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ORIGINAL RESEARCH

On the oscillation of fourth order strongly superlinear and strongly sublinear dynamic equations

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Abstract We establish some new criteria for the oscillation of fourth order nonlinear dynamic equations on a time scale. We investigate the case of strongly superlinear and the case of strongly sublinear equations subject to various conditions. Some examples are given here to illustrate our main results.

Keywords Oscillation · Strongly superlinear · Strongly sublinear · Dynamic equations

Mathematics Subject Classification 34C10 · 34K11 · 34N05

1 Introduction

This paper deals with the oscillatory behavior of solutions of fourth order nonlinear dynamic equations of the form

$$\big(ax^{\Delta^2}\big)^{\Delta^2}(t) + f\big(t, x^\sigma(t)\big) = 0, \quad t \ge t_0 \tag{1.1}$$

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subject to the following conditions:

(i) a is a positive real-valued rd-continuous function satisfying

$$\int_{t_0}^{\infty} \frac{\sigma(s)}{a(s)} \Delta s < \infty. \tag{1.2}$$

(ii) $f : [t_0, \infty) \times \mathbb{R} \to \mathbb{R}$ is continuous satisfying

$$\operatorname{sgn} f(t,x) = \operatorname{sgn} x \quad \text{and} \quad f(t,x) \le f(t,y), \quad x \le y,\ t \ge t_0, \tag{1.3}$$

where $t \ge t_0$ denotes time scale interval $[t_0, +\infty] \cap \mathbb{T}$ on a time scale $\mathbb{T}$ throughout this paper.

By a solution of Eq. (1.1), we mean a nontrivial real-valued function x satisfying Eq. (1.1) for $t \ge t_x \ge t_0$. A solution x of Eq. (1.1) is called oscillatory if it in neither eventually positive nor eventually negative; otherwise it is called nonoscillatory. Equation (1.1) is called oscillatory if all its solution are oscillatory.

Recently, there has been an increasing interest in studying the oscillatory behavior of first, second, third and higher order dynamic equations on time scales, see [1–6]. In [7], the oscillation of fourth order nonlinear dynamic equations

$$\big(a(t)\big(b(t)x^{\Delta}(t)\big)^{\Delta}\big)^{\Delta^2} + f\big(t, x^{\sigma}(t)\big) = 0, \tag{1.4}$$

and

$$\big(a(t)\big(b(t)\big(x(t) + p(t)x(t-\delta)\big)^{\Delta}\big)^{\Delta}\big)^{\Delta^2} + f\big(t, x^{\sigma}(t-\delta)\big) = 0 \tag{1.5}$$

are considered under conditions

$$\int_{t_0}^{\infty} \frac{1}{a(t)} \Delta t = \int_{t_0}^{\infty} \frac{1}{b(t)} \Delta t = \int_{t_0}^{\infty} \frac{t}{a(t)} \Delta t = \infty. \tag{1.6}$$

With respect to dynamic equations on time scales, it is fairly new topic and for general basic ideas and background, we refer to [8, 9].

To the best of our knowledge, there is no results regarding the oscillation of Eq. (1.1) with condition (1.2). Therefore, our main goal of this paper is to establish some new criteria for the oscillation of Eq. (1.1) under condition (1.2).

The obtained results extend our earlier result in [2] from second to fourth order.

2 Preliminaries

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. Since we are interested in oscillatory behavior, we suppose that the time scale under consideration is unbounded above. The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$. For a function $f : \mathbb{T} \to \mathbb{R}$, we write $f^{\sigma} = f \circ \sigma$ and f^{Δ} represents the delta derivative of the function f as defined for example in [8]. For the definition of rd-continuity and further details, we refer to [8].

