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ORIGINAL RESEARCH

Ergodic and weighted pseudo-almost periodic solutions for partial functional differential equations in fading memory spaces

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Abstract We use a new concept of weighted ergodic function based on the measure theory to investigate the existence and uniqueness of weighted pseudo almost periodic solution for a class of partial functional differential equations with infinite delay in fading memory spaces. We illustrate our theoretical results by studying some Lotka-Voltera reaction-diffusion systems with infinite delay.

Keywords Partial functional differential equations · Infinite delay · Fading memory spaces · Exponential dichotomy · Ergodic functions · Weighted pseudo almost periodic functions

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1 Introduction

In the last decade, many authors have produced extensive literature on the theory of almost periodicity and its application to differential equations and partial functional

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differential equations. More details can be found in the books of Corduneanu [10], Fink [20] and the papers [5–7, 11, 13, 14, 16, 17, 24, 26–28]. The notion of weighted pseudo almost periodic function was introduced in 2006 by Diagana [12] (see also [15, 29]). He introduced basic properties of weighted pseudo almost periodic function and investigated the existence and uniqueness of weighted pseudo almost periodic mild solutions for some abstract differential equations.

Recently, Blot, Cieutat and Ezzinbi [15] used some results of the measure theory to establish a new concept of ergodic and weighted pseudo almost periodic functions. They developed some results like completeness and composition theorems to investigate fundamental notions on weighted pseudo almost periodic functions.

First, let us explain the meaning of the notion of weighted pseudo almost periodic function introduced by Diagana [12]. Let ρ be a positive and locally integrable function on $\mathbb{R}$. A continuous function $f : \mathbb{R} \to Y$ (here Y is Banach space) is said ρ-pseudo almost periodic if

$$f = g + \phi,$$

where g is an almost periodic function and ϕ is an ergodic function with respect to ρ, in the sense that

$$\lim_{r\to\infty} \frac{1}{m(r,\rho)} \int_{-r}^{r} \big|\phi(t)\big|\rho(t)\,dt = 0, \quad \text{with } m(r,\rho) = \int_{-r}^{r} \rho(t)\,dt.$$

Second, the new notion of weighted pseudo almost periodic function introduced by Blot, Cieutat and Ezzinbi [15], generalize the concept of Diagana [12]. Let us consider a positive measure μ on $\mathbb{R}$. We say that a function f is μ-pseudo almost periodic if

$$f = g + \phi,$$

where g is almost periodic and ϕ is μ-ergodic in the sense that

$$\lim_{r\to\infty} \frac{1}{\mu([-r,r])} \int_{[-r,r]} \big|\phi(t)\big|\,d\mu(t) = 0,$$

where $\mu([-r,r])$ is the measure of the interval $[-r,r]$ (more details about this notion can be found in [9]). One can observe that a ρ-pseudo almost periodic function is μ-pseudo almost periodic, where the measure μ is absolutely continuous with respect to the Lebesgue measure and its Radon-Nikodym derivative is ρ:

$$\frac{d\mu(t)}{dt} = \rho(t).$$

In this work, we investigate the existence and uniqueness of μ-pseudo almost periodic solutions for the following partial functional differential equation with infinite delay

$$\frac{d}{dt}x(t) = Ax(t) + L(x_t) + f(t) \quad \text{for } t \in \mathbb{R}, \tag{1.1}$$

where $A : D(A) \to X$ is a linear operator (not necessarily densely defined) on a Banach space $(X, |.|)$. For every $t \in \mathbb{R}$, the history function $x_t \in \mathcal{B}$ is defined by

$$x_t(\theta) = x(t+\theta) \quad \text{for } \theta \in (-\infty, 0],$$

where $\mathcal{B}$ is a normed linear space of functions mapping $(-\infty, 0]$ into X and satisfying some fundamental axioms given in [22]. L is a bounded linear operator from $\mathcal{B}$ to X, and f is a given continuous X-valued function on $\mathbb{R}$.

We assume that the unbounded linear operator A satisfies the following Hille-Yosida condition.

($\mathbf{H}_0$) There exist $M_0 \geq 1,\ \omega_0 \in \mathbb{R}$ such that $(\omega_0, +\infty) \subset \rho(A)$ and

$$\left|(\lambda I - A)^{-n}\right| \leq M_0(\lambda - \omega_0)^{-n} \quad \text{for } n \in \mathbb{N} \text{ and } \lambda > \omega_0,$$

where $\rho(A)$ is the resolvent set of A.

Without loss of generality, we can assume that $M_0 = 1$. Otherwise, we can renorm the space X with an equivalent norm such that $M_0 = 1$.

The organization of this paper is as follows. In Sect. 2, we recall some fundamental new results about the notion of μ-pseudo almost periodic function. In Sect. 3, we give tools that we will need in the sequel. In particular, we investigate a variation of constants formula associated to (1.1), and we establish a generalized spectral decomposition technique to solve it. In Sect. 4, we give our main result. We prove existence and uniqueness of μ-pseudo almost periodic mild solutions, in the hyperbolic case, for (1.1). Section 5 is devoted to the study of existence and uniqueness of μ-pseudo almost periodic mild solutions for nonlinear system associated to (1.1). In the last section, we propose an application to some reaction-diffusion equation with infinite delay.

2 Ergodic and μ-pseudo almost periodic functions under the light of measure theory

Throughout this paper, X is a Banach space and $BC(\mathbb{R}, X)$ denotes the Banach space of bounded continuous functions from $\mathbb{R}$ to X provided with the supremum norm

$$|f|_\infty = \sup_{t \in \mathbb{R}} \left|f(t)\right|.$$

We denote by $\mathcal{B}$ the Lebesgue σ-field of $\mathbb{R}$ and by $\mathcal{M}$ the set of all positive measures μ on $\mathcal{B}$ satisfying $\mu(\mathbb{R}) = +\infty$ and $\mu([a, b]) < +\infty$ for all $a,\ b \in \mathbb{R}$ $(a \leq b)$. Let $\mu \in \mathcal{M}$. We formulate the following hypotheses.

($\mathbf{H}_1$) For all $a, b, c \in \mathbb{R}$, such that $0 \leq a < b \leq c$, there exist $\tau_0 \geq 0$ and $\alpha_0 > 0$ such that

$$|\tau| \geq \tau_0 \quad \Longrightarrow \quad \mu\big((a+\tau, b+\tau)\big) \geq \alpha_0\, \mu\big([\tau, c+\tau]\big).$$

($\mathbf{H}_2$) For all $\tau \in \mathbb{R}$, there exist $\beta > 0$ and a bounded interval I such that

$$\mu(\{a+\tau : a \in A\}) \le \beta\mu(A) \quad \text{when } A \in \mathcal{B} \text{ satisfies } A \cap I = \emptyset.$$

Then, we have the following result.

Lemma 2.1 [9] *The property* ($\mathbf{H}_2$) *implies* ($\mathbf{H}_1$).

