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ORIGINAL RESEARCH

Modelling and analysis of an eco-epidemiological model with time delay and stage structure

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Abstract A stage-structured predator–prey model with a transmissible disease spreading in the predator population and a time delay due to the gestation of the predator is formulated and analyzed. By analyzing corresponding characteristic equations, the local stability of each feasible equilibria and the existence of Hopf bifurcations at the disease-free equilibrium and the coexistence equilibrium are addressed, respectively. By using Lyapunov functions and the LaSalle invariant principle, sufficient conditions are derived for the global stability of the trivial equilibrium, the predator–extinction equilibrium and the disease-free equilibrium, respectively. Further, sufficient conditions are derived for the global attractiveness of the coexistence equilibrium of the proposed system. Numerical simulations are carried out to support the theoretical analysis.

Keywords Eco-epidemiological model · Stage structure · Time delay · LaSalle invariant principle · Hopf bifurcation · Stability

Mathematics Subject Classification 34K18 · 34K20 · 34K60 · 92D25

1 Introduction

Since the pioneering work of Kermack–Mckendrick on SIRS [7], epidemiological models have received much attention from scientists(see, for example, [2,3,5–7,10,

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11,13,15,19]). Mathematical models have become important tools in analyzing the spread and control of infectious disease. Eco-epidemiology which is a relatively new branch of study in theoretical biology, tackles such situations by dealing with both ecological and epidemiological issues. Following Anderso and May [2] who were the first to propose an eco-epidemiological model by merging the ecological predator–prey model introduced by Lotka and Volterra, the effect of disease in ecological system is an important issue from mathematical and ecological point of view. Recently, many works have been devoted to the study of the effects of a disease on a predator–prey system [3,5,6,10–13,15,19]. In [12], Venturino formulated two eco-epidemiological models with disease in the predators and mass action and standard incidence rates, respectively. In the two models, it was assumed that the disease spreads among predators only and that the infected individuals do not reproduce, only sound ones do. Analysis of the long-term behavior of solutions of the two models was carried out to show that whether and how the presence of the disease in the predator species affects the behavior of the ecological system, but also whether the introduction of a sound prey can affect the dynamics of the disease in the predator population. Following the work of Venturino [12], in [19], Zhang and Sun considered a predator–prey model with disease in the predator and general functional response. Sufficient conditions were derived for the permanence of the eco-epidemiological system. In [3], Guo et al. considered the following eco-epidemiological model

$$\begin{aligned}
\dot{x}(t) &= rx(t) - ax^2(t) - \frac{a_{12}x(t)S(t)}{1+mx(t)},\\
\dot{S}(t) &= \frac{a_{21}x(t)S(t)}{1+mx(t)} - d_1S(t) - \beta S(t)I(t),\\
\dot{I}(t) &= \beta S(t)I(t) - d_2I(t), \qquad (1.1)
\end{aligned}$$

where $x(t)$, $S(t)$ and $I(t)$ represent the densities of the prey, susceptible (sound) predator and infected predator population at time t, respectively. The parameters t, a_{12}, a_{21}, d_1, d_2, r, m and β are positive constants (see [3]). In system (1.1), the authors assumed that the infectious predator would die of diseases and only the healthy predator had predation capacity, but once infected with the disease, the predator would not be able to recover.

We note that it is assumed in system (1.1) that each individual prey admits the same risk to be attacked by predators. This assumption seems not to be realistic for many animals. In the natural world, there are many species whose individuals pass through an immature stage during which they are raised by their parents, and the rate at which they are attacked by predator can be ignored. Moreover, it can be assumed that their reproductive rate during this stage is zero. Stage-structure is a natural phenomenon and represents, for example, the division of a population into immature and mature individuals. Stage-structured models have received great attention in recent years (see, for example, [14,16–18]).

Time delays of one type or another have been incorporate into biological models by many researchers (see, for example, [1,9,17,18]). In general, delay differential equations exhibit much more complicated dynamics than ordinary differential equations since a time delay could cause the population to fluctuate. Time delay due to the gestation is a common example, because generally the consumption of prey by the predator throughout its past history governs the present birth rate of the predator. Therefore, more realistic models of population interactions should take into account the effect of time delays.

Based on above discussions, in this paper, we incorporate a stage structure for the prey and time delay due to the gestation of predator into the system (1.1). We make the following assumptions:

(**A1**) *The prey population* the birth rate is proportion to the existing mature population with a proportionality $r > 0$; the death rate of the immature population and transformation rate from immature individuals to mature individuals are proportional to the existing immature population with proportionality constants $d_1 > 0$ and $r_1 > 0$, respectively; the intra-specific competition rate of the mature population is $a > 0$; the death rate of the mature population is proportional to the existing mature population with a proportionality $d_2 > 0$.

(**A2**) *The predator population* the disease spreads among the predator species only by contact and the disease can not be transmitted vertically. The infected predator population do not recover or become immune. The disease incidence is assumed to be the simple mass action incidence βSI, where $\beta > 0$ is called the disease transmission coefficient. Only the susceptible predators have ability to capture mature prey with Holling type-II response function $x_2/(1 + mx_2)$, $m > 0$ is the search rate multiplied by the handling time and the infected predator are unable to catch mature prey due to a high infection. The capturing rate of the susceptible predator and the conversion rate of nutrients into the reproduction of the predator by consuming mature prey are $a_1 > 0$ and $a_2/a_1 > 0$, respectively; the natural death rate of the susceptible predator is $d_3 > 0$; the natural and disease-related mortality death rate of the infected predator is $d_4 > 0$.

To this end, we study the following differential equations

$$
\begin{aligned}
\dot{x}_1(t) &= rx_2(t) - (r_1 + d_1)\,x_1(t),\\
\dot{x}_2(t) &= r_1x_1(t) - d_2x_2(t) - ax_2^2(t) - \frac{a_1x_2(t)S(t)}{1 + mx_2(t)},\\
\dot{S}(t) &= \frac{a_2x_2(t-\tau)S(t-\tau)}{1 + mx_2(t-\tau)} - d_3S(t) - \beta S(t)I(t),\\
I(t) &= \beta S(t)I(t) - d_4I(t),
\end{aligned}
\tag{1.2}
$$

where $x_1(t)$ and $x_2(t)$ represent the densities of the immature and the mature prey population at time t, respectively. $\tau \geq 0$ is a constant delay due to the gestation of the predator.

The initial conditions for system (1.2) take the form

$$
\begin{aligned}
&x_1(\theta) = \varphi_1(\theta) \geq 0, \quad x_2(\theta) = \varphi_2(\theta) \geq 0, \quad S(\theta) = \phi_1(\theta) \geq 0,\\
&I(\theta) = \phi_2(\theta) \geq 0, \quad \theta \in [-\tau, 0)\\
&\qquad \varphi_1(0) > 0, \quad \varphi_2(0) > 0, \quad \phi_1(0) > 0,\\
&\qquad \phi_2(0) > 0, \quad (\varphi_1(\theta), \varphi_2(\theta), \phi_1(\theta), \phi_2(\theta)) \in C\left([-\tau, 0], R^4_{+0}\right),
\end{aligned}
\tag{1.3}
$$

where $R^4_{+0} = \{(y_1, y_2, y_3, y_4): y_i \geq 0, i = 1, 2, 3, 4\}$.

The organization of this paper is as follows. In the next section, we discuss the positivity and the boundedness of solutions of system (1.2) with initial conditions (1.3). In Sect. 3, by analyzing the corresponding characteristic equations, we study the local stability of each feasible boundary equilibria of system (1.2) and the existence of Hopf bifurcations of system (1.2) at the disease-free equilibrium. By means of Lyapunov functions and LaSalle invariant principle, we establish sufficient conditions for the global stability of each feasible boundary equilibria of system (1.2). In Sect. 4, by analyzing the corresponding characteristic equation, we discuss the local stability of coexistence equilibrium and the existence of Hopf bifurcations of system (1.2) at the coexistence equilibrium. By means of Lyapunov functions and LaSalle invariant principle, we establish sufficient conditions for the global attractiveness of the coexistence equilibrium. Numerical simulations are carried out in Sect. 5 to support the main theoretical results. A brief discussion is given in Sect. 6 to conclude this work.

2 Positivity and boundedness

It is well known by the fundamental theory of functional differential equations [4] that system (1.2) has a unique solution $(x_1(t), x_2(t), S(t), I(t))$ satisfying initial conditions (1.3). In this section, we show the positivity and the boundedness of solutions of system (1.2) with initial conditions (1.3).

Lemma 2.1 *Solutions of system (1.2) corresponding to initial conditions (1.3) are defined on* $[0, +\infty)$ *and remain positive for all* $t \geq 0$.

Proof Let $(x_1(t), x_2(t), S(t), I(t))$ be a solution of system (1.2) with initial conditions (1.3). First, we show that $x_2(t) > 0$ for all $t > 0$. Notice $x_2(0) > 0$, hence if there exists a t_0 such that $x_2(t_0) = 0$, then $t_0 > 0$. Assume that t_0 is the first such time that $x_2(t) = 0$, that is $t_0 = \inf\{t > 0: x_2(t_0) = 0\}$. By the second equation of system (1.2), we obtain $\dot{x}_2(t_0) = r_1 x_1(t_0) \leq 0$. Hence $x_1(t_0) \leq 0$. By the first equation of system (1.2), we have

$$x_1(t) = \left[x_1(0) + r\int_0^t x_2(s)e^{(r_1+d_1)s}ds\right]e^{-(r_1+d_1)t}. \tag{2.1}$$

By the definition of t_0, $x_2(t) \geq 0$ for $t \in [0, t_0]$. Thus, we have $x_1(t_0) > 0$. This contradiction shows that $x_2(t) > 0$ for all $t > 0$. By (2.1), we have $x_1(t) > 0$ for all $t > 0$.

