

$$g'(t) \le -\sigma(t)g(t), \quad \sigma(t) > 0, \quad \text{for} \quad t > 0,\ \lim_{t\to\infty} \frac{-\alpha'(t)}{\sigma(t)\alpha(t)} = 0.$$

(A_1) τ is a function such that

$$\tau \in W^{2,\infty}([0, T]), \quad \forall\, T > 0, \tag{4}$$
$$0 < \tau_0 \le \tau(t) \le \tau_1, \quad \forall\, t > 0, \tag{5}$$
$$\tau'(t) \le d < 1, \quad \forall\, t > 0, \tag{6}$$

where τ_0 and τ_1 are two positive constants.
(A_2)

$$\mu_2 < \sqrt{1-d}\,\mu_1. \tag{7}$$

As in [17] we choose ξ such that

$$\frac{\mu_2}{\sqrt{1-d}} < \xi < 2\mu_1 - \frac{\mu_2}{\sqrt{1-d}}. \tag{8}$$

As in [16] we denote

$$V = \left\{ \upsilon \in H_0^1(\Omega) : \upsilon = 0 \quad on \quad \Gamma_0 \right\} = H^1_{\Gamma_0}(\Omega),$$

we denote $\langle .,. \rangle$ the scalar product in $L^2(\Omega)$ i.e. $\langle u, \upsilon\rangle = \int_\Omega u(x,t)\upsilon(x,t)dx$. Also we mean by $\|.\|_q$ the $L^q(\Omega)$ norm for $1 \le q \le \infty$, and by $\|.\|_{q.\Gamma_1}$ the $L^q(\Gamma_1)$ norm.

Let $T > 0$ be a real number and X a Banach space endowed with the norm $\|.\|_X$. $L^p(0, T; X)$, $1 \le p < \infty$ denotes the space of functions f which are L^p over $(0, T)$ with values in X, which are measurable and $f \in L^p(0, T; X)$. This space is a Banach space endowed with the norm

$$\|f\|_{L^p(0,T;X)} = \left(\int_0^T \|f\|_X^p dt \right)^{\frac{1}{p}}.$$

$L^\infty(0, T; X)$ denotes the space of functions $f :]0, T[\to X$ which are measurable and $f \in L^\infty(0, T)$. This space is a Banach space endowed with the norm :

$$\|f\|_{L^\infty(0,T;X)} = ess \sup_{0<t<T} \|f\|_X.$$

We recall that if X and Y are two Banach spaces such that $X \hookrightarrow Y$(continuous embedding), then

$$L^p(0,T;X) \hookrightarrow L^p(0,T;Y), \quad 1 \le p \le \infty.$$

We will also use the embedding

$$H^1_{\Gamma_0}(\Omega) \hookrightarrow L^p(\Omega), \quad 2 \le p \le \bar{p} \quad \text{where} \quad \bar{p} = \begin{cases} \frac{2N}{N-2} & \text{if } N \ge 3, \\ +\infty & \text{if } N = 1, 2, \end{cases}$$

and also

$$H^1_{\Gamma_0}(\Omega) \hookrightarrow L^p(\Gamma_1), \quad 2 \le q \le \bar{q} \quad \text{where} \quad \bar{q} = \begin{cases} \frac{2(N-1)}{N-2} & \text{if } N \ge 3, \\ +\infty & \text{if } N = 1, 2. \end{cases}$$

We denote $V = H^1_{\Gamma_0}(\Omega) \cap L^2(\Gamma_1)$.

Lemma 1 (Sobolev–Poincaré inequality). *Let* $2 \le p \le \frac{2n}{n-2}$. *The inequality*

$$\|u\|_p \le c_s \|\nabla u\|_2 \quad for \quad u \in H^1_{\Gamma_0}(\Omega),$$

holds with some positive constant c_s.

Now we give some estimates related to the convolution operator. By direct calculations, as in [18,19] we find

$$\begin{aligned} \sigma(t)(g*u,u_t) = &-\frac{\sigma(t)}{2} g(t)\|u(t)\|_2^2 \\ &- \frac{d}{dt}\left[\frac{\sigma(t)}{2}(g\ o\ u)(t) - \frac{\sigma(t)}{2}\left(\int_0^\infty g(s)ds\right)\|u(t)\|_2^2\right] \\ &+ \frac{\sigma(t)}{2}(g'\ o\ u)(t) + \frac{\sigma'(t)}{2}(g\ o\ u)(t) \\ &- \frac{\sigma'(t)}{2}\int_0^\infty g(s)ds\|u(t)\|_2^2, \end{aligned} \tag{9}$$

where

$$(g*u)(t) = \int_0^\infty g(t-s)u(s)ds, \ g\ o\ u = \int_0^\infty g(t-s)u(s)ds\|u(t)-u(s)\|_2^2 ds, \tag{10}$$

and

$$(g*u,u) \le 2\left(\int_0^t g(s)ds\right)\|u(t)\|_2^2 + \frac{1}{4}(g\ o\ u)(t). \tag{11}$$