Definition 2.2 [10] A continuous function $f : \mathbb{R} \longrightarrow X$ is said to be (in Bohr sense) almost periodic, if for all $\varepsilon > 0$, there exists $\ell > 0$, such that for all $\alpha \in \mathbb{R}$, there exists $\tau \in [\alpha, \alpha + \ell]$ with

$$\sup_{t\in\mathbb{R}} \left| f(t+\tau) - f(t) \right| < \varepsilon.$$

In the sequel $AP(\mathbb{R}, X)$ denotes the space of almost periodic X-valued functions. It is well known that a continuous function $f : \mathbb{R} \longrightarrow X$ is almost periodic if and only if the set $\{f_\tau : \tau \in \mathbb{R}\}$ is relatively compact in $BC(\mathbb{R}, X)$, where the function f_τ is defined by $f_\tau(s) = f(\tau + s)$, for $s \in \mathbb{R}$.

Proposition 2.3 [10] $(AP(\mathbb{R}, X), |\cdot|_\infty)$ *is a Banach space.*

Definition 2.4 [9] Let $\mu \in \mathcal{M}$. A bounded continuous function $f : \mathbb{R} \longrightarrow X$ is said to be μ-ergodic if

$$\lim_{r\to+\infty} \frac{1}{\mu([-r, r])} \int_{[-r,r]} \left| f(t) \right| d\mu(t) = 0.$$

We denote by $\mathcal{E}(\mathbb{R}, X, \mu)$ the space of μ-ergodic bounded continuous functions.

Definition 2.5 [9] Let $\mu \in \mathcal{M}$. A continuous function $f : \mathbb{R} \longrightarrow X$ is said to be μ-pseudo almost periodic if f can be written as

$$f = g + \phi,$$

where $g \in AP(\mathbb{R}, X)$ and $\phi \in \mathcal{E}(\mathbb{R}, X, \mu)$.

In the sequel, $PAP(\mathbb{R}, X, \mu)$ denotes the space of μ-pseudo almost periodic functions from $\mathbb{R}$ to X, it is endowed with the uniform norm topology.

Remark 2.6 A pseudo almost periodic function is μ-pseudo almost periodic, with μ the Lebesgue measure.

Example 2.7 Let ρ be a nonnegative $\mathcal{B}$-measurable function. Denote by μ the positive measure defined by

$$\mu(A) = \int_A \rho(t)\, dt \quad \text{for } A \in \mathcal{B},$$

where dt denotes the Lebesgue measure on $\mathbb{R}$. The function ρ is called the *Radon-Nikodym derivative* of μ with respect to the Lebesgue measure on $\mathbb{R}$. In this case, $\mu \in \mathcal{M}$ if and only if its Radon-Nikodym derivative ρ is locally Lebesgue-integrable on $\mathbb{R}$ and satisfies

$$\int_{-\infty}^{+\infty} \rho(t)\,dt = +\infty.$$

Example 2.8 In [24], to study pseudo almost periodic solutions for a class of differential equations with piecewise constant argument, the authors considered the following spaces

$$E_0 = \left\{ f \in BC(\mathbb{R}, X) : \lim_{r\to+\infty} \frac{1}{2r}\int_{-r}^{r} \big|f(t)\big|\,dt = 0 \text{ and } \lim_{r\to+\infty} \frac{1}{2N+1}\sum_{n=-N}^{N} \big|f(n)\big| = 0 \right\},$$

$$E = \big\{ f = g + \phi : g \in AP(\mathbb{R}, X) \text{ and } \phi \in E_0 \big\}.$$

In [9], the authors proved that

$$E_0 = \mathcal{E}(\mathbb{R}, X, \mu) \quad \text{and} \quad E = PAP(\mathbb{R}, X, \mu),$$

for some measure $\mu \in \mathcal{M}$ defined by

$$\mu(A) = \mu_1(A) + \mu_2(A), \quad \text{for all } A \in \mathcal{B},$$

where μ_1 is the Lebesgue measure on $(\mathbb{R}, \mathcal{B})$ and μ_2 the measure on $(\mathbb{R}, \mathcal{B})$ defined by

$$\mu_2(A) = \begin{cases} card(A \cap \mathbb{Z}), & A \cap \mathbb{Z} \text{ is finite}, \\ \infty, & A \cap \mathbb{Z} \text{ is infinite}. \end{cases}$$

In fact, we have for $f \in BC(\mathbb{R}, X)$,

$$\frac{1}{\mu([-r,r])}\int_{[-r,r]} \big|f(t)\big|\,d\mu(t) = \frac{1}{2r + 2[r] + 1}\left(\int_{-r}^{r} \big|f(t)\big|\,dt + \sum_{k=-[r]}^{[r]} \big|f(k)\big|\right),$$

where [.] denotes the greatest integer function.

Then, one can prove the following equality (see [9])

$$\frac{1}{\mu([-r,r])}\int_{[-r,r]} \big|f(t)\big|\,d\mu(t) = \alpha(r)\left(\frac{1}{2r}\int_{-r}^{r} \big|f(t)\big|\,dt\right) + \beta(r)\left(\frac{1}{2[r]+1}\sum_{k=-[r]}^{[r]} \big|f(k)\big|\right),$$

where $\alpha(r)$ and $\beta(r) \in [\frac{1}{4}, 1]$, for all $r \geq 1$. In that example, one can see that $E_0 = \mathcal{E}(\mathbb{R}, X, \mu)$ and $E = PAP(\mathbb{R}, X, \mu)$.

Proposition 2.9 [9] *Let $\mu \in \mathcal{M}$. Then, $(\mathcal{E}(\mathbb{R}, X, \mu), |\cdot|_\infty)$ is a Banach space.*

Theorem 2.10 [9] *Let $\mu \in \mathcal{M}$ satisfy $(\mathbf{H}_1)$ and $f \in PAP(\mathbb{R}, X, \mu)$ be such that*

$$f = g + \phi,$$

where $g \in AP(\mathbb{R}, X)$ and $\phi \in \mathcal{E}(\mathbb{R}, X, \mu)$. Then,

$$\left\{g(t):\ t \in \mathbb{R}\right\} \subset \overline{\left\{f(t):\ t \in \mathbb{R}\right\}} \text{ (the closure of the range of } f\text{)}.$$

Theorem 2.11 [9] *Let $\mu \in \mathcal{M}$ satisfy $(\mathbf{H}_1)$. Then, the decomposition of a μ-pseudo almost periodic function of the form $f = g + \phi$, where $g \in AP(\mathbb{R}, X)$ and $\phi \in \mathcal{E}(\mathbb{R}, X, \mu)$, is unique.*

Theorem 2.12 [9] *Let $\mu \in \mathcal{M}$ satisfy $(\mathbf{H}_2)$. Then, $PAP(\mathbb{R}, X, \mu)$ is invariant by translation: $f \in PAP(\mathbb{R}, X, \mu)$ implies $f_\tau \in PAP(\mathbb{R}, X, \mu)$, for all $\tau \in \mathbb{R}$.*

Theorem 2.13 [9] *Let $\mu \in \mathcal{M}$ satisfy $(\mathbf{H}_1)$. Then, $PAP(\mathbb{R}, X, \mu)$ is a Banach space.*

Theorem 2.14 *Let $\mu \in \mathcal{M}$ satisfy $(\mathbf{H}_2)$. If $f \in PAP(\mathbb{R}, X, \mu)$ and $G \in L^1(\mathbb{R}, \mathcal{L}(X))$, then the convolution product $f * G$ defined by*

$$(f * G)(t) = \int_{-\infty}^{\infty} G(s) f(t-s)\, ds \quad \text{for } t \in \mathbb{R},$$

is also μ-pseudo almost periodic. In fact, if $f \in AP(\mathbb{R}, X)$ (respectively $f \in \mathcal{E}(\mathbb{R}, X, \mu)$), then $f * G \in AP(\mathbb{R}, X)$ (respectively $f * G \in \mathcal{E}(\mathbb{R}, X, \mu)$).

