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Single machine scheduling problem with multiple due windows assignment in a group technology

Wei-Xuan Li · Chuan-Li Zhao

Abstract We consider group scheduling problem on a single machine with multiple due windows assignment. Jobs are divided into groups in advance according to their processing similarities, and all jobs of the same group are required to be processed contiguously on the machine in order to achieve production efficiency and save time/money resource. A sequence-independent setup time precedes the processing of each group. The goal is to determine the optimal sequence for both groups and jobs, together with an optimal combination of the due windows assignment strategy so as to minimize the total of earliness, tardiness and due windows related costs. We give an $O(n \log n)$ time algorithm for the problem.

1 Introduction

Scheduling problems with due date assignment have attracted attention in recent years due to the just-in-time (JIT) production management. In a JIT system, jobs should be completed as close as possible to their due dates. Completing a job early means having to bear the costs of holding unnecessary inventories, while finishing a job late results in a contractual penalty and a loss of customer goodwill. Kuo and Yang [12] stressed

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a single machine scheduling problem with a common due-date and deteriorating jobs, in which the job processing time is an increasing function of its starting time. They made a concise analysis of the problem and provided a polynomial time algorithm to solve it. Li et al. [13] studied the problem of scheduling deteriorating jobs and due date assignment on a single machine. They presented some polynomial time algorithms to solve the problem with two due date assignment methods: CON and SLK. Moreover, Chand and Chhajed [1] introduced the problem of simultaneous determination of optimal due dates and optimal schedule for the single machine problem with multiple due dates, they provided an efficient optimal algorithm to solve it. For further research on this topic, the reader may refer to Wang and Wang [23], Yang et al. [27], Yang et al. [28], Fan and Zhao [6] and Zhao et al. [33].

Furthermore, job completions can be accepted without penalty within an interval in time in the JIT system. This time interval is often called the due window. A job completed within the due window is considered to be on time and will not be penalized. However, a job completed prior to or after the due window will be penalized according to their earliness/tardiness. Sidney [19] was among the pioneers, who stressed a single machine scheduling problem with due windows assignment. He assumed that each job has its due window and no job's due window is allowed to contain the due window of another job. The goal was to find a schedule that minimizes the total costs of earliness and tardiness. Recently, Janiak et al. [9] considered various models of due window assignment scheduling problems on a single machine such that the objective function including the maximum or total earliness and tardiness and the due window parameters is minimized. They constructed polynomial algorithms for the considered problems. Wang and Wang [22] explored a single machine common due window scheduling problem simultaneously with the learning effect and deteriorating jobs considerations. Cheng et al. [4] investigated a common due window assignment scheduling with linear time-dependent deteriorating jobs and a deteriorating maintenance on a single machine setting. They showed that the proposed model is polynomial solvable. Yang et al. [25] first proposed the multiple due windows assignment scheduling problem with controllable processing times, in which n jobs may have distinct m due windows, where $1 \leq m \leq n$. The objective is to determine the optimal due window positions and sizes, the set of jobs assigned to each due window, the optimal job compressions, and the optimal schedule to minimize a total cost function, which consists of the earliness, the tardiness, the processing time compressions, and the due windows related costs. For the detailed research results on scheduling with due window, the reader may refer to Mor and Mosheiov [16], Yin et al. [30], Chen et al. [3], Ji et al. [11], and Yin et al. [31].

Additionally, an important issue of scheduling problems is characterized by the group technology (GT) assumption. The GT method groups products with similar characteristics into families and sets aside groups of machines for their production. Families may be based on size, shape, geometry, manufacturing requirements and so on. Through decades of application, people have found many advantages of GT. For instance, jobs in the same group tend to move through production in a direct route, reducing the manufacturing lead time; changeover among different jobs in the same group is simplified, reducing the costs or time involved (Ji et al. [10], Li et al. [14]). As a realistic example of GT, Conway et al. [5] stressed paint production on a single

machine. Customer orders vary in colors, but they can be divided into several major color groups, such as red, blue, and green. Within a color group, red, for example, colors may range from very light to dark red. The setup time of the machine to produce paint in colors of the same group is small and can be neglected, since it is natural to produce paint from lighter to darker colors. While the time to switch the machine from the production of paint of one color group to another color group is substantial, which may include the time to clean the machine and change the tools, and the setup time is color-independent in general. Another example can be found in the metal or wood cutting process, the process need to cut various sizes and shapes of product, the products with similar shape or manufacturing requirements are grouping as a product family and the cutting tools used to process the individual product family are grouping as a machine family. Thus, it is clear that dividing the orders into groups according to their processing similarities can significantly increase the production efficiency. Chen et al. [2] considered single machine scheduling with common due date assignment in a group technology environment. The objective is to find an optimal common due date and an optimal sequence of jobs to minimize the sum of the cost of tardy jobs and the cost related to the common due date. Ng et al. [17] explored group scheduling with controllable setup and processing times, the objective is to minimize the total weighted completion time. Wang et al. [20] studied single machine scheduling problems with GT and deteriorating jobs. The job processing time is a decreasing function of its starting time. The objectives are to minimize the makespan and the total completion time. They provided polynomial time algorithms to solve these problems. Shabtay et al. [18] argued that single machine scheduling with optimal due date assignment and resource allocation in a group technology environment. They gave a polynomial algorithm for the considered problem in which each job has a constant processing time, and when jobs have controllable processing times, the complexity of this problem remains an open question. Ji et al. [10] assessed a single-machine group scheduling and job-dependent due window assignment problem in which each job is assigned an individual due window based on a common flow allowance. The goal is to find the optimal sequence for both groups and jobs, together with the optimal due window assignment, to minimize the total cost that comprises the earliness and tardiness penalties, the due window starting time and due window size costs. Apart from above, the reader may refer to the following literatures for a detailed comprehension of group scheduling problem: Ham et al. [7], Li et al. [14], Yang [26], Yang and Yang [29], Zhu et al. [34], Wang et al. [21], Low and Lin [15], Zhang [32], and Xu et al. [24].

Although the due date and due window assignment in JIT scheduling have been extensively investigated in the literature, there are only a few researches on JIT scheduling problem with multiple due dates, especially multiple due windows assignment. Moreover, the multiple due dates and multiple due windows assignment scheduling problems under a GT restriction have never been explored. The group scheduling problems with due date assignment that jobs within the same group are assigned to a common due date and each job within the same group is assigned to a different due date with no restrictions have been studied by Li et al. [14]. Actually, however, jobs within the same group may have not only one due date because of the different needs of customers or other special reasons. Then to consider the case that jobs within the same group G_i have z_i distinct due dates can meet the actual needs better, where

$1 \leq z_i \leq n_i$, and n_i is the number of jobs in group G_i. If $z_i = 1$, it means that all jobs in the same group have one common due date; if $z_i = n_i$, it indicates that each job within the same group is assigned to a different due date. Obviously it is an extension of Li et al. [14]. This paper will further expand the due date problem into the due window problem. That is, jobs within the same group G_i have z_i distinct due windows, $1 \leq z_i \leq n_i$. The extension allows for greater flexibility in modeling real-life problems. For example, in an order picking operation process, the number of orders to be completed may be too great to realistically justify measurement from a single due window for a single customer. By viewing the order picking operation process as being composed of several discrete segments, each team of orders could be made uniform around its own due window. The orders should be ready at their due window in order to avoid staging delays. Moreover, a higher cost in the form of transship fee generally accompanies a later due window (Yang et al. [25]).

In this paper, we consider scheduling problem on a single machine with multiple due windows assignment in a group technology. We seek to determine the optimal group sequence, the optimal job sequence, and the optimal due windows assignment to minimize the total of earliness, tardiness and the due windows related costs.

The remainder of this paper is organized as follows: In Sect. 2 we introduce the notation and formulate the problem. In Sect. 3 we study the group scheduling problem with multiple due windows assignment. Finally, conclusions are given in the last Section.

