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ORIGINAL RESEARCH

Pseudo almost periodic solutions for a class of nonlinear Duffing system with a deviating argument

Bingwen Liu · Cemil Tunç

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Abstract This paper is concerned with a nonlinear Duffing system with pseudo almost periodic coefficients and delay. Under proper conditions, by using theory of exponential dichotomies and contraction mapping principle, some sufficient conditions are established to ensure the existence and uniqueness of pseudo almost periodic solutions, which improve and extend some known relevant results. Moreover, an example is given to illustrate the theoretical results.

Keywords Pseudo almost periodic solution · Duffing system · Exponential dichotomy · Contraction mapping principle · Deviating argument

Mathematics Subject Classification 34C25 · 34K13 · 34K25

1 Introduction

As we know, due to the promising potential in the areas of physics, mechanics and the engineering technique fields, the study of the existence of almost periodic solutions for Duffing equations has attracted many authors (see [1–5] and the references therein).

B. Liu (✉)
College of Mathematics and Computer Science, Hunan University of Arts and Science,
Changde 415000 , Hunan, People's Republic of China
e-mail: liubw007@yahoo.com.cn; liubw007@aliyun.com

C. Tunç
Department of Mathematics, Faculty of Sciences, Yüzüncü Yıl University, 65080 Van, Turkey
e-mail: cemtunc@yahoo.com

Recently, Peng and Wang [6] considered the following model for nonlinear Duffing equation with a deviating argument:

$$x''(t) + cx'(t) - ax(t) + bx^m(t - \tau(t)) = p(t). \tag{1.1}$$

Define

$$y = \frac{dx}{dt} + \xi x - Q_1(t), \quad Q_2(t) = p(t) + (\xi - c)Q_1(t) - Q_1'(t), \tag{1.2}$$

where $\xi > 1$ is a constant, then Peng and Wang [6] transformed (1.1) into the following system:

$$\begin{cases} \dfrac{dx(t)}{dt} = -\xi x(t) + y(t) + Q_1(t), \\ \dfrac{dy(t)}{dt} = -(c - \xi)y(t) + (a - \xi(\xi - c))x(t) - bx^m(t - \tau(t)) + Q_2(t). \end{cases} \tag{1.3}$$

Consequently, the system (1.3) has been naturally extended by Xu [7] to the following Duffing system with time-varying coefficients and delay:

$$\begin{cases} \frac{dx(t)}{dt} = -\delta_1(t)x(t) + y(t) + Q_1(t) \\ \frac{dy(t)}{dt} = \delta_2(t)y(t) + [\alpha(t) - \delta_2^2(t)]x(t) - \beta(t)x^m(t - \tau(t)) + Q_2(t), \end{cases} \tag{1.4}$$

where $\alpha(t)$, $\beta(t)$, $\tau(t)$, $\delta_1(t)$, $\delta_2(t)$, $Q_1(t)$ and $Q_2(t)$ are almost periodic functions on $\mathbb{R}$, $m > 1$ is an integer, $\alpha(t) > 0$, $\beta(t) \neq 0$. In particular, some sufficient conditions for the existence of almost periodic solutions of (1.3) and (1.4) were established in [6,7].

On the other hand, the existence of pseudo almost periodic solutions is among the most attractive topics in qualitative theory of differential equations due to its wide applications, especially in the fields of biology, economics and physics(see [8]). Furthermore, some criteria ensuring the existence of pseudo almost periodic solutions on functional differential equations were established in [9–11] and the references cited therein. Moreover, the properties of the almost periodic functions do not always hold in the set of pseudo almost periodic functions. For example, if $x(t)$ and $\tau(t)$ are almost periodic functions, we can show that $x(t - \tau(t))$ is an almost periodic function. But when $x(t)$ and $\tau(t)$ are pseudo almost periodic functions, $x(t - \tau(t))$ may not be a pseudo almost periodic function. For more details, readers may refer to [8]. Due to the above reasons, to the best of our knowledge, there exist few results on the existence of pseudo almost periodic solutions to the Duffing equation (1.1) and its generalizations, (1.3) and (1.4).

Motivated by the above discussions, the main purpose of this paper is to give some sufficient conditions for the existence of pseudo almost periodic solutions of (1.4). The proof is based on the exponential dichotomy theory and fixed point theorem. Particularly, our results not only generalize the results in the literature [6,7], but also improve them. In fact, one can see the following Remark 4.1 for details.