Let us consider the new variable z as in [12],

$$z(x,k,t) = u_t(x, t - \tau(t)k), \quad x \in \Gamma_1, \quad k \in (0,1),$$

which implies that

$$\tau(t) z_t(x,k,t) + (1 - \tau'(t)k) z_k(x,k,t) = 0, \quad in \quad \Gamma_1 \times (0,1) \times (0,\infty).$$

Therefore, problem (1) is equivalent to:

$$\begin{cases} u_{tt} - \Delta u - \delta \Delta u_t + \alpha(t) \int_0^t g(t-s) \Delta u(s) ds = |u|^{p-2} u, \quad in \quad \Omega \times (0,\infty), \\ u_{tt} = -a \Big[\frac{\partial u}{\partial \upsilon}(x,t) + \delta \frac{\partial u_t}{\partial \upsilon}(x,t) + \alpha(t) \int_0^t g(t-s) \Delta u(s) \frac{\partial u}{\partial \upsilon}(x,s) ds \\ \qquad + \mu_1 u_t(x,t) + \mu_2 z_k(x,1,t) \Big], \qquad on \quad \Gamma_1 \times (0,+\infty), \\ \tau(t) z_t(x,k,t) + z_k(x,k,t) = 0, \quad in \qquad \Gamma_1 \times (0,1) \times (0,\infty), \\ z(x,k,0) = f_0(x, -\tau k), \qquad x \in \Gamma_1, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), \qquad x \in \Omega, \\ u(x,t) = 0, \qquad x \in \Gamma_0,\ t \geq 0. \end{cases} \tag{12}$$

Remark 1 For seeking of simplicity, we take $a = 1$ in (12).

Now inspired by [15,16,18], we define the modified energy functional related with problem (12) by

$$\begin{aligned} E(t) = &\frac{1}{2}\left(1 - \alpha(t)\int_0^t g(s)ds\right) \|\nabla u(t)\|_2^2 + \frac{\xi(t)\tau(t)}{2} \int_{\Gamma_1} \int_0^1 z^2(\gamma,k,s)) dk d\gamma \\ &+ \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\|u_t(t)\|_{2.\Gamma_1}^2 - \frac{1}{p}\|u(t)\|_p^p + \alpha(t)(g\ o\ \nabla u)(t). \end{aligned} \tag{13}$$

Lemma 2 *Let* $2 \leq p \leq \bar{q}$ *and* (u, z) *be a solution of the problem* (12). *Then the energy functional defined by* (12) *satisfies*

$$\begin{aligned} E'(t) \leq &-\left(\frac{\xi(1-\tau'(t))}{2} - \frac{\mu_2\sqrt{1-d}}{2}\right) \int_{\Gamma_1} z^2(\gamma,1,t) d\gamma - \delta \|\nabla u_t(t)\|_2^2 \\ &- \left(\mu_1 - \frac{\xi}{2} - \frac{\mu_2}{2\sqrt{1-d}}\right) \|u_t(t)\|_{2.\Gamma_1}^2 + \frac{\alpha(t)}{2}(g'\ o\ \nabla u)(t) \\ &- \frac{\alpha'(t)}{2}\int_0^t g(s)ds \|\nabla u(t)\|_2^2 - \frac{\alpha(t)}{2} g(t) \|\nabla u(t)\|_2^2. \end{aligned} \tag{14}$$

Proof By multiplying the first and second equation in (12) by $u_t(t)$, and integrating the first equation over Ω and the second equation over Γ_1, using the Green's formula, we get

$$\begin{aligned}&\frac{d}{dt}\left[\frac{1}{2}\|u_t(t)\|_2^2+\frac{1}{2}\|u_t(t)\|_{2.\Gamma_1}^2+\frac{1}{2}\|\nabla u(t)\|_2^2-\frac{1}{p}\|u(t)\|_p^p\right]\\&+\mu_1\int_{\Gamma_1}\|u_t(t)\|_{2.\Gamma_1}^2d\gamma+\int_{\Gamma_1}\mu_2 z(\gamma,1,t)u_t(t)d\gamma\\&+\alpha(t)(g'o\nabla u)(t)-\alpha'(t)\left(\int_0^t g(s)ds\right)\|\nabla u(t)\|_2^2\\&-\frac{\alpha(t)}{2}g(t)\|\nabla u(t)\|_2^2+\frac{\alpha'(t)}{2}(g\ o\ \nabla u)(t)+\delta\|\nabla u_t(t)\|_2^2=0.\end{aligned} \tag{15}$$

We multiply the third equation in (12) by $\xi(t)z$ and integrate over $\Gamma_1\times(0,1)$ to obtain

$$\xi(t)\tau(t)\int_{\Gamma_1}\int_0^1 z_t z(\gamma,k,t)dkd\gamma=-\frac{\xi(t)}{2}\int_{\Gamma_1}\int_0^1(1-\tau'(t)k)\frac{\partial}{\partial k}z^2(\gamma,k,t)dkd\gamma. \tag{16}$$