2 Problem formulation

The following notations will be used throughout the paper and we will introduce additional notations when needed.

n : the total number of jobs;

m : the total number of groups, $1 \leq m \leq n$;

n_i : the number of jobs in group G_i, $i = 1, 2, \ldots, m$, and $\sum_{i=1}^{m} n_i = n$;

s_i : the setup time of group G_i;

p_{ij} : the processing time of job J_j of group G_i;

$p_{i[j]}$: the processing time of job in the jth position of group G_i, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n_i$;

z_i : the number of due windows assigned to the n_i jobs of group G_i, $1 \leq z_i \leq n_i$;

I_{ik} : the set of jobs assigned to the kth due window in group G_i, $k = 1, 2, \ldots, z_i$, $i = 1, 2, \ldots, m$;

n_{ik} : the number of jobs assigned to the kth due window in group G_i, i.e., $|I_{ik}| = n_{ik}$ and $\sum_{k=1}^{z_i} n_{ik} = n_i$;

N_{ik} : the total number of jobs assigned to the first k due windows in group G_i, i.e., $\sum_{j=1}^{k} n_{ij} = N_{ik}$ for $k = 1, 2, \ldots, z_i$, $i = 1, 2, \ldots, m$, and $N_{i0} = 0$.

The problem under consideration is formulated as follows. We are given n independent and non-preemptive jobs that are classified into m groups $G_1, G_2, \ldots, G_m$. Each group G_i, for $i = 1, 2, \ldots, m$, has n_i jobs $\{J_{i1}, J_{i2}, \ldots, J_{in_i}\}$, where $\sum_{i=1}^{m} n_i = n$. Jobs of the same group are required to be processed contiguously. A sequence-

independent machine setup time s_i precedes the processing of the first job of group G_i. All jobs are simultaneously available for processing at time zero.

Let $d_{ik}(\geq 0)$ and $w_{ik}(\geq d_{ik})$ denote the starting time and finishing time of the k th due window in group G_i, respectively. $S_{ik} = w_{ik} - d_{ik}$ denotes the size of the k th due window in group G_i for $k = 1, 2, \ldots, z_i$, $i = 1, 2, \ldots, m$. We assume that the number of distinct due windows z_i assigned to group G_i ($i = 1, 2, \ldots, m$) is given in advance, and n_{ik}, for $k = 1, 2, \ldots, z_i$, is known. If $z_i = 1$, it means that all the jobs in group G_i have one common due window; If $z_i = n_i$, it indicates that there exists n_i distinct due windows for n_i jobs in group G_i. The earliness and tardiness of job $J_{ij} \in I_{ik}$ are $E_{ij} = \max\{0, d_{ik} - C_{ij}\}$ and $T_{ij} = \max\{0, C_{ij} - w_{ik}\}$, respectively. The objective is to find the optimal group sequence, the optimal job sequence, and the optimal due window assignment to minimize the following total cost function

$$g(\pi) = \sum_{i=1}^{m} \sum_{l=1}^{n_i} \left(\alpha_i E_{il} + \beta_i T_{il} + \gamma_i d_{il} + \delta_i S_{il} \right) \tag{1}$$

where $\alpha_i, \beta_i, \gamma_i, \delta_i$ are non-negative parameters representing the unit time costs of earliness, tardiness, due window starting time and due window size, respectively.

Using the three-field notation, the considered problem can be described as

$$1 \,|GT, MDW|\, \sum_{i=1}^{m} \sum_{l=1}^{n_i} \left(\alpha_i E_{il} + \beta_i T_{il} + \gamma_i d_{il} + \delta_i S_{il} \right),$$

where MDW means the multiple due windows.

3 Group scheduling with multiple due windows

In this section, we study the scheduling problem with multiple due windows assignment under a GT restriction, i.e., $1 \,|GT, MDW|\, \sum_{i=1}^{m} \sum_{l=1}^{n_i} (\alpha_i E_{il} + \beta_i T_{il} + \gamma_i d_{il} + \delta_i S_{il})$

3.1 Preliminary results

Some useful lemmas are given in this subsection to solve the group scheduling problem with multiple due windows.

Lemma 1 *In an optimal schedule, there exists no idle time between consecutive jobs on the machine and the first job starts at time zero.*

Proof The proof is obvious and omitted. □

Lemma 2 *Given two sequences of non-negative numbers x_i and y_i, the sum of the products of the corresponding elements $\sum_{i=1}^{n} x_i y_i$ is minimized if the sequences are monotonic in the opposite way.*

Proof See Hardy et al. [8].   □

The following Lemma 3 indicates that for an optimal sequence, the jobs assigned to different due windows are mutually disjoint, that is, there is an optimal solution such that n_{ik} consecutive jobs (in positions $N_{i(k-1)} + 1$ to N_{ik}) in a sequence π are assigned to the k th due window in group G_i.

Lemma 3 *For any given $d_i = \{d_{i1}, d_{i2}, \ldots, d_{iz_i}\}$, $w_i = \{w_{i1}, w_{i2}, \ldots, w_{iz_i}\}$ in group G_i, $i = 1, 2, \ldots, m$, and a schedule π, there exists an optimal I_i such that $I_{ik} = \{J_{i[N_{i(k-1)}+1]}, J_{i[N_{i(k-1)}+2]}, \ldots, J_{i[N_{ik}]}\}$ for $k = 1, 2, \ldots, z_i$, where $J_{i[r]}$ is the job scheduled in position r in group G_i.*

Proof For any given d_i, w_i and π, $i = 1, 2, \ldots, m$, without loss of generality, we assume that in a schedule $S_1 = (\pi_1, J_k, J_j, \pi_2)$ job J_k immediately precedes job J_j, while in another schedule $S_2 = (\pi_1, J_j, J_k, \pi_2)$ jobs J_k and J_j are mutually replaced, where π_1 and π_2 denote partial sequences and jobs J_k and J_j are, respectively, scheduled in the h th and $(h + 1)$ th position in group G_i in the schedule S_1. In addition, we assume that for both schedules S_1 and S_2, J_k is early for the $(r + 1)$ th due window in group G_i, and J_j is tardy for the r th due window in group G_i, where $J_k \in I_{i(r+1)}$ and $J_j \in I_{ir}$.

Let $C_{il}(S_1)$ and $C_{il}(S_2)$ be the completion times of job J_l in group G_i in schedule S_1 and S_2, respectively. And $S_i(\pi)$ denotes the starting time of group G_i. By the definition, the completion times of jobs J_k and J_j in S_1 are

$$C_{ik}(S_1) = S_i(\pi) + s_i + \sum_{l=1}^{h-1} p_{i[l]} + p_k \tag{2}$$

and

$$C_{ij}(S_1) = S_i(\pi) + s_i + \sum_{l=1}^{h-1} p_{i[l]} + p_k + p_j \tag{3}$$

Similarly, the completion times of jobs J_k and J_j in S_2 are

$$C_{ij}(S_2) = S_i(\pi) + s_i + \sum_{l=1}^{h-1} p_{i[l]} + p_j \tag{4}$$

and

$$C_{ik}(S_2) = S_i(\pi) + s_i + \sum_{l=1}^{h-1} p_{i[l]} + p_j + p_k \tag{5}$$

Then,

$$\begin{aligned}
g(S_1) - g(S_2) &= \{\alpha_i[d_{i(r+1)} - C_{ik}(S_1)] + \beta_i[C_{ij}(S_1) - w_{ir}]\} \\
&\quad - \{\alpha_i[d_{i(r+1)} - C_{ik}(S_2)] + \beta_i[C_{ij}(S_2) - w_{ir}]\} \\
&= \alpha_i p_j + \beta_i p_k \\
&> 0
\end{aligned}$$

Obviously,

$$g(S_1) > g(S_2).$$

Similarly, the total cost decreases as repeating this interchange argument for the jobs which assigned to the same due window are not sequenced consecutively. This completes the proof. □