2 Preliminary results

In this section, we shall first recall some basic definitions, lemmas which are used in what follows.

We denote by $\mathbb{R}^n(\mathbb{R}=\mathbb{R}^1)$ the set of all $n-$dimensional real vectors (real numbers). Let $X=(x_1,x_2,\ldots,x_n)^T\in\mathbb{R}^n$ denote a column vector, in which the symbol $(^T)$ denotes the transpose of a vector. Define $|X|=(|x_1|,|x_2|,\ldots,|x_n|)^T$ and $\|X\|=\max\limits_{1\le i\le n}|x_i|$. A vector $X\ge 0$ means that all x_i are greater than or equal to zero. $X>0$ is defined similarly. For vectors X and Y, $X\ge Y$ (resp. $X>Y$) means that $X-Y\ge 0$(resp. $X-Y>0$). $BC(\mathbb{R},\mathbb{R}^n)$ denotes the set of bounded and continuous functions from $\mathbb{R}$ to $\mathbb{R}^n$, and $BUC(\mathbb{R},\mathbb{R}^n)$ denotes the set of bounded and uniformly continuous functions from $\mathbb{R}$ to $\mathbb{R}^n$. Note that $(BC(\mathbb{R},\mathbb{R}^n),\|\cdot\|_\infty)$ is a Banach space, where $\|\cdot\|_\infty$ denotes the supremum norm $\|f\|_\infty:=\sup\limits_{t\in\mathbb{R}}\|f(t)\|$.

Definition 2.1 (*see* [8,12]) Let $u(t)\in BC(\mathbb{R},\mathbb{R}^n)$. $u(t)$ is said to be almost periodic on $\mathbb{R}$ if, for any $\varepsilon>0$, the set $T(u,\varepsilon)=\{\delta:\|u(t+\delta)-u(t)\|<\varepsilon \text{ for all } t\in\mathbb{R}\}$ is relatively dense, i.e., for any $\varepsilon>0$, it is possible to find a real number $l=l(\varepsilon)>0$ with the property that, for any interval with length $l(\varepsilon)$, there exists a number $\delta=\delta(\varepsilon)$ in this interval such that $\|u(t+\delta)-u(t)\|<\varepsilon,\quad \text{for all}\quad t\in\mathbb{R}$.

We denote by $AP(\mathbb{R},\mathbb{R}^n)$ the set of the almost periodic functions from $\mathbb{R}$ to $\mathbb{R}^n$. Besides, the concept of pseudo almost periodicity was introduced by C. Zhang in the early nineties. It is a natural generalization of the classical almost periodicity. Precisely, define the class of functions $PAP_0(\mathbb{R},\mathbb{R}^n)$ as follows:

$$\Big\{f\in BC(\mathbb{R},\mathbb{R}^n)|\lim_{r\to+\infty}\frac{1}{2r}\int_{-r}^{r}|f(t)|dt=0\Big\}.$$

A function $f\in BC(\mathbb{R},\mathbb{R}^n)$ is called pseudo almost periodic if it can be expressed as

$$f=h+\varphi,$$

where $h\in AP(\mathbb{R},\mathbb{R}^n)$ and $\varphi\in PAP_0(\mathbb{R},\mathbb{R}^n)$. The collection of such functions will be denoted by $PAP(\mathbb{R},\mathbb{R}^n)$. In particular, $(PAP(\mathbb{R},\mathbb{R}^n),\|.\|_\infty)$ is a Banach space and $AP(\mathbb{R},\mathbb{R}^n)$ is a proper subspace of $PAP(\mathbb{R},\mathbb{R}^n)$(see [8]).