Besides the usual properties of the time scale integrals, we require in this paper only the use of chain rule ([8], Theorem 1.90)

$$\frac{(x^{1-\alpha})^{\Delta}}{1-\alpha} = x^{\Delta} \int_0^1 \big[hx^{\sigma} + (1-h)x\big]^{-\alpha} dh, \tag{2.1}$$

where $\alpha > 0$ and x is such that the right-hand side of (2.1) is well defined, and the integration by parts ([8], Theorem 1.77)

$$\int_a^b f(\sigma(t)) g^\Delta(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^\Delta(t) g(t)\Delta t, \tag{2.2}$$

where $a, b \in \mathbb{T}$ and $f, g \in \mathbb{C}_{rd}$.

One consequence of (2.1) that will be used in the proof of our results throughout this paper is as follows (see [2], Lemma 2.1]).

Lemma 2.1 *Suppose* $|y|^\Delta > 0$ *on* $[t_0, \infty)$ *and* $\alpha > 0$. *Then*

$$\frac{|y|^\Delta}{(|y|^\sigma)^\alpha} \le \frac{(|y|^{1-\alpha})^\Delta}{1-\alpha} \le \frac{|y|^\Delta}{|y|^\alpha} \quad \text{on } [t_0, \infty). \tag{2.3}$$

We shall obtain results for strongly superlinear and strongly sublinear equations according to the following classification:

Definition 2.1 Equation (1.1) (or the function f) is said to be strongly superlinear if there exists a constant $\beta > 1$ such that

$$\frac{|f(t,x)|}{|x|^\beta} \le \frac{|f(t,y)|}{|y|^\beta} \quad \text{for } |x| \le |y|,\ xy > 0,\ t \ge t_0, \tag{2.4}$$

and it is said to be strongly sublinear if there exists a constant $\gamma \in (0, 1)$ such that

$$\frac{|f(t,x)|}{|x|^\gamma} \ge \frac{|f(t,y)|}{|y|^\gamma} \quad \text{for } |x| \le |y|,\ xy > 0,\ t \ge t_0. \tag{2.5}$$

If (2.4) holds with $\beta = 1$, then Eq. (1.1) is called superlinear and if (2.5) holds with $\gamma = 1$, then Eq. (1.1) is called sublinear.

We may note that condition (1.3) implies

$$\big|f(t,x)\big| \le \big|f(t,y)\big| \quad \text{for } |x| \le |y|,\ xy > 0,\ t \ge t_0. \tag{2.6}$$

Next, we establish some results that will be needed to prove oscillation criteria. Lemmas 2.2 and 2.3 give useful information about the behavior of possible nonoscillatory solutions of Eq. (1.1).

Lemma 2.2 *If x is an eventually positive solution of Eq.* (1.1), *then one of the following four cases hold for all sufficiently large* $t \ge t_0$:

$$\begin{aligned}
&\text{(I)} && \big(ax^{\Delta^2}\big)^\Delta(t) > 0, && x^{\Delta^2}(t) > 0 && \textit{and} && x^\Delta(t) > 0;\\
&\text{(II)} && \big(ax^{\Delta^2}\big)^\Delta(t) > 0, && x^{\Delta^2}(t) < 0 && \textit{and} && x^\Delta(t) > 0;\\
&\text{(III)} && \big(ax^{\Delta^2}\big)^\Delta(t) < 0, && x^{\Delta^2}(t) < 0 && \textit{and} && x^\Delta(t) > 0;\\
&\text{(IV)} && \big(ax^{\Delta^2}\big)^\Delta(t) > 0, && x^{\Delta^2}(t) > 0 && \textit{and} && x^\Delta(t) < 0.
\end{aligned}$$

Proof From Eq. (1.1), we have

$$\left(ax^{\Delta^2}\right)^{\Delta^2}(t) = -f\left(t, x^\sigma(t)\right) < 0 \quad \text{for all } t \geq t_0.$$

It follows that $(ax^{\Delta^2})^{\Delta}(t)$, $x^{\Delta^2}(t)$ and $x^{\Delta}(t)$ are eventually monotonic and of one sign on $[t_0, \infty)$.

