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On a globally convergent trust region algorithm with infeasibility control for equality constrained optimization

Xiaojing Zhu

Abstract Recently, a novel step acceptance mechanism for equality constrained optimization was proposed by Zhu and Pu (Comput. Appl. Math. 31(2):407–432, 2012). This new mechanism uses an infeasibility control technique that is quite different from traditional penalty functions and filters. However, in that paper, global convergence of the algorithm with this new mechanism was proved in a double trust regions framework where a strong assumption on step sizes must be required. In this paper, we improve Zhu and Pu's work and furnish a complete global convergence proof for this infeasibility control mechanism in a standard trust region framework where some mild assumptions are sufficient. In addition, numerical results on a number of CUTEr problems accompanied by comparison with SNOPT show the efficiency of the proposed algorithm.

1 Introduction

In this paper, we consider a numerical algorithm for the following nonlinear equality constrained optimization problem

$$\begin{cases} \min\ f(x) \\ \text{s.t.}\ c(x) = 0, \end{cases} \tag{1}$$

X. Zhu (✉)
School of Mathematics and Physics, Shanghai University of Electric Power,
Shanghai 200090, People's Republic of China
e-mail: 042563zxj@163.com

where the objective $f : \mathbb{R}^n \to \mathbb{R}$ and the constraints $c : \mathbb{R}^n \to \mathbb{R}^m$ with $m \leq n$ are sufficiently smooth.

The proposed algorithm is a member of the class of two-phase trust region algorithms for which the reader is referred to, e.g., [1–6]. More specifically, our approach is based on the Byrd–Omojokun trust region SQP method [4,5], which is recognized as the most practical trust region method for general nonlinear equality constrained optimization problems. In the Byrd–Omojokun method, a complete step is decomposed into a normal step and a tangent step, which are computed by solving a two-phase relaxation QP subproblem.

As a common view, the technique used to accept or reject steps deeply impacts the efficiency of methods for nonlinear constrained optimization. In many traditional algorithms for nonlinear constrained optimization, penalty functions or filters are used to judge the quality of a trial step. The most obvious disadvantage of penalty functions is their over dependence upon penalty parameters. An inappropriate penalty parameter can possibly reject a good step, and even a sophisticatedly designed strategy for updating the penalty parameter is not very efficient in practical use. That is why Fletcher and Leyffer proposed the creative concept of filters [7]. Convergence theories on filter methods can be seen, e.g., in [8,9]. Nevertheless, filter methods also have an Achilles' heel, a restoration phase is used to reduce infeasibility until a feasible subproblem is obtained. Gould, Loh, and Robinson [10] recently proposed a new robust filter method which is free of restoration phases. This method uses a complicated unified step computation process and a mixed step acceptance strategy based on filters and steering penalty functions.

Over the last few years, methods without a penalty function or a filter has been a hot topic in the nonlinear optimization community. Bielschowsky and Gomes [11] introduced an infeasibility control technique based on trust cylinders. This method needs to obtain a possibly computationally expensive restoration step per iteration. Gould and Toint [12] introduced a trust funnel technique for step acceptance. This method uses different trust regions for normal and tangent steps, but the strategy for coordinating normal and tangent trust region radii is very sophisticated and still in exploration. Zhu and Pu [13] proposed a step acceptance mechanism based on a control set. Liu and Yuan [14] proposed a penalty-filter-free technique in the line search framework. Those four methods are designed especially for equality constrained optimization. More recently, Shen et al. [15] proposed a non-monotone SQP method for general nonlinear constrained optimization without a penalty function or a filter. Although the penalty-filter-free methods mentioned above share some similarities, they are quite different from each other.

This paper is mostly based on the method of Zhu and Pu [13]. The most outstanding novelty of Zhu and Pu's work is a new technique controlling infeasibility via a set of constraint violation of some previous iterations. This technique is very useful in practice, but in theory it has a fly in the ointment. Specifically, a strong assumption on the step size must be required to establish global convergence. The primary cause of this assumption is that a double trust regions strategy similar to Gould and Toint's work [12] is used for step computations and the ratio of normal and tangent trust region radii is out of control in theory though in practice it is not the case.

The main contribution of this paper is to complete the global convergence theory of the new infeasibility control technique in [13]. Compared with [13], the most significant modification made in the proposed algorithm is that the double trust regions strategy is replaced by a standard (single) trust region strategy. Global convergence to first order critical points is then proved under mild assumptions. Of course, we also present an extended numerical results on some CUTEr problems to demonstrate the efficiency of the proposed algorithm.

The paper is organized as follows. In Sect. 2, a complete description of the proposed algorithm is introduced. we Assumptions and global convergence analysis are presented in Sect. 3. Section 4 is devoted to some numerical results.

2 The algorithm

2.1 Step computation

We compute steps on the basis of the Byrd–Omojokun trust region method [5]. Each complete step s_k is composed of a normal step n_k and a tangent step t_k, i.e.,

$$s_k = n_k + t_k. \tag{2}$$

The normal step n_k aims at reducing the constraint violation function $h(x)$, where

$$h(x) := \frac{1}{2}||c(x)||^2 \tag{3}$$

with $||\cdot||$ denoting the Euclidean norm. This function can be viewed as an infeasibility measure at a point x. The tangent step t_k aims at reducing the objective as much as possible while preserving the constraint violation. Specifically, n_k and t_k are computed as follows.

For the normal step n_k, we solve the following trust region least squares problem

$$\begin{cases} \min \frac{1}{2}||c_k + A_k v||^2 \\ \text{s.t. } ||v|| \leq \tau \Delta_k, \end{cases} \tag{4}$$

where $\tau \in (0, 1)$, $c_k = c(x_k)$, and $A_k = A(x_k)$ which is the Jacobian of $c(x)$ at x_k. We assume the solution to (4), the normal step n_k, satisfies

$$||n_k|| \leq \kappa_n ||c_k||, \tag{5}$$

where $\kappa_n > 0$. This assumption is actually a regularization condition on the Jacobian of constraints. In fact, suppose A_k has the SVD: $A_k = U_k \Sigma_k V_k^T$. Then $v_k = -V_k \Sigma_k^\dagger U_k^T c_k$, where $\Sigma_k^\dagger$ is the pseudo-inverse of Σ_k, solves the least squares problem

$$\min ||c_k + A_k v||^2.$$

Thus, a sufficient condition for (5) is that the smallest positive singular value of A_k is bounded below away from zero. Notice that $n_k = 0$ when x_k is feasible.