Lemma 4 *For a given schedule π, if the due window starting time d_{ik} and finishing time w_{ik} are within the starting and ending times of group G_i, then the optimal values of d_{ik} and w_{ik} coincide with some jobs' completion times of group G_i for $k = 1, 2, \ldots, z_i$, $i = 1, 2, \ldots, m$.*

Proof Assume that $d_{ik} = S_i(\pi) + s_i + \sum_{l=1}^{b_{ik}} p_{i[l]}$, $w_{ik} = S_i(\pi) + s_i + \sum_{l=1}^{b_{ik}+h_{ik}} p_{i[l]}$, $k = 1, 2, \ldots, z_i$, $i = 1, 2, \ldots, m$, where b_{ik} and $b_{ik} + h_{ik}$ mean the b_{ik} th and $(b_{ik} + h_{ik})$ th positions of group G_i, respectively. We only address G_i here as an example to demonstrate that the result is correct. Consider that there exists an optimal schedule without the stated property, without loss of generality, suppose that there exists $k = r$, $1 \leq r \leq z_i$ such that

$$d_{ir} = S_i(\pi) + s_i + \sum_{l=1}^{b_{ir}} p_{i[l]} + \Delta_1, \quad 0 \leq \Delta_1 \leq p_{i[b_{ir}+1]},$$

and

$$w_{ir} = S_i(\pi) + s_i + \sum_{l=1}^{b_{ir}+h_{ir}} p_{i[l]} + \Delta_2, \quad 0 \leq \Delta_2 \leq p_{i[b_{ir}+h_{ir}+1]}.$$

Note that moving d_{ir} and w_{ir} Δ units of time only change the cost of jobs assigned to the r th due window in group G_i.

The total cost of jobs assigned to the r th due window in group G_i as a function of Δ_1 and Δ_2 is given by

$$g_{ir}(\Delta_1, \Delta_2) = \alpha_i \sum_{l=N_{i(r-1)}+1}^{N_{ir}} E_{i[l]} + \beta_i \sum_{l=N_{i(r-1)}+1}^{N_{ir}} T_{i[l]} + \gamma_i \sum_{l=N_{i(r-1)}+1}^{N_{ir}} d_{i[l]}$$
$$+ \delta_i \sum_{l=N_{i(r-1)}+1}^{N_{ir}} S_{i[l]}$$

We consider each cost component separately as follows:

(a)

$$\alpha_i \sum_{l=N_{i(r-1)}+1}^{N_{ir}} E_{i[l]} = \alpha_i \sum_{l=N_{i(r-1)}+1}^{b_{ir}} E_{i[l]} = \alpha_i \sum_{l=N_{i(r-1)}+1}^{b_{ir}} (d_{ir} - C_{i[l]})$$
$$= \alpha_i \left\{ \left[\left(S_i(\pi) + s_i + \sum_{l=1}^{b_{ir}} p_{i[l]} + \Delta_1 \right) - \left(S_i(\pi) + s_i + \sum_{l=1}^{N_{i(r-1)}+1} p_{i[l]} \right) \right] \right.$$
$$+ \left[\left(S_i(\pi) + s_i + \sum_{l=1}^{b_{ir}} p_{i[l]} + \Delta_1 \right) - \left(S_i(\pi) + s_i + \sum_{l=1}^{N_{i(r-1)}+2} p_{i[l]} \right) \right]$$
$$+ \cdots$$
$$\left. + \left[\left(S_i(\pi) + s_i + \sum_{l=1}^{b_{ir}} p_{i[l]} + \Delta_1 \right) - \left(S_i(\pi) + s_i + \sum_{l=1}^{b_{ir}} p_{i[l]} \right) \right] \right\}$$
$$= \alpha_i \sum_{l=N_{i(r-1)}+1}^{b_{ir}} (l - N_{i(r-1)} - 1) p_{i[l]} + \alpha_i (b_{ir} - N_{i(r-1)}) \Delta_1$$

(b)

$$
\begin{aligned}
\beta_i \sum_{l=N_{i(r-1)}+1}^{N_{ir}} T_{i[l]} &= \beta_i \sum_{l=b_{ir}+h_{ir}+1}^{N_{ir}} T_{i[l]} = \beta_i \sum_{l=b_{ir}+h_{ir}+1}^{N_{ir}} (C_{i[l]} - w_{ir}) \\
&= \beta_i \left\{ \left[\left(S_i(\pi) + s_i + \sum_{l=1}^{b_{ir}+h_{ir}+1} p_{i[l]} \right) - \left(S_i(\pi) + s_i + \sum_{l=1}^{b_{ir}+h_{ir}} p_{i[l]} + \Delta_2 \right) \right] \right. \\
&\quad + \left[\left(S_i(\pi) + s_i + \sum_{l=1}^{b_{ir}+h_{ir}+2} p_{i[l]} \right) - \left(S_i(\pi) + s_i + \sum_{l=1}^{b_{ir}+h_{ir}} p_{i[l]} + \Delta_2 \right) \right] \\
&\quad + \cdots \\
&\quad \left. + \left[\left(S_i(\pi) + s_i + \sum_{l=1}^{N_{ir}} p_{i[l]} \right) - \left(S_i(\pi) + s_i + \sum_{l=1}^{b_{ir}+h_{ir}} p_{i[l]} + \Delta_2 \right) \right] \right\} \\
&= \beta_i \sum_{l=b_{ir}+h_{ir}+1}^{N_{ir}} (N_{ir} - l + 1) p_{i[l]} - \beta_i (N_{ir} - b_{ir} - h_{ir}) \Delta_2
\end{aligned}
$$

(c)

$$
\begin{aligned}
\gamma_i \sum_{l=N_{i(r-1)}+1}^{N_{ir}} d_{i[l]} &= \gamma_i \sum_{l=N_{i(r-1)}+1}^{N_{ir}} d_{ir} \\
&= \gamma_i n_{ir} \left(S_i(\pi) + s_i + \sum_{l=1}^{b_{ir}} p_{i[l]} + \Delta_1 \right) \\
&= \gamma_i n_{ir} \sum_{l=1}^{b_{ir}} p_{i[l]} + \gamma_i n_{ir} (S_i(\pi) + s_i) + \gamma_i n_{ir} \Delta_1
\end{aligned}
$$

(d)

$$
\begin{aligned}
\delta_i \sum_{l=N_{i(r-1)}+1}^{N_{ir}} S_{i[l]} &= \delta_i \sum_{l=N_{i(r-1)}+1}^{N_{ir}} (d_{i[l]} - w_{i[l]}) = \delta_i \sum_{l=N_{i(r-1)}+1}^{N_{ir}} (d_{ir} - w_{ir}) \\
&= \delta_i n_{ir} \left[\left(S_i(\pi) + s_i + \sum_{l=1}^{b_{ir}+h_{ir}} p_{i[l]} + \Delta_2 \right) - \left(S_i(\pi) + s_i + \sum_{l=1}^{b_{ir}} p_{i[l]} + \Delta_1 \right) \right] \\
&= \delta_i n_{ir} \sum_{l=b_{ir}+1}^{b_{ir}+h_{ir}} p_{i[l]} + \delta_i n_{ir} (\Delta_2 - \Delta_1)
\end{aligned}
$$

Therefore the total cost can be expressed as

$$
g_{ir}(\Delta_1, \Delta_2) = A\Delta_1 + B\Delta_2 + C,
$$

Where $A = \alpha_i(b_{ir} - N_{i(r-1)}) + \gamma_i n_{ir} - \delta_i n_{ir}$, $B = \delta_i n_{ir} - \beta_i(N_{ir} - b_{ir} - h_{ir})$, and

$$
\begin{aligned}
C &= \sum_{l=1}^{N_{i(r-1)}} \gamma_i n_{ir} p_{i[l]} + \sum_{l=N_{i(r-1)}+1}^{b_{ir}} \left[\gamma_i n_{ir} + \alpha_i(l - N_{i(r-1)} - 1) \right] p_{i[l]} \\
&\quad + \sum_{l=b_{ir}+1}^{b_{ir}+h_{ir}} \delta_i n_{ir} p_{i[l]} + \sum_{l=b_{ir}+h_{ir}+1}^{N_{ir}} \beta_i (N_{ir} - l + 1) p_{i[l]} + \gamma_i n_{ir} (S_i(\pi) + s_i)
\end{aligned}
$$