Next, we consider the following four cases for all large $t \geq t_0$:

$$\begin{aligned}
&(i_1) \quad \left(ax^{\Delta^2}\right)^{\Delta}(t) > 0, \quad x^{\Delta^2}(t) < 0 \quad \text{and} \quad x^{\Delta}(t) < 0;\\
&(i_2) \quad \left(ax^{\Delta^2}\right)^{\Delta}(t) < 0, \quad x^{\Delta^2}(t) < 0 \quad \text{and} \quad x^{\Delta}(t) < 0;\\
&(i_3) \quad \left(ax^{\Delta^2}\right)^{\Delta}(t) < 0, \quad x^{\Delta^2}(t) > 0 \quad \text{and} \quad x^{\Delta}(t) < 0;\\
&(i_4) \quad \left(ax^{\Delta^2}\right)^{\Delta}(t) < 0, \quad x^{\Delta^2}(t) > 0 \quad \text{and} \quad x^{\Delta}(t) > 0.
\end{aligned}$$

Then two cases (i_1) and (i_2) are not valid. In fact, if $x^{\Delta^2}(t) < 0$ and $x^{\Delta}(t) < 0$ eventually, then $\lim_{t\to\infty} x(t) = -\infty$, which contradicts the fact that $x(t) > 0$ eventually.

In the two cases (i_3) and (i_4), we see that $(ax^{\Delta^2})^{\Delta}(t) < 0$ and $(ax^{\Delta^2})^{\Delta^2}(t) < 0$ and hence one can easily find that $\lim_{t\to\infty} a(t)x^{\Delta^2}(t) = -\infty$, i.e., $x^{\Delta^2}(t) < 0$ for all $t \geq t_0$. Thus the two cases (i_3) and (i_4) are disregarded. This completes the proof. □

Lemma 2.3 *If x is an eventually positive solution of Eq.* (1.1), *then there exist positive constants c_1 and c_2 such that*

$$c_1\xi(t) \leq x(t) \leq c_2 t \quad \textit{for all sufficiently large } t \geq t_0, \tag{2.7}$$

where

$$\xi(t) = \int_t^\infty \int_s^\infty \frac{1}{a(u)} \Delta u \Delta s.$$

Proof Let x be an eventually positive solution. Let $t_1 \leq t_0$, integrating (1.1) twice from t_1 to t and dividing by $a(t)$, we get

$$\begin{aligned}
x^{\Delta^2}(t) &= a(t_1)x^{\Delta^2}(t_1)\frac{1}{a(t)} + \left(ax^{\Delta^2}\right)^{\Delta}(t_1)\frac{(t-t_1)}{a(t)}\\
&\quad - \frac{1}{a(t)}\int_{t_1}^t \int_{t_1}^\eta f\left(\zeta, x^\sigma(\zeta)\right)\Delta\zeta\,\Delta\eta\\
&= a(t_1)x^{\Delta^2}(t_1)\frac{1}{a(t)} + \left(ax^{\Delta^2}\right)^{\Delta}(t_1)\frac{(t-t_1)}{a(t)}\\
&\quad - \frac{1}{a(t)}\int_{t_1}^t \left(t-\sigma(\eta)\right) f\left(\zeta, x^\sigma(\zeta)\right)\Delta\zeta\,\Delta\eta
\end{aligned}$$

where we have applied Lemma 1 in [5] in the last step. Integrating the resulting equation twice from t_1 to t and using the same arguments, we get

$$x(t)=x(t_1)+x^{\Delta}(t_1)(t-t_1)+a(t_1)x^{\Delta^2}(t_1)\int_{t_1}^{t}\frac{(t-\sigma(\eta))}{a(\eta)}\Delta\eta$$
$$+\left(ax^{\Delta^2}\right)^{\Delta}(t_1)\int_{t_1}^{t}\frac{(t-\sigma(\eta))(\eta-t_1)}{a(\eta)}\Delta\eta$$
$$-\int_{t_1}^{t}\frac{(t-\sigma(\eta))}{a(\eta)}\int_{t_1}^{\eta}\left(\eta-\sigma(\zeta)\right)f\left(\zeta,x^{\sigma}(\zeta)\right)\Delta\zeta\,\Delta\eta \tag{2.8}$$