After computing the normal step n_k, we proceed to the next task that is to find a tangent step t_k to improve the optimality of the current iterate x_k. Consider a quadratic model of the Lagrangian at x_k

$$m_k(x_k + d) := f_k + g_k^T d + \frac{1}{2} d^T B_k d,$$

where $f_k = f(x_k), g_k = \nabla f(x_k)$, and B_k is an approximate Hessian of the Lagrangian

$$\mathcal{L}(x, \lambda) = f(x) + \lambda^T c(x)$$

at x_k. It follows that

$$m_k(x_k + n_k) = f_k + g_k^T n_k + \frac{1}{2} n_k^T B_k n_k$$

and

$$m_k(x_k + n_k + t) = m_k(x_k + n_k) + (g_k^n)^T t + \frac{1}{2} t^T B_k t,$$

where $g_k^n = g_k + B_k n_k$. Then the tangent step t_k should be an solution to the following problem

$$\begin{cases} \min \ (g_k^n)^T t + \frac{1}{2} t^T B_k t \\ \text{s.t. } A_k t = 0, \\ \qquad \|n_k + t\| \leq \Delta_k. \end{cases}$$

But, in practice, we solves for t_k this problem

$$\begin{cases} \min \ (Z_k^T g_k^n)^T v + \frac{1}{2} v^T Z_k^T B_k Z_k v \\ \text{s.t. } \|v\| \leq \sqrt{\Delta_k^2 - \|n_k\|^2}, \end{cases} \tag{6}$$

where Z_k is an orthonormal basis matrix of the null space of A_k. Therefore, the dogleg method [16], the CG-Steihaug [17] method, and the generalized Lanczos trust region (GLTR) method [18] can apply. Let v_k be the obtained solution to (6), we set $t_k = Z_k v_k$. Since the tangent step t_k is in the null space of A_k, we have from (2) that

$$c_k + A_k s_k = c_k + A_k n_k, \tag{7}$$

which means that the linearized constraint violation remains unchanged after the tangent step t_k is taken.

The lagrangian multiplier vector λ_{k+1} is obtained by solving the following least squares problem

$$\min_{\lambda} \frac{1}{2} ||g_k + A_k^T \lambda||^2 \tag{8}$$

2.2 Step acceptance

The mechanism for step acceptance is introduced by Zhu and Pu [13]. This mechanism uses a novel infeasibility control technique to promote global convergence to first order critical points. Now we describe this technique in detail as follows.

The key is the innovative concept of "control set" which is a set of l positive numbers and denoted by

$$H_k := \{H_{k,1}, H_{k,2}, \ldots, H_{k,l}\},$$

where the l elements are sorted in a non-increasing order, i.e., $H_{k,1} \geq H_{k,2} \geq \cdots \geq H_{k,l}$. At the beginning, we define H_0 as $H_0 = \{u, \cdots, u\}$ where u is a sufficiently large constant such that

$$u \geq \max\{h(x_0), 1\}. \tag{9}$$

For an arbitrary iteration k, when the complete step s_k is computed, we consider the following three cases.

- $h(x_k) = 0, \quad h(x_k + s_k) \leq H_{k,1},$ (10)
- $h(x_k) > 0, \quad h(x_k + s_k) \leq \beta h(x_k),$ (11)
- $h(x_k) > 0, \quad f(x_k + s_k) \leq f(x_k) - \gamma h(x_k + s_k), \quad h(x_k + s_k) \leq \beta H_{k,2},$ (12)

where β and γ are two constants such that $0 < \gamma < \beta < 1$. If one of (11) and (12) is satisfied, then

$$f(x_k + s_k) \leq f(x_k) - \gamma h(x_k + s_k) \text{ or } h(x_k + s_k) \leq \beta h(x_k). \tag{13}$$

After $x_k + s_k$ is accepted as the next iterate x_{k+1}, we may update the control set H_k by substituting a new element h_k^+ for the biggest element $H_{k,1}$, where

$$h_k^+ := (1 - \theta)h(x_k) + \theta h(x_{k+1}) \tag{14}$$

with $\theta \in (0, 1)$. Of course, all the elements in the new control set H_{k+1} will be rearranged in a non-increasing order as well. The purpose of the control set is evidently to compel the infeasibility of the iterates to approach zero progressively. Although only the first two elements $H_{k,1}$ and $H_{k,2}$ of H_k are involved in conditions (10)–(13), the length l of H_k impacts the strength of infeasibility control. For example, consider two cases that $l = 2$ and $l = 3$ with the same values for the initial number and the entering number:

$H_0 = \{100, 100\}, \qquad H_1 = H_0 \oplus 10 = \{100, 10\}, \qquad H_2 = H_1 \oplus 1 = \{10, 1\},$

$H_0 = \{100, 100, 100\}, \quad H_1 = H_0 \oplus 10 = \{100, 100, 10\}, \quad H_2 = H_1 \oplus 1 = \{100, 10, 1\}.$

Here the notation "$\oplus$" means the control set is updated with some new entry.

It is observed that the first two elements of bigger l change faster.

All iterations are classified into the following three types.

- f-**type.** At least one of (10)–(12) holds and

$$\chi_k > \sigma_1 \|c_k\|^{\sigma_2}, \quad \delta_k^f \geq \zeta \delta_k^{f,t}, \tag{15}$$

where $\sigma_1, \sigma_2, \zeta \in (0, 1)$ and $\chi_k, \delta_k^f, \delta_k^{f,t}$ are defined by

$$\chi_k := \|Z_k^T g_k^n\|, \tag{16}$$

$$\delta_k^f := f(x_k) - m_k(x_k + s_k), \tag{17}$$

$$\delta_k^{f,t} := m_k(x_k + n_k) - m_k(x_k + s_k). \tag{18}$$

- h-**type.** At least one of (10)–(12) holds is but (15) fails.
- c-**type.** None of (10)–(12) holds.

Given some constants $\eta_1, \eta_2, \tau_1, \tau_2, \bar{\Delta}, \hat{\Delta}$ such that $0 < \eta_1 < \eta_2 < 1, 0 < \tau_1 < 1 \leq \tau_2, 0 < \bar{\Delta} < \hat{\Delta}$, we accept or reject the trial step according to the following strategy.

When k is an f-type iteration, we accept $x_k + s_k$ if

$$\rho_k^f := \frac{f(x_k) - f(x_k + s_k)}{\delta_k^f} \geq \eta_1. \tag{19}$$

The corresponding update rule for the trust region radius Δ_k is

$$\Delta_{k+1} = \begin{cases} \min\{\max\{\tau_2 \Delta_k, \bar{\Delta}\}, \hat{\Delta}\} & \text{if } \rho_k^f \geq \eta_2, \\ \max\{\Delta_k, \bar{\Delta}\}, & \text{if } \eta_1 \leq \rho_k^f < \eta_2, \\ \tau_1 \Delta_k, & \text{if } \rho_k^f < \eta_1. \end{cases} \tag{20}$$

When k is an h-type iteration, we always accept $x_k + s_k$ and update the trust region radius Δ_k according to the following rule

$$\Delta_{k+1}^f = \max\{\Delta_k, \bar{\Delta}\}. \tag{21}$$

When k is a c-type iteration, we accept $x_k + s_k$ if

$$\delta_k^c > 0, \quad \rho_k^c := \frac{h(x_k) - h(x_k + s_k)}{\delta_k^c} \geq \eta_1 \tag{22}$$

where

$$\delta_k^c := \frac{1}{2}||c_k||^2 - \frac{1}{2}||c_k + A_k s_k||^2. \tag{23}$$

The trust region radius Δ_k is then updated by

$$\Delta_{k+1} = \begin{cases} \min\{\max\{\tau_2 \Delta_k, \bar{\Delta}\}, \hat{\Delta}\} & \text{if } \rho_k^c \geq \eta_2, \\ \max\{\Delta_k, \bar{\Delta}\}, & \text{if } \eta_1 \leq \rho_k^c < \eta_2, \\ \tau_1 \Delta_k, & \text{if } \rho_k^c < \eta_1. \end{cases} \tag{24}$$

Before we present a formal description of our trust region infeasibility control algorithm, the refer should notice that formulae (20), (21), and (24) imply that

$$\Delta_{k+1} \geq \bar{\Delta} \tag{25}$$

if k is a successful iteration, which is important for the global convergence analysis in the next section.