Let Y be a Banach space.

Definition 2.15 [9] A continuous function $f : \mathbb{R} \times X \longrightarrow Y$ is said to be almost periodic in $t \in \mathbb{R}$ uniformly with respect to $x \in X$, if for each compact set K in X and for all $\varepsilon > 0$, there exists $\ell > 0$ such that for any $\alpha \in \mathbb{R}$, there exists $\tau \in [\alpha, \alpha + \ell]$ such that

$$\sup_{t \in \mathbb{R}} \sup_{x \in K} \left| f(t+\tau, x) - f(t, x) \right| < \varepsilon.$$

Denote by $APU(\mathbb{R} \times X, Y)$ the set of all such functions $f : \mathbb{R} \times X \longrightarrow Y$.

Definition 2.16 [9] Let $\mu \in \mathcal{M}$. A continuous function $f : \mathbb{R} \times X \longrightarrow Y$ is said to be μ-ergodic in $t \in \mathbb{R}$ uniformly with respect to $x \in X$, if the two following conditions are satisfied:

(i) for all $x \in X$, $f(., x) \in \mathcal{E}(\mathbb{R}, Y, \mu)$,

(ii) f is uniformly continuous on each compact set K in X with respect to the second variable x.

Denote by $\mathcal{E}U(\mathbb{R} \times X, Y, \mu)$ the set of all such functions.

Definition 2.17 [9] Let $\mu \in \mathcal{M}$. A continuous function $f : \mathbb{R} \times X \longrightarrow Y$ is said to be μ-pseudo almost periodic in $t \in \mathbb{R}$ uniformly with respect to $x \in X$, if f can be written in the form

$$f = g + \phi,$$

where $g \in APU(\mathbb{R} \times X, Y)$ and $\phi \in \mathcal{E}U(\mathbb{R} \times X, Y, \mu)$.

$PAPU(\mathbb{R} \times X, Y, \mu)$ denotes the set of such functions.

Proposition 2.18 [9] *Let $\mu \in \mathcal{M}$ and $f \in PAPU(\mathbb{R} \times X, Y, \mu)$. We have the following properties.*

(i) *For all $x \in X$, $f(., x) \in PAP(\mathbb{R}, Y, \mu)$,*
(ii) *f is uniformly continuous on each compact set K in X with respect to the second variable x.*

Theorem 2.19 ([9] Composition theorem) *Let $\mu \in \mathcal{M}$, $f \in PAPU(\mathbb{R} \times X, Y, \mu)$ and $x \in PAP(\mathbb{R}, X, \mu)$. Assume that for all bounded subset B of X, f is bounded on $\mathbb{R} \times B$. Then, the function $[t \mapsto f(t, x(t))] \in PAP(\mathbb{R}, Y, \mu)$.*

3 Phase spaces, variation of constants formula and spectral decomposition

We use the axiomatic approach introduced in [22] (see also [19]) to define the phase space $\mathcal{B}$. We assume that $(\mathcal{B}, \|\cdot\|)$ is a normed space of functions mapping $(-\infty, 0]$ into a Banach space X and satisfying the following fundamental axioms.

(**A**) There exist a positive constant N, a locally bounded function $M(\cdot)$ on $[0, +\infty)$ and a continuous function $K(\cdot)$ on $[0, +\infty)$ such that if $x : (-\infty, a] \to X$ is continuous on $[\sigma, a]$ with $x_\sigma \in \mathcal{B}$, for some $\sigma < a$, then for all $t \in [\sigma, a]$,

(i) $x_t \in \mathcal{B}$,
(ii) $t \to x_t$ is continuous with respect to $\|\cdot\|$ on $[\sigma, a]$,
(iii) $N|x(t)| \leq \|x_t\| \leq K(t-\sigma)\sup_{\sigma \leq s \leq t}|x(s)| + M(t-\sigma)\|x_\sigma\|$.

(**B**) $\mathcal{B}$ is a Banach space.

As a consequence of axioms (**A**), we deduce the following result.

Lemma 3.1 [19] *Let $C_{00} := C_{00}((-\infty, 0]; X)$ be the space of continuous functions mapping $(-\infty, 0]$ into X with compact supports. Then, $C_{00}((-\infty, 0]; X) \subset \mathcal{B}$. More precisely, for $a < 0$, we have*

$$\|\varphi\| \leq K(-a)\sup_{\theta \leq 0}\left|\varphi(\theta)\right|,$$

for any $\varphi \in C_{00}((-\infty, 0]; X)$ with the support included in $[a, 0]$.

The following lemma is well known.

Lemma 3.2 [19] *Assume that* $(\mathbf{H}_0)$ *holds. Let* A_0 *be the part of the operator* A *in* $D(A)$, *which is defined by*

$$\begin{cases} D(A_0) = \left\{x \in D(A) : Ax \in \overline{D(A)}\right\}, \\ A_0 x = Ax. \end{cases}$$

Then, A_0 *generates a* C_0*-semigroup* $(T_0(t))_{t\geq 0}$ *on* $\overline{D(A)}$.

To Eq. (1.1), we associate the following Cauchy problem

$$\begin{cases} \dfrac{d}{dt}x(t) = Ax(t) + L(x_t) + f(t) \quad \text{for } t \geq \sigma, \\ x_\sigma = \phi \in \mathcal{B}, \end{cases} \tag{3.1}$$

where $f : [\sigma, \infty) \to X$ is continuous. The following results are taken from [2].

Definition 3.3 [2] Let $\phi \in \mathcal{B}$. A function $u : \mathbb{R} \to X$ is called a mild solution of Eq. (3.1) on $\mathbb{R}$ if the following conditions hold

(i) u is continuous on $[\sigma, \infty)$,
(ii) $u_\sigma = \phi$,
(iii) $\int_\sigma^t u(s)ds \in D(A)$ for $t \geq \sigma$,
(iv) $u(t) = \phi(0) + A\int_\sigma^t u(s)ds + \int_\sigma^t L(u_s)ds + \int_\sigma^t f(s)ds$ for $t \geq \sigma$.