Consequently,

$$\begin{aligned}&\frac{d}{dt}\left(\frac{\xi(t)\tau(t)}{2}\int_{\Gamma_1}\int_0^1 z^2(\gamma,k,t)dkd\gamma\right)\\&=-\frac{\xi(t)}{2}\int_0^1\int_{\Gamma_1}\frac{\partial}{\partial k}((1-\tau'(t)k)z^2(\gamma,k,t))dkd\gamma\\&\quad+\frac{\xi'(t)\tau(t)}{2}\int_0^1\int_{\Gamma_1}z^2(\gamma,k,t)dkd\gamma\\&=\frac{\xi(t)}{2}\int_{\Gamma_1}(z^2(\gamma,0,t)-z^2(\gamma,1,t))d\gamma+\frac{\xi(t)\tau'(t)}{2}\int_{\Gamma_1}z^2(\gamma,1,t)d\gamma\\&\quad+\frac{\xi'(t)\tau(t)}{2}\int_0^1\int_{\Gamma_1}z^2(\gamma,k,t)dkd\gamma\\&\leq\frac{\xi(t)}{2}\int_{\Gamma_1}(z^2(\gamma,0,t)-z^2(\gamma,1,t))d\gamma+\frac{\xi(t)\tau'(t)}{2}\int_{\Gamma_1}z^2(\gamma,1,t)d\gamma.\end{aligned} \tag{17}$$

From (15), (17) and Young's inequality, we get

$$\begin{aligned}E'(t)\leq&-\left(\mu_1-\frac{\xi(t)}{2}\right)\|u_t(t)\|_{2.\gamma_1}^2-\left(\frac{\xi(t)(1-\tau'(t))}{2}\right)\int_{\Gamma_1}z^2(\gamma,k,t)d\gamma\\&-\mu_2\int_{\Gamma_1}z(\gamma,1,t)u_t(\gamma,t)d\gamma+\frac{\alpha(t)}{2}(g'\ o\ \nabla u)(t)-\delta\|\nabla u_t(t)\|_2^2\\&-\frac{\alpha'(t)}{2}\int_0^t g(s)ds\|\nabla u(t)\|_2^2-\frac{\alpha(t)}{2}g(t)\|\nabla u(t)\|_2^2.\end{aligned} \tag{18}$$

Due to Young's inequality, we have

$$\mu_2 \int_{\Gamma_1} z(\gamma,1,t)u_t(\gamma,t)d\gamma \leq \frac{\mu_2}{2\sqrt{1-d}}\|u_t(t)\|^2_{2.\Gamma_1} + \frac{\mu_2\sqrt{1-d}}{2}\int_{\Gamma_1} z^2(\gamma,1,t)d\gamma. \tag{19}$$

Inserting (19) into (18), we obtain

$$\begin{aligned} E'(t) \leq &-\left(\frac{\xi(1-\tau'(t))}{2} - \frac{\mu_2\sqrt{1-d}}{2}\right)\int_{\Gamma_1} z^2(\gamma,1,t)d\gamma - \delta\|\nabla u_t(t)\|_2^2 \\ &-\left(\mu_1 - \frac{\xi}{2} - \frac{\mu_2}{2\sqrt{1-d}}\right)\|u_t(t)\|^2_{2.\Gamma_1} + \frac{\alpha(t)}{2}(g' \, o \, \nabla u)(t) \\ &-\frac{\alpha'(t)}{2}\int_0^t g(s)ds\|\nabla u(t)\|_2^2 - \frac{\alpha(t)}{2}g(t)\|\nabla u(t)\|_2^2. \end{aligned} \tag{20}$$

This completes the proof. □

Remark 2 Since

$$-\frac{\alpha'(t)}{2}\int_0^\infty g(s)ds\|\nabla u(t)\|^2 > 0,$$

E(t) may not be non-increasing.

Remark 3 The following result to problem (12) can be established by combining arguments of [16,17,31].

Theorem 1 *Let* $2 \leq p \leq \bar{q}$ *and then given* $u_0 \in H^1_{\Gamma_0}(\Omega), u_1 \in L^2(\Omega), f_0 \in L^2(\Gamma_1 \times (0,1))$. *Suppose that* $(A_0) - (A_2)$ *hold. Then the problem* (12) *admits a unique weak solution satisfying*

$$\begin{aligned} &u \in L^\infty((0,T); H^1_{\Gamma_0}(\Omega)), \quad u_t \in L^\infty((0,T); H^1_{\Gamma_0}(\Omega)) \cap L^\infty((0,T); L^2(\Gamma_1)), \\ &u_{tt} \in L^\infty((0,T); L^2(\Omega)) \cap L^\infty((0,T); L^2(\Gamma_1)). \end{aligned}$$

3 Asymptotic behavior

In this section, we establish the asymptotic behavior for the solutions. We define the following perturbed function:

$$L(t) = ME(t) + \epsilon\alpha(t)\psi(t) + \epsilon\alpha(t)I(t) + \epsilon\frac{\delta\alpha(t)}{2}\|\nabla u\|_2^2, \tag{21}$$

where

$$\psi(t) = \int_\Omega uu_t dx + \int_{\Gamma_1} uu_t d\gamma, \tag{22}$$

and

$$I(t) = \xi(t) \int_{\Gamma_1} \int_0^1 e^{-k\tau(t)} z^2(\gamma, k, t) dk d\gamma. \tag{23}$$

We need also the following lemma

Lemma 3 *Let (u,z) be a solution of problem (12), then there exists two positive constants λ_1,λ_2 such that*

$$\lambda_1 E(t) \leq L(t) \leq \lambda_2 E(t), \quad t \geq 0, \tag{24}$$

for M sufficiently large .

