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ORIGINAL RESEARCH

The Adomian decomposition method with Green's function for solving nonlinear singular boundary value problems

Randhir Singh · Jitendra Kumar

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Abstract In this paper, we present an efficient numerical algorithm for solving a general class of nonlinear singular boundary value problems. This present algorithm is based on the Adomian decomposition method (ADM) and Green's function. The method depends on constructing Green's function before establishing the recursive scheme. In contrast to the existing recursive schemes based on ADM, the proposed numerical algorithm avoids solving a sequence of transcendental equations for the undetermined coefficients. The approximate series solution is calculated in the form of series with easily computable components. Moreover, the convergence analysis and error estimation of the proposed method is given. Furthermore, the numerical examples are included to demonstrate the accuracy, applicability, and generality of the proposed scheme. The numerical results reveal that the proposed method is very effective.

Keywords Nonlinear singular boundary value problems · Adomian decomposition method · Green's function · Approximations of solution · Adomian's polynomials

Mathematics Subject Classification 34B05 · 34B15 · 34B16 · 34B18 · 34B27

1 Introduction

Two-point singular boundary value problems for ordinary differential equations arise very frequently in many branches of applied mathematics and physics such as chemical reactions, atomic calculations, gas dynamics, nuclear physics, atomic structures,

R. Singh (✉) · J. Kumar
Department of Mathematics, Indian Institute of Technology Kharagpur, Kharagpur 721302, India
e-mail: randhir.math@gmail.com

J. Kumar
e-mail: jkumar@maths.iitkgp.ernet.in

in the theory of shallow membrane caps, and study of positive radial solutions of nonlinear elliptic equations and physiological studies. In most of the cases, singular boundary value problems in general cannot be solved analytically. So these problems must be handled by various numerical techniques. However, the numerical treatment of the singular boundary value problems has always been far from trivial due to the singularity.

The objective of this paper is to propose an effective technique to solve a general nonlinear derivative-dependent singular boundary value problems (SBVPs). The proposed technique is based on the work of Singh et al. [1], where we transform the original nonlinear weakly singular boundary value problem with Dirichlet and Robin boundary conditions into an equivalent nonlinear Fredholm integral equation before establishing the recursive scheme for the solution. In this paper we consider the following class of derivative-dependent nonlinear singular boundary value [2–5]

$$\left.\begin{aligned} &\big(p(x)y'(x)\big)' = q(x)f\big(x, y(x), p(x)y'(x)\big), \quad 0 < x \leq 1,\\ &\lim_{x\to 0+} p(x)y'(x) = 0, \quad ay(1) + by'(1) = c, \end{aligned}\right\} \tag{1.1}$$

where $a > 0$, b and c are any finite real constants. The condition $p(0) = 0$ says that the problem (1.1) is singular and if q is allowed to be discontinuous at $x = 0$ then the problem (1.1) is called doubly singular [2]. Throughout the paper we assume the following conditions on p, q and $f(x, y, py')$:

(E_1) $p \in C[0,1] \cap C^1(0,1]$ with $p > 0$ in $(0,1]$;
(E_2) $q > 0$ in $(0,1]$, $q \in L^1(0,1]$ and q is not identically zero;
(E_3) $\int_0^1 \frac{1}{p(x)} \int_0^x q(s)ds dx < \infty$;
(F_1) let $f(x, y, py')$ is continuous on $D_1 = \{[0,1] \times \mathbb{R} \times \mathbb{R}\}$ and is not identically zero;
(F_2) let $f(x, y, py')$ satisfies the Lipschitz condition

$$\big|f\big(x, y_1, py_1'\big) - f\big(x, y_2, py_2'\big)\big| \leq l_1|y_1 - y_2| + l_2\big|p\big(y_1' - y_2'\big)\big|, \tag{1.2}$$

where l_1 and l_2 are Lipschitz constants.

Past couple of decades, there has been much interest in the study of singular two-point boundary value problems, [1–4, 6–18] and many of the references therein. The main difficulty of (1.1) is that the singular behavior occurs at $x = 0$. In [2, 3, 6], authors have discussed the existence and uniqueness of solution of the problem (1.1). A lot of numerical techniques have been applied to tackle the particular case of singular boundary value problem (1.1). For instant, the cubic spline and finite difference methods were carried out in [7–9]. Although, these numerical techniques have lot of advantages, but a huge amount of computational work is involved which combines some root-finding methods for obtaining accurate numerical solution especially for nonlinear singular boundary value problems.

Recently, some newly developed numerical-approximate methods have also been applied to handle some particular cases of (1.1). Such as, the ADM and MADM have been used in [10, 12]. The homotopy analysis method (HAM) was introduced in [13]. To solve equation (1.1) using ADM, MADM, and HAM is always a computationally involved task as it requires the computation of undetermined coefficients in a sequence of nonlinear algebraic or more difficult transcendental equations which increases the computational work (see [10, 12, 13]). Moreover, the undetermined coefficients may not be uniquely determined and this may be the main disadvantage of these methods for solving nonlinear two-point BVPs.

The variational iteration method (VIM) and its modified version for solving the particular cases of SBVP (1.1) have been employed in [15, 17, 19]. The VIM gives good approximations only when the problem is linear or nonlinear BVPs with nonlinearity of the form $y^n, yy', y'^n \dots$ etc., but the method suffers when the nonlinear function is of the form $e^y, \ln(y), \sin y, \sinh y \dots$ etc., this may be one of the major disadvantage of VIM for solving difficult nonlinear problems (see [17]).

1.1 Review of ADM

In this subsection, we shall briefly describe ADM for nonlinear second order differential equations.

In the recent past, a lot of researchers [1, 10–12, 14, 16, 17, 20–33] have expressed their interest in the study of ADM for various scientific models. Adomian [22] asserted that the ADM provides an efficient and computationally worthy method for generating approximate series solution for a large class of differential equations.

According to the ADM, the operator form of (1.1) can be written as

$$\mathcal{L}y = Ry + Ny, \tag{1.3}$$

where $\mathcal{L} \equiv \frac{d^2}{dx^2}$ is linear second-order differential operator, $Ry = (-p'/p)y'$ and $Ny = (q/p)f(x, y, py')$ represents the nonlinear term.
The inverse operator of $\mathcal{L}$ can be defined as

$$\mathcal{L}^{-1}[\cdot] = \int_0^x \int_0^x [\cdot]dxdx. \tag{1.4}$$

Operating the inverse operator $\mathcal{L}^{-1}[\cdot]$ on both sides of (1.3), we have

$$y = y(0) + y'(0)x + \mathcal{L}^{-1}[Ry + Ny]. \tag{1.5}$$

Then the solution y and the nonlinear function Ny are decomposed by infinite series

$$y = \sum_{n=0}^{\infty} y_n, \qquad Ny = \sum_{n=0}^{\infty} A_n, \tag{1.6}$$

where A_n are Adomian's polynomials that can be constructed for various classes of nonlinear functions with the formula given by Adomian and Rach [23]

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{k=0}^{\infty} y_k\lambda^k\right)\right]_{\lambda=0}, \qquad n = 0, 1, 2, \dots. \tag{1.7}$$

New efficient algorithms and subroutines in 'MATHEMATICA' for rapid computer-generation of the Adomian polynomials to high orders are provided in [34–36]. In [36, 37], authors proposed a new modification of the ADM (MADM) based on a new formula for Adomian's polynomials by rearranging Taylor series components of the analytic function $N(y)$. By defining the series $\psi_n = \sum_{j=0}^{n} y_j$, and according to new MADM, A_k's are obtained as following:

$$A_k = N(\psi_k) - \sum_{j=0}^{k-1} A_j. \tag{1.8}$$

This formula enforces many additional terms to the calculation processes in (1.7), which implying faster convergence [36, 37].