2.3 The algorithm

Now a formal statement of the algorithm is presented as follows.

Algorithm 1 A trust-region algorithm with infeasibility control (TRIC)

> **Initialization:** Choose x_0, B_0 and parameters $\beta, \gamma, \theta, \zeta, \eta_1, \eta_2, \sigma_1, \sigma_2, \tau, \tau_1 \in (0, 1)$, $u, \tau_2 \in [1, +\infty)$, $l \in \{2, 3, \cdots\}$ and $\Delta_0, \bar{\Delta}, \hat{\Delta} \in (0, +\infty)$ such that $\bar{\Delta} < \Delta_0 < \hat{\Delta}$. Set $k = 0$.
> **Step 1:** Stop if x_k is a KT point.
> **Step 2:** Solve (4) for n_k if $c_k \neq 0$ and set $n_k = 0$ if $c_k = 0$.
> **Step 3:** Compute Z_k, solve (6) for v_k, set $t_k = Z_k v_k$, and obtain $s_k = n_k + t_k$.
> **Step 4:** When k is f-type, accept $x_k + s_k$ if (19) holds and update Δ_k according to (20).
> When k is h-type, accept $x_k + s_k$, update Δ_k according to (21), and update H_k.
> When k is c-type, accept $x_k + s_k$ if (22) holds, update Δ_k according to (24), and update H_k.
> **Step 5:** If $x_k + s_k$ has been accepted, set $x_{k+1} = x_k + s_k$, and set $x_{k+1} = x_k$ otherwise.
> **Step 6:** If $x_k + s_k$ has been accepted, solve (8) for λ_{k+1}.
> **Step 7:** If $x_k + s_k$ has been accepted, choose a symmetric matrix B_{k+1}.
> **Step 8:** Increment k by one and go to Step 1.

Remark 1 From step 4 we observe that the control set H_k is updated if k is a successful h or c-type iterations, and left unchanged otherwise. Also, from step 4 we see that k is always a successful iteration if it is h-type.

3 Global convergence

Firtst, we make the following assumptions that are essential for our convergence analysis.

Assumptions

A1. Then objective f and the constraints c are twice continuously differentiable.
A2. The set $\{x_k\} \cup \{x_k + s_k\}$ is contained in a compact and convex set Ω.
A3. There exists a positive constant κ_B such that $||B_k|| \leq \kappa_B$ for all k.
A4. Inequality (5) is satisfied for all k.
A5. There exist two constants $\kappa_h, \kappa_\sigma > 0$ such that

$$h(x) \leq \kappa_h \implies \sigma_{\min}(A(x)) \geq \kappa_\sigma, \tag{26}$$

where $\sigma_{\min}(A)$ represents the smallest singular value of A.

Remark By contrast, these assumptions are weaker than that in [13]. In [13], the authors use a double trust regions strategy and impose

$$||s_k|| \leq \kappa_s \min\{\Delta_k^c, \Delta_k^f\},$$

where κ_s is a positive constant and Δ_k^c and Δ_k^f are the trust regions for the normal step and the tangent step, respectively. This assumption is strong in theory because Δ_k^c and Δ_k^f there are updated independently.

In the rest of this section, we denote the index set of successful iterations by $\mathcal{S}$ and the index sets of f-type, h-type, and c-type iterations by $\mathcal{F}$, $\mathcal{H}$, and $\mathcal{C}$, respectively.

Lemma 1 *Suppose that $k \in \mathcal{S}$ and that x_k is a feasible point which is not a KT point. Then k must be an f-type iteration and therefore all elements of the control set are positive.*

Proof The feasibility of x_k implies that $n_k = 0$, $\delta_k^f = \delta_k^{f,t}$, $\delta_k^c = 0$, and by (22) that k cannot be a successful c-type iteration. The hypothesis that x_k is not a KT point implies by (16) that $\chi_k = ||Z_k^T g_k^n|| > 0$ and therefore (15) holds. Thus, k must be a successful f-type iteration. It follows from the mechanism of the algorithm, the control set H_k is updated only in successful h-type and c-type iterations. Recalling the update rule of the control set is substituting h_k^+ defined by (14) for $H_{k,1}$, we can deduce by induction that $H_{k,i} > 0$, $i = 1, \ldots, l$ for all k. □

Lemma 2 *For all k, we have*

$$h(x_j) \leq H_{k,1}, \quad \forall j \geq k, \tag{27}$$

and $H_{k,1}$ is monotonically non-increasing in k.

Proof Without loss of generality, we can assume that all k are successful iterations. We first prove the inequality

$$h(x_k) \leq H_{k,1} \tag{28}$$

by induction. Obviously, (9) implies that (28) holds for $k = 0$. For $k \geq 1$, we assume that (28) holds for $k - 1$ and consider the following three cases.

The first case is that $k - 1 \in \mathcal{F}$. Then at least one of (10)–(12) holds and therefore, according to the hypothesis $h(x_{k-1}) \leq H_{k-1,1}$, we have from (10)–(12) that

$$h(x_k) \leq \max\{H_{k-1,1}, \beta h(x_{k-1}), \beta H_{k-1,2}\} = H_{k-1,1}.$$

Since the H_k cannot be updated if k is an f-type iteration, we have $H_{k,1} = H_{k-1,1}$. Thus (28) follows.

The second case is that $k - 1 \in \mathcal{H}$. Lemma 1 implies that x_{k-1} is infeasible. Then either (11) or (12) is satisfied and H_{k-1} is updated by substituting h_{k-1}^+ for $H_{k-1,1}$. It follows from (11) and (12) that

$$h(x_k) \leq \beta \max\{h(x_{k-1}), H_{k-1,2}\}.$$

Therefore, by the update rule of the control set together with (14), we obtain

$$H_{k,1} = \max\{h_{k-1}^+, H_{k-1,2}\} = \max\{(1 - \theta)h(x_{k-1}) + \theta h(x_k), H_{k-1,2}\} > h(x_k).$$

Thus (28) follows.

The third case is that $k - 1 \in \mathcal{C}$. Then (22) holds and H_{k-1} is updated by substituting h_{k-1}^+ for $H_{k-1,1}$. According to (14) and (22), we have

$$h(x_k) < (1 - \theta)h(x_{k-1}) + \theta h(x_k) = h_{k-1}^+.$$

Therefore, by the update rule of the control set, we have $h_{k-1}^+ \leq H_{k,1}$. Hence we obtain (28) from the last two inequalities.

Now we can finish the proof of this lemma based on (28). Note that

$$\max\{h(x_{k+1}), h(x_k)\} \leq H_{k,1}$$

according to (10)–(12), (22), (28) and the mechanism of the algorithm. Then we have $h_k^+ \leq H_{k,1}$ from (14). Thus the monotonicity of $H_{k,1}$ follows from the update rule of the control set. Finally, (27) follows immediately from (28) and the monotonicity of $H_{k,1}$. □

Lemma 3 *For all k, we have that*

$$\delta_k^c \geq \kappa_c \|A_k^T c_k\| \min\left\{ \frac{\|A_k^T c_k\|}{\|A_k^T A_k\|}, \Delta_k \right\}, \tag{29}$$

where $\kappa_c = \frac{1}{2}\tau$, and

$$\delta_k^{f,t} \geq \kappa_f \chi_k \min\left\{\frac{\chi_k}{||B_k||}, \Delta_k\right\}, \tag{30}$$

where $\kappa_f = \frac{1}{2}\sqrt{1-\tau^2}$.