Proof Thank's to the Cauchy Schwarz and Young's inequalities, Lemma 1 and using the fact that $\|u\|_{2.\Gamma_1} \leq B\|\nabla u\|_2$, we have

$$|\psi(t)| \leq \frac{1}{\omega}\|u_t\|_2^2 + \frac{1}{4\omega}\|u_t\|_{2.\Gamma}^2 + \omega\|\nabla u\|_2^2 + \omega B^2\|\nabla u\|_2^2, \tag{25}$$

it follows from (23) that $\forall c > 0$:

$$\begin{aligned} |I(t)| &= \left|\xi(t) \int_{\Gamma_1} \int_0^1 e^{-k\tau(t)} z^2(\gamma, k, s)) dk d\gamma\right| \\ &\leq c\xi(t) \int_{\Gamma_1} \int_0^1 z^2(\gamma, k, s)) dk d\gamma. \end{aligned} \tag{26}$$

Hence, combining (25), (26) and using the fact that $\alpha(t) < \alpha(0)$, we get

$$\begin{aligned} |L(t) - ME(t)| &= \epsilon\alpha(t)\psi(t) + \xi(t) \int_{\Gamma_1} \int_0^1 e^{-k\tau(t)} z^2(\gamma, k, t) dk d\gamma \\ &\leq \frac{\epsilon}{\omega}\|u_t\|_2^2 + \frac{\epsilon}{4\omega}\|u_t\|_{2.\Gamma_1}^2 + (\epsilon\omega + \epsilon B^2)\|\nabla u\|_2^2 \\ &\quad + c\xi(t) \int_{\Gamma_1} \int_0^1 z^2(\gamma, k, t) dk d\gamma + \epsilon\frac{\delta\alpha(t)}{2}\|\nabla u\|_2^2. \end{aligned} \tag{27}$$

Where $c_1 = \frac{\epsilon}{\omega},\ c_2 = \frac{\epsilon}{4\omega},\ c_3 = (\epsilon\omega + \epsilon B^2),\ c_4 = c$, then we can write

$$|L(t) - ME(t)| \leq c_5 E(t), \tag{28}$$

where $c_5 = max(c_1, c_2, c_3, c_4)$. Thus, from the definition of E(t) and selecting M sufficiently large,

$$\lambda_1 E(t) \leq L(t) \leq \lambda_2 E(t). \tag{29}$$

Where $\lambda_1 = (M - \epsilon c_5), \lambda_2 = (M + \epsilon c_5)$. This completes the proof. □

Lemma 4 *The functional defined in (23) satisfies*

$$\frac{d}{dt}I(t) \leq \frac{\xi(t)}{2\tau_0}\|u_t\|^2_{2.\Gamma_1} - \xi(t)\left(\frac{1-d}{2\tau_1}\right)\int_{\Gamma_1}\int_0^1 z^2(\gamma,1,t)d\gamma$$
$$-\frac{\tau'(t)\eta_1}{2\tau_1}\int_{\Gamma_1}\int_0^1 z^2(\gamma,k,t)dkd\gamma.$$

where η_1, η_2 , τ_0, τ_1 and d are a positive constants and $\xi(t)$ are positive and bounded functions such that

$$\xi_0 = \sup_{t\geq 0}\xi(t),$$
$$\xi_1 = \inf_{t\geq 0}\xi(t).$$

Proof Taking derivative of (23) produces

$$\begin{aligned}
\frac{d}{dt}I(t) &= \frac{d}{dt}\left(\xi(t)e^{-k\tau(t)}\int_{\Gamma_1}\int_0^1 z^2(\gamma,k,t))dkd\gamma\right)\\
&= \Bigg[\xi'(t)e^{-\tau(t)k}\int_{\Gamma_1}\int_0^1 z^2(\gamma,k,t))dkd\gamma\\
&\quad -\xi(t)ke^{-\tau(t)k}\tau'(t)\int_{\Gamma_1}\int_0^1 z^2(\gamma,k,t))dkd\gamma\Bigg]\\
&\quad +\frac{1}{\tau(t)}e^{-\tau(t)k}\tau(t)\int_{\Gamma_1}\int_0^1 \frac{d}{dt}z^2(\gamma,k,t))dkd\gamma\\
&= \Bigg[\xi'(t)e^{-\tau(t)k}\int_{\Gamma_1}\int_0^1 z^2(\gamma,k,t))dkd\gamma\\
&\quad -\xi(t)ke^{-\tau(t)k}\tau'(t)\int_{\Gamma_1}\int_0^1 z^2(\gamma,k,t))dkd\gamma\Bigg]\\
&\quad +\frac{1}{\tau(t)}e^{-\tau(t)k}\int_{\Gamma_1}\int_0^1 \frac{\partial}{\partial k}(1-\tau'(t)k)z^2(\gamma,k,t))dkd\gamma\\
&\leq \Bigg[\xi'(t)e^{-\tau(t)k}\int_{\Gamma_1}\int_0^1 z^2(\gamma,k,t)dkd\gamma\\
&\quad -\xi(t)ke^{-\tau(t)k}\tau'(t)\int_{\Gamma_1}\int_0^1 z^2(\gamma,k,t))dkd\gamma\Bigg]\\
&\quad +\frac{1}{\tau(t)}\Bigg[\xi(t)\int_{\Gamma_1}[z^2(\gamma,0,t))d\gamma - z^2(\gamma,1,t)d\gamma]\\
&\quad +\xi(t)\tau'(t)\int_{\Gamma_1} z^2(\gamma,1,t)d\gamma\Bigg]\\
&\leq \frac{\xi(t)}{2\tau_0}\|u_t\|^2_{2.\Gamma_1} - \xi(t)\left(\frac{1-d}{2\tau_1}\right)\int_{\Gamma_1} z^2(\gamma,1,t)d\gamma\\
&\quad -\tau'(t)\eta_1\int_{\Gamma_1}\int_0^1 z^2(\gamma,k,t)dkd\gamma.
\end{aligned} \tag{30}$$