Substituting the series (1.6) into (1.5), we obtain

$$\sum_{n=0}^{\infty} y_n = y(0) + y'(0)x + \mathcal{L}^{-1}[Ry] + \mathcal{L}^{-1}\left[\sum_{n=0}^{\infty} A_n\right]. \tag{1.9}$$

On comparing both sides of (1.9), the ADM admits the following recursive scheme

$$y_0 = y(0) + y'(0)x, \qquad y_{n+1} = \mathcal{L}^{-1}[Ry] + \mathcal{L}^{-1}[A_n], \quad n \geq 0, \tag{1.10}$$

that will lead to the complete determination of components y_n, and the series solution of y follows immediately with the undetermined coefficient $y(0)$, and the unknown constant $y(0)$ will be determined later imposing the boundary conditions at $x = 1$ (see [12, 17, 24, 25]). Then the n-term truncated approximate series solution is given by $\psi_n(x) = \sum_{m=0}^{n} y_m(x)$.

In [11, 12, 16, 21, 24, 25], the researchers have used the ADM or MADM for solving nonlinear boundary value problems for ordinary differential equations. To solve two-point boundary value problems using ADM or MADM is always a computationally involved task as it requires the computation of undetermined coefficients in a sequence of nonlinear algebraic equations which increases the computational work, (see [10, 12, 16, 17, 21, 24, 25]). In other words, major disadvantage of the ADM for solving nonlinear boundary value problems is to solve a sequence of growingly higher order polynomials or more difficult transcendental equations for undetermined coefficients [21]. In order to avoid solving a sequence of difficult transcendental equations for a two-point boundary value problems. In [36, 38], authors proposed a new modified ADM for solving second order as well as higher order non-singular boundary value problems with Dirichlet as well as Robin boundary conditions, where they transformed the original nonlinear boundary value problem into an equivalent nonlinear Fredholm-Volterra integral equation for the solution before designing the recursion scheme. Later, in [1, 14, 20, 32], authors proposed some new modification of ADM for solving second-order singular boundary value problems with Dirichlet, Dirichlet and Robin, Neumann and Robin boundary conditions.

In this paper, we propose an efficient numerical algorithm to overcome the difficulties occur in ADM or MADM for solving nonlinear singular boundary value problems. To establish this algorithm, we first construct Green's function to establish the recursion scheme for solution components. The proposed method provides a direct scheme for obtaining approximations for the solution of the singular boundary value problem without solving a sequence of difficult transcendental equations for unknown constant. In addition, the convergence analysis and error estimation of the proposed method is established. Finally, some numerical examples are included to demonstrate the accuracy of the proposed method.

2 Adomian decomposition method with Green's function

In this section, we propose an efficient numerical algorithm, which is based on Green's function and Adomian's polynomial for solving a general nonlinear singular two-point boundary value problems of the form (1.1). To do this, we first consider the corresponding homogeneous problem of (1.1)

$$\left.\begin{aligned}&\big(p(x)y'(x)\big)'=0,\quad 0<x\le 1,\\&\lim_{x\to 0+}p(x)y'(x)=0,\quad ay(1)+by'(1)=c.\end{aligned}\right\}\tag{2.1}$$

The unique solution of (2.1) can easily be found and given by

$$\hat{y}(x)=\frac{c}{a}.\tag{2.2}$$

Now we again consider the SBVP (1.1) with homogeneous boundary conditions

$$\left.\begin{aligned}&\big(p(x)y'(x)\big)'=q(x)f\big(x,y(x),p(x)y'(x)\big),\quad 0<x\le 1,\\&\lim_{x\to 0}p(x)y'(x)=0,\quad ay(1)+by'(1)=0.\end{aligned}\right\}\tag{2.3}$$

The Green's function of (2.3) can easily be constructed as

$$G(x,\xi)=\begin{cases}\int_\xi^1\frac{dx}{p(x)}+\frac{b}{ap(1)}, & 0<x\le\xi\le 1,\\ \int_\xi^1\frac{dx}{p(x)}-\int_\xi^x\frac{dx}{p(x)}+\frac{b}{ap(1)}, & 0<\xi\le x\le 1.\end{cases}\tag{2.4}$$

The derivation of Green's function is provided in the appendix.
Now, using (2.2) and (2.4), the SBVP (1.1) can be converted into following integral equation as

$$y(x)=\frac{c}{a}+\int_0^1 G(x,\xi)q(\xi)f\big(\xi,y(\xi),p(\xi)y'(\xi)\big)d\xi.\tag{2.5}$$

In other words, the integral equation (2.5) is equivalent to SBVP (1.1). It should also be noted that the right hand side of (2.5) does not involve any undetermined coefficients.

We next decompose the solution y by an infinite series as:

$$y=\sum_{n=0}^{\infty}y_n,\tag{2.6}$$

and the nonlinear function $f(x, y, py')$ by an infinite series

$$f\left(x, y, py'\right) = \sum_{n=0}^{\infty} A_n, \tag{2.7}$$

where A_n are Adomian's polynomials [23].

Substituting the series (2.6) and (2.7) in (2.5), we obtain

$$\sum_{n=0}^{\infty} y_n = \frac{c}{a} + \int_0^1 G(x,\xi)q(\xi) \sum_{n=0}^{\infty} A_n d\xi. \tag{2.8}$$

Upon comparing both sides of (2.8), we obtain the following scheme as follows:

$$\left.\begin{aligned} y_0 &= \frac{c}{a}, \\ y_{n+1} &= \int_0^1 G(x,\xi)q(\xi)A_n d\xi, \quad n \geq 0. \end{aligned}\right\} \tag{2.9}$$

The solution components y_n can be calculated using the scheme (2.9) and hence, the n-term approximate series solution is obtained as

$$\psi_n(x) = \sum_{j=0}^{n} y_j. \tag{2.10}$$

Unlike existing ADM or MADM, the proposed scheme (2.9) avoids solving a sequence of nonlinear algebraic or transcendental equations for the undetermined coefficients.

3 Convergence analysis

In this section, we shall suggest an alternative approach for proving the convergence analysis of proposed recursive scheme for singular boundary value problem (1.1). We remark that many authors [37, 39–42] have also established the convergence of ADM for differential and integral equations. To do this, let $\mathbb{X} = C[0,1] \cap C^1(0,1]$ be a Banach space with the norm

$$\|y\| = \max\left\{\|y\|_0, \|y\|_1\right\}, \quad y \in \mathbb{X}, \tag{3.1}$$

where,

$$\|y\|_0 = \max_{0<x\leq 1} \left|y(x)\right| \quad \text{and} \quad \|y\|_1 = \max_{0<x\leq 1} \left|p(x)y'(x)\right|.$$

It is well known that $\mathbb{X}$ is Banach space with the norm (3.1) (see, pp. 45, [2]).