for all $t\geq t_1\geq t_0$. By using the Leibniz rule ([8], Theorem 1.117) and the L'Hôpital's ([8], Theorem 1.120), we have

$$\lim_{t\to\infty}\frac{\int_{t_1}^{t}\frac{(t-\sigma(\eta))(\eta-t_1)}{a(\eta)}\Delta\eta}{(t-t_1)}=\lim_{t\to\infty}\int_{t_1}^{t}\frac{(\eta-t_1)}{a(\eta)}\Delta\eta=\text{constant},$$

which shows the asymptotic relation

$$\int_{t_1}^{t}\frac{(t-\sigma(\eta))(\eta-t_1)}{a(\eta)}\Delta\eta\sim(t-t_1). \tag{2.9}$$

Now consider the discussion above in 4 different cases of Lemma 2.2 to prove the claim.

For the right-hand side, if (I) in Lemma 2.2 holds then

$$x(t)\leq x(t_1)+x^{\Delta}(t_1)(t-t_1)+a(t_1)x^{\Delta^2}(t_1)\int_{t_1}^{t}\frac{(t-\sigma(\eta))}{a(\eta)}\Delta\eta$$
$$+\left(ax^{\Delta^2}\right)^{\Delta}(t_1)\int_{t_1}^{t}\frac{(t-\sigma(\eta))(\eta-t_1)}{a(\eta)}\Delta\eta,$$

then the right-hand side of (2.7) trivially holds as the coefficients in the inequality above are all positive and (2.9) holds; if (II) in Lemma 2.2 holds then

$$x(t)\leq x(t_1)+x^{\Delta}(t_1)(t-t_1)+\left(ax^{\Delta^2}\right)^{\Delta}(t_1)\int_{t_1}^{t}\frac{(t-\sigma(\eta))(\eta-t_1)}{a(\eta)}\Delta\eta,$$

hence (2.7) holds from (2.9); if (III) in Lemma 2.2 holds then

$$x(t)\leq x(t_1)+x^{\Delta}(t_1)(t-t_1),$$

hence (2.7) holds; if (IV) in Lemma 2.2 holds then

$$x(t)\leq x(t_1)+a(t_1)x^{\Delta^2}(t_1)\int_{t_1}^{t}\frac{(t-\sigma(\eta))}{a(\eta)}\Delta\eta$$
$$+\left(ax^{\Delta^2}\right)^{\Delta}(t_1)\int_{t_1}^{t}\frac{(t-\sigma(\eta))(\eta-t_1)}{a(\eta)}\Delta\eta,$$

then the right-hand side of (2.7) trivially holds as the coefficients in the inequality above are all positive and (2.9) holds;

For the left-hand side, if (I) or (IV) in Lemma 2.2 holds, $(ax^{\Delta^2})(t)$ is positive and increasing and hence we conclude that $x(t)\geq\bar{c}\xi(t)$ for $t\geq t_1$ and $\bar{c}$ is a positive constant.

If (II) in Lemma 2.2 holds, $(ax^{\Delta^2})(t)$ is negative and increasing and hence we conclude that $x(t)\geq\bar{\bar{c}}\xi(t)$ for $t\geq t_1$ and $\bar{\bar{c}}$ is a positive constant.

If (III) in Lemma 2.2 holds, $(ax^{\Delta^2})(t)$ is negative and decreasing and hence we conclude that $x(t) \geq \tilde{c}\xi(t)$ for $t \geq t_1$ and $\tilde{c}$ is a positive constant. This completes the proof. □

We may note that the inequality parallel to those listed in Lemmas 2.2 and 2.3 hold for an eventually negative solution of Eq. (1.1).

Remark 2.1 Based on Lemma 2.3, all nonoscillatory solutions of Eq. (1.1) which are asymptotic to $c_1 t$, $c_1 \neq 0$ as $t \to \infty$ can be classified an maximal solutions and which are asymptotic to $c_2\xi(t)$, $c_2 \neq 0$ as $t \to \infty$ can be regarded as the minimal solutions.