Now, we show that $S(t) > 0$ for all $t > 0$. Let us consider $S(t)$ for $t \in [0, \tau]$. Since $\phi_1(\theta) \geq 0$ for $\theta \in [-\tau, 0]$, we derive from the third equation of system (1.2) that

$$\dot{S}(t) \geq -(d_3 + \beta I(t))\, S(t).$$

A standard comparison argument shows that

$$S(t) \geq S(0) \exp\left\{-\int_0^t (d_3 + \beta I(s))\, ds\right\} > 0,$$

i.e., $S(t) > 0$ for $t \in [0, \tau]$. In a similar way, we treat the intervals $[\tau, 2\tau], \ldots,$ $[n\tau, (n+1)\tau]$, $n \in N$. Thus, we have $S(t) > 0$ for all $t > 0$.

It follows from the fourth equation of (1.2) that

$$I(t) = I(0) \exp\left\{\int_0^t (\beta S(u) - d_4)\, du\right\} > 0.$$

Thus, we have $I(t) > 0$ for all $t > 0$. □

Lemma 2.2 *Positive solutions of system (1.2) with initial conditions (1.3) are ultimately bounded.*

Proof Let $(x_1(t), x_2(t), S(t), I(t))$ be any positive solution of system (1.2) with initial conditions (1.3). Denote $d = \min\{d_1, d_2, d_3, d_4\}$. Define

$$V(t) = x_1(t-\tau) + x_2(t-\tau) + \frac{a_1}{a_2}(S(t) + I(t)).$$

Calculating the derivative of $V(t)$ along positive solutions of system (1.2), it follows that

$$\begin{aligned}\dot{V}(t) = &-d_1 x_1(t-\tau) - d_2 x_2(t-\tau) - a x_2^2(t-\tau) + r x_2(t-\tau) - \frac{a_1}{a_2}[d_3 S(t) + d_4 I(t)] \\ &\leq -dV(t) - a\left[x_2(t-\tau) - \frac{r}{2a}\right]^2 + \frac{r^2}{4a} \\ &\leq -dV(t) + \frac{r^2}{4a}\end{aligned}$$

which yields

$$\limsup_{t\to\infty} V(t) \leq \frac{r^2}{4ad}.$$

If we choose $M_1 = r^2/(4ad),\ M_2 = a_2 r^2/(4aa_1 d)$, then

$$\limsup_{t\to\infty} x_i(t) \leq M_1\ (i = 1, 2), \quad \limsup_{t\to\infty} S(t) \leq M_2, \quad \limsup_{t\to\infty} I(t) \leq M_2.$$

□

Lemma 2.3 *For any positive solution* $(x_1(t), x_2(t), S(t), I(t))$ *of system* (*1.2*) *with initial conditions* (*1.3*), *we have that*

$$\liminf_{t\to+\infty} x_2(t) \geq \underline{x}_2 := \frac{rr_1 - (r_1+d_1)(d_2+a_1M_2)}{a(r_1+d_1)},$$

where M_2 *is defined in Lemma 2.2.*

Proof Let $(x_1(t), x_2(t), S(t), I(t))$ be any positive solution of system (1.2) with initial conditions (1.3). By Lemma 2.2, it follows that $\limsup_{t\to+\infty} S(t) \leq M_2$. Hence, for $\varepsilon > 0$ being sufficiently small, there is a $T_0 > 0$ such that if $t > T_0$, $S(t) < M_2 + \varepsilon$. Accordingly, for $\varepsilon > 0$ being sufficiently small, we derive from the first and the second equations of system (1.2) that, for $t > T_0$,

$$\begin{aligned} \dot{x}_1(t) &= rx_2(t) - (r_1+d_1)\,x_1(t), \\ \dot{x}_2(t) &\geq r_1x_1(t) - d_2x_2(t) - ax_2^2(t) - a_1\,(M_2+\varepsilon)\,x_2(t), \end{aligned} \tag{2.2}$$

which yields

$$\liminf_{t\to+\infty} x_2(t) \geq \underline{x}_2 := \frac{rr_1 - (r_1+d_1)(d_2+a_1M_2)}{a(r_1+d_1)}. \tag{2.3}$$

The proof is complete. □

3 Boundary equilibria and their stability

In this section, we discuss the stability of the boundary equilibria and the existence of a Hopf bifurcation at the disease-free equilibrium.

System (1.2) always has a trivial equilibrium $E_0(0, 0, 0, 0)$. If $rr_1 > d_2(r_1+d_1)$, then system (1.2) has a predator–extinction equilibrium $E_1(x_1^0, x_2^0, 0, 0)$, where

$$x_1^0 = \frac{r[rr_1 - d_2(r_1+d_1)]}{a(r_1+d_1)^2}, \quad x_2^0 = \frac{rr_1 - d_2(r_1+d_1)}{a(r_1+d_1)}.$$

Further, if the following holds

$$(H_1) \quad \frac{rr_1 - d_2(r_1+d_1)}{a(r_1+d_1)} > \frac{d_3}{a_2 - md_3} > 0,$$

then system (1.2) has a disease-free equilibrium $E_2(x_1^+, x_2^+, S^+, 0)$, where

$$\begin{aligned} x_1^+ &= \frac{rd_3}{(r_1+d_1)(a_2-md_3)}, \quad x_2^+ = \frac{d_3}{a_2-md_3}, \\ S^+ &= \frac{aa_2}{a_1(a_2-md_3)}\left[\frac{rr_1 - d_2(r_1+d_1)}{a(r_1+d_1)} - \frac{d_3}{a_2-md_3}\right]. \end{aligned}$$

The characteristic equation of system (1.2) at the equilibrium $E_0(0,\ 0,\ 0,\ 0)$ takes the form

$$(\lambda + d_3)(\lambda + d_4)\left[\lambda^2 + (r_1 + d_1 + d_2)\lambda + d_2(r_1 + d_1) - rr_1\right] = 0. \tag{3.1}$$

It is readily seen from Eq. (3.1) that if $rr_1 < d_2(r_1 + d_1)$, then E_0 is locally asymptotically stable; if $rr_1 > d_2(r_1 + d_1)$, then E_0 is unstable.

Theorem 3.1 *If $rr_1 < d_2(r_1 + d_1)$, then the trivial equilibrium $E_0(0,\ 0,\ 0,\ 0)$ of system (1.2) is globally asymptotically stable; if $rr_1 > d_2(r_1+d_1)$, then the equilibrium E_0 is unstable.*

Proof Based on above discussions, we only prove that all positive solutions of system (1.2) with initial conditions (1.3) converge to E_0. Let $(x_1(t),\ x_2(t),\ S(t),\ I(t))$ be any positive solution of system (1.2) with initial conditions (1.3). Define

$$V_0(t) = \frac{r_1}{r_1 + d_1}x_1(t) + x_2(t) + \frac{a_1}{a_2}[S(t) + I(t)].$$

Calculating the derivative of $V_0(t)$ along positive solutions of system (1.2), it follows that

$$\dot{V}_0(t) = -\frac{d_2(r_1 + d_1) - rr_1}{r_1 + d_1}x_2(t) - ax_2^2(t) - \frac{a_1}{a_2}\left[d_3S(t) + d_4I(t)\right]. \tag{3.2}$$

Let Λ be the largest invariant subset of $\{\dot{V}_0(t) = 0\}$. Clearly, if $rr_1 < d_2(r_1 + d_1)$, it then follows from (3.2) that $\dot{V}_0(t) \leq 0$ with equality if and only if $x_2(t) = 0,\ S(t) = 0$ and $I(t) = 0$. Noting that Λ is invariant, for each element in Λ, we have $x_2(t) = 0$. It therefore follows from the second equation of system (1.2) that

$$0 = \dot{x}_2(t) = r_1x_1(t),$$

which yields $x_1(t) = 0$. Hence, $\dot{V}_0(t) = 0$ if and only if $(x_1(t),\ x_2(t),\ y_1(t),\ y_2(t)) = (0,\ 0,\ 0,\ 0)$. Accordingly, the global asymptotic stability of E_0 follows from LaSalle invariant principle for delay differential systems. □

The characteristic equation of system (1.2) at the equilibrium $E_1(x_1^0,\ x_2^0,\ 0,\ 0)$ is of the form

$$(\lambda + d_4)\left[\lambda^2 + \left(r_1 + d_1 + d_2 + 2ax_2^0\right)\lambda + rr_1 - d_2(r_1 + d_1)\right]\left(\lambda + d_3 - \frac{a_2x_2^0}{1 + mx_2^0}e^{-\lambda\tau}\right) = 0. \tag{3.3}$$

Equation (3.3) always has a negative real root: $\lambda_1 = -d_4$. If $rr_1 > d_2(r_1 + d_1)$, then the equation

$$\lambda^2 + \left(r_1 + d_1 + d_2 + 2ax_2^0\right)\lambda + rr_1 - d_2(r_1 + d_1) = 0,$$

has two roots with negative real parts. All other roots of Eq. (3.3) are determined by the equation

$$\lambda + d_3 - \frac{a_2 x_2^0}{1 + m x_2^0} e^{-\lambda\tau} = 0. \tag{3.4}$$

Denote $f(\lambda) = \lambda + d_3 - \frac{a_2 x_2^0}{1+m x_2^0} e^{-\lambda\tau}$. If (H_1) holds, it is easy to show that, for λ real,

$$f(0) = d_3 - \frac{a_2 x_2^0}{1 + m x_2^0} = -\frac{(a_2 - m d_3)[r r_1 - d_2(r_1 + d_1)] - d_3 a(r_1 + d_1)}{a(r_1 + d_1) + m[r r_1 - d_2(r_1 + d_1)]} < 0,$$
$$\lim_{\lambda \to +\infty} f(\lambda) = +\infty.$$

Hence, $f(\lambda) = 0$ has a positive real root. Therefore, if (H_1) holds, the equilibrium $E_1(x_1^0,\ x_2^0,\ 0,\ 0)$ is unstable.

If the following holds:

$$(H_2)\ \ 0 < \frac{r r_1 - d_2(r_1 + d_1)}{a(r_1 + d_1)} < \frac{d_3}{a_2 - m d_3},$$

we claim that E_1 is locally asymptotically stable. Otherwise, there is a root λ satisfying $Re\lambda \geq 0$. It follows from (3.4) that

$$Re\lambda = \frac{a_2 x_2^0}{1 + m x_2^0} e^{-\tau Re\lambda} \cos(\tau Im\lambda) - d_3 \leq \frac{a_2 x_2^0}{1 + m x_2^0} - d_3 < 0,$$

which is a contradiction. Hence, if (H_2) holds, then the equilibrium E_1 is locally asymptotically stable.