Theorem 3.4 [2] *Assume that* $(\mathbf{H}_0)$, $(\mathbf{A})$, $(\mathbf{B})$ *hold and* $f : [\sigma, \infty) \to X$ *is continuous. Then, for all* $\phi \in \mathcal{B}$ *such that* $\phi(0) \in \overline{D(A)}$, *Eq.* (3.1) *has a unique mild solution* $u = u(\cdot, \sigma, \phi, L, f)$ *on* $\mathbb{R}$ *which is given by*

$$u(t) = \begin{cases} T_0(t-\sigma)\phi(0) + \lim_{\lambda\to+\infty} \int_\sigma^t T_0(t-s)\lambda R(\lambda, A)[L(u_s) + f(s)]ds \\ \quad \text{for } t \geq \sigma, \\ \phi(t) \quad \text{for } t \leq \sigma. \end{cases}$$

where the operator $R(\lambda, A) = (\lambda I - A)^{-1}$ *for* λ *large enough.*

Let

$$\mathcal{B}_A := \left\{\phi \in \mathcal{B} : \phi(0) \in \overline{D(A)}\right\} \tag{3.2}$$

be the phase space corresponding to Eq. (3.1). We define, for $t \geq 0$, the operator $U(t)$ by

$$U(t)\phi = u_t(\cdot, 0, \phi, L, 0) \quad \text{for } \phi \in \mathcal{B}_A,$$

where $u(\cdot, \phi, L, 0)$ is the mild solution of Eq. (3.1), with $f = 0$ and $\sigma = 0$.

Theorem 3.5 [2] *Assume that* $(\mathbf{H}_0)$, $(\mathbf{A})$ *and* $(\mathbf{B})$ *hold. Then* $(U(t))_{t\geq 0}$ *is a* C_0*-semigroup on* $\mathcal{B}_A$. *That is*

(i) $U(0) = Id$,
(ii) $U(t+s) = U(t)U(s)$ *for* $t, s \geq 0$,
(iii) *for all* $\phi \in \mathcal{B}_A$, $t \mapsto U(t)\phi$ *is continuous from* $[0, \infty)$ *to* $\mathcal{B}_A$.

Moreover, $(U(t))_{t \geq 0}$ *satisfies, for* $t \geq 0$ *and* $\phi \in \mathcal{B}_A$, *the translation property*

$$\big(U(t)\phi\big)(\theta) = \begin{cases} (U(t+\theta)\phi)(0) & \textit{for } t+\theta \geq 0, \\ \phi(t+\theta) & \textit{for } t+\theta \leq 0. \end{cases}$$

In order to give a variation-of-constant formula, we need to introduce the following sequence of linear operators $(\widetilde{B}_n)_{n\in\mathbb{N}}$ mapping X into $\mathcal{B}$, defined for $n > \omega$ and $x \in X$, by

$$(\widetilde{B}_n x)(\theta) = \begin{cases} n(n\theta + 1)R(n, A)x & \text{for } -\frac{1}{n} \leq \theta \leq 0, \\ 0 & \text{for } \theta < -\frac{1}{n}. \end{cases}$$

For each $x \in X$ and $n > \omega$, the function $\widetilde{B}_n x$ belongs to $C_{00}((-\infty, 0]; X)$ with the support included in $[-1, 0]$. By Lemma 3.1, we deduce that

$$|\widetilde{B}_n x| \leq \widetilde{N} K(1)|x| \quad \text{for } x \in X \text{ and } n > \omega,$$

where

$$\widetilde{N} = \sup\big\{\lambda \big|R(\lambda, A)\big| : \lambda > \omega\big\}.$$

The variation of constants formula is the principal working tools in partial functional differential equations, the qualitative analysis of solutions is based on that formula. In literature, we have many works dealing with formula and using many approaches based on sun-star theory [18], integrated semigroups and operator theory [1, 3, 21, 22, 25] and [23].

In the following result, we have developed a new variation of constants formula.

Theorem 3.6 [4] *Assume that* $(\mathbf{H}_0)$ *holds and* $f : [\sigma, \infty) \to X$ *is continuous. Then, for all* $\varphi \in \mathcal{B}_A$, *the mild solution* $u(., \sigma, \varphi, L, f)$ *of Eq.* (3.1) *satisfies the following variation-of-constants formula*

$$u_t(., \sigma, \varphi, L, f) = U(t-\sigma)\varphi + \lim_{n\to\infty} \int_\sigma^t U(t-s)\widetilde{B}_n f(s)ds, \quad \textit{for } t \geq \sigma. \tag{3.3}$$

Moreover, for any $T > \sigma$, *the limit in* (3.3) *exists uniformly for* $t \in [\sigma, T]$.

Note that the semigroup $(U(t))_{t\geq 0}$ *is acting on the phase space* $\mathcal{B}_A$ *and we cannot put the limit inside of the integral, since the limit inside does not exist, more details can be found in* [4].

We suppose the following axiom.

(**C**) If a uniformly bounded sequence $(\varphi_n)_n$ in $C_{00}((-\infty, 0]; X)$ converges to a function φ compactly in $(-\infty, 0]$, then φ is in $\mathcal{B}$ and $\|\varphi_n - \varphi\| \to 0$ as $n \to \infty$.

Let $(S_0(t))_{t\geq 0}$ be the strongly continuous C_0-semigroup defined on the subspace

$$\mathcal{B}_0 = \big\{\phi \in \mathcal{B} : \phi(0) = 0\big\},$$

by

$$\big(S_0(t)\phi\big)(\theta)=\begin{cases}\phi(t+\theta) & \text{for } t+\theta\le 0,\\ 0 & \text{for } t+\theta\ge 0.\end{cases}$$

Definition 3.7 Assume that the space $\mathcal{B}$ satisfies the axioms (**A**), (**B**) and (**C**). $\mathcal{B}$ is said to be a fading memory space, if for all $\phi\in\mathcal{B}_0$,

$$S_0(t)\phi \underset{t\to\infty}{\longrightarrow} 0 \quad \text{in } \mathcal{B}.$$

Moreover, $\mathcal{B}$ is said to be a uniform fading memory space, if

$$\big\|S_0(t)\big\| \underset{t\to\infty}{\longrightarrow} 0.$$

The following results give some properties of fading memory spaces.

Lemma 3.8 [22] *The following statements hold.*

(i) *If $\mathcal{B}$ is a fading memory space, then the functions $K(\cdot)$ and $M(\cdot)$ in the axiom* (**A**) *can be chosen to be constants.*
(ii) *If $\mathcal{B}$ is a uniform fading memory space, then the functions $K(\cdot)$ and $M(\cdot)$ can be chosen such that $K(\cdot)$ is constant and $M(t)\to 0$ as $t\to\infty$.*

Proposition 3.9 [22] *If $\mathcal{B}$ is a fading memory space, then the space $\mathcal{BC}((-\infty,0];X)$ of all bounded and continuous X-valued functions on $(-\infty,0]$, endowed with the uniform norm topology, is continuously embedding in $\mathcal{B}$.*

In order to study the qualitative behavior of the C_0-semigroup $(U(t))_{t\ge 0}$, we suppose the following property.

($\mathbf{H}_3$) $T_0(t)$ is compact on $\overline{D(A)}$, for each $t>0$.