Clearly, A, B and C are independent of Δ_1 and Δ_2. We can see that $f_{ir}(\Delta_1, \Delta_2)$ is a linear function of Δ_1 and Δ_2. Thus we can either decrease Δ_1 and Δ_2 to zero, or increase them to $p_{i[b_{ir}+1]}$ and $p_{i[b_{ir}+h_{ir}1]}$, respectively, to obtain a lower cost. In any case, we can see that d_{ik} and w_{ik} coincide with some jobs' completion times of group G_i. □

Lemma 5 *For any given $I = \{I_{11}, \ldots, I_{1z_1}, I_{21}, \ldots, I_{2z_2}, \ldots, I_{m1}, \ldots, I_{mz_m}\}$ and a schedule π, if the values of d_{ik} and w_{ik} are within the starting time and ending times*

of group G_i, $k = 1, 2, \ldots, z_i$, $i = 1, 2, \ldots, m$, then there exists optimal due windows such that $d_{ik} = C_{i[b_{ik}]}$ and $w_{ik} = C_{i[b_{ik}+h_{ik}]}$, where

$$b_{ik} = N_{i(k-1)} + \left\lceil \frac{n_{ik}(\delta_i - \gamma_i)}{\alpha_i} \right\rceil \text{ and } b_{ik} + h_{ik} = N_{i(k-1)} + \left\lceil \frac{n_{ik}(\beta_i - \delta_i)}{\beta_i} \right\rceil.$$

Proof This lemma can be easily proved by the standard technique of small perturbations. Without loss of generality, we only consider group G_i here as an example to demonstrate that the result is correct. Suppose that $k = r$, $1 \le r \le z_i$, $d_{ir} = C_{i[b_{ir}]}$ and $w_{ir} = C_{i[b_{ir}+h_{ir}]}$. The effect of moving d_{ik} Δ units of time to the left is

$$- \alpha_i (b_{ir} - N_{i(r-1)} - 1)\Delta - \gamma_i n_{ir} \Delta + \delta_i n_{ir} \Delta \tag{6}$$

The effect of moving d_{ik} Δ units of time to the right is

$$\alpha_i (b_{ir} - N_{i(r-1)})\Delta + \gamma_i n_{ir} \Delta - \delta_i n_{ir} \Delta \tag{7}$$

Both (6) and (7) are non-negative due to the optimality of the original schedule. Then from $-\alpha_i(b_{ir} - N_{i(r-1)} - 1)\Delta - \gamma_i n_{ir}\Delta + \delta_i n_{ir}\Delta \ge 0$ and $\alpha_i(b_{ir} - N_{i(r-1)})\Delta + \gamma_i n_{ir}\Delta - \delta_i n_{ir}\Delta \ge 0$, we have $b_{ik} \le N_{i(k-1)} + n_{ik}(\delta_i - \gamma_i)/\alpha_i + 1$ and $b_{ik} \ge N_{i(k-1)} + n_{ik}(\delta_i - \gamma_i)/\alpha_i$. By Lemma 4, we know that b_{ik} is an integer, therefore, b_{ik} is the smallest integer greater than or equal to $N_{i(k-1)} + \lceil n_{ik}(\delta_i - \gamma_i)/\alpha_i \rceil$, i.e., $b_{ik} = N_{i(k-1)} + \lceil n_{ik}(\delta_i - \gamma_i)/\alpha_i \rceil$. Using the same method, we obtain $b_{ik} + h_{ik} = N_{i(k-1)} + \lceil n_{ik}(\beta_i - \delta_i)/\beta_i \rceil$. □

Remarks For any given $I = \{I_{11}, \ldots, I_{1z_1}, I_{21}, \ldots, I_{2z_2}, \ldots, I_{m1}, \ldots, I_{mz_m}\}$ and a schedule π, if $0 < \lceil n_{ik}(\delta_i - \gamma_i)/\alpha_i \rceil < \lceil n_{ik}(\beta_i - \delta_i)/\beta_i \rceil$, we can determine the optimal due window assignments for a group by Lemmas 4 and 5. But the ratios may not meet the above inequality, e.g., for the given parameters, there may exist inequality $\lceil n_{ik}(\beta_i - \delta_i)/\beta_i \rceil \le \lceil n_{ik}(\delta_i - \gamma_i)/\alpha_i \rceil$, then we need further analysis. Note that $\lceil n_{ik}(\beta_i - \delta_i)/\beta_i \rceil \le \lceil n_{ik}(\delta_i - \gamma_i)/\alpha_i \rceil$ means $n_{ik}(\beta_i - \delta_i)/\beta_i \le n_{ik}(\delta_i - \gamma_i)/\alpha_i$.

Similar to the analysis of Ji et al. [10], we have the following discussion.

(i) With a shift of d_{ik} (without changing w_{ik}) by Δ units of time to the right, resulting in a change of total costs of $\Delta g = \alpha_i(b_{ik} - N_{i(k-1)})\Delta + \gamma_i n_{ik}\Delta - \delta_i n_{ik}\Delta$. With a shift of w_{ik} (without changing d_{ik}) by Δ units of time to the right, resulting in a change of total costs of $\Delta g = \delta_i n_{ik}\Delta - \beta_i(N_{ik} - b_{ik} - h_{ik})\Delta$.

If $b_{ik} + h_{ik} < N_{i(k-1)} + n_{ik}(\beta_i - \delta_i)/\beta_i$, a shift of w_{ik} by Δ units of time to the right till $b_{ik} + h_{ik} = N_{i(k-1)} + \lceil n_{ik}(\beta_i - \delta_i)/\beta_i \rceil$ can only reduce the total cost. If $b_{ik} < N_{i(k-1)} + n_{ik}(\beta_i - \delta_i)/\beta_i$, then $b_{ik} < N_{i(k-1)} + n_{ik}(\beta_i - \delta_i)/\beta_i \le N_{i(k-1)} + n_{ik}(\delta_i - \gamma_i)/\alpha_i$, so a shift of d_{ik} by Δ units of time to the right till $b_{ik} = N_{i(k-1)} + \lceil n_{ik}(\beta_i - \delta_i)/\beta_i \rceil$ can only reduce the total cost.

(ii) With a shift of d_{ik} (without changing w_{ik}) by Δ units of time to the left, resulting in a change of total costs of $\Delta g = -\alpha_i(b_{ik} - N_{i(k-1)} - 1)\Delta - \gamma_i n_{ik}\Delta + \delta_i n_{ik}\Delta$. With a shift of w_{ik} (without changing d_{ik}) by Δ units of time to the left, resulting in a change of total costs of $\Delta g = \beta_i(N_{ik} - b_{ik} - h_{ik} + 1)\Delta - \delta_i n_{ik}\Delta$.

If $b_{ik} > N_{i(k-1)} + n_{ik}(\delta_i - \gamma_i)/\alpha_i + 1$, a shift of d_{ik} by Δ units of time to the left till $b_{ik} = N_{i(k-1)} + \lceil n_{ik}(\delta_i - \gamma_i)/\alpha_i \rceil$ can only reduce the total cost. If $b_{ik} + h_{ik} > N_{i(k-1)} + n_{ik}(\delta_i - \gamma_i)/\alpha_i + 1$, then $b_{ik} + h_{ik} > N_{i(k-1)} + n_{ik}(\delta_i - \gamma_i)/\alpha_i + 1 \geq N_{i(k-1)} + n_{ik}(\beta_i - \delta_i)/\beta_i + 1$, so a shift of w_{ik} by Δ units of time to the left till $b_{ik} + h_{ik} = N_{i(k-1)} + \lceil n_{ik}(\delta_i - \gamma_i)/\alpha_i \rceil$ can only reduce the total cost.