Remark 2.2 Based on Lemma 2.3, all nonoscillatory solutions of Eq. (1.1) which are asymptotic to $c_1 t$, $c_1 \neq 0$ as $t \to \infty$ can be classified an maximal solutions and which are asymptotic to $c_2\xi(t)$, $c_2 \neq 0$ as $t \to \infty$ can be regarded as the minimal solutions.

Remark 2.3 From Lemma 1 in [5] $\xi(t)$ can be rewritten as

$$\xi(t) = \int_t^\infty \int_s^\infty \frac{1}{a(u)} \Delta u \Delta s = \int_t^\infty \int_t^{\sigma(u)} \frac{1}{a(u)} \Delta s \Delta u = \int_t^\infty \frac{\sigma(u) - t}{a(u)} \Delta u.$$

3 Main results

We shall investigate the oscillatory behavior of all solutions of Eq. (1.1) when it is either strongly superlinear or strongly sublinear. We begin with the strongly superlinear case of Eq. (1.1).

Theorem 3.1 *Suppose that Eq.* (1.1) *is strongly superlinear. If*

$$\int_{t_0}^\infty \sigma(s) \big| f\big(s, c\xi^\sigma(s)\big)\big| \Delta s = \infty, \quad \textit{for all } c \neq 0, \tag{3.1}$$

then Eq. (1.1) *is oscillatory.*

Proof Let x be a nonoscillatory solution of Eq. (1.1). Suppose that $x(t) > 0$ for $t \geq t_0 \in \mathbb{T}$, where t_0 is chosen so large that the four cases of Lemma 2.2 and (2.7) in Lemma 2.3 hold with $\xi(t) < 1$ for all $t \geq t_0$.

Next, we distinguish the following four cases:

Case (I). In this case, we see that $x^\Delta(t) > 0$ and $x^{\Delta^2}(t) > 0$ for $t \geq t_0$, and in view of (2.7), there exists a constant $b > 0$ such that

$$x^\sigma(t) \leq b\sigma(t) \quad \text{and} \quad x^\sigma(t) \geq b\xi^\sigma(t) \quad \text{for } t \geq t_0. \tag{3.2}$$

Integrating Eq. (1.1) from t_0 to t and using the fact that $(ax^{\Delta^2})^\Delta(t) > 0$ for $t \geq t_0$, we have

$$\big(ax^{\Delta^2}\big)^\Delta(t) + \int_{t_0}^t f\big(s, x^\sigma(s)\big)\Delta s = \big(ax^{\Delta^2}\big)^\Delta(t_0) < \infty.$$

Thus,

$$\int_{t_0}^{\infty} f(s, x^{\sigma}(s)\Delta s < \infty. \tag{3.3}$$

Using (3.2) and the superlinearity of f in (3.3), we get

$$\begin{aligned}\infty > \int_{t_0}^{\infty} \left(\frac{f(s, x^{\sigma}(s))}{(x^{\sigma}(s))^{\beta}}\right)\left(x^{\sigma}(s)\right)^{\beta}\Delta s \geq \int_{t_0}^{\infty}\left(\frac{\sigma(s)}{\xi^{\sigma}(s)}\right)^{\beta} f\left(s, b\xi^{\sigma}(s)\right)\Delta s \\ > \int_{t_0}^{\infty} \sigma(s) f\left(s, b\xi^{\sigma}(s)\right)\Delta s, \end{aligned}\tag{3.4}$$

which contradicts condition (3.1).

Case (II). In this case, we have $(ax^{\Delta^2})^{\Delta}(t) > 0$, $x^{\Delta^2}(t) < 0$ and $x^{\Delta}(t) > 0$ for $t \geq t_0$. By (2.7), there exists a constant $\alpha > 0$ such that

$$x^{\sigma}(t) \geq \alpha \quad \text{and} \quad x^{\sigma}(t) \geq \alpha\xi^{\sigma}(t) \quad \text{for } t \geq t_0. \tag{3.5}$$