Let V be a bounded subset of a Banach space Y. The Kuratowski measure of noncompactness $\alpha(V)$ of V is defined by

$$\alpha(V)=\inf\left\{\begin{array}{l} d>0\text{: there exists a finite number of sets } V_1,\dots,V_n \text{ with}\\ diam(V_i)\le d \text{ such that } V\subseteq\bigcup_{i=1}^n V_i\end{array}\right\}.$$

Moreover, for a bounded linear operator P on Y, we define $|P|_\alpha$ by

$$|P|_\alpha=\inf\big\{k>0:\alpha\big(P(V)\big)\le k\alpha(V)\text{ for any bounded set } V \text{ of } Y\big\}.$$

For the C_0-semigroup $(U(t))_{t\ge 0}$, its essential growth bound $\omega_{ess}(U)$ is given by

$$\omega_{ess}(U)=\inf\big\{\omega\in\mathbb{R}:\sup e^{-\omega t}\big\|U(t)\big\|_\alpha<\infty\big\}.$$

It is well known that

$$\omega_{ess}(U)=\lim_{t\to\infty}\frac{1}{t}\log\big\|U(t)\big\|_\alpha.$$

We have the following fundamental result.

Theorem 3.10 [8] *Assume that* $(\mathbf{H}_0)$, $(\mathbf{H}_3)$ *hold and* $\mathcal{B}$ *is a uniform fading memory space. Then,*

$$\omega_{ess}(U) < 0.$$

Definition 3.11 Let $\mathcal{C}$ be a densely defined operator on Y. The essential spectrum of $\mathcal{C}$ denoted by $\sigma_{ess}(\mathcal{C})$ is the set of $\lambda \in \sigma(\mathcal{C})$ such that one of the following conditions holds.

(i) $\operatorname{Im}(\lambda I - \mathcal{C})$ is not closed,
(ii) the generalized eigenspace $M_\lambda(\mathcal{C}) = \bigcup_{k\geq 1} \operatorname{Ker}(\lambda I - \mathcal{C})^k$ is of infinite dimension,
(iii) λ is a limit point of $\sigma(\mathcal{C}) \setminus \{\lambda\}$.

The essential radius of any bounded operator $\mathcal{T}$ is defined by

$$r_{ess}(\mathcal{T}) = \sup\big\{|\lambda| : \lambda \in \sigma_{ess}(\mathcal{T})\big\}.$$

In the sequel, A_U denotes the infinitesimal generator of the C_0-semigroup $(U(t))_{t\geq 0}$.

Theorem 3.12 [4] *Assume that* $(\mathbf{H}_0)$, $(\mathbf{H}_3)$ *hold and* $\mathcal{B}$ *is a uniform fading memory space. Then* $\sigma^+(A_U) = \{\lambda \in \sigma(A_U) : \mathcal{R}e(\lambda) \geq 0\}$ *is a finite set of the eigenvalues of* A_U *which are not in the essential spectrum. More precisely,* $\lambda \in \sigma^+(A_U)$ *if and only if there exists* $x \in D(A)\backslash\{0\}$ *solving the following characteristic equation*

$$\Delta(\lambda)x := \lambda x - Ax - L\big(e^{\lambda \cdot} x\big) = 0.$$

Definition 3.13 The C_0-semigroup $(U(t))_{t\geq 0}$ is hyperbolic if

$$\sigma(A_U) \cap i\mathbb{R} = \emptyset.$$

Since $\omega_{ess}(U) < 0$, then we get the following result on the spectral decomposition of the phase space:

$$\mathcal{B}_A := \big\{\phi \in \mathcal{B} : \phi(0) \in \overline{D(A)}\big\}.$$

Theorem 3.14 [4] *Assume that* $(\mathbf{H}_0)$, $(\mathbf{H}_3)$ *hold and* $\mathcal{B}$ *is a uniform fading memory space. If the* C_0*-semigroup* $(U(t))_{t\geq 0}$ *is hyperbolic then the space* $\mathcal{B}_A$ *is decomposed as a direct sum* $\mathcal{B}_A = \mathcal{S} \oplus \mathcal{U}$ *of two* $U(t)$*-invariant closed subspaces* $\mathcal{S}$ *and* $\mathcal{U}$ *such that the restricted* C_0*-semigroup on* $\mathcal{U}$ *is a group and there exist positive constants* M *and* c *such that*

$$\begin{aligned} \big\|U(t)\varphi\big\| &\leq Me^{-ct}\|\varphi\| \quad \text{for } t \geq 0 \text{ and } \varphi \in \mathcal{S}, \\ \big\|U(t)\varphi\big\| &\leq Me^{ct}\|\varphi\| \quad \text{for } t \leq 0 \text{ and } \varphi \in \mathcal{U}. \end{aligned}$$

Consequently, we deduce the following interesting result on the existence and uniqueness of bounded mild solutions of Eq. (1.1).

Theorem 3.15 [19] *Assume that* $(\mathbf{H}_0)$, $(\mathbf{H}_3)$ *hold*, $\mathcal{B}$ *is a uniform fading memory space and* $f \in BC(\mathbb{R}, X)$. *If the* C_0*-semigroup* $(U(t))_{t\geq 0}$ *is hyperbolic*, *then*, *Eq*. (1.1) *has a unique bounded mild solution* y *on* $\mathbb{R}$ *such that*

$$y_t = \lim_{n\to+\infty} \int_{-\infty}^{t} U^{\mathcal{S}}(t-s)\Pi^{\mathcal{S}}\big(\widetilde{B}_n f(s)\big)ds + \lim_{n\to+\infty} \int_{+\infty}^{t} U^{\mathcal{U}}(t-s)\Pi^{\mathcal{U}}\big(\widetilde{B}_n f(s)\big)ds \quad \text{for } t \in \mathbb{R}, \tag{3.4}$$

where $\Pi^{\mathcal{S}}$, $\Pi^{\mathcal{U}}$ *denote respectively the projections on* $\mathcal{S}$ *and* $\mathcal{U}$, *and* $U^{\mathcal{S}} = \Pi^{\mathcal{S}}(U)$, $U^{\mathcal{U}} = \Pi^{\mathcal{U}}(U)$.

4 Existence and uniqueness of μ-pseudo almost periodic solution

In this section, we give our main result: if the input function f is μ-pseudo almost periodic then, (1.1) has a unique μ-pseudo almost periodic mild solution.