Throughout this paper, it will be assumed that $m>1$, and

$$\delta_1,\delta_2\in AP(\mathbb{R},\mathbb{R})\quad \alpha,\beta,\tau,Q_1,Q_2\in PAP(\mathbb{R},\mathbb{R}),\quad \alpha(t)>0,\quad \beta(t)\ne 0,\quad \text{for all } t\in\mathbb{R}. \tag{2.1}$$

Let $\underline{\delta}_1$, $\underline{\delta}_2$, δ^*, l, θ and q be constants such that

$$\underline{\delta}_1=\inf_{t\in\mathbb{R}}|\delta_1(t)|,\quad \underline{\delta}_2=\inf_{t\in\mathbb{R}}|\delta_2(t)|,\quad \delta^*=\min\{\underline{\delta}_1,\underline{\delta}_2\}, \tag{2.2}$$

$$l=\max\left\{\frac{\sup_{t\in\mathbb{R}}|Q_1(t)|}{\delta^*},\frac{\sup_{t\in\mathbb{R}}|Q_2(t)|}{\delta^*}\right\},\theta=\max\left\{\frac{1}{\delta^*},\frac{\sup_{t\in\mathbb{R}}[|\alpha(t)-\delta_2^2(t)|+|\beta(t)|]}{\delta^*}\right\},\tag{2.3}$$

$$q=\max\left\{\frac{1}{\delta^*},\frac{\sup_{t\in\mathbb{R}}[|\alpha(t)-\delta_2^2(t)|+|\beta(t)|m(\frac{2l}{1-\theta})^{m-1}]}{\delta^*}\right\}.\tag{2.4}$$

Set

$$B^*=\left\{\varphi\middle|\|\varphi-\varphi_0\|_\infty|\le\frac{\theta l}{1-\theta},\ \varphi=(\varphi_1,\varphi_2)^T\in PAP(\mathbb{R},\mathbb{R}^2)\bigcap BUC(\mathbb{R},\mathbb{R}^2)\right\},$$

where

$$\varphi_0=\left(\int_{-\infty}^t e^{-\int_s^t\delta_1(w)dw}Q_1(s)ds,\ -\int_t^{+\infty}e^{-\int_t^s\delta_2(w)dw}Q_2(s)ds\right)^T.$$

Lemma 2.1 *B^* is a closed subset of $PAP(\mathbb{R},\mathbb{R}^2)$.*

Proof Suppose that $\{x_p\}_{p=1}^{+\infty}\subseteq B^*$ satisfies

$$\lim_{p\to+\infty}\|x_p-\varphi\|_\infty=0.\tag{2.5}$$

Obviously, $\varphi\in PAP(\mathbb{R},\mathbb{R}^2)$, and $||\varphi-\varphi_0||_\infty\le\frac{\theta l}{1-\theta}$. We next show that

$$\varphi=(\varphi_1,\varphi_2)^T\in BUC(\mathbb{R},\mathbb{R}^2).$$

In fact, for any $\varepsilon>0$, from (2.5), we can choose $p>0$ such that

$$\|x_p-\varphi\|_\infty<\frac{\varepsilon}{3}.\tag{2.6}$$

Note that $x_p=(x_{1_p},\ x_{2_p})^T$ is uniformly continuous on $\mathbb{R}$. Then, there exists $\delta=\delta(\varepsilon)$ such that

$$|x_{1_p}(t_1)-x_{1_p}(t_2)|<\frac{\varepsilon}{3},\quad |x_{2_p}(t_1)-x_{2_p}(t_2)|<\frac{\varepsilon}{3},\quad \text{where } t_1,t_2\in\mathbb{R} \text{ and } |t_1-t_2|<\delta,$$

which, together with (2.6), implies that

$$\begin{aligned}&|\varphi_i(t_1)-\varphi_i(t_2)|\\&\quad\le|\varphi_i(t_1)-x_{i_p}(t_1)|+|x_{i_p}(t_1)-x_{i_p}(t_2)|+|x_{i_p}(t_2)-\varphi_i(t_2)|\end{aligned}$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}$$
$$= \varepsilon \quad \text{where } t_1, t_2 \in \mathbb{R} \text{ and } |t_1 - t_2| < \delta, \ i = 1, \ 2,$$

i.e., φ is uniformly continuous on $\mathbb{R}^2$ and $\varphi \in B^*$. Hence, B^* is a closed subset of $PAP(\mathbb{R}, \mathbb{R}^2)$. This completes the proof of Lemma 2.1. □

Definition 2.2 (*see* [8,12]) Let $x \in \mathbb{R}^n$ and $Q(t)$ be an $n \times n$ continuous matrix defined on $\mathbb{R}$. The linear system

$$x'(t) = Q(t)x(t) \tag{2.7}$$

is said to admit an exponential dichotomy on $\mathbb{R}$ if there exist positive constants k, α, projection P and the fundamental solution matrix $X(t)$ of (2.7) satisfying

$$\|X(t)PX^{-1}(s)\| \le ke^{-\alpha(t-s)} \quad \text{for } t \ge s,$$
$$\|X(t)(I-P)X^{-1}(s)\| \le ke^{-\alpha(s-t)} \quad \text{for } t \le s.$$