Note that Eq. (2.5) can be written in the operator equation form as

$$y = \mathcal{N} y, \tag{3.2}$$

where $\mathcal{N}y$ is given by

$$\mathcal{N}y = \frac{c}{a} + \int_0^1 G(x,\xi)q(\xi)f\big(\xi, y(\xi), p(\xi)y'(\xi)\big)d\xi. \tag{3.3}$$

We next discuss the existence of the unique solution of Eq. (3.2). To do this, we first prove the following Lemma.

Lemma 3.1 *Let the assumptions* (E_1)–(E_3) *hold and the Green's function of* (2.3) *is given by* $G(x,\xi)$, *then*

(i) $m_1 := \max_{0<x\le 1} |\int_0^1 G(x,\xi)q(\xi)d\xi| < \infty$,
(ii) $m_2 := \max_{0<x\le 1} |\int_0^1 p(x)G_x(x,\xi)q(\xi)d\xi| < \infty$, *where* $G_x(x,\xi) = \frac{\partial G(x,\xi)}{\partial x}$.

Proof (i) This is trivial, it follows from the assumptions (E_1)–(E_3) and from the Green's function (2.4).

(ii) From (6.4), we see that

$$p(x)G_x(x,\xi) = \begin{cases} 0, & 0 < x \le \xi \le 1, \\ -1, & 0 < \xi \le x \le 1. \end{cases} \tag{3.4}$$

Hence, we obtain $c_2 = \max_{0<x,\xi\le 1} |p(x)G_x(x,\xi)| < \infty$.

Now again using the assumption (E_2), we have

$$\begin{aligned} &\left|\int_0^1 p(x)G_x(x,\xi)q(\xi)d\xi\right| \\ &\quad \le \max_{0<x,\ \xi\le 1}\big|p(x)G_x(x,\xi)\big|\int_0^1 \big|q(\xi)\big|d\xi = c_2\int_0^1 \big|q(\xi)\big|d\xi < \infty. \end{aligned} \tag{3.5}$$

Hence it follows that $m_2 = \max_{0<x\le 1} |\int_0^1 p(x)G_x(x,\xi)q(\xi)d\xi| < \infty$. □

Theorem 3.1 *Let* $\mathbb{X}$ *be Banach space with norm given by* (3.1). *Also assume that the nonlinear function* $f(x,y,py')$ *satisfies the Lipschitz condition* (F_2). *Let* $m = \max\{m_1, m_2\}$ *and* $l = \max\{l_1, l_2\}$, *where the constants* m_1 *and* m_2 *given as in Lemma* 3.1 *and* l_1 *and* l_2 *are Lipschitz constants. If* $\delta = 2lm < 1$, *then Eq.* (3.2) *has a unique solution in* $\mathbb{X}$.

Proof Using the Lemma 3.1, we have for any $y_1, y_2 \in \mathbb{X}$,

$$\begin{aligned} \big|\mathcal{N}y_1(x) - \mathcal{N}y_2(x)\big| &= \left|\int_0^1 G(x,\xi)q(\xi)\big[f\big(\xi, y_1, py_1'\big) - f\big(\xi, y_2, py_2'\big)\big]d\xi\right| \\ &\le \max_{0<\xi\le 1}\big|f\big(\xi, y_1, py_1'\big) - f\big(\xi, y_2, py_2'\big)\big| \\ &\quad \times \max_{0<x\le 1}\left|\int_0^1 G(x,\xi)q(\xi)d\xi\right| \\ &= m_1 \max_{0<\xi\le 1}\big|f\big(\xi, y_1, py_1'\big) - f\big(\xi, y_2, py_2'\big)\big|. \end{aligned}$$

Now using the Lipschitz continuity of f, we have

$$\max_{0<x\le 1}\left|\mathcal{N}y_1(x)-\mathcal{N}y_2(x)\right| \le m_1 \max_{0<\xi\le 1}\left[l_1|y_1-y_2|+l_2\left|p\left(y_1'-y_2'\right)\right|\right]$$
$$\le 2lm_1 \max\left\{\|y_1-y_2\|_0, \|y_1-y_2\|_1\right\}$$

where $l=\max\{l_1,l_2\}$. Thus, we have

$$\left\|\mathcal{N}y_1-\mathcal{N}y_2\right\|_0 \le 2lm_1\|y_1-y_2\|. \tag{3.6}$$

Similarly, we have

$$\begin{aligned}\left|p(x)(\mathcal{N}y_1-\mathcal{N}y_2)'(x)\right| &= \left|p(x)\int_0^1 G_x(x,\xi)q(\xi)\right.\\ &\qquad \left.\times\left[f\left(\xi,y_1,py_1'\right)-f\left(\xi,y_2,py_2'\right)\right]d\xi\right|\\ &\le \max_{0<\xi\le 1}\left|f\left(\xi,y_1,py_1'\right)-f\left(\xi,y_2,py_2'\right)\right|\\ &\qquad \times \max_{0<x\le 1}\left|\int_0^1 p(x)G_x(x,\xi)q(\xi)d\xi\right|,\\ &= m_2\max_{0<\xi\le 1}\left|f\left(\xi,y_1,py_1'\right)-f\left(\xi,y_2,py_2'\right)\right|.\end{aligned}$$

Using the Lipschitz continuity of f, we obtain

$$\max_{0<x\le 1}\left|p(x)(\mathcal{N}y_1-\mathcal{N}y_2)'(x)\right| \le 2lm_2\max\left\{\|y_1-y_2\|_0, \|y_1-y_2\|_1\right\}.$$

Hence

$$\|\mathcal{N}y_1-\mathcal{N}y_2\|_1 \le 2lm_2\|y_1-y_2\|. \tag{3.7}$$

Combining the estimates (3.6) and (3.7), we obtain

$$\begin{aligned}\|\mathcal{N}y_1-\mathcal{N}y_2\| &= \max\left\{\|\mathcal{N}y_1-\mathcal{N}y_2\|_0, \|\mathcal{N}y_1-\mathcal{N}y_2\|_1\right\}\\ &\le \max\left\{2lm_1\|y_1-y_2\|, 2lm_2\|y_1-y_2\|\right\}=\delta\|y_1-y_2\|.\end{aligned}$$

Thus, we have

$$\|\mathcal{N}y_1-\mathcal{N}y_2\| \le \delta\|y_1-y_2\|, \tag{3.8}$$

where $\delta=2lm$ and $m=\max\{m_1,m_2\}$. If $\delta<1$, then $\mathcal{N}:\mathbb{X}\to\mathbb{X}$ is contraction mapping and hence by the Banach contraction mapping theorem, Eq. (3.2) has a unique solution in $\mathbb{X}$. □