Theorem 3.2 *If (H_2) holds, then the predator–extinction equilibrium $E_1(x_1^0,\ x_2^0,\ 0,\ 0)$ of system (1.2) is globally asymptotically stable; if (H_1) holds, then the equilibrium E_1 is unstable.*

Proof Based on above discussions, we only prove that all positive solutions of system (1.2) with initial conditions (1.3) converge to E_1. Let $(x_1(t),\ x_2(t),\ S(t),\ I(t))$ be any positive solution of system (1.2) with initial conditions (1.3). System (1.2) can be rewritten as

$$\begin{aligned}
\dot{x}_1(t) &= \frac{r}{x_1^0}\left[-x_2(t)\left(x_1(t) - x_1^0\right) + x_1(t)\left(x_2(t) - x_2^0\right)\right],\\
\dot{x}_2(t) &= \frac{r_1}{x_2^0}\left[-x_1(t)\left(x_2(t) - x_2^0\right) + x_2(t)\left(x_1(t) - x_1^0\right)\right]\\
&\quad + x_2(t)\left[-a\left(x_2(t) - x_2^0\right)\right] - \frac{a_1 x_2(t) S(t)}{1 + m x_2(t)},\\
\dot{S}(t) &= \frac{a_2 x_2(t-\tau) S(t-\tau)}{1 + m x_2(t-\tau)} - d_3 S(t) - \beta S(t) I(t),\\
\dot{I}(t) &= \beta S(t) I(t) - d_4 I(t).
\end{aligned} \tag{3.5}$$

Define

$$V_{11}(t) = c_1\left(x_1 - x_1^0 - x_1^0 \ln \frac{x_1}{x_1^0}\right) + x_2 - x_2^0 - x_2^0 \ln \frac{x_2}{x_2^0} + k_1(S + I),$$

where $c_1 = r_1 x_1^0/(r x_2^0)$, $k_1 = a_1(1 + m x_2^0)/a_2$. Calculating the derivative of $V_{11}(t)$ along positive solutions of system (1.2), it follows that

$$\begin{aligned}
\dot{V}_{11}(t) &= \frac{c_1(x_1(t) - x_1^0)}{x_1(t)}\dot{x}_1(t) + \frac{x_2(t) - x_2^0}{x_2(t)}\dot{x}_2(t) + k_1(\dot{S}(t) + \dot{I}(t)) \\
&= -\frac{r_1}{x_2^0}\left(\sqrt{\frac{x_2(t)}{x_1(t)}}\left(x_1(t) - x_1^0\right) - \sqrt{\frac{x_1(t)}{x_2(t)}}\left(x_2(t) - x_2^0\right)\right)^2 \\
&\quad - a\left(x_2(t) - x_2^0\right)^2 - k_1 d_4 I(t) - \frac{a_1(1 + m x_2^0)x_2(t)S(t)}{1 + m x_2(t)} \\
&\quad + \frac{a_1(1 + m x_2^0)x_2(t-\tau)S(t-\tau)}{1 + m x_2(t-\tau)} - \left(k_1 d_3 - a_1 x_2^0\right) S(t). \qquad (3.6)
\end{aligned}$$

Define

$$V_1(t) = V_{11}(t) + a_1\left(1 + m x_2^0\right)\int_{t-\tau}^{t} \frac{x_2(u)S(u)}{1 + m x_2(u)} du.$$

By calculation, we have that

$$\begin{aligned}
\dot{V}_1(t) = &-\frac{r_1}{x_2^0}\left(\sqrt{\frac{x_2(t)}{x_1(t)}}\left(x_1(t) - x_1^0\right) - \sqrt{\frac{x_1(t)}{x_2(t)}}\left(x_2(t) - x_2^0\right)\right)^2 - a\left(x_2(t) - x_2^0\right)^2 \\
&- \left(k_1 d_3 - a_1 x_2^0\right) S(t) - k_1 d_4 I(t). \qquad (3.7)
\end{aligned}$$

It follows from (3.7) that if $k_1 d_3 > a_1 x_2^0$, i.e., (H_2) holds, then $\dot{V}_1(t) \le 0$ with equality if and only if $x_1(t) = x_1^0$, $x_2(t) = x_2^0$, $S(t) = 0$ and $I(t) = 0$. Hence, the only invariant set $M = \{(x_1^0, x_2^0, 0, 0)\}$. Using LaSalle invariant principle for delay differential systems, the global asymptotic stability of E_1 follows. □

The characteristic equation of system (1.2) at the equilibrium $E_2(x_1^+, x_2^+, S^+, 0)$ takes the form

$$\left(\lambda + d_4 - \beta S^+\right)\left[\lambda^3 + g_2\lambda^2 + g_1\lambda + g_0 + \left(f_2\lambda^2 + f_1\lambda + f_0\right)e^{-\lambda\tau}\right] = 0, \qquad (3.8)$$

where

$$g_0 = d_3\left[\alpha\left(r_1 + d_1\right) - rr_1\right], \quad g_1 = d_3\left(r_1 + d_1 + \alpha\right) + \alpha\left(r_1 + d_1\right) - rr_1,$$
$$g_2 = r_1 + d_1 + d_3 + \alpha, \quad \alpha = d_2 + 2ax_2^+ + \frac{a_1 S^+}{(1 + mx_2^+)^2},$$
$$f_0 = -d_3\left[\alpha\left(r_1 + d_1\right) - rr_1 - \frac{a_1(r_1 + d_1)S^+}{(1 + mx_2^+)^2}\right],$$
$$f_1 = -d_3\left[r_1 + d_1 + \alpha - \frac{a_1 S^+}{(1 + mx_2^+)^2}\right], \quad f_2 = -d_3.$$

Clearly, Eq. (3.8) always has a root $\lambda_1 = \beta S^+ - d_4$. All other roots of Eq. (3.8) are determined by the following equation

$$\lambda^3 + g_2\lambda^2 + g_1\lambda + g_0 + \left(f_2\lambda^2 + f_1\lambda + f_0\right)e^{-\lambda\tau} = 0. \tag{3.9}$$

When $\tau = 0$, Eq. (3.9) reduces to

$$\lambda^3 + \left(g_2 + f_2\right)\lambda^2 + \left(g_1 + f_1\right)\lambda + g_0 + f_0 = 0. \tag{3.10}$$

It is easy to show that

$$\Delta_1 = g_2 + f_2 = r_1 + d_1 + \alpha > 0,$$
$$\Delta_2 = \left(g_1 + f_1\right)\left(g_2 + f_2\right) - \left(g_0 + f_0\right) = \left(r_1 + d_1 + \alpha\right)\left[\alpha\left(r_1 + d_1\right) - rr_1\right] + \frac{a_1 d_3 \alpha S^+}{(1 + mx_2^+)^2},$$
$$\Delta_3 = \frac{a_1 d_3 (r_1 + d_1) S^+}{(1 + mx_2^+)^2}\Delta_2.$$

Hence, if $\beta S^+ < d_4$ and the following hold:

$$(H_3)\ \left(r_1 + d_1 + \alpha\right)\left[\alpha\left(r_1 + d_1\right) - rr_1\right] + \frac{a_1 d_3 \alpha S^+}{(1 + mx_2^+)^2} > 0,$$

the equilibrium E_2 is locally asymptotically stable when $\tau = 0$.

If $i\omega(\omega > 0)$ is a solution of (3.9), separating real and imaginary parts, we have

$$\begin{aligned} f_1\omega\sin\omega\tau + \left(f_0 - f_2\omega^2\right)\cos\omega\tau &= g_2\omega^2 - g_0, \\ f_1\omega\cos\omega\tau - \left(f_0 - f_2\omega^2\right)\sin\omega\tau &= \omega^3 - g_1\omega. \end{aligned} \tag{3.11}$$

Squaring and adding the two equations of (3.11), it follows that

$$\omega^6 + h_2\omega^4 + h_1\omega^2 + h_0 = 0. \tag{3.12}$$

By calculation, we derive that

$$\begin{aligned} h_2 &= g_2^2 - 2g_1 - f_2^2 = (r_1 + d_1)^2 + \alpha^2 + 2rr_1 > 0, \\ h_1 &= g_1^2 - 2g_0g_2 + 2f_0f_2 - f_1^2 = [\alpha\,(r_1 + d_1) - rr_1]^2 \\ &\quad + \frac{a_1(d_3)^2 S^+}{(1 + mx_2^+)^2}\left(\alpha + d_2 + 2ax_2^+\right) > 0, \\ h_0 &= g_0^2 - f_0^2 = \frac{a_1(d_3)^2(r_1 + d_1)S^+}{(1 + mx_2^+)^2}\left[2\,(\alpha\,(r_1 + d_1) - rr_1) - \frac{a_1(r_1 + d_1)S^+}{(1 + mx_2^+)^2}\right]. \end{aligned}$$

Note that if $g_0 > f_0$, then (H_3) holds. Hence, if $g_0 > f_0$, Eq. (3.12) has no positive real roots. Accordingly, by Theorem 3.4.1 in Kuang [8], we see that if $g_0 > f_0$ and $\beta S^+ < d_4$, then E_2 is locally asymptotically stable for all $\tau \geq 0$. If $g_0 < f_0$, then Eq. (3.12) has a unique positive root ω_0. That is, Eq. (3.9) has a pair of purely imaginary roots of the form $\pm i\omega_0$. Denote

$$\tau_k = \frac{1}{\omega_0} \arccos \frac{f_1\omega_0(\omega_0^3 - g_1\omega_0) + (f_0 - f_2\omega_0^2)(g_2\omega_0^2 - g_0)}{(f_1\omega_0)^2 + (f_0 - f_2\omega_0^2)^2} + \frac{2k\pi}{\omega_0}, \quad k = 0,\ 1,\ 2, \ldots \tag{3.13}$$

By Theorem 3.4.1 in Kuang [8], we see that if $\beta S^+ < d_4,\ g_0 < f_0$ and (H_3) hold, then E_2 remains stable for $\tau < \tau_0$.