Form (i) and (ii), we can see that

$$N_{i(k-1)} + \lceil n_{ik}(\beta_i - \delta_i)/\beta_i \rceil \leq b_{ik} \leq b_{ik} + h_{ik} \leq N_{i(k-1)} + \lceil n_{ik}(\delta_i - \gamma_i)/\alpha_i \rceil,$$

if $b_{ik} < b_{ik} + h_{ik}$, a further shift of d_{ik} to the right and a further shift of w_{ik} to the left till $b_{ik} = b_{ik} + h_{ik}$ can only reduce the total cost, which means that $d_{ik} = w_{ik}, k = 1, 2, \ldots, z_i$. And the multiple due windows assignment problem reduces to the multiple due dates assignment problem. Then we have the following Lemma 6.

Lemma 6 *If $\lceil n_{ik}(\beta_i - \delta_i)/\beta_i \rceil \leq \lceil n_{ik}(\delta_i - \gamma_i)/\alpha_i \rceil$ for $i \in \{1, 2, \ldots, m\}$, then the multiple due windows assignment problem reduces to the multiple due dates assignment problem. The optimal due date assignment is given as follows. For any $k = 1, 2, \ldots, z_i$,*

$$D_{ik} = \begin{cases} C_{i[b_{ik}]} & \text{where } b_{ik} = N_{i(k-1)} + \left\lceil \frac{n_{ik}(\beta_i - \gamma_i)}{\alpha_i + \beta_i} \right\rceil \text{ if } \beta_i > \gamma_i, \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

Where D_{ik} represents the due date of job J_{ik}.

Proof When all jobs belong to a single group and setup time equals to zero, Chand and Chhajed [1] showed that there exists an optimal schedule for $1 |MD| \sum_{i=1}^{m} \sum_{J_j \in I_i} (\alpha E_j + \beta T_j + \gamma D_i)$ in which the due date can be assigned according to (8), where MD means the multiple due dates. Because their proof is independent of the job distribution on the time axis, the result can immediately be generalized to the problem $1 |GT, MD| \sum_{i=1}^{m} \sum_{l=1}^{n_i} (\alpha_i E_{il} + \beta_i T_{il} + \gamma_i D_{il})$. Then the above lemma holds.

From Lemma 6, without loss of generality, we only consider the case $\lceil n_{ik}(\delta_i - \gamma_i) / \alpha_i \rceil < \lceil n_{ik}(\beta_i - \delta_i)/\beta_i \rceil$ in the following Lemmas 7 and 8. □

Lemma 7 *If $\lceil n_{ik}(\delta_i - \gamma_i)/\alpha_i \rceil \leq 0 < \lceil n_{ik}(\beta_i - \delta_i)/\beta_i \rceil$, the problem becomes a special due window assignment problem, with $d_{ik} = 0$, $w_{ik} = S_i(\pi) + s_i + \sum_{l=1}^{b_{ik}+h_{ik}} p_{i[l]}$, and $b_{ik} + h_{ik} = N_{i(k-1)} + \left\lceil \frac{n_{ik}(\beta_i - \delta_i)}{\beta_i} \right\rceil$.*

Proof If $\lceil n_{ik}(\delta_i - \gamma_i)/\alpha_i \rceil \leq 0$, meaning $\delta_i \leq \gamma_i$, then a shift of d_{ik} by Δ units of time to the left will result in a change of the total cost $\Delta f \leq 0$, so result d_{ik} can be shifted to the left until it equals time zero. Using the same analytical method of Lemmas 4 and 5, we can get the result about w_{ik}. □

Lemma 8 *If $\lceil n_{ik}(\delta_i - \gamma_i)/\alpha_i \rceil < \lceil n_{ik}(\beta_i - \delta_i)/\beta_i \rceil \leq 0$ for $i \in \{1, 2, \ldots, m\}$, then the multiple due windows assignment problem also reduces to the multiple due dates assignment problem. And for any $k = 1, 2, \ldots, z_i$, the due date $D_{ik} = 0$.*

Proof If $\lceil n_{ik}(\delta_i - \gamma_i)/\alpha_i \rceil < \lceil n_{ik}(\beta_i - \delta_i)/\beta_i \rceil \leq 0$, which means $\beta_i \leq \delta_i \leq \gamma_i$, then a shift of w_{ik} by Δ units of time to the left until d_{ik} will result in a change of the total cost $\Delta f \leq 0$, so w_{ik} can be shifted to d_{ik}, and the multiple due windows assignment problem reduces to the multiple due dates assignment problem. Since $\beta_i \leq \gamma_i$, according to (8), we can get the result that $D_{ik} = 0$, for $k = 1, 2, \ldots, z_i$.

From Lemmas 4–8, according to the ratios of $\lceil n_{ik}(\delta_i - \gamma_i)/\alpha_i \rceil$ and $\lceil n_{ik}(\beta_i - \delta_i)/\beta_i \rceil$, there exist three cases:

Case 1.1: $0 < \lceil n_{ik}(\delta_i - \gamma_i)/\alpha_i \rceil < \lceil n_{ik}(\beta_i - \delta_i)/\beta_i \rceil$. This will be the normal multiple due windows assignment problem, which can be determined by Lemmas 4 and 5.

Case 1.2: $\lceil n_{ik}(\delta_i - \gamma_i)/\alpha_i \rceil \leq 0 < \lceil n_{ik}(\beta_i - \delta_i)/\beta_i \rceil$. This is a special multiple due windows assignment problem, which can be determined by Lemma 7.

Case 1.3: $\lceil n_{ik}(\beta_i - \delta_i)/\beta_i \rceil \leq \lceil n_{ik}(\delta_i - \gamma_i)/\alpha_i \rceil$ or $\lceil n_{ik}(\delta_i - \gamma_i)/\alpha_i \rceil < \lceil n_{ik}(\beta_i - \delta_i)/\beta_i \rceil \leq 0$. The problem reduces to the multiple due dates assignment problem, which can be determined by Lemmas 6 and 8.

$\square$

Lemma 9 *For any given sequence π, the cost function of all jobs within group G_i under the optimal due windows assignment strategy of Case 1.1, denoted by $d_i^*(\pi) = \{d_{i1}^*, d_{i2}^*, \ldots, d_{iz_i}^*\}$ and $w_i^*(\pi) = \{w_{i1}^*, w_{i2}^*, \ldots, w_{iz_i}^*\}$, can be expressed as follows:*

$$g_i\left(\pi, d_i^*(\pi), w_i^*(\pi)\right) = \sum_{l=1}^{n_i} w_{il} \, p_{i[l]} + n_i \gamma_i \left(S_i(\pi) + s_i\right) \tag{9}$$

Where

$$w_{il} = \begin{cases} \alpha_i(l - N_{i(k-1)} - 1) + \gamma_i(n_i - N_{i(k-1)}), & N_{i(k-1)} + 1 \leq l \leq b_{ik} \\ n_{ik}\delta_i + \gamma_i(n_i - N_{ik}), & b_{ik} + 1 \leq l \leq b_{ik} + h_{ik} \\ \beta_i(N_{ik} - l + 1) + \gamma_i(n_i - N_{ik}), & b_{ik} + h_{ik} + 1 \leq l \leq N_{ik} \end{cases} \tag{10}$$