Theorem 4.1 *Assume that* $(\mathbf{H}_0)$, $(\mathbf{H}_2)$, $(\mathbf{H}_3)$ *hold and* $\mathcal{B}$ *is a uniform fading memory space*. *If the* C_0*-semigroup* $(U(t))_{t\geq 0}$ *is hyperbolic and the function* f *is* μ*-pseudo almost periodic*, *then*, *Eq*. (1.1) *has one and only one* μ*-pseudo almost periodic mild solution* y *such that the following formula holds*

$$y_t = \lim_{n\to+\infty} \int_{-\infty}^{t} U^{\mathcal{S}}(t-s)\Pi^{\mathcal{S}}\big(\widetilde{B}_n f(s)\big)ds + \lim_{n\to+\infty} \int_{+\infty}^{t} U^{\mathcal{U}}(t-s)\Pi^{\mathcal{U}}\big(\widetilde{B}_n f(s)\big)ds \quad \text{for } t \in \mathbb{R}.$$

Proof By the help of Theorem 3.15, Eq. (1.1) has one and only one bounded mild solution on $\mathbb{R}$. Moreover, this solution is given by (3.4). Since the function f is μ-pseudo almost periodic, then it is decomposed as follows

$$f = g + \phi,$$

where $g \in AP(\mathbb{R}, X)$ and $\phi \in \mathcal{E}(\mathbb{R}, X, \mu)$. Consequently, we can define the operators Q^S and Q^U from $BC(\mathbb{R}, X)$ to $BC(\mathbb{R}, \mathcal{B})$, for $e \in BC(\mathbb{R}, X)$ and $t \in \mathbb{R}$, by

$$\begin{cases} (Q^S e)(t) := \lim_{n\to+\infty} \int_{-\infty}^{t} U^{\mathcal{S}}(t-\tau)\ \Pi^{\mathcal{S}}(\widetilde{B}_n f(s))\, d\tau, \\ (Q^U e)(t) := \lim_{n\to+\infty} \int_{-\infty}^{t} U^{\mathcal{U}}(t-\tau)\ \Pi^{\mathcal{U}}(\widetilde{B}_n f(s))\, d\tau. \end{cases}$$

Since Q^S and Q^U are bounded linear operators from $BC(\mathbb{R}, X)$ to $BC(\mathbb{R}, \mathcal{B})$, then the unique bounded mild solution x of (1.1) such that

$$x_t = \big(Q^S f\big)(t) + \big(Q^U f\big)(t).$$

We will show that both $Q^S f$ and $Q^U f$ are μ-pseudo almost periodic functions. In fact, we have

$$Q^S f = Q^S g + Q^S \phi \quad \text{and} \quad Q^U f = Q^U g + Q^U \phi.$$

On the other hand, we have $(Q^S g)_\tau = (Q^S g_\tau)$, for $\tau \in \mathbb{R}$. By using the continuity of the operator Q^S, we deduce that $Q^S(\{g_\tau : \tau \in \mathbb{R}\})$ is relatively compact on $BC(\mathbb{R}, \mathcal{B})$. This implies that $Q^S g \in AP(\mathbb{R}, \mathcal{B})$. Using a same argument as above, we can prove that $Q^U g \in AP(\mathbb{R}, \mathcal{B})$. It remains to prove that $Q^S \phi \in \mathcal{E}(\mathbb{R}, X, \mu)$ and $Q^U \phi \in \mathcal{E}(\mathbb{R}, X, \mu)$. By using the Hille-Yosida condition on A, one can find a positive constant $\widetilde{K}$ such that

$$\left\| \left(Q^S \phi\right)(t) \right\| \le \widetilde{K} \int_{-\infty}^{t} e^{-c(t-\tau)} \left|\phi(\tau)\right| d\tau. \tag{4.1}$$

Let $G : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by

$$G(t) = e^{-ct} \quad \text{for } t \ge 0 \quad \text{and} \quad G(t) = 0 \quad \text{for } t < 0.$$

Then, we have

$$\int_{-\infty}^{t} e^{-c(t-\tau)} \left|\phi(\tau)\right| d\tau = \int_{0}^{\infty} e^{-c\tau} \left|\phi(t-\tau)\right| d\tau = \int_{-\infty}^{\infty} G(\tau) \left|\phi(t-\tau)\right| d\tau. \tag{4.2}$$

Since $t \mapsto |\phi(t)| \in \mathcal{E}(\mathbb{R}, \mathbb{R}, \mu)$, we deduce from (4.2) and Theorem 2.14 that

$$t \mapsto \int_{-\infty}^{t} e^{-c(t-\tau)} \left|\phi(\tau)\right| d\tau \in \mathcal{E}(\mathbb{R}, \mathbb{R}, \mu).$$

Then, we obtain from (4.1) that $Q^S \phi \in \mathcal{E}(\mathbb{R}, \mathcal{B}, \mu)$. Arguing as above, we prove also that $Q^U \phi \in \mathcal{E}(\mathbb{R}, \mathcal{B}, \mu)$. □

5 Composition theorem and nonlinear functional differential equation

Let $r > 0$. We use the exponential dichotomy to study the existence of a unique μ-pseudo almost periodic mild solution of the following nonlinear equation

$$\frac{d}{dt} u(t) = Au(t) + L(u_t) + g\big(t, u(t-r)\big) \quad \text{for } t \in \mathbb{R}. \tag{5.1}$$

We make the following assumptions.

(**H**$_4$) $g : \mathbb{R} \times X \to X$ is continuous and Lipschitzian with respect to the second argument: there exists $\sigma > 0$ such that

$$\left|g(t, u_1) - g(t, u_2)\right| \le \sigma |u_1 - u_2| \quad \text{for } t \in \mathbb{R} \text{ and } u_1, u_2 \in X.$$

(**H**$_5$) $g \in PAPU(\mathbb{R} \times X, X, \mu)$.

Theorem 5.1 *Assume that* ($\mathbf{H}_0$), ($\mathbf{H}_2$), ($\mathbf{H}_3$), ($\mathbf{H}_4$), ($\mathbf{H}_5$), *hold and the* C_0*-semigroup* $(U(t))_{t\geq 0}$ *is hyperbolic. Then, there exists* $\sigma_0 > 0$ *such that for* $\sigma < \sigma_0$, *Eq.* (5.1) *has a unique* μ*-pseudo almost periodic mild solution.*

Proof Let $v \in PAP(\mathbb{R}, X, \mu)$. Assumption ($\mathbf{H}_5$) and Theorem 2.19 imply that the function $t \to g(t, v(t-r))$ is in $PAP(\mathbb{R}, X, \mu)$. Consider the equation

$$\frac{d}{dt}u(t) = Au(t) + L(u_t) + g\big(t, v(t-r)\big) \quad \text{for } t \in \mathbb{R}. \tag{5.2}$$

Applying Theorem 4.1, we deduce that Eq. (5.2) has a unique μ-pseudo almost periodic mild solution $\widetilde{K}v$ which is defined, for $t \in \mathbb{R}$, by the following formula

$$\Big[\lim_{n\to+\infty}\int_{-\infty}^{t} U^{\mathcal{S}}(t-s)\Pi^{\mathcal{S}}\big(\widetilde{B}_n g\big(s, v(s-r)\big)\big)ds + \lim_{n\to+\infty}\int_{+\infty}^{t} U^{\mathcal{U}}(t-s)\Pi^{\mathcal{U}}\big(\widetilde{B}_n g\big(s, v(s-r)\big)\big)ds\Big](0).$$