□

Lemma 5 *The functional $\psi(t)$ defined in (22) satisfies*

$$\frac{d}{dt}\psi(t) \leq \|u_t\|_2^2 + \|u_t\|_{2.\Gamma_1}^2 - (1-2n-\eta_1)\|\nabla u\|_2^2 + \|u\|_p^p + \frac{\alpha(t)}{4}(g\ o\ \nabla u)(t) + \frac{c}{4\eta}\int_{\Gamma_1}|u_t(\gamma,t))|^2 d\gamma + \frac{c}{4\eta}\int_{\Gamma_1}|z(\gamma,1,t))|^2 d\gamma, \tag{31}$$

where $n = \left(1 - \frac{2\alpha(t)}{\lambda}\int_0^t g(s)ds\right) > 0,\quad \eta_1 = 2\epsilon\eta c_s^2 B^2 > 0$ *and* $(1-2n-\eta_1) > 0$.

Proof Taking derivative of ψ and using the problem (11) and (12), we have

$$\frac{d}{dt}\psi(t) \leq \|u_t\|_2^2 + \|u_t\|_{2.\Gamma_1}^2 - \|\nabla u\|_2^2 + \|u\|_p^p + \alpha(t)(g * \nabla u.\nabla u) - \mu_1\int_{\Gamma_1} u_t u d\gamma - \mu_2\int_{\Gamma_1} z(\gamma,1,t)u d\gamma. \tag{32}$$

Young's inequality produces $\forall\ \ \epsilon > 0$ and put $|\sigma(t)| \leq c$

$$\left|\int_{\Gamma_1} u_t(\gamma,t)u(\gamma,t)d\gamma\right| \leq \eta c_s^2 B^2\epsilon\|\nabla u\|_2^2 + \frac{c}{4\eta}\int_{\Gamma_1}|u_t|^2 d\gamma \tag{33}$$

$$\left|\int_{\Gamma_1} z(\gamma,1,t))u(\gamma,t)d\gamma\right| \leq \eta c_s^2 B^2\epsilon\|\nabla u\|_2^2 + \frac{c}{4\eta}\int_{\Gamma_1}|z(\gamma,1,t))|^2 d\gamma, \tag{34}$$

$$\alpha(t)(g * \nabla u.\nabla u) \leq \frac{2\alpha(t)}{\lambda}\int_0^t g(s)ds\|\nabla u\|_2^2 + \frac{\alpha(t)}{4}(g\ o\ \nabla u)(t), \tag{35}$$

inserting (33)–(35) in (32) gives

$$\frac{d}{dt}\psi(t) \leq \|u_t\|_2^2 + \|u_t\|_{2.\Gamma_1}^2 - \left[1 - \frac{2\alpha(t)}{\lambda}\int_0^t g(s)ds - 2\epsilon\eta c_s^2 B^2\right]\|\nabla u\|_2^2 + \|u\|_p^p + \frac{\alpha(t)}{4}(g\ o\ \nabla u)(t) + \frac{c}{4\eta}\int_{\Gamma_1}|u_t(\gamma,t)|^2 d\gamma + \frac{c}{4\eta}\int_{\Gamma_1}|z(\gamma,1,t)|^2 d\gamma, \tag{36}$$

then

$$\frac{d}{dt}\psi(t) \leq \|u_t\|_2^2 + \|u_t\|_{2.\Gamma_1}^2 - (1-2n-\eta_1)\|\nabla u\|_2^2 + \|u\|_p^p + \frac{\alpha(t)}{4}(g\ o\ \nabla u)(t) + \frac{c}{4\eta}\int_{\Gamma_1}|u_t(\gamma,t))|^2 d\gamma + \frac{c}{4\eta}\int_{\Gamma_1}|z(\gamma,1,t))|^2 d\gamma, \tag{37}$$

where $n = \left(1 - \frac{2\alpha(t)}{\lambda}\int_0^t g(s)ds\right) > 0$, $\eta_1 = 2\epsilon\eta c_s^2 B^2 > 0$ and $(1 - 2n - \eta_1) > 0$, which completes the proof. □