Proof Let π be a given job sequence, and group G_i with Case 1.1, the optimal d_{ik}^* and w_{ik}^* for $k = 1, 2, \ldots, z_i$, can be determined by Lemma 5. Then for any $k \in \{1, 2, \ldots, z_i\}$, we have

$$E_{i[l]} = \sum_{j=l+1}^{b_{ik}} p_{i[j]} \text{ for } N_{i(k-1)} + 1 \leq l \leq b_{ik} - 1, \, E_{i[l]} = 0 \text{ for } b_{ik} \leq l \leq N_{ik}.$$

$$T_{i[l]} = 0 \text{ for } N_{i(k-1)} + 1 \leq l \leq b_{ik} + h_{ik}, \quad T_{i[l]} = \sum_{j=b_{ik}+h_{ik}+1}^{l} p_{i[j]} \text{ for } b_{ik} + h_{ik} + 1 \leq l \leq N_{ik}.$$

$$d_{i[l]} = d_{ik} = C_{i[b_{ik}]} = S_i(\pi) + s_i + \sum_{l=1}^{b_{ik}} p_{i[l]} \text{ for } N_{i(k-1)} + 1 \leq l \leq N_{ik}.$$

$$w_{i[l]} = w_{ik} = C_{i[b_{ik}+h_{ik}]} = S_i(\pi) + s_i + \sum_{l=1}^{b_{ik}+h_{ik}} p_{i[l]} \text{ for } N_{i(k-1)} + 1 \leq l \leq N_{ik}.$$

$$S_{i[l]} = w_{i[l]} - d_{i[l]} = w_{ik} - d_{ik} = \sum_{l=b_{ik}+1}^{b_{ik}+h_{ik}} p_{i[l]} \text{ for } N_{i(k-1)} + 1 \leq l \leq N_{ik}.$$

Then the cost function of all the cost function of all jobs within group G_i can be written as follows:

$$g_i\left(\pi, d_i^*(\pi), w_i^*(\pi)\right) = \sum_{k=1}^{z_i} \sum_{l=N_{i(k-1)}+1}^{N_{ik}} \left(\alpha_i E_{i[l]} + \beta_i T_{i[l]} + \gamma_i d_{i[l]} + \delta_i S_{i[l]}\right)$$

$$= \sum_{k=1}^{z_i} \left[\sum_{l=N_{i(k-1)}+1}^{b_{ik}} \left(\alpha_i \sum_{j=l+1}^{b_{ik}} p_{i[j]}\right) + \sum_{l=b_{ik}+h_{ik}+1}^{N_{ik}} \left(\beta_i \sum_{j=b_{ik}+h_{ik}+1}^{l} p_{i[j]}\right)\right.$$

$$\left. + n_{ik}\gamma_i \left(S_i(\pi) + s_i + \sum_{l=1}^{N_{i(k-1)}} p_{i[l]} + \sum_{l=N_{i(k-1)}+1}^{b_{ik}} p_{i[l]}\right) + n_{ik}\delta_i \sum_{l=b_{ik}+1}^{b_{ik}+h_{ik}} p_{i[l]}\right]$$

$$= \sum_{k=1}^{z_i} \left[\sum_{l=N_{i(k-1)}+1}^{b_{ik}} \alpha_i(l - N_{i(k-1)} - 1)p_{i[l]} + \sum_{l=b_{ik}+h_{ik}+1}^{N_{ik}} \beta_i(N_{ik} - l + 1)p_{i[l]}\right.$$

$$\left. + n_{ik}\gamma_i \left(S_i(\pi) + s_i + \sum_{l=1}^{N_{i(k-1)}} p_{i[l]}\right) + n_{ik}\gamma_i \sum_{l=N_{i(k-1)}+1}^{b_{ik}} p_{i[l]} + n_{ik}\delta_i \sum_{l=b_{ik}+1}^{b_{ik}+h_{ik}} p_{i[l]}\right]$$

$$= \sum_{k=1}^{z_i} \left\{\sum_{l=N_{i(k-1)}+1}^{b_{ik}} \left[\alpha_i(l - N_{i(k-1)} - 1) + \gamma_i(n_i - N_{i(k-1)})\right] p_{i[l]}\right.$$

$$\left. + \sum_{l=b_{ik}+1}^{b_{ik}+h_{ik}} \left[n_{ik}\delta_i + \gamma_i(n_i - N_{ik})\right] p_{i[l]} + \sum_{l=b_{ik}+h_{ik}+1}^{N_{ik}} \left[\beta_i(N_{ik} - l + 1) + \gamma_i(n_i - N_{ik})\right] p_{i[l]}\right\}$$

$$+ n_i\gamma_i\left(S_i(\pi) + s_i\right)$$

Therefore, the result holds.   □

Lemma 10 *For any given sequence π, the cost function of all jobs within group G_i under the optimal due windows assignment strategy of Case 1.2, denoted by $d_i^*(\pi) = \{0, 0, \ldots, 0\}$ and $w_i^*(\pi) = \{w_{i1}^*, w_{i2}^*, \ldots, w_{iz_i}^*\}$, can be expressed as follows:*

$$g_i\left(\pi, d_i^*(\pi), w_i^*(\pi)\right) = \sum_{l=1}^{n_i} w_{il}\, p_{i[l]} + n_i\delta_i\left(S_i(\pi) + s_i\right) \tag{11}$$

Where,

$$w_{il} = \begin{cases} n_{ik}\delta_i, & N_{i(k-1)} + 1 \leq l \leq b_{ik} + h_{ik} \\ \beta_i(N_{ik} - l + 1), & b_{ik} + h_{ik} + 1 \leq l \leq N_{ik} \end{cases}. \tag{12}$$

Proof Let π be a given job sequence, and group G_i with Case 1.2, the optimal d_{ik}^* and w_{ik}^* for $k = 1, 2, \ldots, z_i$, can be determined by Lemma 7. Then for any $k \in \{1, 2, \ldots, z_i\}$, we have

$E_{i[l]} = 0$ for $N_{i(k-1)} + 1 \leq l \leq N_{ik}$.

$T_{i[l]} = 0$ for $N_{i(k-1)} + 1 \leq l \leq b_{ik} + h_{ik}$,

$T_{i[l]} = \sum_{j=b_{ik}+h_{ik}+1}^{l} p_{i[j]}$ for $b_{ik} + h_{ik} + 1 \leq l \leq N_{ik}$.

$d_{i[l]} = d_{ik} = 0$ for $N_{i(k-1)} + 1 \leq l \leq N_{ik}$.

$w_{i[l]} = w_{ik} = C_{i[b_{ik}+h_{ik}]} = S_i(\pi) + s_i + \sum_{l=1}^{b_{ik}+h_{ik}} p_{i[l]}$ for $N_{i(k-1)}+1 \leq l \leq N_{ik}$.

$S_{i[l]} = w_{i[l]} - d_{i[l]} = w_{ik} = S_i(\pi) + s_i + \sum_{l=1}^{b_{ik}+h_{ik}} p_{i[l]}$ for $N_{i(k-1)}+1 \leq l \leq N_{ik}$.

Then the cost function of all jobs within group G_i can be written as follows:

$$g_i\left(\pi, d_i^*(\pi), w_i^*(\pi)\right) = \sum_{k=1}^{z_i} \sum_{l=N_{i(k-1)}+1}^{N_{ik}} \left(\alpha_i E_{i[l]} + \beta_i T_{i[l]} + \gamma_i d_{i[l]} + \delta_i S_{i[l]}\right)$$

$$= \sum_{k=1}^{z_i} \left[\sum_{l=b_{ik}+h_{ik}+1}^{N_{ik}} \left(\beta_i \sum_{j=b_{ik}+h_{ik}+1}^{l} p_{i[j]} \right) + n_{ik}\delta_i \left(S_i(\pi) + s_i + \sum_{l=1}^{b_{ik}+h_{ik}} p_{i[l]} \right) \right]$$

$$= \sum_{k=1}^{z_i} \left[\sum_{l=b_{ik}+h_{ik}+1}^{N_{ik}} \beta_i (N_{ik} - l + 1) p_{i[l]} + n_{ik}\delta_i \sum_{l=1}^{b_{ik}+h_{ik}} p_{i[l]} + n_{ik}\delta_i (S_i(\pi) + s_i) \right]$$

$$= \sum_{k=1}^{z_i} \left[\sum_{l=1}^{b_{ik}+h_{ik}} n_{ik}\delta_i p_{i[l]} + \sum_{l=b_{ik}+h_{ik}+1}^{N_{ik}} \beta_i (N_{ik} - l + 1) p_{i[l]} \right] + n_i \delta_i (S_i(\pi) + s_i)$$

Therefore, the result holds.