The operator $\widetilde{K}$ is well defined on $PAP(\mathbb{R}, X, \mu)$. Moreover, by using the assumption ($\mathbf{H}_4$) and the fact that the C_0-semigroup $(U(t))_{t\geq 0}$ is hyperbolic, we deduce that there exists a positive constant μ_0 such that

$$\sup_{t\in\mathbb{R}}\big|\widetilde{K}(v_1)(t) - \widetilde{K}(v_2)(t)\big| \leq (\sigma\mu_0)\sup_{t\in\mathbb{R}}\big|v_1(t) - v_2(t)\big|.$$

If we choose $\sigma < \frac{1}{\mu_0}$, then the mapping $v \to \widetilde{K}(v)$ is a strict contraction form $PAP(\mathbb{R}, X, \mu)$ to $PAP(\mathbb{R}, X, \mu)$. Since by Theorem 2.13, we deduce that $\widetilde{K}$ has a unique fixed point in $PAP(\mathbb{R}, X, \mu)$. Then, Eq. (5.1) has a unique μ-pseudo almost periodic mild solution. □

In the sequel, we study the existence of μ-pseudo almost periodic mild solution in a special case when the Lipschitz coefficient of g is time-dependent. More precisely, we make the following assumption on the nonlinear function g.

($\mathbf{H}_6$) $g : \mathbb{R} \times X \to X$ is continuous and

$$\big|g(t, u_1) - g(t, u_2)\big| \leq \sigma(t)|u_1 - u_2| \quad \text{for } t \in \mathbb{R} \text{ and } u_1, u_2 \in X,$$

where $\sigma \in L^p(\mathbb{R}, \mathbb{R}^+)$ for some $1 \leq p < \infty$.

We also need the following assumption.

($\mathbf{H}_7$) For all bounded subsets B of X, g is bounded on $\mathbb{R} \times B$.

Theorem 5.2 *Assume that* ($\mathbf{H}_0$), ($\mathbf{H}_2$), ($\mathbf{H}_3$), ($\mathbf{H}_5$), ($\mathbf{H}_6$), ($\mathbf{H}_7$) *hold. Moreover, assume that the* C_0*-semigroup* $(U(t))_{t\geq 0}$ *is hyperbolic and the unstable space is reduced to zero. Then, Eq.* (5.1) *has a unique* μ*-pseudo almost periodic mild solution.*

Proof (i) *First case:* $\sigma \in L^1(\mathbb{R}, \mathbb{R}^+)$.

Let $v \in PAP(\mathbb{R}, X, \mu)$. Then, Theorem 2.19 implies that the function $t \to g(t, v(t-r))$ is in $PAP(\mathbb{R}, X, \mu)$. Consider now the equation

$$\frac{d}{dt}u(t) = Au(t) + L(u_t) + g\big(t, v(t-r)\big) \quad \text{for } t \in \mathbb{R}. \tag{5.3}$$

Since the unstable space is reduced to $\{0\}$, then by Theorem 4.1 we deduce that Eq. (5.3) has a unique μ-pseudo almost periodic mild solution denoted by $\widetilde{K}y$. In fact, we have

$$\widetilde{K}y(t) = \left(\lim_{n\to+\infty}\int_{-\infty}^{t} U^{\mathcal{S}}(t-s)\Pi^{\mathcal{S}}\big(\widetilde{B}_n g\big(s, v(s-r)\big)\big)ds\right)(0) \quad \text{for } t \in \mathbb{R}.$$

The operator $\widetilde{K}$ is well defined on $PAP(\mathbb{R}, X, \mu)$. Let $v_1, v_2 \in PAP(\mathbb{R}, X, \mu)$. Then, for some positive constant η, we have

$$\big|\widetilde{K}(v_1)(t) - \widetilde{K}(v_2)(t)\big| \le \eta \int_{-\infty}^{t} e^{-c(t-s)}\sigma(s)\big|v_1(s) - v_2(s)\big|ds$$

and

$$\big|\widetilde{K}(v_1)(t) - \widetilde{K}(v_2)(t)\big| \le \eta \int_{-\infty}^{t} \sigma(s)ds|v_1 - v_2|.$$

Therefore,

$$\begin{aligned}\big|\widetilde{K}^2(v_1)(t) - \widetilde{K}^2(v_2)(t)\big| &\le \eta^2 \int_{-\infty}^{t} \sigma(s)\int_{-\infty}^{s} \sigma(\tau)d\tau ds|v_1 - v_2| \\ &= \frac{\eta^2}{2}\left(\int_{-\infty}^{t} \sigma(s)ds\right)^2 |v_1 - v_2|.\end{aligned}$$

Consequently, for all $n \ge 1$ we have

$$\big|\widetilde{K}^n(v_1)(t) - \widetilde{K}^n(v_2)(t)\big| \le \frac{\eta^n}{n!}\left(\int_{-\infty}^{t} \sigma(s)ds\right)^n |v_1 - v_2|.$$

We choose n such that

$$\frac{\eta^n}{n!}\left(\int_{-\infty}^{+\infty} \sigma(s)ds\right)^n < 1.$$

So, $\widetilde{K}^n$ is a strict contraction in $PAP(\mathbb{R}, X, \mu)$. Then, it has a unique fixed point. This is also a fixed point of the operator $\widetilde{K}$. Hence, Eq. (5.1) has a unique μ-pseudo almost periodic mild solution.

(ii) *Second case:* $\sigma \in L^p(\mathbb{R}, \mathbb{R}^+)$, for $p > 1$.

We have to renorm the space $PAP(\mathbb{R}, X, \mu)$ with the following equivalent norm

$$|v|_c = \sup_{t\in\mathbb{R}} e^{-c\lambda(t)}\big|v(t)\big| \quad \text{with } c > 0$$

and

$$\lambda(t) = \int_{-\infty}^{t} \sigma^p(s)ds \quad \text{for } t \in \mathbb{R}.$$

It follows that

$$|\widetilde{K}(v_1)(t) - \widetilde{K}(v_2)(t)| \leq \eta\left(\int_{-\infty}^{t} e^{-c(t-s)} e^{c\lambda(s)} \sigma(s)ds\right)|v_1 - v_2|_c.$$

Using the Holder inequality, we obtain for $\frac{1}{p} + \frac{1}{q} = 1$ that

$$\begin{aligned}\int_{-\infty}^{t} e^{-c(t-s)} e^{c\lambda(s)} \sigma(s)ds &\leq \left(\int_{-\infty}^{t} e^{-qc(t-s)}ds\right)^{\frac{1}{q}} \left(\int_{-\infty}^{t} e^{cp\lambda(s)} \sigma^p(s)ds\right)^{\frac{1}{p}} \\ &= \frac{1}{(qc)^{\frac{1}{q}}} \frac{1}{(pc)^{\frac{1}{p}}} \left(\int_{-\infty}^{t} cpe^{cp\lambda(s)} \lambda'(s)ds\right)^{\frac{1}{p}} \\ &= \frac{1}{(qc)^{\frac{1}{q}}} \frac{1}{(pc)^{\frac{1}{p}}} e^{c\lambda(t)}.\end{aligned}$$