We now claim that

$$\left.\frac{d(Re(\lambda))}{d\tau}\right|_{\tau=\tau_0} > 0.$$

This will show that there exists at least one eigenvalue with positive real part for $\tau > \tau_0$. Moreover, the conditions for the existence of a Hopf bifurcation [4] are then satisfied yielding a periodic solution. To this end, differentiating Eq. (3.9) with respect to τ, it follows that

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{3\lambda^2 + 2g_2\lambda + g_1}{-\lambda(\lambda^3 + g_2\lambda^2 + g_1\lambda + g_0)} + \frac{2f_2\lambda + f_1}{\lambda(f_2\lambda^2 + f_1\lambda + f_0)} - \frac{\tau}{\lambda}.$$

Hence, a direct calculation shows that

$$\begin{aligned} sgn\left\{\frac{d(Re\lambda)}{d\tau}\right\}_{\lambda=i\omega_0} &= sgn\left\{Re\left(\frac{d\lambda}{d\tau}\right)^{-1}\right\}_{\lambda=i\omega_0} \\ &= sgn\left\{\frac{3\omega_0^4 + 2(g_2^2 - 2g_1)\omega_0^2 + g_1^2 - 2g_0g_2}{(\omega_0^3 - g_1\omega_0)^2 + (g_0 - g_2\omega_0^2)^2} + \frac{-2f_2^2\omega_0^2 + 2f_2f_0 - f_1^2}{(f_1\omega_0)^2 + (f_2\omega_0^2 - f_0)^2}\right\}. \end{aligned}$$

We derive from (3.11) that

$$\left(\omega_0^3 - g_1\omega_0\right)^2 + \left(g_0 - g_2\omega_0^2\right)^2 = (f_1\omega_0)^2 + \left(f_2\omega_0^2 - f_0\right)^2.$$

Hence, it follows that

$$sgn\left\{\frac{d(Re\lambda)}{d\tau}\right\}_{\lambda=i\omega_0} = sgn\left\{\frac{3\omega_0^4 + 2h_2\omega_0^2 + h_1}{(f_1\omega_0)^2 + (f_2\omega_0^2 - f_0)^2}\right\} > 0.$$

Therefore, the transversal condition holds and a Hopf bifurcation occurs at $\omega = \omega_0,\ \tau = \tau_0$.

In conclusion, we have the following results.

Theorem 3.3 *For system (1.2), assume (H_1) and $\beta S^+ < d_4$ hold, we have the following*:

(i) *If $g_0 > f_0$, then the disease-free equilibrium $E_2(x_1^+, x_2^+, S^+, 0)$ is locally asymptotically stable for all $\tau \geq 0$;*
(ii) *If $g_0 < f_0$ and (H_3) hold, then there exists a positive number τ_0, such that E_2 is locally asymptotically stable if $0 < \tau < \tau_0$ and is unstable if $\tau > \tau_0$. Further, system (1.2) undergoes a Hopf bifurcation at E_2 when $\tau = \tau_0$.*

Theorem 3.4 *Let (H_1) hold, if $\beta S^+ < d_4$, then the disease-free equilibrium E_2 is globally asymptotically stable provided*

$$(H_4)\ \ \underline{x}_2 > \frac{rr_1 - d_2(r_1 + d_1)}{a(r_1 + d_1)} - \frac{d_3}{a_2 - md_3}.$$

Here, $\underline{x}_2 > 0$ is defined in Lemma 2.3.

Proof It is easy to see that if (H_4) holds, then $x_2^+ > \frac{rr_1 - d_2(r_1+d_1)}{a(r_1+d_1)} - \frac{d_3}{a_2 - md_3}$. It follows from (3.12) that $g_0 > f_0$ holds. By Theorem 3.3, we see that if $\beta S^+ < d_4$, (H_1) and (H_4) hold, then the equilibrium $E_2(x_1^+ \cdot x_2^+, S^+, 0)$ is locally asymptotically stable. Hence, it suffices to show that all positive solutions of system (1.2) with initial conditions (1.3) converge to E_2. We achieve this by constructing a Lyapunov function. Let $(x_1(t), x_2(t), S(t), I(t))$ be any positive solution of system (1.2) with initial conditions (1.3).

System (1.2) can be rewritten as

$$\begin{aligned}
\dot{x}_1(t) &= \frac{r}{x_1^+}\left[-x_2(t)\left(x_1(t) - x_1^+\right) + x_1(t)\left(x_2(t) - x_2^+\right)\right],\\
\dot{x}_2(t) &= \frac{r_1}{x_2^+}\left[-x_1(t)\left(x_2(t) - x_2^+\right) + x_2(t)\left(x_1(t) - x_1^+\right)\right] + x_2(t)\left[-a\left(x_2(t) - x_2^+\right)\right]\\
&\quad + \frac{a_1 S^+}{1 + mx_2^+}x_2(t) - \frac{a_1 x_2(t) S(t)}{1 + mx_2(t)},\\
\dot{S}(t) &= \frac{a_2 x_2(t-\tau)S(t-\tau)}{1 + mx_2(t-\tau)} - d_3 S(t) - \beta S(t) I(t),\\
\dot{I}(t) &= \beta S(t) I(t) - d_4 I(t).
\end{aligned} \tag{3.14}$$

Define

$$V_{21}(t) = c_2\left(x_1 - x_1^+ - x_1^+ \ln \frac{x_1}{x_1^+}\right) + x_2 - x_2^+ - x_2^+ \ln \frac{x_2}{x_2^+} + k_2\left(S - S^+ - S^+ \ln \frac{S}{S^+}\right) + k_2 I,$$

where $c_2 = r_1 x_1^+/(r x_2^+),\ k_2 = a_1(1 + m x_2^+)/a_2$.

Calculating the derivative of $V_{21}(t)$ along positive solutions of system (1.2), it follows that

$$\begin{aligned}\dot{V}_{21}(t) &= c_2 \frac{x_1(t) - x_1^+}{x_1(t)}\dot{x}_1(t) + \frac{x_2(t) - x_2^+}{x_2(t)}\dot{x}_2(t) + k_2 \frac{S(t) - S^+}{S(t)}\dot{S}(t) + k_2 \dot{I}(t)\\ &= -\frac{r_1}{x_2^+}\left(\sqrt{\frac{x_2(t)}{x_1(t)}}\left(x_1(t) - x_1^+\right) - \sqrt{\frac{x_1(t)}{x_2(t)}}\left(x_2(t) - x_2^+\right)\right)^2 - a\left(x_2(t) - x_2^+\right)^2\\ &\quad - \frac{a_1(1 + m x_2^+)x_2(t)S(t)}{1 + m x_2(t)} + \frac{a_1(1 + m x_2^+)x_2(t-\tau)S(t-\tau)}{1 + m x_2(t-\tau)}\\ &\quad - k_2\left(d_4 - \beta S^+\right) I(t) - a_1\left(1 + m x_2^+\right)\frac{S^+}{S(t)}\frac{x_2(t-\tau)S(t-\tau)}{1 + m x_2(t-\tau)}\\ &\quad + \frac{a_1 S^+}{1 + m x_2^+}\left(x_2(t) - x_2^+\right) + k_2 d_3 S^+. \end{aligned} \tag{3.15}$$

Define

$$V_2(t) = V_{21}(t) + a_1\left(1 + m x_2^+\right)\int_{t-\tau}^{t}\left[\frac{x_2(u)S(u)}{1 + m x_2(u)} - \frac{x_2^+ S^+}{1 + m x_2^+} - \frac{x_2^+ S^+}{1 + m x_2^+}\ln \frac{(1 + m x_2^+)x_2(u)S(u)}{x_2^+ S^+(1 + m x_2(u))}\right] du.$$

By calculation, we have that

$$\begin{aligned}\dot{V}_2(t) &= -\frac{r_1}{x_2^+}\left(\sqrt{\frac{x_2(t)}{x_1(t)}}\left(x_1(t) - x_1^+\right) - \sqrt{\frac{x_1(t)}{x_2(t)}}\left(x_2(t) - x_2^+\right)\right)^2 - a\left(x_2(t) - x_2^+\right)^2\\ &\quad - a_1 x_2^+ S^+\left[\frac{(1 + m x_2^+)x_2(t-\tau)S(t-\tau)}{x_2^+ S(t)(1 + m x_2(t-\tau))} - 1 - \ln \frac{(1 + m x_2^+)x_2(t-\tau)S(t-\tau)}{x_2^+ S(t)(1 + m x_2(t-\tau))}\right]\\ &\quad - a_1 x_2^+ S^+\left[\frac{x_2^+(1 + m x_2(t))}{(1 + m x_2^+)x_2(t)} - 1 - \ln \frac{x_2^+(1 + m x_2(t))}{(1 + m x_2^+)x_2(t)}\right]\\ &\quad - k_2\left(d_4 - \beta S^+\right) I(t) - \left(x_2(t) - x_2^+\right)^2\left[a - \frac{a_1 S^+}{1 + m x_2^+}\frac{1}{x_2(t)}\right]. \end{aligned} \tag{3.16}$$

It follows from (3.16) that if $\beta S^+ < d_4$ and (H_4) hold true, then $\dot{V}_2(t) \le 0$ with equality if and only if $x_1(t) = x_1^+$, $x_2(t) = x_2^+$, $I(t) = 0$ and $\frac{(1+mx_2^+)S^+x_2(t-\tau)S(t-\tau)}{x_2^+S^+S(t)(1+mx_2(t-\tau))} = 1$. We now look for the invariant subset M within the set

$$\bigwedge = \left\{(x_1,\ x_2,\ S,\ I) : x_2(t) = x_2^+,\ x_1(t) = x_1^+,\ I(t) = 0,\ \frac{(1+mx_2^+)S^+x_2(t-\tau)S(t-\tau)}{x_2^+S^+S(t)(1+mx_2(t-\tau))} = 1\right\}.$$

Since $x_1(t) = x_1^+$ and $x_2(t) = x_2^+$ on M and consequently, $0 = \dot{x}_2(t) = r_1x_1^+ - d_2x_2^+ - a(x_2^+)^2 - \frac{a_1x_2^+S(t)}{1+mx_2^+}$, which yields $S(t) = S^+$. Hence, the only invariant set in $\bigwedge$ is $M = \{(x_1^+,\ x_2^+,\ S^+,\ 0)\}$. Using LaSalle invariant principle for delay differential systems, the global asymptotic stability of the equilibrium E_2 of system (1.2) follows. □

4 Coexistence equilibrium and its stability

In this section, we discuss the stability of the coexistence equilibrium and the existence of a Hopf bifurcation.