If group G_i is under the optimal due windows assignment strategy of Case 1.3, then the multiple due windows assignment problem of group G_i reduces to the multiple due dates assignment problem, i.e., $\{d_{i1}^*, d_{i2}^*, \ldots, d_{iz_i}^*\} = d_i^*(\pi) = D_i^*(\pi) = w_i^*(\pi) = \{w_{i1}^*, w_{i2}^*, \ldots, w_{iz_i}^*\}$. The corresponding objective function $g(\pi, d(\pi), w(\pi)) = \sum_{i=1}^{m} \sum_{l=1}^{n_i} (\alpha_i E_{il} + \beta_i T_{il} + \gamma_i d_{il} + \delta_i S_{il})$ is translated into $f(\pi, D(\pi)) = \sum_{i=1}^{m} \sum_{l=1}^{n_i} (\alpha_i E_{il} + \beta_i T_{il} + \gamma_i D_{il})$. Then we contain the following lemma. □

Lemma 11 *For any given sequence π, the cost function of all jobs within group G_i under the optimal due windows assignment strategy of Case 1.3, denoted by $d_i^*(\pi) = w_i^*(\pi) = D_i^*(\pi) = \{D_{i1}^*, D_{i2}^*, \ldots, D_{iz_i}^*\}$, can be expressed as follows:*

$$f_i(\pi, D_i^*(\pi)) = \sum_{l=1}^{n_i} w_{il} p_{i[l]} + n_i \varpi_i (S_i(\pi) + s_i) \tag{13}$$

where

$$\varpi_i = \min\{\beta_i, \gamma_i\}, \tag{14}$$

$$w_{il} = \begin{cases} \alpha_i (l - N_{i(k-1)} - 1) + \gamma_i (n_i - N_{i(k-1)}), & N_{i(k-1)} + 1 \leq l \leq b_{ik} \\ \beta_i (N_{ik} - l + 1) + \gamma_i (n_i - N_{ik}), & b_{ik} + 1 \leq l \leq N_{ik} \end{cases} \text{ for } \beta_i > \gamma_i, \tag{15}$$

and

$$w_{il} = \beta_i (n_i - l + 1), \quad 1 \leq l \leq n_i \text{ for } \beta_i \leq \gamma_i, \tag{16}$$

Proof Let π be a given job sequence. By Lemma 6, the optimal due date D_{ik}^* for $k = 1, 2, \ldots, z_i$, $i = 1, 2, \ldots, m$ can be determined according to (8). We split the proof into two cases with respect to the relation of β_i and γ_i.

Case 2.1: $\beta_i > \gamma_i$. For any $k \in \{1, 2, \ldots, z_i\}$ in this case, we have

$$E_{i[l]} = \sum_{j=l+1}^{b_{ik}} p_{i[j]} \text{ for } N_{i(k-1)} + 1 \leq l \leq b_{ik} - 1 \text{ and } E_{i[l]} = 0 \text{ for } b_{ik} \leq l \leq N_{ik}.$$

$$T_{i[l]} = 0 \text{ for } N_{i(k-1)} + 1 \leq l \leq b_{ik} \text{ and } T_{i[l]} = \sum_{j=b_{ik}+1}^{l} p_{i[j]} \text{ for } b_{ik} + 1 \leq l \leq N_{ik}.$$

$$D_{i[l]} = D_{ik} = C_{i[b_{ik}]} = S_i(\pi) + s_i + \sum_{l=1}^{b_{ik}} p_{i[l]} \text{ for } N_{i(k-1)} + 1 \leq l \leq N_{ik} \tag{17}$$

Then the cost function of all jobs within group G_i can be written as follows:

$$f_i(\pi, D_i^*(\pi)) = \sum_{l=1}^{n_i} (\alpha_i E_{il} + \beta_i T_{il} + \gamma_i D_{il})$$

$$= \sum_{k=1}^{z_i} \sum_{l=N_{i(k-1)}+1}^{N_{ik}} \left(\alpha_i E_{i[l]} + \beta_i T_{i[l]} + \gamma_i D_{i[l]} \right)$$

$$= \sum_{k=1}^{z_i} \left[\sum_{l=N_{i(k-1)}+1}^{b_{ik}} \left(\alpha_i \sum_{j=l+1}^{b_{ik}} p_{i[j]} \right) + \sum_{l=b_{ik}+1}^{N_{ik}} \left(\beta_i \sum_{j=b_{ik}+1}^{l} p_{i[j]} \right) \right.$$

$$\left. + n_{ik}\gamma_i \left(S_i(\pi) + s_i + \sum_{l=1}^{N_{i(k-1)}} p_{i[l]} + \sum_{l=N_{i(k-1)}+1}^{b_{ik}} p_{i[l]} \right) \right]$$

$$= \sum_{k=1}^{z_i} \left[\sum_{l=N_{i(k-1)}+1}^{b_{ik}} \alpha_i \left(l - N_{i(k-1)} - 1 \right) p_{i[l]} + \sum_{l=b_{ik}+1}^{N_{ik}} \beta_i \left(N_{ik} - l + 1 \right) p_{i[l]} \right.$$

$$\left. + n_{ik}\gamma_i \left(S_i(\pi) + s_i + \sum_{l=1}^{N_{i(k-1)}} p_{i[l]} \right) + n_{ik}\gamma_i \sum_{l=N_{i(k-1)}+1}^{b_{ik}} p_{i[l]} \right]$$

$$= \sum_{k=1}^{z_i} \left\{ \sum_{l=N_{i(k-1)}+1}^{b_{ik}} \left[\alpha_i \left(l - N_{i(k-1)} - 1 \right) + \gamma_i \left(n_i - N_{i(k-1)} \right) \right] p_{i[l]} \right.$$

$$\left. + \sum_{l=b_{ik}+1}^{N_{ik}} \left[\beta_i \left(N_{ik} - l + 1 \right) + \gamma_i \left(n_i - N_{ik} \right) \right] p_{i[l]} \right\} + n_i\gamma_i \left(S_i(\pi) + s_i \right)$$

Therefore, the result retains in this case.

Case 2.2: $\beta_i \leq \gamma_i$. For any $k \in \{1, 2, \ldots, z_i\}$ in this case, we have

$$E_{i[l]} = 0 \text{ for } N_{i(k-1)} + 1 \leq l \leq N_{ik}.$$

$$T_{i[l]} = C_{i[l]} = S_i(\pi) + s_i + \sum_{j=1}^{N_{i(k-1)}} p_{i[j]}$$

$$+ \sum_{j=N_{i(k-1)}+1}^{l} p_{i[j]} \text{ for } N_{i(k-1)} + 1 \leq l \leq N_{ik}.$$

$$D_{i[l]} = D_{ik} = 0 \text{ for } N_{i(k-1)} + 1 \leq l \leq N_{ik} \tag{18}$$

Then the cost function of all jobs within group G_i can be written as follows:

$$f_i(\pi, D_i^*(\pi)) = \sum_{l=1}^{n_i} \left(\alpha_i E_{il} + \beta_i T_{il} + \gamma_i D_{il} \right)$$

$$= \sum_{k=1}^{z_i} \sum_{l=N_{i(k-1)}+1}^{N_{ik}} \left(\alpha_i E_{i[l]} + \beta_i T_{i[l]} + \gamma_i D_{i[l]} \right)$$

$$= \sum_{k=1}^{z_i} \sum_{l=N_{i(k-1)}+1}^{N_{ik}} \beta_i \left(S_i(\pi) + s_i + \sum_{j=1}^{N_{i(k-1)}} p_{i[j]} + \sum_{j=N_{i(k-1)}+1}^{l} p_{i[j]} \right)$$

$$= \sum_{k=1}^{z_i} \sum_{l=N_{i(k-1)}+1}^{N_{ik}} \beta_i \left(n_i - N_{ik} \right) p_{i[j]} + \sum_{k=1}^{z_i} \sum_{j=N_{i(k-1)}+1}^{N_{ik}} \beta_i \left(N_{ik} - l + 1 \right) p_{i[j]}$$

$$+ \sum_{k=1}^{z_i} \sum_{l=N_{i(k-1)}+1}^{N_{ik}} \beta_i \left(S_i(\pi) + s_i \right)$$

$$= \sum_{k=1}^{z_i} \sum_{l=N_{i(k-1)}+1}^{N_{ik}} \beta_i \left(n_i - l + 1 \right) p_{i[j]} + n_i\beta_i \left(S_i(\pi) + s_i \right)$$

Therefore, the result also holds for the second case. $\qquad\square$

3.2 A unified optimization algorithm

In this subsection, we will present an $O(n \log n)$ time unified optimization algorithm for the problem $1 \,|GT, MDW| \sum_{i=1}^{m} \sum_{l=1}^{n_i} (\alpha_i E_{il} + \beta_i T_{il} + \gamma_i d_{il} + \delta_i S_{il})$.