Lemma 2.2 (see [8]) *Assume that $Q(t)$ is an almost periodic matrix function and $g(t) \in PAP(\mathbb{R}, \mathbb{R}^n)$. If the linear system* (2.7) *admits an exponential dichotomy, then pseudo almost periodic system*

$$x'(t) = Q(t)x(t) + g(t) \tag{2.8}$$

has a unique pseudo almost periodic solution $x(t)$, and

$$x(t) = \int_{-\infty}^{t} X(t)PX^{-1}(s)g(s)ds - \int_{t}^{+\infty} X(t)(I-P)X^{-1}(s)g(s)ds. \tag{2.9}$$

Lemma 2.3 (see [8,12]) *Let $Q(t) = (q_{ij})_{n\times n}$ be an almost periodic matrix defined on $\mathbb{R}$, and there exists a positive constant ν such that*

$$|q_{ii}(t)| - \sum_{j=1, j\neq i}^{n} |q_{ij}(t)| \ge \nu, \quad \forall\, t \in \mathbb{R}, \quad i = 1, 2, \ldots, n,$$

Then the linear system (2.7) *admits an exponential dichotomy on $\mathbb{R}$.*

3 Existence and uniqueness of pseudo almost periodic solutions

Theorem 3.1 *Assume $\delta_i(t) > 0, i = 1, 2, \ t \in \mathbb{R}$, and let positive constants l, θ and q satisfy*

$$\theta < 1, \quad \frac{l}{1-\theta} < 1, \quad q < 1. \tag{3.1}$$

Then there exists a unique pseudo almost periodic solution of system (1.4) *in the region B^*.*

Proof Let $\varphi = (\varphi_1, \varphi_2)^T \in B^*$ and $f(t, z) = \varphi_1(t - z)$. From Theorem 5.3 in [8, p. 58] and Definition 5.7 in [8, p. 59], the uniform continuity of φ_2 implies that $f \in PAP(\mathbb{R} \times \Omega)$ and f is continuous in $z \in L$ and uniformly in $t \in \mathbb{R}$ for all compact subset L of $\Omega \subset \mathbb{R}$. This, together with $\tau \in PAP(\mathbb{R}, \mathbb{R})$ and Theorem 5.11 in [8, p. 60], yields

$$\varphi_1(t - \tau(t)) \in PAP(\mathbb{R}, \mathbb{R}).$$

According to Corollary 5.4 in [8, p. 58] and the composition theorem of pseudo almost periodic functions, we have

$$\varphi_2(t) + Q_1(t), (\alpha(t) - \delta_2^2(t))\varphi_1(t) - \beta(t)\varphi_1^m(t - \tau(t)) + Q_2(t) \in PAP(\mathbb{R}, \mathbb{R}).$$

We next consider an auxiliary two-dimensional system

$$\begin{pmatrix} \frac{dx(t)}{dt} \\ \frac{dy(t)}{dt} \end{pmatrix} = \begin{pmatrix} -\delta_1(t) & 0 \\ 0 & \delta_2(t) \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \begin{pmatrix} \varphi_2(t) + Q_1(t) \\ \tilde{\varphi}_1(t) \end{pmatrix}, \tag{3.2}$$

where

$$\tilde{\varphi}_1(t) = (\alpha(t) - \delta_2^2(t))\varphi_1(t) - \beta(t)\varphi_1^m(t - \tau(t)) + Q_2(t).$$

Then, from Lemma 2.3, we obtain that the linear system

$$\begin{pmatrix} \frac{dx(t)}{dt} \\ \frac{dy(t)}{dt} \end{pmatrix} = \begin{pmatrix} -\delta_1(t) & 0 \\ 0 & \delta_2(t) \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \tag{3.3}$$

admits an exponential dichotomy on $\mathbb{R}$. Define a projection P by setting