We multiply Eq. (1.1) by $\sigma(t)$ and integrate from t_0 to t we obtain

$$\begin{aligned}\int_{t_0}^{t} \sigma(s) f\left(s, x^{\sigma}(s)\right)\Delta s &= -t\left(ax^{\Delta^2}\right)^{\Delta}(t) + t_0\left(ax^{\Delta^2}\right)^{\Delta}(t_0) + \int_{t_0}^{t}\left(ax^{\Delta^2}\right)^{\Delta}\Delta s \\ &\leq t_0\left(ax^{\Delta^2}\right)^{\Delta}(t_0) + \int_{t_0}^{t}\left(ax^{\Delta^2}\right)^{\Delta}(s)\Delta s \\ &= t_0\left(ax^{\Delta^2}\right)^{\Delta}(t_0) + ax^{\Delta^2}(t) - ax^{\Delta^2}(t_0) \\ &\leq t_0\left(ax^{\Delta^2}\right)^{\Delta}(t_0) - ax^{\Delta^2}(t_0) < \infty.\end{aligned}$$

Thus, we find

$$\int_{t_0}^{\infty} \sigma(s) f\left(s, x^{\sigma}(s)\right)\Delta s < \infty. \tag{3.6}$$

From (3.5) and (3.6), we obtain (3.4) which contradicts (3.1).

Case (III). In this case (3.5) holds and since $(ax^{\Delta^2})^{\Delta}(t) < 0$ and is decreasing on $[t_0, \infty)$, we have

$$\left(ax^{\Delta^2}\right)^{\Delta}(u) \leq \left(ax^{\Delta^2}\right)^{\Delta}(t) \quad \text{for } u \geq t \geq t_0.$$

Integrating this inequality from t to $s \geq t$ we get

$$\left(ax^{\Delta^2}\right)(s) \leq (s-t)\left(ax^{\Delta^2}\right)^{\Delta}(t) \leq s\left(ax^{\Delta^2}\right)^{\Delta}(t),$$

or

$$x^{\Delta^2}(s) \leq \left(\frac{s}{a(s)}\right)\left(ax^{\Delta^2}\right)^{\Delta}(t) \quad \text{for } s \geq t \geq t_0. \tag{3.7}$$

Integrating (3.7) from t to $\upsilon \geq s \geq t \geq t_0$ and letting $\upsilon \to \infty$, we get

$$x^{\Delta}(t) \geq \left(\int_{t}^{\infty}\left(\frac{s}{a(s)}\right)\Delta s\right) y(t) \geq \xi(t) y(t) \quad \text{for } t \geq t_0, \tag{3.8}$$

where $y(t) = -(ax^{\Delta^2})^{\Delta}(t) > 0$ for $t \geq t_0$. Since $x^{\Delta}(t) > 0$ and $x^{\Delta^2}(t) < 0$ for $t \geq t_0$, there exists a constant $k > 0$ such that

$$x(t) \geq ktx^{\Delta}(t) \quad \text{for } t \geq t_0 \tag{3.9}$$

Using (3.9) in (3.8), we obtain

$$x(t) \geq kt\xi(t)y(t) \quad \text{for } t \geq t_0,$$

or

$$x^{\sigma}(t) \geq k\sigma(t)\xi^{\sigma}(t)y^{\sigma}(t) \quad \text{for } t \geq t_0. \tag{3.10}$$

Now, Eq. (1.1) takes the form

$$y^{\Delta}(t) = f\big(t, x^{\sigma}(t)\big) = \frac{f(t, x^{\sigma}(t))}{(x^{\sigma}(t))^{\beta}}\big(x^{\sigma}(t)\big)^{\beta}, \quad t \geq t_0. \tag{3.11}$$

Using (3.5), (3.10) and the superlinearity property in (3.11) we have

$$y^{\Delta}(t) \geq \bar{c}\sigma^{\beta}(t)f\big(t, d\xi^{\sigma}(t)\big)\big(y^{\sigma}(t)\big)^{\beta} \quad \text{for } t \geq t_0,$$

where $\bar{c} = (\frac{k}{d})^{\beta}$. Thus, we can easily see that

$$\frac{y^{\Delta}(t)}{(y^{\sigma}(t))^{\beta}} \geq \bar{c}\sigma(t)f\big(t, d\xi^{\sigma}(t)\big) \quad \text{for } t \geq t_0. \tag{3.12}$$