Proof It follows from Lemma 4.3 in [19] that a solution d^* to the problem

$$\begin{cases} \min \ g^T d + \frac{1}{2}d^T B d \\ \text{s.t. } ||d|| \leq \Delta \end{cases}$$

must satisfy the Cauchy condition

$$-g^T d^* - \frac{1}{2}(d^*)^T B d^* \geq \frac{1}{2}||g|| \min\left\{\frac{||g||}{||B||}, \Delta\right\}. \tag{31}$$

Then, according to (4), (7), (23), and (31), we have

$$\begin{aligned} \delta_k^c &= \frac{1}{2}||c_k||^2 - \frac{1}{2}||c_k + A_k n_k||^2 \\ &\geq \frac{1}{2}||A_k^T c_k|| \min\left\{\frac{||A_k^T c_k||}{||A_k^T A_k||}, \tau\Delta_k\right\} \\ &\geq \frac{1}{2}\tau||A_k^T c_k|| \min\left\{\frac{||A_k^T c_k||}{||A_k^T A_k||}, \Delta_k\right\}. \end{aligned}$$

Similarly, according to (6), (16), (18), and (31), we have

$$\begin{aligned} \delta_k^{f,t} &\geq \frac{1}{2}\chi_k \min\left\{\frac{\chi_k}{||Z_k^T B_k Z_k||}, \sqrt{\Delta_k^2 - ||n_k||^2}\right\} \\ &\geq \frac{1}{2}\chi_k \min\left\{\frac{\chi_k}{||B_k||}, \sqrt{1-\tau^2}\Delta_k\right\} \\ &\geq \frac{1}{2}\sqrt{1-\tau^2}\chi_k \min\left\{\frac{\chi_k}{||B_k||}, \Delta_k\right\}. \end{aligned}$$

The proof is complete.   □

Lemma 4 *For all k, we have that*

$$|f(x_k + s_k) - m_k(x_k + s_k)| \leq \kappa_D ||s_k||^2, \tag{32}$$

and

$$|\ ||c(x_k + s_k)||^2 - ||c_k + A_k s_k||^2\ | \leq \kappa_D ||s_k||^2, \tag{33}$$

where $\kappa_D > 0$ is a constant.

Proof Inequalities (32) and (33) are just two consequences of the assumptions at the beginning of this section and Taylor's theorem. □

Lemma 5 *If $k \in \mathcal{F}$ and*

$$\Delta_k \leq \kappa_\Delta^f \chi_k, \tag{34}$$

where $\kappa_\Delta^f = \min\{\frac{1}{\kappa_B}, \frac{(1-\eta_1)\zeta\kappa_f}{2\kappa_D}\}$, then $\rho_k^f > \eta_1$. Similarly, if $k \in \mathcal{C}$, $c_k \neq 0$, and

$$\Delta_k \leq \kappa_\Delta^c \|A_k^T c_k\|, \tag{35}$$

where $\kappa_\Delta^c = \min\{\frac{1}{\kappa_A}, \frac{(1-\eta_1)\kappa_c}{2\kappa_D}\}$ with κ_A being a positive constant, then $\rho_k^c > \eta_1$.

Proof It follows from (15), (30), and A4 that

$$\delta_k^f \geq \zeta\kappa_f \chi_k \min\left\{\frac{\chi_k}{\|B_k\|}, \Delta_k\right\} \geq \zeta\kappa_f \chi_k \min\left\{\frac{\chi_k}{\kappa_B}, \Delta_k\right\}.$$

This, together with (32) and the fact that

$$\|s_k\| \leq \|n_k\| + \|t_k\| \leq \left(\tau + \sqrt{1 - \tau^2}\Delta_k\right) \leq \sqrt{2}\Delta_k, \tag{36}$$

implies if (34) holds then

$$|1 - \rho_k^f| = \left|\frac{f(x_k + s_k) - m_k(x_k + s_k)}{\delta_k^f}\right| \leq \frac{\kappa_D\|s_k\|^2}{\zeta\kappa_f \chi_k \min\left\{\frac{\chi_k}{\kappa_B}, \Delta_k\right\}}$$

$$\leq \frac{2\kappa_D\Delta_k^2}{\zeta\kappa_f \chi_k \Delta_k} \leq 1 - \eta_1.$$

Hence, the first assertion follows. Similarly, using (29) and assumptions A1 and A2, we have

$$\delta_k^c \geq \kappa_c\|A_k^T c_k\| \min\left\{\frac{\|A_k^T c_k\|}{\kappa_A}, \Delta_k\right\},$$

where $\kappa_A = \max_k\{\|A_k^T A_k\|\}$. This, together with (33) and (36), implies that if (35) holds then

$$|1 - \rho_k^c| = \left|\frac{\|c(x_k + s_k)\|^2 - \|c_k + A_k s_k\|^2}{2\delta_k^c}\right| \leq \frac{\kappa_D\|s_k\|^2}{\kappa_c\|A_k^T c_k\| \min\left\{\frac{\|A_k^T c_k\|}{\kappa_A}, \Delta_k\right\}}$$

$$\leq \frac{2\kappa_D\Delta_k^2}{\kappa_c\|A_k^T c_k\|\Delta_k} \leq 1 - \eta_1.$$

Then the second assertion follows as well. $\square$

We show below that our algorithm can eventually make a step forward at any iterate which is not an infeasible stationary point of $h(x)$. We recall beforehand the definition of an infeasibility stationary point of $h(x)$.

Definition 1 A point $\hat{x}$ is an infeasible stationary point of $h(x)$ if $\hat{x}$ satisfies

$$A(\hat{x})^T c(\hat{x}) = 0 \quad \text{and} \quad c(\hat{x}) \neq 0.$$

Lemma 6 *Suppose that KT points and infeasible stationary points never occur. Then we have $|\mathcal{S}| = +\infty$.*

Proof According to the mechanism of the algorithm, $x_k + s_k$ must be accepted if k is an h-type iteration, we only consider the cases $k \in \mathcal{F}$ and $k \in \mathcal{C}$.

Suppose that x_k is infeasible. Since by the hypothesis x_k is not an infeasible stationary point, we have $||A_k^T c_k|| > 0$. It then follows from (29) that $\delta_k^c > 0$. Therefore, when $k \in \mathcal{C}$, Lemma 5 ensures that $\rho_k^c \geq \eta_1$ for all Δ_k such that $\Delta_k \leq \kappa_\Delta^c ||A_k^T c_k||$. Thus, k is a successful c-type iteration. When $k \in \mathcal{F}$, we know by (15) that $\chi_k > \sigma_1 ||c_k||^{\sigma_2}$ and by Lemma 5 that $\rho_k^f \geq \eta_1$ for all Δ_k such that $\Delta_k \leq \kappa_\Delta^f \chi_k$. Note that (16) implies χ_k depends on $g_k^n = B_k n_k + g_k$ and therefore depends on n_k which may change as Δ_k decreases. Since $||A_k^T c_k|| > 0$, it follows from Theorem 4.1 in [19] that $||n_k|| = \tau \Delta_k$ for all sufficiently small Δ_k. Using the arguments above and (5), we have

$$\chi_k \geq O(||n_k||^{\sigma_2}) = O(\Delta_k^{\sigma_2}).$$

Thus, (34) must be satisfied for all sufficiently small Δ_k. Therefore, a successful f-type iteration will eventually be finished at x_k.