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus, by Lemma 2.2, we obtain that the system (3.2) has exactly one pseudo almost periodic solution:

$$\begin{pmatrix} x^\varphi(t) \\ y^\varphi(t) \end{pmatrix} = \begin{pmatrix} \int_{-\infty}^t e^{-\int_s^t \delta_1(w)dw}(\varphi_2(s) + Q_1(s))ds \\ -\int_t^{+\infty} e^{-\int_t^s \delta_2(w)dw}[(\alpha(s) - \delta_2^2(s))\varphi_1(s) - \beta(s)\varphi_1^m(s - \tau(s)) + Q_2(s)]ds \end{pmatrix}.$$

Define a mapping $T : B^* \to PAP(\mathbb{R}, \mathbb{R}^2)$ by setting

$$(T\varphi)(t) = \begin{pmatrix} x^\varphi(t) \\ y^\varphi(t) \end{pmatrix}, \quad \forall \varphi \in B^*.$$

According to the definition of the norm in Banach space $PAP(\mathbb{R}, \mathbb{R}^2)$, we derive

$$\begin{aligned}||\varphi_0||_\infty &\le \sup_{t\in\mathbb{R}} \max\{\textstyle\int_{-\infty}^t e^{-\int_s^t \delta_1(w)dw}|Q_1(s)|ds, \int_t^{+\infty} e^{-\int_t^s \delta_2(w)dw}|Q_2(s)|ds\}\\ &\le \max\left\{\frac{\sup_{t\in\mathbb{R}}|Q_1(t)|}{\delta^*}, \quad \frac{\sup_{t\in\mathbb{R}}|Q_2(t)|}{\delta^*}\right\} = l.\end{aligned}$$

Therefore, for any $\varphi \in B^*$, we have

$$||\varphi||_\infty \le ||\varphi - \varphi_0||_\infty + ||\varphi_0||_\infty \le \frac{\theta l}{1-\theta} + l = \frac{l}{1-\theta} < 1. \tag{3.4}$$

Now, we prove that the mapping T is a self-mapping from B^* to B^* . In fact, for any $\varphi \in B^*$, from (3.4), we obtain

$$\begin{aligned}&||T\varphi - \varphi_0||_\infty\\ &= \sup_{t\in\mathbb{R}} \max\left\{\left|\int_{-\infty}^t e^{-\int_s^t \delta_1(w)dw}\varphi_2(s)ds\right|, \left|\int_t^{+\infty} e^{-\int_t^s \delta_2(w)dw}[(\alpha(s)-\delta_2^2(s))\varphi_1(s)\right.\right.\\ &\quad \left.\left. -\beta(s)\varphi_1^m(s-\tau(s))]ds pg\right|\right\}\\ &\le \sup_{t\in\mathbb{R}} \max\left\{\int_{-\infty}^t e^{-\int_s^t \delta_1(w)dw}ds, \sup_{t\in\mathbb{R}}\left[\left|\alpha(t)-\delta_2^2(t)\right|+\left|\beta(t)\right|\right]\int_t^{+\infty} e^{-\int_t^s \delta_2(w)dw}ds\right\}||\varphi||_\infty\\ &\le \max\left\{\frac{1}{\delta^*}, \frac{\sup_{t\in\mathbb{R}}[|\alpha(t)-\delta_2^2(t)|+|\beta(t)|]}{\delta^*}\right\}||\varphi||_\infty = \theta||\varphi||_\infty \le \frac{\theta l}{1-\theta},\end{aligned} \tag{3.5}$$

and

$$||T\varphi||_\infty \le ||T\varphi - \varphi_0||_\infty + ||\varphi_0||_\infty \le \frac{\theta l}{1-\theta} + l = \frac{l}{1-\theta} < 1. \tag{3.6}$$

which, together with (3.2), imply that $((T\varphi)(t))'$ is bounded on $\mathbb{R}$. Thus, $(T\varphi)(t)$ is uniformly continuous on $\mathbb{R}$, and $T\varphi \in B^*$. So, the mapping T is a self-mapping from B^* to B^*.