Lemma 6 *Let $L(t)$ the functional defined in (21), then $L(t)$ satisfies*

$$\frac{d}{dt}L(t) \leq -\alpha(t)C_1E(t) + C_2\alpha(t)(g\ o\ \nabla u)(t), \quad \forall t \geq 0. \tag{38}$$

Proof We take the derivative of (21), we get

$$\frac{d}{dt}L(t) = ME'(t) + \epsilon\alpha(t)\psi'(t) + \epsilon\alpha'(t)\psi(t) + \epsilon\alpha'(t)I(t) + \epsilon\alpha(t)I'(t) + \epsilon\frac{\delta\alpha'(t)}{2}\|\nabla u\|_2^2 + \epsilon\delta\alpha(t)\int_\Omega \nabla u\nabla u_t dx, \tag{39}$$

making use of the inequalities

$$\alpha'(t)\left|\int_\Omega uu_t dx\right| \leq \alpha'(t)\frac{c_s^2}{\alpha_1}\|\nabla u\|_2^2 + \alpha'(t)\alpha_1^2\|u_t\|_2^2, \tag{40}$$

and

$$\alpha'(t)\left|\int_{\Gamma_1} uu_t d\gamma\right| \leq \alpha'(t)\frac{c_s^2B^2}{\alpha_1}\|\nabla u\|_2^2 + \alpha'(t)\alpha_1^2\|u_t\|_{2.\Gamma_1}^2, \tag{41}$$

using Lemmas 3, 4, so $L'(t)$ gives the form:

$$\begin{aligned}
L'(t) = &-Ma_1\int_{\Gamma_1} z^2(\gamma, 1, t)d\gamma - Ma_2\|u_t\|_{2.\Gamma_1}^2 + \frac{M\alpha(t)}{2}(g'\ o\ \nabla u)(t) \\
&-\frac{M\alpha'(t)}{2}\int_0^t g(s)ds\|\nabla u\|_2^2 - \frac{M\alpha(t)}{2}g(t)\|\nabla u\|_2^2 - M\delta\|\nabla u_t\|_2^2 \\
&+\epsilon\alpha(t)\|u_t\|_2^2 + \epsilon\alpha(t)\|u_t\|_{2.\Gamma_1}^2 - \epsilon\alpha(t)(1 - 2n - \eta_1)\|\nabla u\|_2^2 \\
&+\epsilon\alpha(t)\|u\|_p^p + \epsilon\frac{\alpha(t)^2}{4}(g\ o\ \nabla u)(t) + \epsilon\frac{\alpha(t)}{4\eta}\|u_t\|_{2.\Gamma_1}^2 \\
&+\epsilon\frac{\alpha(t)}{4\eta}\|z(\gamma, 1, t)\|_{2.\Gamma_1}^2 + \epsilon\frac{\alpha'(t)c_s^2}{\alpha_1}\|\nabla u\|_2^2 + \epsilon\alpha'(t)\alpha_1^2\|u_t\|_2^2 \\
&+\epsilon\frac{\alpha'(t)c_s^2B^2}{\alpha_1}\|\nabla u\|_2^2 + \epsilon\alpha'(t)\alpha_1^2\|u_t\|_{2.\Gamma_1}^2 + \epsilon\frac{\alpha(t)\xi(t)}{2\tau_0}\|u_t\|_{2.\Gamma_1}^2 \\
&+\epsilon\alpha'(t)\xi(t)\int_{\Gamma_1}\int_0^1 e^{-k\tau(t)}z^2(\gamma, k, t)dkd\gamma + \epsilon\frac{\delta\alpha'(t)}{2}\|\nabla u\|_2^2 \\
&-\epsilon\tau(t)\xi(t)\alpha(t)\frac{\tau'(t)\eta_1}{2\tau_1\xi_0}\int_{\Gamma_1}\int_0^1 z^2(\gamma, k, t)dkd\gamma \\
&-\epsilon\alpha(t)\xi(t)\left(\frac{1-d}{2\tau_1}\right)\int_{\Gamma_1} z^2(\gamma, 1, t)d\gamma,
\end{aligned} \tag{42}$$