It is easy to show that if $\beta S^+ > d_4$, system (1.2) has a unique coexistence equilibrium $E^*(x_1^*,\ x_2^*,\ S^*,\ I^*)$, where

$$x_1^* = \frac{r}{r_1+d_1}x_2^*,\quad x_2^* = \frac{1}{2}\Delta + \sqrt{\frac{1}{4}\Delta^2 + \frac{1}{m}\left[\frac{rr_1 - d_2(r_1+d_1)}{a(r_1+d_1)} - \frac{a_1d_4}{a\beta}\right]},$$
$$\Delta = \frac{rr_1 - d_2(r_1+d_1)}{a(r_1+d_1)} - \frac{1}{m},\quad S^* = \frac{d_4}{\beta},\quad I^* = \frac{1}{\beta}\left[\frac{a_2x_2^*}{1+mx_2^*} - d_3\right].$$

The characteristic equation of system (1.2) at the equilibrium E^* is of the form

$$\lambda^4 + p_3\lambda^3 + p_2\lambda^2 + p_1\lambda + p_0 + \left(q_3\lambda^3 + q_2\lambda^2 + q_1\lambda\right)e^{-\lambda\tau} = 0, \tag{4.1}$$

where

$$\begin{aligned}
p_3 &= \hat{\alpha} + r_1 + d_1 + d_3 + \beta I^*,\\
p_2 &= \left(d_3 + \beta I^*\right)\left(\hat{\alpha} + r_1 + d_1\right) + \hat{\alpha}\left(r_1 + d_1\right) - rr_1 + \beta^2S^*I^*,\\
p_1 &= \beta^2S^*I^*\left(\hat{\alpha} + r_1 + d_1\right) + \left(d_3 + \beta I^*\right)\left[\hat{\alpha}\left(r_1 + d_1\right) - rr_1\right],\\
p_0 &= \beta^2S^*I^*\left[\hat{\alpha}\left(r_1 + d_1\right) - rr_1\right],\\
q_3 &= -\frac{a_2x_2^*}{1+mx_2^*},\quad q_2 = -\frac{a_2x_2^*}{1+mx_2^*}\left(d_2 + 2ax_2^* + r_1 + d_1\right),\\
q_1 &= -\frac{a_2x_2^*}{1+mx_2^*}\left[\left(r_1 + d_1\right)\left(d_2 + 2ax_2^*\right) - rr_1\right],\\
\hat{\alpha} &= d_2 + 2ax_2^* + \frac{a_1S^*}{(1+mx_2^*)^2}.
\end{aligned}$$

When $\tau = 0$, Eq. (4.1) becomes

$$\lambda^4 + (p_3 + q_3)\lambda^3 + (p_2 + q_2)\lambda^2 + (p_1 + q_1)\lambda + p_0 = 0. \tag{4.2}$$

If the following holds

$$(H_5)\ \ \hat{\alpha}(r_1 + d_1) > rr_1,$$

then it is easy to show that

$$\begin{aligned}
\Delta_1 &= p_3 + q_3 = \hat{\alpha} + r_1 + d_1 > 0,\\
\Delta_2 &= (p_3 + q_3)(p_2 + q_2) - (p_1 + q_1) = \left(\hat{\alpha} + r_1 + d_1\right)\left[\hat{\alpha}(r_1 + d_1) - rr_1\right]\\
&\quad + \hat{\alpha}\frac{a_2 x_2^* a_1 S^*}{(1 + m x_2^*)^3} > 0,\\
\Delta_3 &= (p_3 + q_3)(p_2 + q_2)(p_1 + p_2) - (p_1 + p_2)^2 - p_0(p_3 + q_3)^2\\
&= \beta^2 S^* I^* \frac{a_2 x_2^* a_1 S^*}{(1 + m x_2^*)^3}\left(\hat{\alpha} + r_1 + d_1\right) + \left(\hat{\alpha} + r_1 + d_1\right)\left[\hat{\alpha}(r_1 + d_1) - rr_1\right]\\
&\quad \left[\beta^2 S^* I^*\left(\hat{\alpha} + r_1 + d_1\right) + (r_1 + d_1)\frac{a_2 x_2^* a_1 S^*}{(1 + m x_2^*)^3}\right] > 0,\\
\Delta_4 &= p_0 \Delta_3 > 0.
\end{aligned}$$

Hence, by the Routh–Hurwitz criterion we know that if (H_5) holds, the coexistence equilibrium E^* of system (1.2) is locally asymptotically stable when $\tau = 0$.

Substituting $\lambda = i\omega(\omega > 0)$ into (4.1) and separating the real and imaginary parts, one obtains that

$$\begin{aligned}
\left(q_3\omega^3 - q_1\omega\right)\sin\omega\tau + q_2\omega^2\cos\omega\tau &= \omega^4 - p_2\omega^2 + p_0,\\
\left(q_3\omega^3 - q_1\omega\right)\cos\omega\tau - q_2\omega^2\sin\omega\tau &= p_1\omega - p_3\omega^3.
\end{aligned} \tag{4.3}$$

Squaring and adding the two equations of (4.3), it follows that

$$\omega^8 + \hat{h}_3\omega^6 + \hat{h}_2\omega^4 + \hat{h}_1\omega^2 + p_0^2 = 0, \tag{4.4}$$

where

$$\hat{h}_3 = p_3^2 - 2p_2 - q_3^2,\quad \hat{h}_2 = p_2^2 + 2p_0 - 2p_1 p_3 - q_2^2 + 2q_1 q_3,\quad \hat{h}_1 = p_1^2 - 2p_0 p_2 - q_1^2.$$

Letting $z = \omega^2$, Eq. (4.4) can be written as

$$\hat{h}(z) := z^4 + \hat{h}_3 z^3 + \hat{h}_2 z^2 + \hat{h}_1 z + p_0^2 = 0. \tag{4.5}$$

If $\hat{h}_3 > 0$ and $\hat{h}_2^2 - 4\hat{h}_1\hat{h}_3 < 0$, then $\hat{h}(z)$ has always no positive roots. Hence, under these conditions, Eq. (4.4) has no purely imaginary roots for any $\tau > 0$ and accordingly, the equilibrium E^* is locally asymptotically stable for all $\tau \geq 0$.

If Eq. (4.5) has at least one positive root, without loss of generality, we assume that (4.5) has four positive roots, namely, $z_1, \ z_2, \ z_3$ and z_4, respectively. Accordingly, Eq. (4.4) has four positive roots $\omega_k = \sqrt{z_k}$ $(k = 1, 2, 3, 4)$.

For $k = 1, 2, 3, 4$, from (4.3) one can get the corresponding $\tau_k^j > 0$ such that (4.1) has a pair of purely imaginary roots $\pm i\omega_k$ given by

$$\tau_k^j = \frac{2\pi j}{\omega_k} + \frac{1}{\omega_k} \arccos \frac{q_2\omega_k^2(\omega_k^4 - p_2\omega_k^2 + p_0) + (q_3\omega_k^3 - q_1\omega_k)(p_1\omega_k - p_3\omega_k^3)}{(q_2\omega_k^2)^2 + (q_3\omega_k^3 - q_1\omega_k)^2}, \quad j = 0, 1, 2, \ldots$$

Differentiating the two sides of (4.1) with respect to τ, it follows that

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{4\lambda^3 + 3p_3\lambda^2 + 2p_2\lambda + p_1}{-\lambda(\lambda^4 + p_3\lambda^3 + p_2\lambda^2 + p_1\lambda + p_0)} + \frac{3q_3\lambda^2 + 2q_2\lambda + q_1}{\lambda(q_3\lambda^3 + q_2\lambda^2 + q_1\lambda)} - \frac{\tau}{\lambda}.$$

After some algebra, one obtains that

$$\begin{aligned} sign\left\{\frac{dRe\lambda}{d\tau}\right\}_{\tau=\tau_k^j} &= sign\left\{Re\left(\frac{d\lambda}{d\tau}\right)^{-1}\right\}_{\tau=\tau_k^j} \\ &= sign\left\{-\frac{(p_1 - 3p_3\omega_k^2)(p_3\omega_k^2 - p_1) + 2(p_2 - 2\omega_k^2)(\omega_k^4 - p_2\omega_k^2 + p_0)}{\omega_k^2(p_1 - p_3\omega_k^2)^2 + (\omega_k^4 - p_2\omega_k^2 + p_0)^2}\right. \\ &\quad \left. + \frac{(q_1 - 3q_3\omega_k^2)(q_3\omega_k^2 - q_1) - 2q_2^2\omega_k^2}{(q_2\omega_k)^2 + (q_1\omega_k - q_3\omega_k^3)^2}\right\}. \end{aligned}$$

We derive from (4.3) that

$$\omega_k^2\left(p_1 - p_3\omega_k^2\right)^2 + \left(\omega_k^4 - p_2\omega_k^2 + p_0\right)^2 = (q_2\omega_k)^2 + \left(q_1\omega_k - q_3\omega_k^3\right)^2.$$

Hence, it follows that

$$\begin{aligned} sgn\left\{\frac{dRe\lambda}{d\tau}\right\}_{\tau=\tau_k^j} &= sgn\left\{\frac{4\omega_k^6 + 3\hat{h}_3\omega_k^4 + 2\hat{h}_2\omega_k^2 + \hat{h}_1}{(q_2\omega_k)^2 + (q_1\omega_k - q_3\omega_k^3)^2}\right\} \\ &= sgn\left\{\frac{\hat{h}'(z_k)}{(q_2\omega_k)^2 + (q_1\omega_k - q_3\omega_k^3)^2}\right\}. \end{aligned}$$

From what has been discussed above, we have the following results.