In view of Lemmas 9–11, for any given job sequence π, the unified cost function under the optimal due windows assignment strategy, denoted by $d^*(\pi) = \{d_{11}^*, \ldots, d_{1z_1}^*, \ldots, d_{m1}^*, \ldots, d_{mz_m}^*\}$ and $w^*(\pi) = \{w_{11}^*, \ldots, w_{1z_1}^*, \ldots, w_{m1}^*, \ldots, w_{mz_m}^*\}$, can be formulated as follows:

$$g(\pi, d^*(\pi), w^*(\pi)) = \sum_{i=1}^{m} \sum_{l=1}^{n_i} w_{il} p_{i[l]} + \sum_{i=1}^{m} n_i \psi_i (S_i(\pi) + s_i) \quad (19)$$

where,

$$\psi_i = \begin{cases} \gamma_i, & \text{if group } G_i \text{ is Case 1.1} \\ \delta_i, & \text{if group } G_i \text{ is Case 1.2} \\ \varpi_i, & \text{if group } G_i \text{ is Case 1.3} \end{cases}, \varpi_i = \min\{\beta_i, \gamma_i\},$$

and $w_{il}(1 \leq i \leq m, 1 \leq l \leq n_i)$ is defined by (10) for Case 1.1, by (12) for Case 1.2, and by (15) and (16) for Case 1.3.

From (19) we can observe that under an optimal due window assignment policy, the total cost is the sum of $m + 1$ separable terms. The first m terms, $\sum_{l=1}^{n_i} w_{il} p_{i[l]}$ for $i = 1, 2, \ldots, m$, is dependent only on the internal job sequence within group G_i, while the last term, $\sum_{i=1}^{m} n_i \psi_i (S_i(\pi) + s_i)$, is only concerned with the starting time of the group and is independent of the internal job sequence of each group. As a result of this separable characteristic of the objective function, our problem reduces to $m + 1$ subproblems. The first m subproblems are to find an optimal job sequence which minimize $\sum_{l=1}^{n_i} w_{il} p_{i[l]}$ for $i = 1, 2, \ldots, m$, while the last subproblem is to find an optimal group sequence which minimizes $\sum_{i=1}^{m} n_i \psi_i (S_i(\pi) + s_i)$. Then the following two lemmas are immediately obtained.

Lemma 12 *The optimal job sequence within group G_i can be obtained by matching the smallest w_{il} value to the largest p_{il} value, the second smallest w_{il} value to the second largest p_{il} value, and so on.*

Proof It can be easily proved by Lemma 2. □

Lemma 13 *In an optimal schedule, the groups are ordered in non-decreasing order of $(s_i + \sum_{l=1}^{n_i} p_{il})/n_i \psi_i$.*

Proof we prove the result by contradiction. Let π be an optimal schedule that does not satisfy the property of the lemma. Then π must contain at least two adjacent groups, say G_j followed by G_k, such that $(s_k + \sum_{l=1}^{n_i} p_{kl})/n_k \psi_k < (s_j + \sum_{l=1}^{n_i} p_{jl})/n_j \psi_j$. Swapping G_j and G_k, while leaving the other groups in their original order, we obtain a new schedule π'. Then we have

$$S_k(\pi) = S_j(\pi) + s_j + \sum_{l=1}^{n_j} p_{jl}, \quad (20)$$

$$S_k(\pi') = S_j(\pi), \quad (21)$$

$$S_j(\pi') = S_j(\pi) + s_k + \sum_{l=1}^{n_k} p_{kl}. \quad (22)$$

Because the completion times of jobs processed before (after) G_j and G_k are the same under schedule π and π', the cost function for groups processed before (after)

G_j and G_k are not changed. As a consequence, by submitting (20–22) into (19), the difference between the objective values of the two schedules is

$$g\left(\pi, d^*(\pi), w^*(\pi)\right) - g\left(\pi', d^*(\pi'), w^*(\pi')\right)$$

$$= n_k \psi_k \left(s_j + \sum_{l=1}^{n_j} p_{jl}\right) - n_j \psi_j \left(s_k + \sum_{l=1}^{n_k} p_{kl}\right)$$

$$= n_k n_j \psi_k \psi_j \left(\frac{s_j + \sum_{l=1}^{n_j} p_{jl}}{n_j \psi_j} - \frac{s_k + \sum_{l=1}^{n_k} p_{kl}}{n_k \psi_k}\right) > 0.$$

This contradicts the optimality of π, and completes the proof of the lemma.

Based on the above analysis, now we present the following algorithm to solve the $1|GT, MDW| \sum_{i=1}^{m} \sum_{l=1}^{n_i} (\alpha_i E_{il} + \beta_i T_{il} + \gamma_i d_{il} + \delta_i S_{il})$ problem.   □

Algorithm

Step 1 Calculate the ratios of $\lceil n_{ik}(\delta_i - \gamma_i)/\alpha_i \rceil$ and $\lceil n_{ik}(\beta_i - \delta_i)/\beta_i \rceil$, and according to Lemmas 4–8, determine which case each group is.

Step 2 For the groups of Case 1.1, calculate w_{il} according to (10), and b_{ik} and $b_{ik}+h_{ik}$ according to Lemma 5. For the groups of Case 1.2, calculate w_{il} according to (12), and $b_{ik}+h_{ik}$ according to Lemma 7. For the groups of Case 1.3, calculate w_{il} according to (15) and (16), and b_{ik} according to Lemma 6 or 8.

Step 3 Sequence the jobs within each group by Lemma 12 and arrange the groups in non-decreasing order of $(s_i + \sum_{l=1}^{n_i} p_{il})/n_i \psi_i$ by Lemma 13.

Step 4 For the groups of Case 1.1, assign the due windows according to Lemma 5. For the groups of Case 1.2, assign the due windows according to Lemma 7. For the groups of Case 1.3, assign the due windows according to Lemma 6 or 8.

To determine the computational complexity of the algorithm, note that Step 1, Step 2 and Step 4 can be performed in linear time; Step 3 requires $O(\sum_{i=1}^{m} n_i \log n_i + m \log m)$ time. Since $\sum_{i=1}^{m} n_i = n$ and $m \leq n$, the time complexity of the algorithm is $O(n \log n)$.

Theorem 1 *The* $1|GT, MDW| \sum_{i=1}^{m} \sum_{l=1}^{n_i} (\alpha_i E_{il} + \beta_i T_{il} + \gamma_i d_{il} + \delta_i S_{il})$ *problem can be solved by the above algorithm in* $O(n \log n)$ *time.*

4 Conclusions

We considered the problem of the simultaneous determination of multiple due windows assignment and job scheduling on a single machine under a GT restriction, where the objective is to determine an optimal combination of the job schedule and due windows assignment strategy to minimize the total related cost that comprises the earliness, tardiness and the due windows. We presented an $O(n \log n)$ time algorithm for the problem. It would be an interesting and valuable topic to investigate the case where the job processing times are time-dependent or resource-dependent in the future research.

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