In order to establish the convergence of proposed scheme (2.9), we write the scheme (2.9) in operator form as follows. For that, we define the sequence $\{\psi_n\}$ such that $\psi_n = y_0 + y_1 + \cdots + y_n$, is a sequence of partial sums of the series solution $\sum_{j=0}^{\infty} y_j$. Since

$$f\left(\sum_{j=0}^{\infty} y_j\right) = \sum_{j=0}^{\infty} A_j,$$

or as an approximation, we have

$$f(\psi_n) = \sum_{j=0}^{n} A_j. \tag{3.9}$$

Now, by using (2.8) and (3.9), we have

$$\sum_{j=0}^{n+1} y_j = \frac{c}{a} + \int_0^1 G(x,\xi) q(\xi) \sum_{n=0}^{n} A_j d\xi,$$

$$\psi_{n+1} = \frac{c}{a} + \int_0^1 G(x,\xi) q(\xi) f(\psi_n) d\xi.$$

Thus the operator form of the scheme can be written as

$$\psi_{n+1} = \mathcal{N}\psi_n. \tag{3.10}$$

Note that the formulation (3.10) is used to prove the Theorems 3.2 and 3.3. Next, we give the convergence of the sequence $\{\psi_n\}$ to the exact solution y of (3.2).

Theorem 3.2 *Let $\mathcal{N}y$ be the nonlinear operator defined by (3.3) which satisfies the Lipschitz condition with Lipschitz constant $\delta < 1$. If $\|y_0\| < \infty$, then there holds $\|y_{k+1}\| \le \delta \|y_k\|$, $k = 0, 1, 2, \ldots$ and the sequence $\{\psi_n\}$ defined by (3.10) converges to the exact solution y.*

Proof Since

$$\psi_1 = y_0 + y_1, \quad \psi_2 = y_0 + y_1 + y_2, \quad \ldots, \quad \psi_n = y_0 + y_1 + y_2 + \cdots + y_n, \quad \ldots,$$

we see that $y_{k+1} = \psi_{k+1} - \psi_k$, $k = 1, 2, \ldots$.

Now we show that the sequence $\{\psi_n\}$ is convergent sequence.

Using the estimate (3.8) and (3.10), we have

$$\|y_{n+1}\| = \|\psi_{n+1} - \psi_n\| = \|\mathcal{N}\psi_n - \mathcal{N}\psi_{n-1}\| \le \delta \|\psi_n - \psi_{n-1}\| = \delta \|y_n\|.$$

Hence, we obtain

$$\|\psi_{n+1} - \psi_n\| = \|y_{n+1}\| \le \delta \|y_n\| \le \delta^2 \|y_{n-1}\| \cdots \le \delta^{n+1} \|y_0\|.$$

Now for all $n, m \in \mathbb{N}$, with $n \ge m$, we have

$$\|\psi_n - \psi_m\| = \big\|(\psi_n - \psi_{n-1}) + (\psi_{n-1} - \psi_{n-2}) + \cdots + (\psi_{m+1} - \psi_m)\big\|$$

$$\leq \|\psi_n - \psi_{n-1}\| + \|\psi_{n-1} - \psi_{n-2}\| + \cdots + \|\psi_{m+1} - \psi_m\|$$
$$\leq \delta^n \|y_0\| + \delta^{n-1}\|y_0\| + \cdots + \delta^{m+1}\|y_0\|$$
$$\leq \delta^{m+1}\big(1 + \delta + \delta^2 + \cdots + \delta^{n-m-1}\big)\|y_0\|$$
$$\leq \frac{\delta^{m+1}(1-\delta^{n-m})}{1-\delta}\|y_0\|.$$

Since $0 \leq \delta < 1$, implies $(1-\delta^{n-m}) \leq 1$ and since $\|y_0\| < \infty$, it follows that

$$\|\psi_n - \psi_m\| \leq \frac{\delta^{m+1}}{1-\delta}\|y_0\|, \tag{3.11}$$

which converges to zero, that is, $\|\psi_n - \psi_m\| \to 0$, as $m \to \infty$. Hence $\{\psi_n\}$ is Cauchy sequence in $\mathbb{X}$. Since $\mathbb{X}$ is Banach space, the sequence $\{\psi_n\}$ must be convergent in $\mathbb{X}$. Hence there exits ψ in $\mathbb{X}$ such that $\lim_{n\to\infty}\psi_n = \psi$. But, we have $y = \sum_{n=0}^{\infty} y_n = \lim_{n\to\infty}\psi_n$, that is, $y = \psi$ which is exact solution of Eq. (3.2). □

Theorem 3.3 *Let y be the exact solution of* (3.2). *Let* ψ_m *be the sequence of approximate series solution obtained by* (3.10). *Then there holds*

$$\max_{0\leq x\leq 1}\left| y - \sum_{j=0}^{m} y_j \right| \leq \frac{\delta^{m+1}}{1-\delta}\|y_0\|.$$

Proof Using the inequality (3.11) for $n \geq m,\ n,\ m \in \mathbb{N}$, we have

$$\|\psi_n - \psi_m\| \leq \frac{\delta^{m+1}}{1-\delta}\|y_0\|.$$

Since $\lim_{n\to\infty}\psi_n = y$, fixing m and letting $n \to \infty$ in above estimate, we obtain

$$\|y - \psi_m\| \leq \frac{\delta^{m+1}}{1-\delta}\|y_0\|,$$

which completes the proof using $\psi_m = \sum_{j=0}^{m} y_j$. □

4 Numerical illustrations

In this section, the proposed scheme (2.9) is applied to solve singular two point boundary value problems. In order to check the accuracy of the proposed method, we have consider three nonlinear singular examples. All the numerical results obtained by proposed method are compared with known results.

Example 4.1 Consider the following nonlinear SBVP

$$\left.\begin{aligned} &\big(x^3 y'(x)\big)' = x^3\left(\frac{1}{8y^2(x)} - \frac{a_0}{y(x)} - b_0 x^{2\gamma-4}\right), \quad 0 < x \leq 1, \\ &\lim_{x\to 0+} x^3 y'(x) = 0, \quad y(1) = 1, \end{aligned}\right\} \tag{4.1}$$

where $a_0 \geq 0$, $b_0 > 0$ and $\gamma > 1$ are any real constants, arises in the theory of shallow membrane caps [3, 5, 15].