Theorem 4.1 *Assume that $\beta S^+ > d_4$ and $\hat{\alpha}(r_1 + d_1) > rr_1$ hold, we have*

(i) *If $\hat{h}_3 > 0$, and $\hat{h}_2^2 - 4\hat{h}_1\hat{h}_3 < 0$, then the coexistence equilibrium E^* is locally asymptotically stable for all $\tau \geq 0$.*
(ii) *If $\hat{h}(z)$ has at least one positive root z_k, then all roots of (4.1) have negative real parts for $\tau \in [0, \tau_k^0]$, and the equilibrium E^* of system (1.2) is locally asymptotically stable for $\tau \in [0, \tau_k^0]$.*
(iii) *If all conditions as stated in (ii) hold true and $\hat{h}'(z_k) \neq 0$, then system (1.2) undergoes a Hopf bifurcation at E^* when $\tau = \tau_k^j$ $(j = 0, 1, \ldots)$*

Now, we are concerned with the global attractiveness of the coexistence equilibrium E^*.

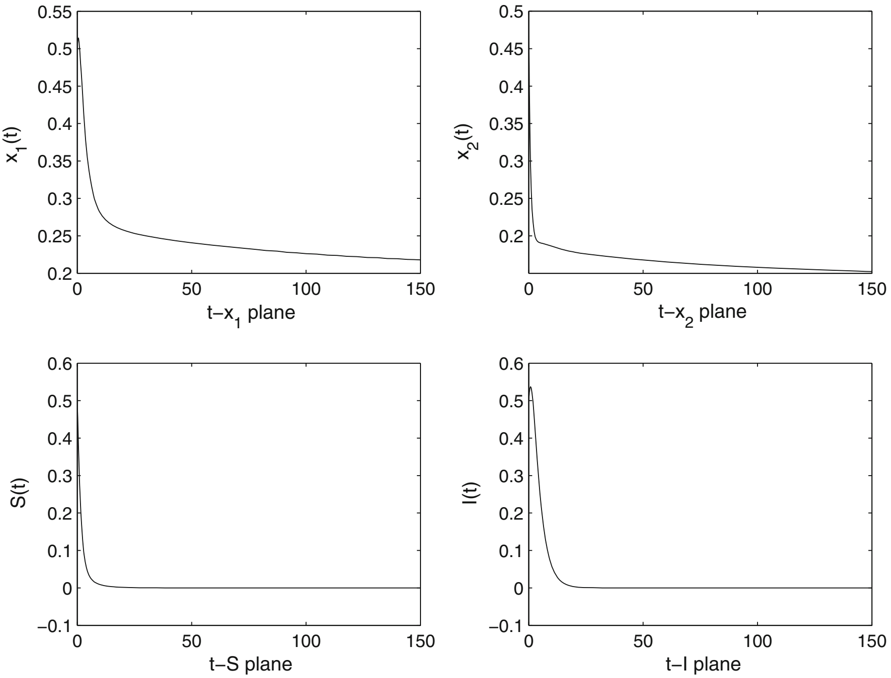

Fig. 1 The numerical solution of system (1.2) with $\tau = 0.01$, $(\phi_1, \phi_2, \varphi_1, \varphi_2) = (0.5, 0.5, 0.5, 0.5)$

Theorem 4.2 *Assume that* $\beta S^+ > d_4$. *If* (H_4) *holds, then the coexistence equilibrium* $E^*(x_1^*, x_2^*, S^*, I^*)$ *of system (1.2) is globally attractive.*

Proof Let $(x_1(t), x_2(t), S(t), I(t))$ be any positive solution of system (1.2) with initial conditions (1.3). System (1.2) can be rewritten as

$$
\begin{aligned}
\dot{x}_1(t) &= \frac{r}{x_1^*}\left[-x_2(t)\left(x_1(t)-x_1^*\right)+x_1(t)\left(x_2(t)-x_2^*\right)\right],\\
\dot{x}_2(t) &= \frac{r_1}{x_2^*}\left[-x_1(t)\left(x_2(t)-x_2^*\right)+x_2(t)\left(x_1(t)-x_1^*\right)\right]+x_2(t)\left[-a\left(x_2(t)-x_2^*\right)\right]\\
&\quad+\frac{a_1S^*}{1+mx_2^*}x_2(t)-\frac{a_1x_2(t)S(t)}{1+mx_2(t)},\\
\dot{S}(t) &= \frac{a_2x_2(t-\tau)S(t-\tau)}{1+mx_2(t-\tau)}-d_3S(t)-\beta S(t)I(t),\\
\dot{I}(t) &= \beta S(t)I(t)-d_4I(t).
\end{aligned} \tag{4.6}
$$

Define

$$
\begin{aligned}
V_{31}(t) &= c_3\left(x_1-x_1^*-x_1^*\ln\frac{x_1}{x_1^*}\right)+x_2-x_2^*-x_2^*\ln\frac{x_2}{x_2^*}+k_3\left(S-S^*-S^*\ln\frac{S}{S^*}\right)\\
&\quad+k_3\left(I-I^*-I^*\ln\frac{I}{I^*}\right).
\end{aligned}
$$

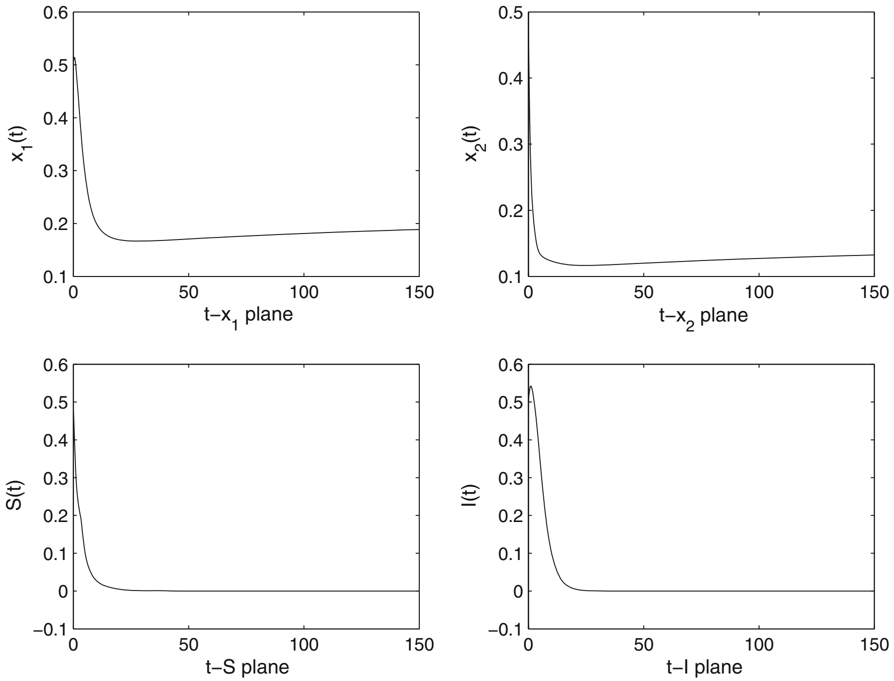

Fig. 2 The numerical solution of system (1.2) with $\tau = 3$, $(\phi_1, \phi_2, \varphi_1, \varphi_2) = (0.5, 0.5, 0.5, 0.5)$

where $c_3 = r_1 x_1^*/(r x_2^*)$, $k_3 = a_1(1 + m x_2^*)/a_2$. Calculating the derivative of $V_{31}(t)$ along positive solutions of system (1.2), it follows that

$$\begin{aligned}
\dot{V}_{31}(t) &= c_3 \frac{x_1(t) - x_1^*}{x_1(t)} \dot{x}_1(t) + \frac{x_2(t) - x_2^*}{x_2(t)} \dot{x}_2(t) + k_3 \frac{S(t) - S^*}{S(t)} \dot{S}(t) + k_3 \frac{I(t) - I^*}{I(t)} \dot{I}(t) \\
&= -\frac{r_1}{x_2^*} \left(\sqrt{\frac{x_2(t)}{x_1(t)}} \left(x_1(t) - x_1^*\right) - \sqrt{\frac{x_1(t)}{x_2(t)}} \left(x_2(t) - x_2^*\right) \right)^2 - a \left(x_2(t) - x_2^*\right)^2 \\
&\quad + \frac{a_1 S^*}{1 + m x_2^*} \left(x_2(t) - x_2^*\right) - \frac{a_1(1 + m x_2^*) x_2(t) S(t)}{1 + m x_2(t)} \\
&\quad + \frac{a_1(1 + m x_2^*) x_2(t - \tau) S(t - \tau)}{1 + m x_2(t - \tau)} - a_1 \left(1 + m x_2^*\right) \frac{S^*}{S(t)} \frac{x_2(t - \tau) S(t - \tau)}{1 + m x_2(t - \tau)} \\
&\quad + k_3 d_3 S^* + k_3 d_4 I^*. \qquad (4.7)
\end{aligned}$$

Define

$$V_3(t) = V_{31}(t) + a_1 \left(1 + m x_2^*\right) \int_{t-\tau}^{t} \left[\frac{x_2(u) S(u)}{1 + m x_2(u)} - \frac{x_2^* S^*}{1 + m x_2^*} - \frac{x_2^* S^*}{1 + m x_2^*} \ln \frac{(1 + m x_2^*) x_2(u) S(u)}{x_2^* S^* (1 + m x_2(u))} \right] du.$$