using the fact that $\alpha(t) < \alpha(0)$, we conclude

$$\begin{aligned}
L'(t) = &-\alpha(t)\epsilon\left((1-2n-\eta_1) - \left(c_s^2(1+B^2) - \frac{\delta}{2}\right)\frac{\alpha'(t)}{\alpha(t)}\right)\|\nabla u\|_2^2 \\
&+\epsilon\alpha(t)\left(1+\alpha_1^2\frac{\alpha'(t)}{\alpha(t)}+\frac{1}{4\eta}+\frac{\xi(t)}{2\tau_0}\right)\|u_t\|_{2.\Gamma_1}^2 + \epsilon\alpha(t)\left(1+\alpha_1^2\frac{\alpha'(t)}{\alpha(t)}\right)\|u_t\|_2^2 \\
&+\epsilon\alpha(t)\|u\|_p^p - \delta M\|\nabla u_t\|_2^2 + \epsilon\frac{\alpha(t)^2}{4}(g\ o\ \nabla u)(t) + \epsilon\frac{\alpha(t)}{4\eta}\|z(\gamma,1,t)\|_{2.\Gamma_1}^2 \\
&-\alpha(t)\left(\frac{Ma_2\sigma(t)}{\alpha(0)} - \epsilon\frac{\xi(1-d)}{2\tau_1\alpha(0)}\right)\int_{\Gamma_1} z^2(\gamma,1,t))d\gamma \\
&-\alpha(t)\left(\frac{Ma_1\sigma(t)}{\alpha(0)} - \epsilon\frac{\xi\alpha_2}{\tau_0}\right)\|u_t\|_{2.\Gamma_1}^2. \\
&-\epsilon\tau(t)\xi(t)\alpha(t)\frac{\tau'(t)\eta_1}{2\tau_1\xi_0}\int_{\Gamma_1}\int_0^1 z^2(\gamma,k,t)dkd\gamma. \qquad (43)
\end{aligned}$$

Consequently, using the definition of the energy (13), for any positive constant M, we obtain:

$$\begin{aligned}
L'(t) = &-\alpha(t)\epsilon\left((1-2n-\eta_1) - (c_s^2(1+B^2) - \frac{\delta}{2})\frac{\alpha'(t)}{\alpha(t)}\right)\|\nabla u\|_2^2 \\
&-\epsilon\alpha(t)\left(\frac{M}{2}-1\right)\|u\|_p^p - \epsilon\alpha(t)\left(\frac{M}{2} - \alpha_1^2\left(1+\alpha_1^2\frac{\alpha'(t)}{\alpha(t)}\right)\right)\|u_t\|_2^2 \\
&-\epsilon\alpha(t)\left(\frac{M}{2} - \left(1+\alpha_1^2\frac{\alpha'(t)}{\alpha(t)}+\frac{1}{4\eta}+\frac{\xi(t)}{2\tau_0}\right)\right)\|u_t\|_{2.\Gamma_1}^2 + \frac{\alpha(t)M}{2}\|u_t\|_2^2 \\
&+\epsilon\frac{\alpha(t)M}{2}\|u_t\|_{2.\Gamma_1}^2 - \epsilon\frac{M\alpha(t)^2}{4}(g\ o\ \nabla u)(t) + \epsilon\frac{M\alpha(t)^2}{2}(g\ o\ \nabla u)(t) \\
&-M\delta\|\nabla u_t\|_2^2 + \epsilon\frac{\alpha(t)}{4\eta}\|u_t\|_{2.\Gamma_1}^2 + \epsilon\frac{\alpha(t)^2}{4\eta}\|z(\gamma,1,t)\|_{2.\Gamma_1}^2 \\
&-\alpha(t)\left(\frac{Ma_2\sigma(t)}{\alpha(0)} - \epsilon\frac{\xi\alpha_2}{\tau_0}\right)\|u_t\|_{2.\Gamma_1}^2 \\
&-\alpha(t)\left(\frac{Ma_1\sigma(t)}{\alpha(0)} - \epsilon\frac{\xi(1-d)}{2\tau_1\alpha(0)}\right)\int_{\Gamma_1} z^2(\gamma,1,t)d\gamma \\
&-\epsilon\alpha(t)\tau(t)\xi(t)\frac{\tau'(t)\eta_1}{2\tau_1\xi_0}\int_{\Gamma_1}\int_0^1 z^2(\gamma,k,t)dkd\gamma. \qquad (44)
\end{aligned}$$

First, we fix $n - \eta_1 > 0$ such that $1 - 2n - \eta_1 > 0$ and then take $M > 0$ such that $\left(\frac{M}{2} - 1\right) > 0$, since

$$\lim_{t\to\infty}\frac{\alpha'(t)}{\alpha(t)} = 0,$$

we can choose $t_0 > 0$ sufficiently large so that

$$\left(\frac{M}{2}-\alpha_1^2\left(1+\frac{\alpha'(t)}{\alpha(t)}\right)\right)>0, \left((1-2n-\eta_1)-(c_s^2(1+B^2)-\frac{\delta}{2})\frac{\alpha'(t)}{\alpha(t)}\right)>0,$$
$$\frac{\tau'(t)\eta_1}{2\tau_1\xi_0}>0$$
$$\left(\frac{Ma_2\sigma(t)}{\alpha(0)}-\epsilon\frac{\bar{\xi}\alpha_2}{\tau_0}\right)>0, \left(\frac{Ma_1\sigma(t)}{\alpha(0)}-\epsilon\frac{\bar{\xi}(1-d)}{2\tau_1\alpha(0)}\right)>0,$$
$$\left(\frac{M}{2}-\left(1+\alpha_1^2\frac{\alpha'(t)}{\alpha(t)}+\frac{1}{4\eta}+\frac{\xi(t)}{2\tau_0}\right)\right)>0.$$

By using the Poincaré and trace inequalities

$$\|u_t\|_2^2 \le C\|\nabla u_t\|_2^2, \quad \|u_t\|_{2.\Gamma_1}^2 \le C\|\nabla u_t\|_2^2.$$