According to proposed scheme (2.9), we have $p(x) = q(x) = x^3$, $a = 1$, $b = 0$ and $c = 1$. Consequently:

$$\left.\begin{aligned} y_0 &= 1, \\ y_{n+1} &= \int_0^1 G(x,\xi)\xi^3 A_n(y_0, y_1, \dots, y_n)d\xi, \quad n \geq 0, \end{aligned}\right\} \tag{4.2}$$

where the Green's function is

$$G(x,\xi) = \begin{cases} \frac{1-\xi^{-2}}{-2} + \frac{b}{a}, & 0 < x \leq \xi \leq 1, \\ \frac{1-x^{-2}}{-2} + \frac{b}{a}, & 0 < \xi \leq x \leq 1. \end{cases} \tag{4.3}$$

The Adomian's polynomials for $f(y) = \frac{1}{8y^2(x)} - \frac{a_0}{y(x)} - b_0 x^{2\gamma-4}$ about y_0, are given as:

$$\left.\begin{aligned} A_0 &= \left(\frac{1}{8y_0^2} - \frac{a_0}{y_0} - b_0 x^{2\gamma-4}\right), \\ A_1 &= \left(-\frac{0.25}{y_0^3} + \frac{a_0}{y_0^2}\right)y_1, \\ A_2 &= \left(-\frac{0.25}{y_0^3} + \frac{a_0}{y_0^2}\right)y_2 + \left(\frac{0.75}{y_0^4} - \frac{2a_0}{y_0^3}\right)\frac{y_1^2}{2!}, \\ &\vdots \end{aligned}\right\} \tag{4.4}$$

Using (4.2), (4.3) and (4.4), for $a_0 = 0$, $b_0 = 0.5$ and $\gamma = 2$, we obtain components as

$$\begin{aligned} y_0 &= 1, \\ y_1 &= -0.046875 + 0.046875x^2, \\ y_2 &= 0.000976563 - 0.00146484x^2 + 0.000488281x^4, \\ y_3 &= 0.000033696 - 0.000072479x^2 + 0.000053405x^4 - 0.000014623x^6, \\ &\vdots \end{aligned}$$

In similar manner, for $a_0 = 0.5$, $b_0 = 1$ and $\gamma = 1.5$, using (4.2), (4.7) and (4.4), we have

$$\begin{aligned} y_0 &= 1, \\ y_1 &= -0.380208 + 0.333333x + 0.046875x^2, \\ y_2 &= -0.005837 + 0.011881x^2 - 0.005555x^3 - 0.000488x^4, \end{aligned}$$

Table 1 Numerical solution and residual error for Example 4.1, when $a_0 = 0$, $b_0 = 0.5$, $\gamma = 2$

x	ψ_6	In [15]	R_2	R_4	R_6
0.0	0.95413530	0.95214843	5.9714E-04	1.1665E-06	1.7063E-09
0.2	0.95594964	0.95408104	5.4431E-04	1.0645E-06	1.0287E-09
0.4	0.96140303	0.95986967	4.0091E-04	7.8006E-07	1.7401E-10
0.6	0.97052624	0.96948658	2.1125E-04	3.9735E-07	5.8980E-10
0.8	0.98336934	0.98288524	4.6368E-05	8.4986E-08	2.2468E-10
1.0	1.00000000	1.00000000	0.0000E-00	0.0000E-00	0.0000E-00

Table 2 Numerical solution and residual error for Example 4.1, when $a_0 = 0.5$, $b_0 = 1$, $\gamma = 1.5$

x	ψ_6	R_2	R_4	R_6
0.0	0.61323147	1.2722E-02	7.3558E-03	6.9297E-04
0.2	0.68227253	1.1209E-02	2.6714E-03	4.4977E-04
0.4	0.75583421	7.7281E-03	7.3862E-04	1.4598E-04
0.6	0.83355517	4.0198E-03	1.2127E-04	1.5719E-05
0.8	0.91506470	1.2110E-03	3.6880E-06	8.0259E-07
1.0	1.00000000	0.0000E-00	0.0000E-00	0.0000E-00

$$y_3 = -0.000757 + 0.002441x^2 - 0.002112x^3 + 0.000269x^4 + 0.000151x^5 + 8.265177 \times 10^{-5}x^6,$$

$$\vdots$$

The comparison of approximate solution obtained by proposed recursive scheme and VIM used in [15] is presented in Table 1. From these results, we see that proposed recursive scheme provides good approximations which is comparable with those in [15].

Furthermore, since the exact solution of (4.1) is not known, we instead investigate the absolute residual error function, which is a measure of how well the approximation satisfies the original nonlinear differential equation as

$$R_n(x) = \left| \left(x^3\psi_n'(x)\right)' - x^3\left(\frac{1}{8\psi_n^2(x)} - \frac{a_0}{\psi_n(x)} - b_0 x^{2\gamma-4}\right) \right|, \quad 0 < x \le 1$$

where $\psi_n(x)$ is a sequence of approximate solution. Finally, the residual error R_n, $n = 2, 4, 6$ is presented in Tables 1 and 2 for various values of the parameters a_0, b_0 and γ. Also, in Figs. 1 and 2, we plot absolute residual error function R_n, and observe that as the number of iterations increases the residual error decreases.

Example 4.2 Consider nonlinear derivative-dependent singular two-point boundary value problem

$$\left.\begin{aligned} &\left(x^\alpha y'\right)' = x^{\alpha+\beta-2}\beta e^y\left(-xy' - \alpha - \beta + 1\right), \quad 0 < x \le 1, \\ &\lim_{x\to 0+} x^\alpha y'(x) = 0, \quad y(1) = -\ln(5), \end{aligned}\right\} \tag{4.5}$$

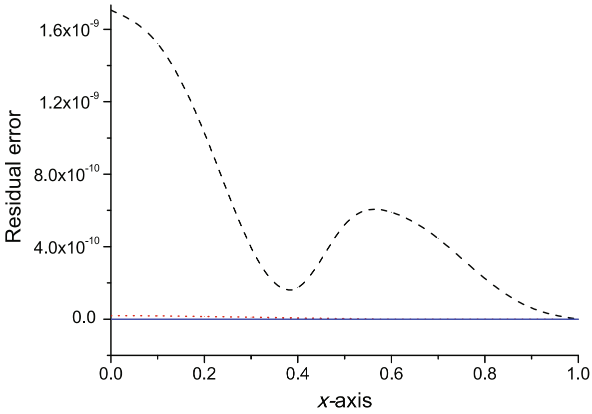

Fig. 1 Residual error functions $R_8(x)$ (*dashed line*), $R_{10}(x)$ (*dotted line*) and $R_{12}(x)$ (*solid line*) of Example 4.1, when $a_0 = 0, b_0 = 0.5, \gamma = 2$

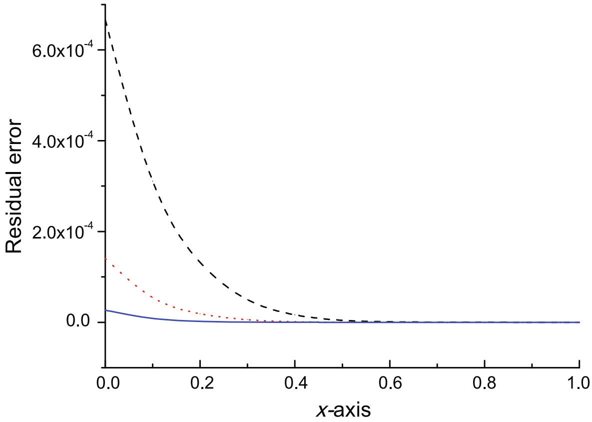

Fig. 2 Residual error functions $R_8(x)$ (*dashed line*), $R_{10}(x)$ (*dotted line*) and $R_{12}(x)$ (*solid line*) of Example 4.1, when $a_0 = 0.5, b_0 = 1, \gamma = 1.5$

with exact solution $y(x) = -\ln(4 + x^{\beta})$, where $\alpha \geq 0$ and $\beta > 0$ are any constants.