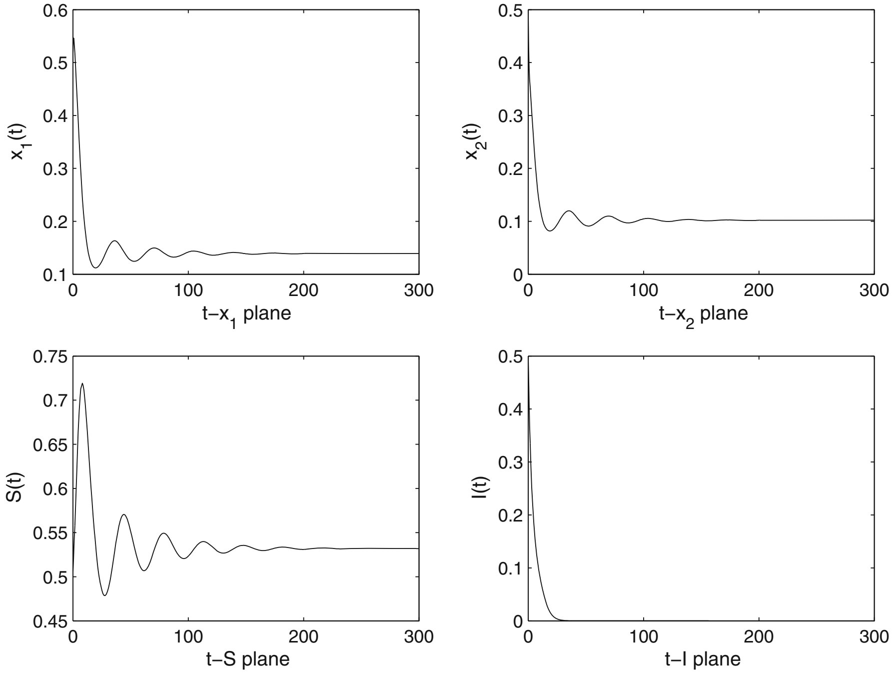

Fig. 3 The numerical solution of system (1.2) with $\tau = 0.01$, $(\phi_1, \phi_2, \varphi_1, \varphi_2) = (0.5, 0.5, 0.5, 0.5)$

By calculation, we have that

$$\begin{aligned}
\dot{V}_3(t) =& -\frac{r_1}{x_2^*}\left(\sqrt{\frac{x_2(t)}{x_1(t)}}\left(x_1(t)-x_1^*\right)-\sqrt{\frac{x_1(t)}{x_2(t)}}\left(x_2(t)-x_2^*\right)\right)^2 - a\left(x_2(t)-x_2^*\right)^2 \\
&+\frac{a_1 S^*}{1+mx_2^*}\left(x_2(t)-x_2^*\right) - a_1\left(1+mx_2^*\right)\frac{S^*}{S(t)}\frac{x_2(t-\tau)S(t-\tau)}{1+mx_2(t-\tau)} \\
&+a_1 x_2^* S^* + a_1 x_2^* S^* \ln\frac{(1+mx_2(t))x_2(t-\tau)S(t-\tau)}{x_2(t)S(t)(1+mx_2(t-\tau))} \\
=& -\frac{r_1}{x_2^*}\left(\sqrt{\frac{x_2(t)}{x_1(t)}}\left(x_1(t)-x_1^*\right)-\sqrt{\frac{x_1(t)}{x_2(t)}}\left(x_2(t)-x_2^*\right)\right)^2 \\
&-a_1 x_2^* S^*\left[\frac{(1+mx_2^*)x_2(t-\tau)S(t-\tau)}{x_2^* S(t)(1+mx_2(t-\tau))}-1\right. \\
&\left.-\ln\frac{(1+mx_2^*)x_2(t-\tau)S(t-\tau)}{x_2^* S(t)(1+mx_2(t-\tau))}\right] \\
&-a_1 x_2^* S^*\left[\frac{x_2^*(1+mx_2(t))}{x_2(t)(1+mx_2^*)}-1-\ln\frac{x_2^*(1+mx_2(t))}{x_2(t)(1+mx_2^*)}\right] \\
&-\left(x_2(t)-x_2^*\right)^2\left[a-\frac{a_1 S^*}{1+mx_2^*}\cdot\frac{1}{x_2(t)}\right].
\end{aligned} \tag{4.8}$$

Note that the function $g(x) = x - 1 - \ln x$ is always non-negative for any $x > 0$, and $g(x) = 0$ if and only if $x = 1$. Hence, if $x_2(t) > \frac{rr_1 - d_2(r_1+d_1)}{a(r_1+d_1)} - \frac{d_3}{a_2 - md_3}$ for $t \geq T$, we have $-(x_2(t) -$

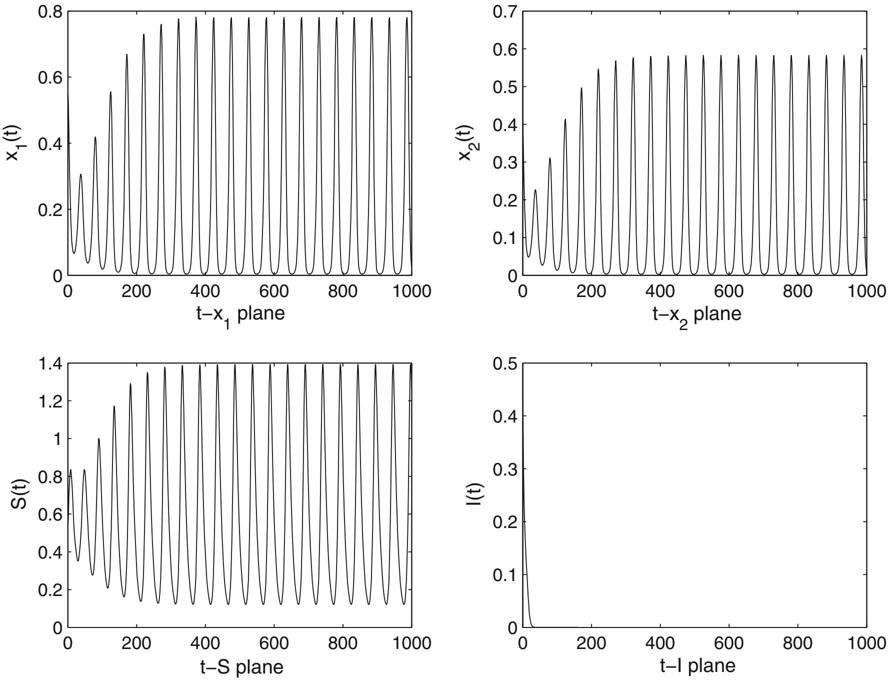

Fig. 4 The numerical solution of system (1.2) with $\tau = 3$, $(\phi_1, \phi_2, \varphi_1, \varphi_2) = (0.5, 0.5, 0.5, 0.5)$

$x_2^*)^2 \left[a - \frac{a_1 S^*}{1+mx_2^*} \frac{1}{x_2(t)}\right] \leq 0$ with equality if and only if $x_2(t) = x_2^*$. This, together with (4.8), implies that if $x_2(t) > \frac{rr_1-d_2(r_1+d_1)}{a(r_1+d_1)} - \frac{d_3}{a_2-md_3}$ for $t \geq T$, $\dot{V}_3(t) \leq 0$ with equality if and only if $x_1(t) = x_1^*$, $x_2(t) = x_2^*$, $\frac{(1+mx_2^*)x_2(t-\tau)S(t-\tau)}{x_2^* S(t)(1+mx_2(t-\tau))} = 1$. We now look for the invariant subset M within the set

$$\Lambda = \left\{(x_1, x_2, S, I) : x_1 = x_1^*, \ x_2 = x_2^*, \ \frac{(1+mx_2^*)x_2(t-\tau)S(t-\tau)}{x_2^* S(t)(1+mx_2(t-\tau))} = 1\right\}.$$

Since $x_1 = x_1^*$, $x_2 = x_2^*$ on M and consequently, $0 = \dot{x}_2(t) = r_1 x_1^* - d_2 x_2^* - a(x_2^*)^2 - \frac{a_1 x_2^*}{1+mx_2^*} S(t)$, which yields $S(t) = S^*$. It follows from the second equation of system (1.2) that $0 = \dot{S}(t) = \frac{a_2 x_2^* S^*}{1+mx_2^*} - d_3 S^* - \beta S^* I(t)$, which leads to $I = I^*$. Hence, the only invariant set M in Λ is $M = \{(x_1^*, x_2^*, S^*, I^*)\}$. Therefore, the global attractiveness of E^* follows from LaSalle invariant principle for delay differential systems. This completes the proof. □

5 Numerical simulations

In this section, we give some examples to illustrate the main results in Sects. 3 and 4.

Example 1 In system (1.2), let $r = 0.5$, $r_1 = 0.15$, $a = 0.1$, $a_1 = 1.5$, $a_2 = 0.5$, $d_1 = 0.2$, $d_2 = 0.2$, $d_3 = 0.25$, $d_4 = 0.3$, $\beta = 0.95$ and $m = 0.01$. It is easy to show that $rr_1 - d_2(r_1 + d_1) \approx 0.005$, $\frac{rr_1-d_2(r_1+d_1)}{a(r_1+d_1)} - \frac{d_3}{a_2-md_3} \approx -0.3597$, that is $rr_1 > d_2(r_1 + d_1)$ and (H_2)

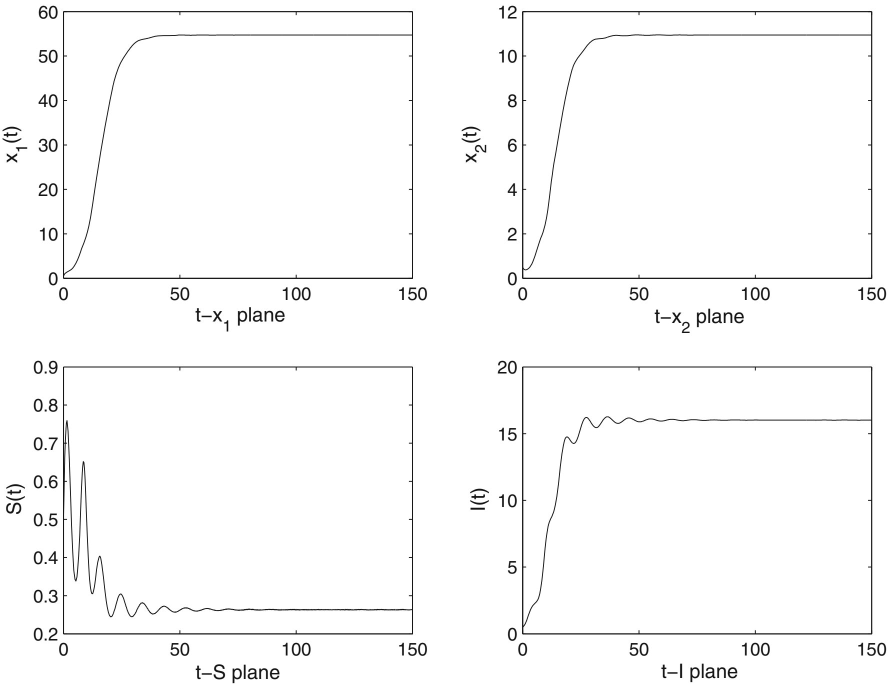

Fig. 5 The numerical solution of system (1.2) with $\tau = 0.01$, $(\phi_1, \phi_2, \varphi_1, \varphi_2) = (0.5, 0.5, 0.5, 0.5)$

hold true. Hence, system (1.2) has a predator–extinction equilibrium $E_1(0.2041, 0.1429, 0, 0)$. By Theorem 3.2, we see that E_1 is locally asymptotically stable for all $\tau \geq 0$. Numerical simulation illustrate the result above (see Figs. 1, 2).