Next, we prove that the mapping T is a contraction mapping of the B^*. In deed, in view of (2.3), (2.4), (3.1), (3.4) and differential mean-value theorem, for all $\varphi, \psi \in B^*$, we have

$$\begin{aligned}&|(T\varphi)(t) - (T\psi)(t)|\\ &\quad = (|((T\varphi)(t) - (T\psi)(t))_1|, |((T\varphi)(t) - (T\psi)(t))_2|)^T\\ &\quad = \left(\left|\int_{-\infty}^t e^{-\int_s^t \delta_1(w)dw}[\varphi_2(s)-\psi_2(s)]ds\right|, \left|\int_t^{+\infty} e^{-\int_t^s \delta_2(w)dw}[(\alpha(s)-\delta_2^2(s))(\varphi_1(s)-\psi_1(s))\right.\right.\\ &\quad\quad \left.\left. -\beta(s)(\varphi_1^m(s-\tau(s)) - \psi_1^m(s-\tau(s)))]ds\right|\right)^T\end{aligned}$$

$$= \left(\left| \int_{-\infty}^{t} e^{-\int_s^t \delta_1(w)dw} [\varphi_2(s) - \psi_2(s)] ds \right|, \left| \int_{t}^{+\infty} e^{-\int_t^s \delta_2(w)dw} [(\alpha(s) - \delta_2^2(s))(\varphi_1(s) - \psi_1(s)) \right.\right.$$
$$-\beta(s)m(\psi_1(s-\tau(s)) + h(s)(\varphi_1(s-\tau(s)) - \psi_1(s-\tau(s)))^{m-1}$$
$$\left.\left. (\varphi_1(s-\tau(s)) - \psi_1(s-\tau(s)))] ds \right| \right)^T$$
$$= \left(\left| \int_{-\infty}^{t} e^{-\int_s^t \delta_1(w)dw} [\varphi_2(s) - \psi_2(s)] ds \right|, \left| \int_{t}^{+\infty} e^{-\int_t^s \delta_2(w)dw} [(\alpha(s) - \delta_2^2(s))(\varphi_1(s) - \psi_1(s)) \right.\right.$$
$$-\beta(s)m((1-h(s))\psi_1(s-\tau(s)) + h(s)\varphi_1(s-\tau(s)))^{m-1}$$
$$\left.\left. (\varphi_1(s-\tau(s)) - \psi_1(s-\tau(s)))] ds \right| \right)^T,$$

where $h(s) \in (0,\ 1)$ is the mean point in Lagrange's mean value theorem. Then,

$$|(T\varphi)(t) - (T\psi)(t)|$$
$$\leq \Big(\int_{-\infty}^{t} e^{-\int_s^t \delta_1(w)dw} ds \sup_{s\in R} \Big|\varphi_2(s) - \psi_2(s)\Big|, \int_{t}^{+\infty} e^{-\int_t^s \delta_2(w)dw} \{|\alpha(s) - \delta_2^2(t)| \cdot$$
$$\sup_{s\in\mathbb{R}} \Big|\varphi_1(s) - \psi_1(s)\Big| + |\beta(s)|m\Big[\sup_{s\in\mathbb{R}} \Big|\psi_1(s-\tau(s))\Big| + \sup_{s\in\mathbb{R}} \Big|\varphi_1(s-\tau(s))\Big|\Big]^{m-1} \cdot$$
$$\sup_{s\in R} \Big|\varphi_1(s-\tau(s)) - \psi_1(s-\tau(s))\Big|\} ds|\Big)^T$$
$$\leq \Big(\int_{-\infty}^{t} e^{-\int_s^t \delta_1(w)dw} ds ||\varphi - \psi||_\infty, \int_{t}^{+\infty} e^{-\int_t^s \delta_2(w)dw} ds \sup_{s\in\mathbb{R}} \Big[|\alpha(s) - \delta_2^2(s)|$$
$$+|\beta(s)|m(\frac{2l}{1-\theta})^{m-1}\Big] ||\varphi - \psi||_\infty| \Big)^T,$$

which yields

$$||T\varphi - T\psi||_\infty \leq \max \left\{ \frac{1}{\delta^*}, \frac{\sup_{t\in\mathbb{R}} [|\alpha(t) - \delta_2^2(t)| + |\beta(t)|m(\frac{2l}{1-\theta})^{m-1}]}{\delta^*} \right\} ||\varphi - \psi||_\infty$$
$$= q||\varphi - \psi||_\infty.$$

It follows from $q < 1$ that the mapping T is a contraction. Therefore, according to Lemma 2.1, we get that the mapping T possesses a unique fixed point $z^* = (x^*(t), y^*(t))^T \in B^*$, $Tz^* = z^*$. By (3.2), z^* satisfies (1.4). So z^* is a pseudo almost periodic solution of system (1.4) in B^* . The proof of Theorem 3.1 is now completed. □