Comparing with the proposed recursive scheme (2.9), we have $p(x) = x^{\alpha}$, $q(x) = x^{\alpha+\beta-2}$ and $a = 1$, $b = 0$, and $c = -\ln(5)$. Consequently, we have following scheme

$$\left.\begin{aligned} y_0 &= -\ln(5), \\ y_{n+1} &= \int_0^1 G(x,\xi)\xi^{\alpha+\beta-2} A_n(y_0, y_1, \ldots, y_n)d\xi, \quad n \geq 1, \end{aligned}\right\} \tag{4.6}$$

where the Green's function is

$$G(x,\xi) = \begin{cases} \frac{1-\xi^{1-\alpha}}{1-\alpha} + \frac{b}{a}, & 0 < x \leq \xi \leq 1, \\ \frac{1-x^{1-\alpha}}{1-\alpha} + \frac{b}{a}, & 0 < \xi \leq x \leq 1. \end{cases} \tag{4.7}$$

We now calculate the Adomian's polynomials for nonlinear term $f = -\beta(xe^y y' + e^y(\alpha+\beta-1))$ about y_0 using the formula (1.7) as:

$$\left.\begin{aligned}
A_0 &= -\beta\big(xy_0' + (\alpha+\beta-1)\big)e^{y_0},\\
A_1 &= -\beta\big(xy_1' + y_1(\alpha+\beta-1)\big)e^{y_0},\\
A_2 &= -\beta\bigg(x\big(y_2' + y_1y_1'\big) + \bigg(y_2 + \frac{y_1^2}{2}\bigg)(\alpha-\beta+1)\bigg)e^{y_0},\\
&\vdots
\end{aligned}\right\} \tag{4.8}$$

For the demonstration purpose, we pick some specific values of α and β.

For $\alpha = 2$, $\beta = 1$, using (4.6), (4.7) and (4.8), we obtain the components y_n as

$$\begin{aligned}
y_0 &= -1.60944,\\
y_1 &= 0.20000 - 0.20000x,\\
y_2 &= 0.02 - 0.04x + 0.02x^2,\\
y_3 &= 0.00266667 - 0.008x + 0.008x^2 - 0.00266667x^3,\\
&\vdots
\end{aligned}$$

For $\alpha = 2$, $\beta = 3$, making use of (4.6), (4.7) and (4.8), we have:

$$\begin{aligned}
y_0 &= -1.60944,\\
y_1 &= 0.2000 - 0.2000x^3,\\
y_2 &= 0.02000 - 0.0400x^3 + 0.0200x^6,\\
y_3 &= 0.002666 - 0.0080x^3 + 0.00800x^6 - 0.00266667x^9,\\
&\vdots
\end{aligned}$$

We now define error function as $E_n(x) = |\psi_n(x) - y(x)|$ and the maximum absolute errors by

$$E^n = \max_{0<x\le 1} E_n(x) \tag{4.9}$$

where $y(x)$ is exact solution, and $\psi_n(x)$ is n-term approximate series solution.

In order to verify the efficiency of proposed recursive scheme, the maximum absolute error $E^{(n)}$, for $n = 3, 4, \ldots, 10$ are listed in Tables 3, 4, 5 and 6 for various values of α and β.

Table 3 Maximum absolute error of Example 4.2, when $\beta=1$

α	$E^{(3)}$	$E^{(4)}$	$E^{(5)}$	$E^{(6)}$
2	2.2509E-03	3.0693E-04	4.4498E-05	6.7073E-06
3	1.5533E-03	1.5533E-04	4.4497E-05	3.2422E-06
5	2.3143E-02	0.0031E-03	4.7688E-04	7.6884E-05

Table 4 Maximum absolute error of Example 4.2, when $\beta=1$

α	$E^{(7)}$	$E^{(8)}$	$E^{(9)}$	$E^{(10)}$
2	1.0386E-06	1.6405E-07	2.6310E-08	4.2703E-09
3	4.4608E-07	6.2609E-08	8.9224E-09	1.2869E-09
5	3.2422E-06	4.4608E-07	6.2609E-08	8.9224E-09

Table 5 Maximum absolute error of Example 4.2, when $\beta=3$

α	$E^{(3)}$	$E^{(4)}$	$E^{(5)}$	$E^{(6)}$
2	9.5756E-04	1.8818E-04	1.5094E-05	3.0380E-06
3	9.2274E-04	2.1914E-04	1.5820E-05	3.5052E-06
5	7.4929E-04	2.3771E-04	2.8953E-05	3.7577E-06

Table 6 Maximum absolute error of Example 4.2, when $\beta=3$

α	$E^{(7)}$	$E^{(8)}$	$E^{(9)}$	$E^{(10)}$
2	3.1337E-07	4.7824E-08	6.4671E-09	1.1272E-09
3	3.2712E-07	6.5307E-08	6.9580E-09	1.3277E-09
5	6.2457E-07	6.9262E-08	1.2991E-08	1.3861E-09

Example 4.3 Consider the following singular derivative-dependent two point boundary value problem

$$\left.\begin{aligned} &\left(x^\alpha y'\right)' = x^{\alpha+\beta-2}\beta\left(xy' + y(\alpha+\beta-1)\right), \quad 0 < x \le 1, \\ &\lim_{x\to 0+} x^\alpha y'(x) = 0, \quad y(1) = e, \end{aligned}\right\} \tag{4.10}$$

with exact solution $y(x) = e^{x^\beta}$.

According to proposed scheme (2.9), we have $p(x) = x^\alpha$, $q(x) = x^{\alpha+\beta-2}$, $a = 1$, $b = 0$, and $c = e$. Consequently

$$\left.\begin{aligned} &y_0 = e, \\ &y_{n+1} = \int_0^1 G(x,\xi)\xi^{\alpha+\beta-2}A_n(y_0, y_1, \ldots, y_n)d\xi, \quad n \ge 1, \end{aligned}\right\} \tag{4.11}$$

where,

$$A_n = \beta\left(xy_n' + y_n(\alpha+\beta-1)\right), \tag{4.12}$$

Table 7 Maximum absolute error of Example 4.3, when $\beta = 1$

α	$E^{(2)}$	$E^{(4)}$	$E^{(6)}$	$E^{(8)}$
2	1.6213E-01	6.5357E-03	1.0272E-04	9.3045E-07
3	1.9210E-01	7.4353E-03	1.0471E-04	9.5041E-07
5	1.3160E-01	3.4038E-03	4.0813E-05	2.8239E-07

Table 8 Maximum absolute error of Example 4.3, when $\beta = 1$

α	$E^{(10)}$	$E^{(12)}$	$E^{(14)}$	$E^{(16)}$
2	5.4823E-09	2.2696E-11	6.9500E-14	2.2204E-16
3	5.5821E-09	2.3697E-11	6.9499E-14	2.2204E-16
5	1.2720E-09	4.0272E-12	8.8817E-15	2.2204E-16

and the Green's function is

$$G(x,\xi) = \begin{cases} \frac{1-\xi^{1-\alpha}}{1-\alpha} + \frac{b}{a}, & 0 < x \le \xi \le 1, \\ \frac{1-x^{1-\alpha}}{1-\alpha} + \frac{b}{a}, & 0 < \xi \le x \le 1. \end{cases} \tag{4.13}$$