Then (44) takes the form:

$$\frac{d}{dt}L(t) \le -M\alpha(t)c\epsilon E(t)-(M\delta-\epsilon M\alpha(0)C)\,\|\nabla u_t\|_2^2+\epsilon\frac{\alpha(0)M}{2}\alpha(t)(g\ o\ \nabla u)(t), \tag{45}$$

then, choosing ϵ small enough such that $(M\delta-\epsilon M\alpha(0)C)>0$, we obtain

$$\frac{d}{dt}L(t) \le -M\alpha(t)c\epsilon E(t)+\epsilon\frac{\alpha(0)M}{2}\alpha(t)(g\ o\ \nabla u)(t), \tag{46}$$

setting $\theta=\frac{M\epsilon}{\lambda_2}$, $C_1=c\theta$, $C_2=\epsilon\frac{\alpha(0)M}{2}$ and

$$\frac{d}{dt}L(t) \le -\alpha(t)C_1E(t)+C_2\alpha(t)(g\ o\ \nabla u)(t),\ \forall t\ge 0. \tag{47}$$

The proof is completed. □

Theorem 2 *There exist two positive constants C_0, θ and t_1 such that*

$$E(t) \le C_0 e^{-\theta\int_{t_1}^t \alpha(s)\sigma(s)ds} \tag{48}$$

Proof Multiplying (47) by $\sigma(t)$ and using the Lemma 1, we get

$$\begin{aligned}\sigma(t)\frac{d}{dt}L(t) &\le -C_1\alpha(t)\sigma(t)E(t)+C_2\alpha(t)\sigma(t)(g\ o\ \nabla u)(t)\\ &\le -C_1\alpha(t)\sigma(t)E(t)-C_2\alpha(t)\sigma(t)(g'\ o\ \nabla u)(t)\\ &\le -C_1\alpha(t)\sigma(t)E(t)+C_2\left(-2\frac{d}{dt}E(t)-\alpha'(t)\int_0^t g(s)ds\|\nabla u\|_2^2\right).\end{aligned} \tag{49}$$

Since σ is nonincreasing, from the definition of $E(t)$ and assumption (A_0), we have

$$\frac{d}{dt}\left(\sigma(t)L(t)+2C_2E(t)\right)\leq-\alpha(t)\sigma(t)\left(C_1+\frac{2C_2l_0\alpha'(t)}{\lambda l\alpha(t)\sigma(t)}\right)E(t)\quad\text{for}\quad t>t_0,$$

as we have

$$\lim_{t\to\infty}\frac{2C_2l_0\alpha'(t)}{\lambda l\alpha(t)\sigma(t)}=0,$$

we can choose $t_1>t_0$ such that $C_3=C_1+\frac{2C_2l_0\alpha'(t)}{\lambda l\alpha(t)\sigma(t)}>0$ for $t>t_1$. Now let $\chi(t)=\sigma(t)L(t)+2C_2E(t)$. Then we can verify that

$$\theta_1E(t)\leq\chi(t)\leq\theta_2E(t). \tag{50}$$

where θ_1,θ_2 are two positive constants, thus we arrive at

$$\frac{d}{dt}\chi(t)\leq-C_4\alpha(t)\sigma(t)\chi(t)\quad\text{for}\quad t>t_1.$$

Integrating the previous differential inequality between t_1 and t gives the following estimate for the function χ

$$\chi(t)\leq\chi(t_1)e^{-C_4\int_{t_1}^{t}\alpha(s)\sigma(s)ds},\quad\forall t\geq t_1.$$

Consequently, by using (50), we conclude

$$E(t)\leq\hat{C}e^{-C_4\int_{t_1}^{t}\alpha(s)\sigma(s)ds},\quad\forall t\geq t_1.$$

This completes the proof. □

Remark 4 We illustrate the energy decay rate given by Theorem 2 through the following examples which are introduced in [19,27].

1. If $g(t)=ae^{-b(1+t)^{\nu}}$, $\alpha(t)=\frac{1}{1+t}$ for $a,b>0$ and $0<\nu\leq1$, then $\sigma(t)=b\nu(1+t)^{\nu-1}$ satisfies (A_0). Thus (48) gives the estimate

$$E(t)\leq C_0e^{-\theta(1+t)^{\nu-1}}.$$

2. If $g(t)=ae^{-b\ln^{\nu}(1+t)}$, $\alpha(t)=\frac{1}{\ln(1+t)}$ for $a,b>0$ and $1<\nu$, then $\sigma(t)=\frac{b\nu\ln^{\nu-1}(1+t)}{(1+t)}$ satisfies (A_0). Thus (48) gives the estimate

$$E(t)\leq C_0e^{-\theta\ln^{\nu}(1+t)}.$$

3. If $g(t)=e^{-at}$, $\alpha(t)=\frac{b}{(1+t)}$ for $a,b>0$ then $\sigma(t)\equiv a$ satisfies (A_0). Thus (48) gives the estimate

$$E(t)\leq C_0(1+t)^{-\theta ab}.$$

4. If $g(t) = e^{-at}$, $\alpha(t) \equiv b$. Note that in this case (48) reduces to one of [13].

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