Similarly, we can obtain the following theorem:

Theorem 3.2 *If* $\delta_i(t) < 0,\ i = 1, 2,\ t \in \mathbb{R}$, *and* (2.1) *holds, then, there exists a unique pseudo almost periodic solution of system* (1.4) *in the region*

$$B^{**} = v\left\{\varphi \middle| \|\varphi - \varphi_0^*\|_\infty| \leq \frac{\theta l}{1-\theta},\ \varphi = (\varphi_1, \varphi_2)^T \in PAP(\mathbb{R}, \mathbb{R}^2) \bigcap BUC(\mathbb{R}, \mathbb{R}^2)\right\},$$

where

$$\varphi_0^* = \left(-\int_t^{+\infty} e^{-\int_t^s \delta_1(w)dw} Q_1(s)ds,\ \int_{-\infty}^t e^{-\int_s^t \delta_2(w)dw} Q_2(s)ds\right)^T.$$

4 Example

Example 4.1 The nonlinear Duffing equation with a deviating argument

$$\begin{aligned} x''(t) &+ (\sin t - \cos t)x'(t) + (626 + 27\cos t + 25\sin t + \cos^2 t + \cos t \sin t)x(t)\\ &+ (1+\sin^2 t)x^3(t - \cos^2\sqrt{2}t - \cos^2 t - e^{-t^4 \sin^2 t})\\ &= \cos\sqrt{2}t + \cos\sqrt{3}t - 50\cos t - 2\cos^2 t, \end{aligned} \tag{4.1}$$

has a pseudo almost periodic solution $x^*(t)$ satisfying

$$\max\left\{\sup_{t\in\mathbb{R}}\left|x^*(t)\right|, \sup_{t\in\mathbb{R}}\left|\frac{dx^*(t)}{dt} + (25+\sin t)x^*(t) - 2\cos t\right|\right\} \le \frac{1}{11}.$$

Proof Set

$$\delta_1(t) = 25+\sin t, \quad \delta_2(t) = 25+\cos t, \quad y(t) = \frac{dx(t)}{dt} + (25+\sin t)x(t) - 2\cos t. \tag{4.2}$$

Then, we can transform (4.1) into the following system: hen, we can transform (4.1) into the following system:

$$\begin{cases} \frac{dx(t)}{dt} = -(25+\sin t)x(t) + y(t) + 2\cos t,\\ \frac{dy(t)}{dt} = (25+\cos t)y(t) + (2-\sin^2 t)x(t)\\ \qquad -(2-\cos^2 t)x^3(t - \cos^2\sqrt{2}t - \cos^2 t - e^{-t^4\sin^2 t})\\ \qquad +\cos\sqrt{2}t + \cos\sqrt{3}t. \end{cases} \tag{4.3}$$

Since $\alpha(t) = 626 + 50\cos t + 2\cos^2 t$, $\beta(t) = 2 - \cos^2 t$, $Q_1(t) = -2\cos t$, $Q_2(t) = \cos\sqrt{2}t + \cos\sqrt{3}t$, $m = 3$, $\delta^* = 25$, then $l = \frac{2}{25}$, $\theta = \frac{3}{25}$, $q = \frac{254}{3025}$. It is straight forward to check that all assumptions needed in Theorem 3.1 are satisfied. Hence, Eq. (4.1) has a pseudo almost periodic solution $x^*(t)$ such that

$$\max\left\{\sup_{t\in\mathbb{R}}\left|x^*(t)\right|, \sup_{t\in\mathbb{R}}\left|\frac{dx^*(t)}{dt} + (25+\sin t)x^*(t) - 2\cos t\right|\right\} \le \frac{l}{1-\theta} = \frac{1}{11}.$$

The proof of Example 4.1 is now completed.

Remark 4.1 For the nonlinear Duffing equation (4.1), the time-varying delay

$$\tau(t) = \cos^2\sqrt{2}t + \cos^2 t + e^{-t^4\sin^2 t}$$

is pseudo almost periodic, but not almost periodic. Thus, the results obtained in [6,7] are invalid for the above example. Moreover, one can find that main theorem in [7] is a special one of Theorem 3.1. This implies that the results of this paper substantially extend and improve the main results of [6,7].

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