For $\alpha = 2$, $\beta = 1$, we use (4.11), (4.13) and (4.12), to obtain the components y_n:

$$\begin{aligned} y_0 &= 2.71828, \\ y_1 &= -2.71828 + 2.71828x, \\ y_2 &= 1.35914 - 2.71828x + 1.35914x^2, \\ y_3 &= -0.453047 + 1.35914x - 1.35914x^2 + 0.453047x^3, \\ &\vdots \end{aligned}$$

The maximum absolute error $E^{(n)}$, for $n = 2, 4, 6, 8, 10, 12, 14, 16$ is listed in Tables 7, 8, 9 and 10 for various values of α and β. From these numerical results, we see that our approximate series solution converges to exact solution as the number of iterations increase.

For $\alpha = 2$, $\beta = 4$, using (4.11), the components y_n are obtained as:

$$\begin{aligned} y_0 &= 2.71828, \\ y_1 &= -2.71828 + 2.71828x^4, \\ y_2 &= 1.35914 - 2.71828x^4 + 1.35914x^8, \\ y_3 &= -0.453047 + 1.35914x^4 - 1.35914x^8 + 0.453047x^{12}, \\ &\vdots \end{aligned}$$

Table 9 Maximum absolute error of Example 4.3, when $\beta = 4$

α	$E^{(2)}$	$E^{(4)}$	$E^{(6)}$	$E^{(8)}$
2	3.5904E-01	1.9346E-02	4.7839E-04	6.7985E-06
3	3.5754E-01	1.9205E-02	4.7347E-04	6.7082E-06
5	3.5109E-01	1.8606E-02	4.5265E-04	6.3290E-06

Table 10 Maximum absolute error of Example 4.3, when $\beta = 4$

α	$E^{(10)}$	$E^{(12)}$	$E^{(14)}$	$E^{(16)}$
2	6.2762E-08	4.0677E-10	1.9531E-12	7.3274E-15
3	6.1741E-08	3.9895E-10	1.9098E-12	7.1054E-15
5	5.7490E-08	3.6663E-10	1.7321E-12	6.2172E-15

5 Conclusion

In this work, we have shown the application of the proposed recursive scheme for solving nonlinear derivative-dependent singular boundary value problems. The accuracy of the computed numerical results measured using the maximum absolute error or absolute residual error shows that the proposed method is suitable for such singular boundary value problems. It provides a direct recursive scheme to obtain a sequence of approximate solutions whose limiting value is the exact solution of the problem. Unlike the existing methods such as ADM or MDAM, the proposed method does not require the computation of undermined coefficients. In addition, the proposed scheme does not require any linearization or discretization of variables. The proposed method is capable for solving a variety of nonlinear boundary value problems whereas the other methods like VIM suffers. The proposed method requires less computational work as compared to other existing methods for solving such equations. The approximate solution of the examples are presented and only a few terms are required to obtain accurate approximations for solutions. The convergence and error estimation of the proposed recursive scheme are also discussed. Finally, it is pointed out that the procedure described in this paper can be extended for higher order boundary value problems.

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Appendix

We construct the Green's function for singular boundary value problem whenever the assumptions (E_1)–(E_3) hold. To construct Green's function we consider following the problem as:

$$\big(p(x)y(x)'\big)' = q(x)F(x), \tag{6.1}$$

$$\lim_{x\to 0+} p(x)y'(x) = 0, \qquad ay(1) + by'(1) = 0, \tag{6.2}$$

where F is continuous. Integrating above Eq. (6.1) from 0 to x and using $\lim_{x\to 0+} p(x)y'(x) = 0$, we have

$$y'(x) = \frac{1}{p(x)} \int_0^x q(\xi)F(\xi)d\xi. \tag{6.3}$$

Again integrating from x to 1, then changing the order of integration, and applying the boundary conditions, we obtain

$$\begin{aligned}
y(x) &= \frac{b}{ap(1)} \int_0^1 q(\xi)F(\xi)d\xi + \int_0^1 \left(\int_\xi^1 \frac{dx}{p(x)} \right) q(\xi)F(\xi)d\xi \\
&\quad - \int_0^x \left(\int_\xi^x \frac{dx}{p(x)} \right) q(s)F(s)d\xi, \quad \xi > 0, \\
y(x) &= \frac{b}{ap(1)} \int_0^x q(\xi)F(\xi)d\xi + \frac{b}{ap(1)} \int_x^1 q(\xi)F(\xi)d\xi \\
&\quad + \int_0^x \left(\int_\xi^1 \frac{dx}{p(x)} \right) q(\xi)F(\xi)d\xi \\
&\quad + \int_x^1 \left(\int_\xi^1 \frac{dx}{p(x)} \right) q(\xi)F(\xi)d\xi - \int_0^x \left(\int_\xi^x \frac{dx}{p(x)} \right) q(\xi)F(\xi)d\xi, \quad \xi > 0, \\
y(x) &= \int_0^x \left(\int_\xi^1 \frac{dx}{p(x)} - \int_\xi^x \frac{dx}{p(x)} + \frac{b}{ap(1)} \right) q(\xi)F(\xi)d\xi \\
&\quad + \int_x^1 \left(\int_\xi^1 \frac{dx}{p(x)} + \frac{b}{ap(1)} \right) q(\xi)F(\xi)d\xi, \quad \xi > 0.
\end{aligned}$$

Finally, we obtain

$$y(x) = \int_0^1 q(\xi)G(x,\xi)F(\xi)d\xi,$$

where the Green's function $G(x,\xi)$ is given by

$$G(x,\xi) = \begin{cases} \int_\xi^1 \frac{dx}{p(x)} + \frac{b}{ap(1)}, & 0 < x \le \xi \le 1, \\ \int_\xi^1 \frac{dx}{p(x)} - \int_\xi^x \frac{dx}{p(x)} + \frac{b}{ap(1)}, & 0 < \xi \le x \le 1. \end{cases}$$

$$G_x(x,\xi) = \begin{cases} 0, & 0 < x \le \xi \le 1, \\ -\frac{1}{p(x)}, & 0 < \xi \le x \le 1. \end{cases} \tag{6.4}$$

It is easy to see that the function $G(x,\xi)$ satisfies all the properties of Green's function. Hence $G(x,\xi)$ is the Green's function for above boundary value problem.