Now we suppose that x_k is feasible. Since x_k is not a KT point, we have

$$\chi_k = ||Z_k^T g_k^n|| = ||Z_k^T g_k|| > 0.$$

So, (15) is satisfied. It follows from $c_k + A_k s_k = 0$, (36), and Taylor's theorem that

$$h(x_k + s_k) = \frac{1}{2} \sum_{i=1}^m c_i^2(x_k + s_k) = \frac{1}{8} \sum_{i=1}^m (s_k^T \nabla^2 c_i(\xi_i) s_k)^2 \leq \frac{1}{8} m \kappa_C^2 ||s_k||^4 \leq \frac{1}{2} m \kappa_C^2 \Delta_k^4, \tag{37}$$

where

$$\kappa_C = \max_{x \in \Omega, 1 \leq i \leq m} \{\nabla^2 c_i(x)\}.$$

So, (10) holds whenever $\Delta_k \leq \left(\frac{2 H_{k,1}}{m \kappa_C^2} \right)^{1/4}$. Applying Lemma 5 once again, we have $\rho_k^f \geq \eta_1$ when Δ_k is sufficiently small. Hence, a successful f-type iteration will be finished at x_k in the end. $\square$

Lemma 7 *If x_k is infeasible but not a stationary point, then*

$$\Delta_k \geq \min \left\{ \tau_1 \kappa_\Delta^f \max(\chi_k, \sigma_1 ||c_k||^{\sigma_2}), \bar{\Delta} \right\}, \tag{38}$$

or

$$\Delta_k \geq \min \left\{ \tau_1 \kappa_\Delta^c ||A_k^T c_k||, \bar{\Delta} \right\}. \tag{39}$$

Proof The results follows immediately from (15), (25), Lemma 4.5, the proof of Lemma 6 and the mechanism of the algorithm. □

Lemma 8 *Suppose $x^* \in \Omega$ is a feasible point but not a KT point. Then there exists a neighbourhood $\mathcal{N}(x^*)$ of x^* and positive constants δ, μ, κ such that for any $x \in \mathcal{N}(x^*) \cap \Omega$, if $\Delta_k \geq \mu ||c_k||$, then $c_k + A_k s_k = 0$ and (15) holds, and moreover, if*

$$\mu ||c_k|| \leq \Delta_k \leq \min\{\kappa, (\kappa_H H_{k,2})^{1/4}\},$$

where $\kappa_H = \frac{2\beta}{m\kappa_C^2}$, then (12) and (19) hold and $\delta_k^f \geq \delta \Delta_k$.

Proof Assumptions A1, A2, and A5 imply that when x_k is sufficiently close to x^*, $(A_k A_k^T)^{-1}$ exists and

$$||A_k^T (A_k A_k^T)^{-1} c_k|| \leq \kappa_I ||c_k|| \tag{40}$$

for some constant $\kappa_I > 0$. Therefore, if

$$\Delta_k \geq \frac{\kappa_I}{\tau} ||c_k||, \tag{41}$$

we have

$$n_k = -A_k^T (A_k A_k^T)^{-1} c_k \tag{42}$$

and $c_k + A_k s_k = 0$.

Because x^* is a feasible point but not a KT point, there exists a constant $\epsilon > 0$ such that, for all x_k sufficiently close to x^*,

$$\chi_k \geq \epsilon > \sigma_1 ||c_k||^{\sigma_2}, \tag{43}$$

and therefore by (30) and assumption A3

$$\delta_k^{f,t} \geq \kappa_f \epsilon \min \left\{ \frac{\epsilon}{\kappa_B}, \Delta_k \right\}. \tag{44}$$

Define

$$\delta_k^{f,n} := f(x_k) - m_k(x_k + n_k) = -g_k^T n_k - \frac{1}{2} n_k^T B_k n_k.$$

It follows from (40)–(42) and assumptions A1–A3 that if x_k is sufficiently close to x^* and (41) is satisfied,

$$|\delta_k^{f,n}| \leq \left(||g_k|| + \frac{1}{2}\kappa_B||n_k|| \right) ||n_k|| \leq \left(||g_k|| + \frac{1}{2}\kappa_B\kappa_I||c_k|| \right) \kappa_I||c_k|| \leq \kappa_G||c_k||, \tag{45}$$

where $\kappa_G = \kappa_I \max\limits_{x \in \Omega} \left\{ ||\nabla f(x)|| + \frac{1}{2}\kappa_B\kappa_I||c(x)|| \right\}$. Then, applying (41), (44), and (45), we have that if x_k is sufficiently close to x^* and

$$\max\left\{ \frac{\kappa_I}{\tau}, \frac{\kappa_G}{(1-\zeta)\kappa_f\epsilon} \right\} ||c_k|| \leq \Delta_k \leq \frac{\epsilon}{\kappa_B},$$

then

$$(1-\zeta)\delta_k^{f,t} \geq -\delta_k^{f,n},$$

and therefore

$$\delta_k^f = \delta_k^{f,t} + \delta_k^{f,n} \geq \zeta\delta_k^{f,t},$$

which together with (43) implies (15).

We deduce from Lemma 5 and (43) that if

$$\Delta_k \leq \kappa_\Delta^f \epsilon, \tag{46}$$

then (19) holds. If, in addition, Δ_k satisfies

$$\Delta_k \leq \min\left\{ \frac{\epsilon}{\kappa_B}, \left(\frac{2\eta_1\zeta\kappa_f\epsilon}{m\gamma\kappa_C^2} \right)^{1/3}, \left(\frac{2\beta H_{k,2}}{m\kappa_C^2} \right)^{1/4} \right\},$$

then by (19), (37), and (44), we have

$$f(x_k) - f(x_k + s_k) \geq \eta_1\delta_k^f \geq \eta_1\zeta\kappa_f\epsilon\Delta_k \geq \gamma h(x_k + s_k),$$

and

$$h(x_k + s_k) \leq \beta H_{k,2}.$$

The last two inequalities mean (12) holds. Finally, defining $\delta = \zeta\kappa_f\epsilon$, $\mu = \max\left\{ \frac{\kappa_I}{\tau}, \frac{\kappa_G}{(1-\zeta)\kappa_f\epsilon} \right\}$, $\kappa = \min\left\{ \frac{\epsilon}{\kappa_B}, \kappa_\Delta^f\epsilon\left(\frac{2\eta_1\zeta\kappa_f\epsilon}{m\gamma\kappa_C^2} \right)^{1/3} \right\}$, and choosing a sufficiently small neighbourhood $\mathcal{N}(x^*)$, we complete the proof. $\square$

Now we consider convergence of the case that successful c-type and h-type iterations are finitely many.

Lemma 9 *Suppose that $|\mathcal{S}| = +\infty$ and $|(\mathcal{H} \cup \mathcal{C}) \cap \mathcal{S}| < +\infty$. Then there exists a subsequence $\mathcal{K} \subset \mathcal{S}$ such that*

$$\lim_{k \to \infty, k \in \mathcal{K}} h(x_k) = 0, \qquad (47)$$

and any limit point of $\{x_k\}_{k \in \mathcal{K}}$ is a KT point.