It follows that

$$|\widetilde{K}v_1 - \widetilde{K}v_2|_c \leq \eta\left(\frac{1}{(qc)^{\frac{1}{q}}} \frac{1}{(pc)^{\frac{1}{p}}}\right)|v_1 - v_2|_c.$$

If we choose c such that

$$\eta\left(\frac{1}{(qc)^{\frac{1}{q}}} \frac{1}{(pc)^{\frac{1}{p}}}\right) < 1,$$

then $\widetilde{K}$ is a strict contraction in $PAP(\mathbb{R}, X, \mu)$ and it has a unique fixed point. We conclude that Eq. (5.1) has a unique μ-pseudo almost periodic mild solution. □

6 Example

To illustrate our previous results, we consider the following Lotka-Volterra model with diffusion and infinite delay

$$\begin{cases} \frac{\partial}{\partial t}v(t,\xi) = \frac{\partial^2}{\partial \xi^2}v(t,\xi) + \int_{-\infty}^{0} \eta(\theta)v(t+\theta,\xi)d\theta + \sigma(t)F(\xi) \\ \quad \text{for } t \in \mathbb{R} \text{ and } 0 \leq \xi \leq \pi, \\ v(t,0) = v(t,\pi) = 0 \quad \text{for } t \in \mathbb{R} \end{cases} \tag{6.1}$$

where η is a positive function on $(-\infty, 0]$ and $\sigma : \mathbb{R} \to \mathbb{R}$ is μ-pseudo almost periodic for some positive measure μ in $\mathcal{M}$ satisfying $(\mathbf{H}_2)$. $F : [0,\pi] \to \mathbb{R}$ is a continuous function. Let $X = C([0,\pi];\mathbb{R})$ be the space of continuous functions from $[0,\pi]$ to $\mathbb{R}$ endowed with the uniform norm topology. Consider the operator $A : D(A) \subset X \to X$ defined by

$$\begin{cases} D(A) = \{z \in C^2([0,\pi];\mathbb{R}) : z(0) = z(\pi) = 0\}, \\ Az = z''. \end{cases}$$

Lemma 6.1 [4] *The operator A satisfies the Hille-Yosida condition on X.*

On the other hand, one can see that

$$\overline{D(A)} = \{\psi \in C([0,\pi];\mathbb{R}) : \psi(0) = \psi(\pi) = 0\}.$$

Let $\gamma > 0$. We define the following space

$$\mathcal{B} = C_\gamma = \left\{\phi \in C((-\infty,0];X) : \lim_{\theta\to-\infty} e^{\gamma\theta}\phi(\theta) \text{ exists in } X\right\}$$

provided with the norm

$$|\phi|_\gamma = \sup_{-\infty<\theta\le 0} e^{\gamma\theta}|\phi(\theta)|, \quad \text{for } \phi \in C_\gamma.$$

Lemma 6.2 [22] *The space C_γ satisfies the axioms* (**A**) *and* (**B**). *Moreover, C_γ is a uniform fading memory space.*

We add the following assumption.

($\mathbf{E}_1$) $\eta(\cdot)e^{-\gamma\cdot}$ is integrable on $(-\infty,0]$.

Define

$$\begin{cases} (L(\phi))(\xi) = \int_{-\infty}^0 \eta(\theta)\phi(\theta)(\xi)d\theta & \text{for } \xi \in [0,\pi] \text{ and } \phi \in \mathcal{B}, \\ f(t)(\xi) = \sigma(t)F(\xi) & \text{for } t \in \mathbb{R} \text{ and } \xi \in [0,\pi]. \end{cases}$$

Assumption ($\mathbf{E}_1$) implies that L is a bounded linear operator from $\mathcal{B}$ to X. Furthermore, $f : \mathbb{R} \to X$ is μ-pseudo almost periodic function. We put

$$x(t)(\xi) = v(t,\xi) \quad \text{for } t \in \mathbb{R} \text{ and } \xi \in [0,\pi].$$

Then, Eq. (6.1) takes the following abstract form

$$\frac{d}{dt}x(t) = Ax(t) + L(x_t) + f(t) \quad \text{for } t \in \mathbb{R}. \tag{6.2}$$

The part A_0 of the operator A in $\overline{D(A)}$ is given by

$$\begin{cases} D(A_0) = \{z \in C^2([0,\pi];\mathbb{R}) : z(0) = z(\pi) = z''(0) = z''(\pi) = 0\}, \\ A_0 z = z''. \end{cases}$$

Then, it is well known that A_0 generates a compact C_0-semigroup on $\overline{D(A)}$. In order to prove the existence and uniqueness of μ-pseudo almost periodic solution of Eq. (6.2) under the hyperbolicity condition, we make the following assumption.

($\mathbf{E}_2$) $\int_{-\infty}^{0} \eta(\theta)d\theta < 1$.

Theorem 6.3 *Assume that* ($\mathbf{E}_1$) *and* ($\mathbf{E}_2$) *are satisfied. Then, the C_0-semigroup solution of* (6.2) *with $f = 0$ is hyperbolic, that is, there exist $M \geq 1$ and $\omega > 0$ such that*

$$\left\|U(t)\right\| \leq Me^{-\omega t} \quad \textit{for } t \geq 0.$$

Proof By Theorem 3.12, it suffices to show that $\sigma^+(A_U) = \emptyset$. We proceed by contradiction and we assume that there exists $\lambda \in \sigma^+(A_U)$. Then, there exists $\vartheta \in D(A)\backslash\{0\}$ such that $\Delta(\lambda)\vartheta = 0$. This is equivalent to

$$\left(\lambda - A - \int_{-\infty}^{0} \eta(\theta)e^{\lambda\theta}d\theta\right)\vartheta = 0. \tag{6.3}$$

On the other hand, the spectrum $\sigma(A)$ is reduced to the point spectrum $\sigma_p(A)$ and it is given by $\sigma_p(A) = \{-n^2 : n \in \mathbb{N}^*\}$. Then, λ is a solution of the characteristic equation (6.3) with $\mathcal{R}e(\lambda) \geq 0$ if and only if λ satisfies

$$\lambda - \int_{-\infty}^{0} \eta(\theta)e^{\lambda\theta}d\theta = -n^2 \quad \text{for some } n \in \mathbb{N}^*.$$

It follows that

$$\begin{aligned}\mathcal{R}e(\lambda) &= \int_{-\infty}^{0} \eta(\theta)e^{\mathcal{R}e(\lambda)\theta}\cos\bigl(\mathrm{Im}(\lambda)\theta\bigr)d\theta - n^2 \\ &\leq \int_{-\infty}^{0} \eta(\theta)d\theta - n^2.\end{aligned}$$

Since $\int_{-\infty}^{0} \eta(\theta)d\theta < 1$, then a contradiction is obtained with the fact that $\mathcal{R}e(\lambda) \geq 0$. Consequently, the C_0-semigroup solution associated to (6.2) with $f = 0$ is hyperbolic. □

Consequently, we have the following result.

Proposition 6.4 *Equation* (6.2) *has a unique μ-pseudo almost periodic mild solution.*

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