Example 2 In system (1.2), let $r = 1.5$, $r_1 = 1$, $a = 0.8$, $a_1 = 1.5$, $a_2 = 1$, $d_1 = 0.1$, $d_2 = 0.5$, $d_3 = 0.1$, $d_4 = 0.5$, $\beta = 0.5$ and $m = 0.2$. It is easy to show that $\beta S^+ - d_4 \approx -0.2540$, $\frac{rr_1 - d_2(r_1+d_1)}{a(r_1+d_1)} - \frac{d_3}{a_2 - md_3} \approx 0.9775$, $(\alpha + r_1 + d_1)[\alpha(r_1 + d_1) - rr_1] + \frac{a_1 d_3 S^+}{(1+mx_2^+)^2} \approx 0.2603$, $g_0 - f_0 \approx -0.0698$, that is $\beta S^+ < d_4$, $g_0 < f_0$, (H_1), and (H_3) hold true. Hence, system (1.2) has a disease-free equilibrium $E_2(0.1391, 0.1020, 0.5320, 0)$. By Theorem 3.3, we see that there exists a positive constant $\tau_0 \approx 0.0556$ such that E_2 is locally asymptotically stable if $0 < \tau < \tau_0$ and is unstable if $\tau > \tau_0$. Further, system (1.2) undergoes a Hopf bifurcation at E_2 when $\tau = \tau_0$. An investigation of system (1.2) with the coefficients above can be conducted via a numerical integration using the standard Matlab algorithm (see Figs. 3, 4).

Example 3 In system (1.2), let $r = 3$, $r_1 = 0.3$, $a = 0.1$, $a_1 = 1$, $a_2 = 2.95$, $d_1 = 0.3$, $d_2 = 0.28$, $d_3 = 0.2$, $d_4 = 0.25$, $\beta = 0.95$ and $m = 0.1$. By calculation we have $\beta S^+ - d_4 \approx 0.9104$. Hence, system (1.2) has a unique coexistence equilibrium $E^*(54.7174, 10.9435, 0.2632, 16.0152)$. A direct calculation shows that $\hat{h}_3 \approx 0.9470$, $\hat{h}_2^2 - 4\hat{h}_1\hat{h}_4 \approx -55.1220$. By Theorem 4.1, we see that E^* is locally asymptotically stable for all $\tau \geq 0$. Numerical simulation illustrate the result above (see Figs. 5, 6).

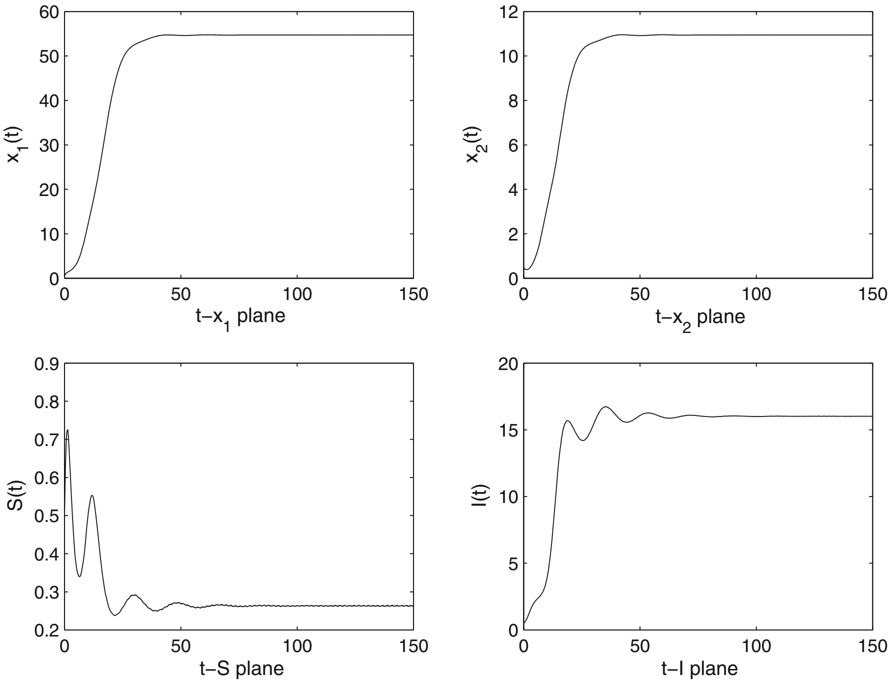

Fig. 6 The numerical solution of system (1.2) with $\tau = 3$, $(\phi_1, \phi_2, \varphi_1, \varphi_2) = (0.5, 0.5, 0.5, 0.5)$

6 Conclusion

In this paper, we have incorporated a stage structure for the prey and time delay due to the gestation of the predator into an eco-epidemiological model. The prey individuals were classified as belonging either the immature or the mature and it was assumed that the immature prey is not at risk of being attacked by the predator. This seems reasonable for a number of mammals, where the immature prey concealed in the mountain cave, are raised by their parents; they do not necessarily go out for seeking food, the rate they are attacked by the predators can be ignored. By comparison argument, a priori lower bound of the density of the mature prey population was derived. By analyzing the corresponding characteristic equations, the local stability of each of feasible equilibria of system (1.2) has been established, respectively. It has been shown that, under some conditions, the time delay may destabilize both the disease-free equilibrium and the coexistence equilibrium of the eco-epidemiological system and cause the population to fluctuate. By means of Lyapunov functions and LaSalle invariant principle, sufficient conditions were obtained for the global asymptotic stability of each of feasible equilibria of system (1.2), respectively. By Theorem 4.2, we see that if the prey population is always abundant enough and the disease transmission coefficient β is sufficiently large, the coexistence equilibrium is a global attractor of the system (1.2). In this case, the disease spreading in the predator becomes endemic and the prey, sound predator and the infected predator coexist. By Theorem 3.4, we see that if the disease transmission coefficient β is sufficiently small and the prey population is always abundant enough, the disease among the predator population dies out and in this case, the prey and the sound predator coexist. By Theorem 3.2, we see that if (H_2) holds, that is the carrying capacity of the prey and the conversion rate of the predator are sufficiently small, and the death rate of the sound predator and the half saturation rate of the predator are sufficiently large, the prey population persists and the predator population goes to extinction.

By Theorem 3.1, we see that if $rr_1 < d_2(r_1 + d_1)$, then both the prey population and the predator population go to extinction.

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References

1. Aiello, W.G., Freedman, H.I.: A time delay model of single species growth with stage-structure. Math. Biosci. **101**, 139–156 (1990)
2. Anderso, R.M., May, R.M.: Infections Disease of Humans Dynamics and Control. Oxford University Press, Oxford (1991)
3. Guo, Z., Li, W., Cheng, L., Li, Z.: Eco-epidemiological model with epidemic and response function in the predator. J. Lanzhou Univ. (Nat. Sci.) **45**(3), 117–121 (2009)
4. Hale, J.: Theory of Functional Differential Equation. Springer, Heidelberg (1977)
5. Haque, M.: A predator–prey model with disease in the predator species only. Nonlinear Anal. Real World Appl. **11**, 2224–2236 (2010)
6. Haque, M., Venturino, E.: An eco-epidemiological model with disease in predator: the ratio-dependent case. Math. Methods Appl. Sci. **30**, 1791–1809 (2007)
7. Kermack, W.O., Mckendrick, A.G.: A contribution on the mathematical theory of epidemics. Proc. R. Soc. Lond. A **115**, 700–721 (1927)
8. Kuang, Y.: Delay Differential Equation with Application in Population Dynamics. Academic Press, New York (1993)
9. Kuang, Y., So, J.W.H.: Analysis of a delayed two-stage population with space-limited recruitment. SIAM J. Appl. Math. **55**, 1675–1695 (1995)
10. Pal, Ak, Samanta, G.P.: Stability analysis of an eco-epidemiological model incorporating a prey refuge. Nonlinear Anal. Model. Control **15**(4), 473–491 (2010)
11. Shi, X., Cui, J., Zhou, X.: Stability and Hopf bifurcation analysis of an eco-epidemiological model with a stage structure. Nonlinear Anal. **74**, 1088–1106 (2011)
12. Venturino, E.: Epidemics in predator–prey models: disease in the predators. IMA J. Math. Appl. Med. Biol. **19**, 185–205 (2002)
13. Wang, S.: The research of eco-epidemiological of models incorporating prey refuges. PhD Thesis, Lanzhou University (2012)
14. Wang, W., Chen, L.: A predator–prey system with stage-structure for predator. Comput. Math. Appl. **33**, 83–91 (1997)
15. Xiao, Y., Chen, L.: A ratio-dependent predator–prey model with disease in the prey. Appl. Math. Comput. **131**, 397–414 (2002)
16. Xiao, Y., Chen, L.: Global stability of a predator–prey system with stage structure for the predator. Acta Math. Sin. **19**, 1–11 (2003)
17. Xu, R., Ma, Z.: Stability and Hopf bifurcation in a predator–prey model with stage structure for the predator. Nonlinear Anal. Real World Appl. **9**(4), 1444–1460 (2008a)
18. Xu, R., Ma, Z.: Stability and Hopf bifurcation in a ratio-dependent predator–prey system with stage structure. Chaos Solitons Fractals **38**, 669–684 (2008b)
19. Zhang, J., Sun, S.: Analysis of eco-epidemiological model with disease in the predator. J. Biomath. **20**(2), 157–164 (2005)