Proof Suppose that x_k is infeasible for all sufficiently large k for otherwise (47) must hold for some subsequence $\mathcal{K}$. The hypothesis of this lemma implies $k \in \mathcal{F}$ for all $k \in \mathcal{S}$ sufficiently large. Then $\{f(x_k)\}$ is monotonically non-increasing from (15) and (19). It follows from (13) and Lemma 1 of [8] that $\lim_{k \to \infty} h(x_k) = 0$. Thus, (47) follows immediately.

Let x^* be an arbitrary limit point of $\{x_k\}_{k \in \mathcal{K}}$. From (47), we deduce that x^* is feasible. Without loss of generality, suppose that $\lim_{k \to \infty, k \in \mathcal{K}} x_k = x^*$. To derive a contradiction, we assume x^* is not a KT point. Then, for sufficiently large $k \in \mathcal{K}$, we have $x_k \in \mathcal{N}(x^*)$, where $\mathcal{N}(x^*)$ is a neighbourhood of x^* characterized in Lemma 8. Applying Lemma 8, if

$$\mu \|c_k\| \leq \Delta_k \leq \min\{\kappa, (\kappa_H H_{k,2})^{1/4}\}, \qquad (48)$$

$x_k + s_k$ must satisfies all the conditions for a successful f-type iteration. Note that the control set is not updated in a successful f-type iteration. Therefore, we can find an index k_0 such that $H_{k_0} = H_k$ for all $k \geq k_0$. Hence, for all sufficiently large $k \in \mathcal{K}$, the interval in (48) becomes

$$\mu \|c_k\| \leq \Delta_k \leq \min\{\kappa, (\kappa_H H_{k_0,2})^{1/4}\},$$

where the lower bound approaches zero and the upper bound is a positive constant. It then follows from the mechanism of the algorithm that, for all sufficiently large $k \in \mathcal{K}$,

$$\Delta_k \geq \Delta_{\min} := \min\left\{\tau_1 \kappa, \tau_1 (\kappa_H H_{k_0,2})^{1/4}, \bar{\Delta}\right\}.$$

Therefore, by Lemma 8 and the non-decreasing monotonicity of δ_k^f in Δ_k on the interval $[\mu \|c_k\|, +\infty)$, we have

$$f(x_k) - f(x_k + s_k) \geq \eta_1 \delta_k^f \geq \eta_1 \delta \Delta_{\min},$$

this together with the non-increasing monotonicity of $\{f(x_k)\}$, implies $f(x_k) \to -\infty$ as $k \to \infty$. This contradicts assumptions A1 and A2. So, the proof is complete. $\square$

Next we consider convergence of the case that successful c-type or h-type iterations are infinitely many.

Lemma 10 *Suppose* $|\mathcal{H}| = +\infty$. *Then* $\lim_{k \to \infty} h(x_k) = 0$.

Proof Denote $\mathcal{H} = \{k_i\}$. Since at least one of (10)–(12) holds in h-type iterations and by Lemma x_{k_i} is infeasible, we deduce from (11), (12), (14), and (27) that

$$
\begin{aligned}
h_{k_i}^+ &= (1 - \theta)h(x_{k_i}) + \theta h(x_{k_i+1}) \\
&\leq (1 - \theta)H_{k_i,1} + \theta\beta \max(H_{k_i,2}, h(x_{k_i})) \\
&\leq (1 - \theta + \theta\beta)H_{k_i,1}.
\end{aligned}
$$

It then follows from Lemma 2 and the update rule of the control set that

$$
H_{k_{i+l},1} \leq (1 - \theta + \beta\theta)H_{k_i,1}. \tag{49}
$$

Applying Lemma 2 once again together with (49), we have

$$
\lim_{k \to \infty} H_{k,1} = 0. \tag{50}
$$

Thus, the result follows from (27) and (50). $\qquad\square$

In what follows, to obtain global convergence, we will rule out a bad scenario of successful c-type iterations that is

$$
\lim_{k \to \infty, k \in \mathcal{C} \cap \mathcal{S}} ||A_k^T c_k|| = 0 \quad \text{with} \quad \liminf_{k \to \infty, k \in \mathcal{C} \cap \mathcal{S}} ||c_k|| > 0. \tag{51}
$$

Lemma 11 *Suppose* $|\mathcal{C} \cap \mathcal{S}| = +\infty$ *and* (51) *is avoided. Then* $\lim_{k \to \infty} h(x_k) = 0$.

Proof We first prove

$$
\lim_{k \to \infty, k \in \mathcal{C} \cap \mathcal{S}} ||A_k^T c_k|| = 0. \tag{52}
$$

Denote $\mathcal{C} \cap \mathcal{S} = \{k_i\}$. From (14), (22), (27), (29), we have

$$
\begin{aligned}
H_{k_i,1} - h_{k_i}^+ &\geq h(x_{k_i}) - h_{k_i}^+ = \theta(h(x_{k_i}) - h(x_{k_i}+1)) \geq \theta\eta_1\delta_{k_i}^c \\
&\geq \theta\eta_1\kappa_c ||A_{k_i}^T c_{k_i}|| \min\left\{\frac{||A_{k_i}^T c_{k_i}||}{||A_{k_i}^T A_{k_i}||}, \Delta_{k_i}\right\} \\
&\geq \theta\eta_1\kappa_c ||A_{k_i}^T c_{k_i}|| \min\left\{\frac{||A_{k_i}^T c_{k_i}||}{\kappa_A}, \Delta_{k_i}\right\},
\end{aligned} \tag{53}
$$

where κ_A is still defined by $\kappa_A = \max_k\{||A_k^T A_k||\}$ as in the proof of Lemma 5.

Since $k_i \in \mathcal{C} \cap \mathcal{S}$, x_k is an infeasible point by Lemma 1. Lemma 7 implies

$$
\Delta_{k_i} \geq \min\left\{\tau_1\kappa_\Delta^f\sigma_1 ||c_{k_i}||^{\sigma_2}, \tau_1\kappa_\Delta^c ||A_{k_i}^T c_{k_i}||, \bar{\Delta}\right\}. \tag{54}
$$

From (53), (54), and $\kappa_\Delta^c \leq \frac{1}{\kappa_A}$, we then have

$$H_{k_i,1} - h_{k_i}^+ \geq \theta \eta_1 \kappa_c \|A_{k_i}^T c_{k_i}\| \min \left\{ \tau_1 \kappa_\Delta^f \sigma_1 \|c_{k_i}\|^{\sigma_2}, \tau_1 \kappa_\Delta^c \|A_{k_i}^T c_{k_i}\|, \bar{\Delta} \right\}. \quad (55)$$

It therefore follows from (55), Lemma 2, and the update rule of the control set that

$$H_{k_i,1} - H_{k_{i+l},1} \geq \theta \eta_1 \kappa_c \|A_{k_i}^T c_{k_i}\| \min \left\{ \tau_1 \kappa_\Delta^f \sigma_1 \|c_{k_i}\|^{\sigma_2}, \tau_1 \kappa_\Delta^c \|A_{k_i}^T c_{k_i}\|, \bar{\Delta} \right\},$$

which implies (52) immediately.