References

1. Singh, R., Kumar, J., Nelakanti, G.: Numerical solution of singular boundary value problems using Green's function and improved decomposition method. J. Appl. Math. Comput. (2013). doi:10.1007/s12190-013-0670-4
2. Bobisud, L.: Existence of solutions for nonlinear singular boundary value problems. Appl. Anal. **35**(1–4), 43–57 (1990)
3. Rachunkov', I., Pulverer, G., Weinmüller, E.: A unified approach to singular problems arising in the membrane theory. Appl. Math. **55**(1), 47–75 (2010)
4. Amit, K.V., et al.: On a constructive approach for derivative-dependent singular boundary value problems. Int. J. Differ. Equ. (2011)
5. Baxley, J.V., Robinson, S.B.: Nonlinear boundary value problems for shallow membrane caps, II. J. Comput. Appl. Math. **88**(1), 203–224 (1998)
6. Pandey, R., Verma, A.K.: On solvability of derivative dependent doubly singular boundary value problems. J. Appl. Math. Comput. **33**(1–2), 489–511 (2010)
7. Chawla, M., Katti, C.: Finite difference methods and their convergence for a class of singular two point boundary value problems. Numer. Math. **39**(3), 341–350 (1982)
8. Kumar, M., Aziz, T.: A uniform mesh finite difference method for a class of singular two-point boundary value problems. Appl. Math. Comput. **180**(1), 173–177 (2006)
9. Ravi Kanth, A., Reddy, Y.: Cubic spline for a class of singular two-point boundary value problems. Appl. Math. Comput. **170**(2), 733–740 (2005)
10. Kumar, M., Singh, N.: Modified Adomian decomposition method and computer implementation for solving singular boundary value problems arising in various physical problems. Comput. Chem. Eng. **34**(11), 1750–1760 (2010)
11. Ebaid, A.: A new analytical and numerical treatment for singular two-point boundary value problems via the Adomian decomposition method. J. Comput. Appl. Math. **235**(8), 1914–1924 (2011)
12. Khuri, S., Sayfy, A.: A novel approach for the solution of a class of singular boundary value problems arising in physiology. Math. Comput. Model. **52**(3), 626–636 (2010)
13. Danish, M., Kumar, S., Kumar, S.: A note on the solution of singular boundary value problems arising in engineering and applied sciences: use of OHAM. Comput. Chem. Eng. **36**, 57–67 (2012)
14. Singh, R., Kumar, J., Nelakanti, G.: New approach for solving a class of doubly singular two-point boundary value problems using Adomian decomposition method. Adv. Numer. Anal. **2012**, 22 (2012). doi:10.1155/2012/541083
15. Ravi Kanth, A., Aruna, K.: He's variational iteration method for treating nonlinear singular boundary value problems. Comput. Math. Appl. **60**(3), 821–829 (2010)
16. Inc, M., Evans, D.: The decomposition method for solving of a class of singular two-point boundary value problems. Int. J. Comput. Math. **80**(7), 869–882 (2003)
17. Wazwaz, A., Rach, R.: Comparison of the Adomian decomposition method and the variational iteration method for solving the Lane-Emden equations of the first and second kinds. Kybernetes **40**(9/10), 1305–1318 (2011)
18. Kumar, M., Gupta, Y.: Methods for solving singular boundary value problems using splines: a review. J. Appl. Math. Comput. **32**(1), 265–278 (2010)
19. Wazwaz, A.: The variational iteration method for solving nonlinear singular boundary value problems arising in various physical models. Commun. Nonlinear Sci. Numer. Simul. **16**(10), 3881–3886 (2011)
20. Singh, R., Kumar, J.: Solving a class of singular two-point boundary value problems using new modified decomposition method. ISRN Comput. Math. **2013**, 1–11 (2013). doi:10.1155/2013/262863
21. Benabidallah, M., Cherruault, Y.: Application of the Adomian method for solving a class of boundary problems. Kybernetes **33**(1), 118–132 (2004)
22. Adomian, G.: Solving Frontier Problems of Physics: The Decomposition Methoc [ie Method]. Kluwer Academic, Dordrecht (1994)
23. Adomian, G., Rach, R.: Inversion of nonlinear stochastic operators. J. Math. Anal. Appl. **91**(1), 39–46 (1983)
24. Wazwaz, A.: Approximate solutions to boundary value problems of higher order by the modified decomposition method. Comput. Math. Appl. **40**(6), 679–691 (2000)
25. Wazwaz, A.: A reliable algorithm for obtaining positive solutions for nonlinear boundary value problems. Comput. Math. Appl. **41**(10–11), 1237–1244 (2001)
26. Haldar, K.: Application of Adomian's approximation to blood flow through arteries in the presence of a magnetic field. J. Appl. Math. Comput. **12**(1–2), 267–279 (2003)

27. Al-Khaled, K., Allan, F.: Decomposition method for solving nonlinear integro-differential equations. J. Appl. Math. Comput. **19**(1–2), 415–425 (2005)
28. Momani, S., Moadi, K.: A reliable algorithm for solving fourth-order boundary value problems. J. Appl. Math. Comput. **22**(3), 185–197 (2006)
29. El-Kalla, I.: Error estimates for series solutions to a class of nonlinear integral equations of mixed type. J. Appl. Math. Comput. **38**(1–2), 341–351 (2012)
30. Mamaloukas, C., Haldar, K., Mazumdar, H.: Application of double decomposition to pulsatile flow. J. Appl. Math. Comput. **10**(1–2), 193–207 (2002)
31. El-Sayed, A., Saleh, M., Ziada, E.: Analytical and numerical solution of multi-term nonlinear differential equations of arbitrary orders. J. Appl. Math. Comput. **33**(1–2), 375–388 (2010)
32. Wazwaz, A.M., Rach, R., Duan, J.-S.: Adomian decomposition method for solving the Volterra integral form of the Lane–Emden equations with initial values and boundary conditions. Appl. Math. Comput. **219**(10), 5004–5019 (2013)
33. Al-Hayani, W.: Adomian decomposition method with green's function for sixth-order boundary value problems. Comput. Math. Appl. **61**(6), 1567–1575 (2011)
34. Duan, J.: Recurrence triangle for Adomian polynomials. Appl. Math. Comput. **216**(4), 1235–1241 (2010)
35. Duan, J.: An efficient algorithm for the multivariable Adomian polynomials. Appl. Math. Comput. **217**(6), 2456–2467 (2010)
36. Duan, J.: New recurrence algorithms for the nonclassic Adomian polynomials. Comput. Math. Appl. **62**(8), 2961–2977 (2011)
37. Rach, R.C.: A new definition of the Adomian polynomials. Kybernetes **37**(7), 910–955 (2008)
38. Duan, J., Rach, R., Wazwaz, A.M., Chaolu, T., Wang, Z.: A new modified Adomian decomposition method and its multistage form for solving nonlinear boundary value problems with robin boundary conditions. Appl. Math. Model. (2013). doi:10.1016/j.apm.2013.02.002
39. Abbaoui, K., Cherruault, Y.: Convergence of Adomian's method applied to differential equations. Comput. Math. Appl. **28**(5), 103–109 (1994)
40. Cherruault, Y.: Convergence of Adomian's method. Kybernetes **18**(2), 31–38 (1989)
41. Cherruault, Y., Adomian, G.: Decomposition methods: a new proof of convergence. Math. Comput. Model. **18**(12), 103–106 (1993)
42. Hosseini, M., Nasabzadeh, H.: On the convergence of Adomian decomposition method. Appl. Math. Comput. **182**(1), 536–543 (2006)