Since (51) is avoided, it follows from (52) that $\liminf\limits_{k \to \infty, k \in \mathcal{C} \cap \mathcal{S}} \|c_k\| = 0$. So, there exists a subsequence $\mathcal{J} \subset \mathcal{C} \cap \mathcal{S}$ such that

$$\lim_{k \to \infty, k \in \mathcal{J}} \|c_k\| = 0. \quad (56)$$

Remembering $h(x_{k+1}) < h(x_k)$ for all $k \in \mathcal{C} \cap \mathcal{S}$, we have from (14) and (56) that $\lim_{k \to \infty, k \in \mathcal{J}} h_k^+ = 0$, which, together with Lemma 2 and the update rule of the control set, implies (50). The result then follows from (27) and (50). □

Lemma 12 *Suppose $|(\mathcal{H} \cup \mathcal{C}) \cap \mathcal{S}| = +\infty$ and (51) is avoided. Then*

$$\lim_{k \to \infty} h(x_k) = 0 \quad (57)$$

and there exists a constant $\kappa_\beta \in (0, 1)$ such that at least one limit point of $\{x_k\}$ is a KT point whenever $\beta \in [\kappa_\beta, 1)$.

Proof Equality (57) follows immediately from Lemma 10 and Lemma 11. It follows from (14) and (57) that $\lim_{k \to \infty} h_k^+ = 0$. Denote $\mathcal{K} = (\mathcal{H} \cup \mathcal{C}) \cap \mathcal{S}$. Therefore, by $|\mathcal{K}| = +\infty$, the positivity of any $H_{k,i}$, and the update rule of the control set, we can find a subsequence $\{k_i\} \subset \mathcal{K}$ such that

$$h_{k_i}^+ < H_{k,2}. \quad (58)$$

Suppose x^* is a limit point of $\{x_{k_i}\}$, which by (57) is a feasible point. To derive a contradiction, we assume x^* is not a KT point. Without loss of generality, we further assume $\lim_{i \to \infty} x_{k_i} = x^*$. Thus, for sufficiently large k_i, we have $x_{k_i} \in \mathcal{N}(x^*)$ and

$$\chi_{k_i} \geq \epsilon, \quad (59)$$

where $\mathcal{N}(x^*)$ is a neighbourhood of x^* characterized in Lemma 8, and $\epsilon > 0$ is a constant. According to (14) and (58), we have

$$h(x_{k_i}) \leq \frac{1}{1-\theta} h_{k_i}^+ < \frac{1}{1-\theta} H_{k,2},$$

and therefore

$$||c(x_{k_i})|| \leq \left(2h(x_{k_i})\right)^{1/2} \leq O\left((H_{k_i,2})^{1/2}\right). \tag{60}$$

We investigate the interval described in Lemma 8

$$\mu||c_{k_i}|| \leq \Delta_{k_i} \leq \min\left\{\kappa, (\kappa_H H_{k_i,2})^{1/4}\right\}. \tag{61}$$

It follows from (60) and $c(x_{k_i}) \to 0$ that the lower bound in (61) is eventually smaller than τ_1 times of the upper bound in (61). Thus, we have from (25) and Lemma 8 that, for all sufficiently large k_i,

$$\Delta_{k_i} \geq \tau_1 \left(\kappa_H H_{k_i,2}\right)^{1/4}. \tag{62}$$

In addition, Lemma 8 ensures that, in this situation, (15) must hold, and therefore k cannot be an h-type iteration.

Now we consider any sufficiently large k_i such that $x_{k_i} \in \mathcal{N}(x^*)$, (59) holds, and

$$h(x_{k_i}) \leq \kappa_h. \tag{63}$$

Using the arguments above, we know $k_i \in \mathcal{C} \cap \mathcal{S}$, which implies by Lemma 1 that x_{k_i} is infeasible. It then follows from (29) and assumptions A1, A2, and A5 that

$$\delta_{k_i}^c \geq \kappa_c ||A_{k_i}^T c_{k_i}|| \min\left\{\frac{||A_{k_i}^T c_{k_i}||}{||A_{k_i}^T A_{k_i}||}, \Delta_{k_i}\right\} \geq \kappa_c \kappa_\sigma ||c_{k_i}|| \min\left\{\frac{\kappa_\sigma ||c_{k_i}||}{\kappa_A}, \Delta_{k_i}\right\}. \tag{64}$$

According to (60) and (62), we have $\Delta_{k_i} \geq O(||c_{k_i}||^{1/2})$, which together with (64), implies for all sufficiently large k_i,

$$\delta_{k_i}^c \geq \kappa_c \kappa_\sigma ||c_{k_i}|| \min\left\{\frac{\kappa_\sigma ||c_{k_i}||}{\kappa_A}, O(||c_{k_i}||^{1/2})\right\} \geq \frac{\kappa_c \kappa_\sigma^2}{\kappa_A} ||c_{k_i}||^2. \tag{65}$$

Since $k_i \in \mathcal{C} \cap \mathcal{S}$, (22) holds, and therefore, by (65), we have

$$h(x_{k_i+1}) \leq h(x_{k_i}) - \eta_1 \delta_{k_i}^c \leq \left(1 - \frac{2\kappa_c \kappa_\sigma^2}{\kappa_A}\right) h(x_{k_i}).$$

This implies that if

$$\kappa_\beta := \left(1 - \frac{2\kappa_c \kappa_\sigma^2}{\kappa_A}\right) \leq \beta < 1,$$

then $h(x_{k_i+1}) \leq \beta h(x_{k_i})$ and therefore (11) holds for k_i. Thus, k_i cannot be a c-type iteration, which produces a contradiction. Hence, x^* is a KT point. $\qquad\square$

Table 1 Numerical results

Problem			TRIC	SNOPT
Name	n	m	Nit–Nf–Ng	Nit–Nf–Ng
aircrfta	8	5	3–4–4	5–5–4
alsotame	2	1	20–21–21	4–6–5
booth	2	2	3–4–4	2–2–2
bt2	3	1	16–19–17	20–17–16
bt3	5	3	8–9–9	5–5–5
bt4	3	2	7–8–8	10–11–10
bt5	3	2	8–9–9	10–11–10
bt6	5	2	13–15–14	17–17–16
bt7	5	3	30–35–31	21–37–36
bt8	5	2	13–14–14	13–14–13
bt9	4	2	21–25–22	20–31–30
bt10	2	2	6–7–7	1–24–23
bt11	5	3	12–16–13	15–15–14
bt12	5	3	7–9–8	9–10–9
byrdsphr	3	2	8–10–9	15–15–14
cb2	3	3	5–6–6	6–13–12
cb3	3	3	5–6–6	1–17–16
cbratu2d	882	882	1–2–2	441–3–2
cbratu3d	1,024	1,024	1–2–2	1,024–3–2
cluster	2	2	8–9–9	2–10–9
coolhans	9	9	11–13–12	5–5–4
coshfun	61	20	259–319–260	173–137–136
deconvc	61	1	210–246–211	251–77–76
dixchlng	10	5	33–50–34	40–31–30
dtoc1nd	735	490	42–51–43	397–50–49
dual1	85	1	165–362–366	83–83–83
dual2	96	1	129–550–130	99–99–99
dual3	111	1	185–759–186	118–118–118
dual4	75	1	42–197–43	67–67–67
eigena2	110	55	2–5–3	11–6–5
eigenaco	110	55	2–3–3	11–5–4
eigenb2	110	55	2–5–3	12–6–5
eigenbco	110	55	2–4–3	12–5–4
eigencco	30	15	33–45–34	38–37–36
fccu	19	8	21–28–22	19–19–19
fletcher	4	4	35–98–36	0–2–1
genhs28	10	8	6–7–7	11–11–11
gigomez1	3	3	5–6–6	6–24–23
gilbert	1,000	1	97–115–98	1,245–57–56

Table 1 continued

Problem			TRIC	SNOPT
Name	n	m	Nit–Nf–Ng	Nit–Nf–Ng
goffin	51	50	4–5–5	25–25–25
gottfr	2	2	5–9–6	2–9–8
haifas	13	9	14–15–15	29–26–25
hatfldg	25	25	8–11–9	44–26–25
heart8	8	8	9–12–10	31–31–30
himmelbc	2	2	5–6–6	2–7–6
himmelp2	2	1	17–22–18	21–26–25
hong	4	1	29–31–30	31–31–30
hs006	2	1	10–12–11	5–9–8
hs007	2	1	13–14–14	18–31–30
hs008	2	2	6–7–7	2–7–6
hs009	2	1	7–8–8	9–9–8

Table 2 Numerical results

Problem			TRIC	SNOPT
Name	n	m	Nit–Nf–Ng	Nit–Nf–Ng
hs026	3	1	18–19–19	25–25–24
hs027	3	1	20–26–21	22–24–23
hs028	3	1	8–9–9	4–4–4
hs039	4	2	21–25–22	20–31–30
hs040	4	3	6–7–7	7–8–7
hs042	4	2	10–11–11	8–9–8
hs046	5	2	29–30–30	28–27–26
hs047	5	3	24–26–25	24–32–31
hs048	5	2	9–13–10	6–6–6
hs049	5	2	20–22–21	37–33–32
hs050	5	3	14–16–15	31–22–21
hs051	5	3	6–8–7	6–6–6
hs052	5	3	7–11–8	5–5–5
hs056	7	4	23–28–24	13–15–14
hs061	3	2	7–8–8	69–169–168
hs077	5	2	11–15–12	15–15–14
hs078	5	3	7–8–8	9–8–7
hs079	5	3	8–10–9	14–15–14
hs100lnp	7	2	21–28–22	19–20–19
hs111lnp	10	3	187–235–188	126–295–294
hypcir	2	2	5–6–6	2–6–5
integreq	100	100	2–3–3	100–4–3
loadbal	31	31	4–5–5	84–55–54

Table 2 continued

Problem			TRIC	SNOPT
Name	n	m	Nit–Nf–Ng	Nit–Nf–Ng
lootsma	3	2	4–5–5	3–9–8
lotschd	12	7	41–42–42	8–8–8
lsnnodoc	5	4	14–16–15	12–10–9
maratos	2	1	4–5–5	8–14–13
mwright	5	3	11–14–12	12–11–10
odfits	10	6	35–36–36	35–33–32
orthrds2	203	100	31–36–32	185–157–156
orthrega	517	256	22–38–23	327–69–68
orthregb	27	6	7–8–8	27–9–8
polak3	12	10	49–54–50	103–302–301
portfl1	12	1	61–62–62	12–12–12
portfl2	12	1	61–62–62	12–12–12
portfl3	12	1	62–63–63	13–13–13
portfl4	12	1	61–62–62	11–11–11
portfl6	12	1	62–63–63	11–11–11
powellbs	2	2	25–29–26	2–16–15
powellsq	2	2	19–31–20	16–39–38
robot	14	2	6–7–7	83–136–135
simpllpa	2	2	1–2–2	3–3–3
supersim	2	2	1–2–2	1–1–1
tame	2	1	1–2–2	1–1–1
try-b	2	1	6–7–7	0–10–9
twobars	2	2	309–310–310	8–15–14
womflet	3	3	15–17–16	20–31–30
zangwil3	3	3	4–5–5	3–3–3
zy2	3	2	11–13–12	5–9–8

We now present our main result below.

Theorem 1 *Suppose that KT points and infeasible stationary points never occur and that* (51) *is avoided. Then at least one limit point of* $\{x_k\}$ *is a KT point whenever the parameter* β *in* (11) *and* (12) *satisfies* $\beta \in [\kappa_\beta, 1)$ *where* $\kappa_\beta \in (0, 1)$ *is a constant.*

Proof The result follows immediately from from Lemmas 6, 9, and 12. □

4 Numerical results

In this section, preliminary numerical results are shown to demonstrate the potential of the new trust region infeasibility control algorithm. All the codes of the new algorithm were written in MATLAB7.9. Details about our implementation are described as follows.
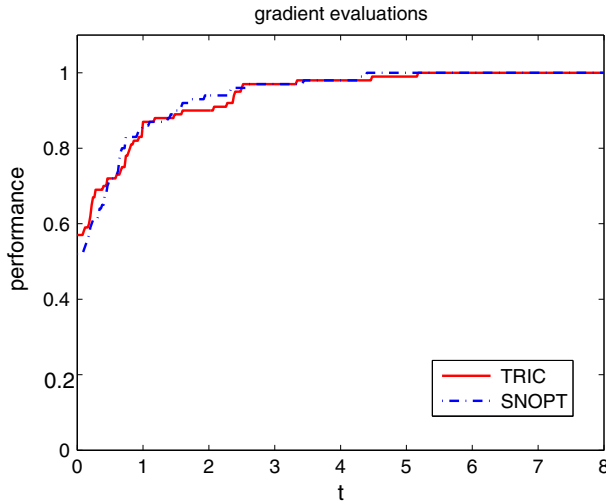
Fig. 1 Performance profile

A standard stopping criterion

$$||c_k||_\infty \le 10^{-6}(1 + ||x_k||_2),$$

and

$$||g_k + A_k^T \lambda_{k+1}||_\infty \le 10^{-6}(1 + ||\lambda_{k+1}||_2)$$

was used for our algorithm. The approximate Hessian B_k was initialized to the identity and updated by the damped BFGS formula. The dogleg method was applied to compute both normal and tangent steps. The Lagrangian multipliers were computed via MATLAB's lsqlin function. All the parameters were chosen as:

$$\tau = 0.8, \tau_1 = 0.5, \tau_2 = 1.2, \beta = 0.9999, \gamma = \theta = \zeta = \eta_1 = 10^{-4}, \eta_2 = 0.7,$$
$$\sigma_1 = 10^{-8}, \sigma_2 = 0.5, l = \max\{\min\{15, \lceil n/5 \rceil\}, 3\}, u = \max\{1000, 1.5h(x_0)\},$$
$$\Delta_0 = \max\{0.4||x_0||, 1.2\sqrt{n}\}, \hat{\Delta} = 10\Delta_0, \bar{\Delta} = 10^{-4}.$$

We compared our algorithm with the famous nonlinear optimization solver SNOPT [20]. The corresponding results are shown in Tables 1 and 2, where "TRIC" denotes our trust region infeasibility control algorithm, "Nit" denotes the number of iterations, "Nf" denotes the number of function evaluations, and "Ng" denotes the number of gradient evaluations. The test problems were a number of equality constrained problems chosen from the CUTEr collection [21].

We also plot the logarithmic performance profile of Dolan and Moré [22] in Fig. 1. In the plots, the performance profile is defined by

$$\pi_s(t) \triangleq \frac{\text{no. of problems where } \log_2(r_{p,s}) \leq t}{\text{total no. of problems}},$$

where $r_{p,s}$ is the ratio of Nf or Ng required to solve problem p by solver s and the lowest value of Ng required by any solver on this problem. The ratio $r_{p,s}$ is set to infinity whenever solver s fails to solve problem p. It can be observed from Fig. 1 that TRIC is comparable with SNOPT.

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