By Lemma 2.1, we obtain

$$\frac{(y^{1-\beta}(t))^{\Delta}}{\beta - 1} \geq \frac{y^{\Delta}(t)}{(y^{\sigma}(t))^{\beta}} \geq \bar{c}\sigma(t)f\big(t, d\xi^{\sigma}(t)\big) \quad \text{for } t \geq t_0.$$

Integrating this inequality from t_0 to t we get

$$\infty > y^{1-\beta}(t_0) \geq y^{1-\beta}(t) + \bar{c}(\beta - 1)\int_{t_0}^{t} \sigma(s)f\big(s, d\xi^{\sigma}(s)\big)\Delta s,$$

which contradicts condition (3.1).

Case (IV). In this case, $a(t)x^{\Delta^2}(t) > 0$ and is increasing on $[t_0, \infty)$. Thus

$$a(s)x^{\Delta^2}(s) \geq a(t)x^{\Delta^2}(t) \quad \text{for } s \geq t \geq t_0.$$

Integrating this inequality twice from t to $u \geq t \geq t_0$ and letting $u \to \infty$, one can easily find

$$x(t) \geq \xi(t)\big(ax^{\Delta^2}\big)(t) \quad \text{for } t \geq t_0. \tag{3.13}$$

From the fact that $(ax^{\Delta^2})^{\Delta^2}(t) < 0$, $(ax^{\Delta^2})^{\Delta}(t) > 0$ and $ax^{\Delta^2}(t) > 0$ for $t \geq t_0$, there exists a constant $b_1 > 0$ such that

$$\begin{aligned}\big(ax^{\Delta^2}\big)(t) &\geq b_1 t\big(ax^{\Delta^2}\big)^{\Delta}(t) \quad \text{for } t \geq t_0 \\ &:= b_1 Z(t),\end{aligned} \tag{3.14}$$

where $Z(t) = (ax^{\Delta^2})^{\Delta}(t) > 0$ on $[t_0, \infty)$. Using (3.14) in (3.13), we get

$$x^{\sigma}(t) \geq b_1\sigma(t)\xi^{\sigma}(t)Z^{\sigma}(t) \quad \text{for } t \geq t_0. \tag{3.15}$$

Integrating Eq. (1.1) from t to $u \geq t \geq t_0$ and letting $u \to \infty$, we have

$$Z(t) = \left(ax^{\Delta^2}\right)^{\Delta}(t) \geq \int_t^{\infty} f\left(s, x^{\sigma}(s)\right)\Delta s, \quad t \geq t_0. \tag{3.16}$$

Also, (2.5) holds and so there exists a constant $d_1 > 0$ such that

$$x^{\sigma}(t) \geq d_1 \xi^{\sigma}(t) \quad \text{for } t \geq t_0. \tag{3.17}$$

Now, using (3.15), (3.17) and superlinearity of f in (3.16) we get

$$\begin{aligned} Z(t) &\geq \int_t^{\infty} \frac{f(s, x^{\sigma}(s))}{(x^{\sigma}(s))^{\beta}} \left(x^{\sigma}(s)\right)^{\beta} \Delta s \geq \left(\frac{b_1}{d_1}\right)^{\beta} \int_t^{\infty} \sigma^{\beta}(s) f\left(s, d_1 \xi^{\sigma}(s)\right) Z^{\sigma}(s)\Delta s \\ &\geq \left(\frac{b_1}{d_1}\right)^{\beta} Z^{\beta}(t) \int_t^{\infty} \sigma(s) f\left(s, d_1 \xi^{\sigma}(s)\right)\Delta s. \end{aligned}$$

Thus

$$Z^{1-\beta}(t) \geq \left(\frac{b_1}{d_1}\right)^{\beta} \int_t^{\infty} \sigma(s) f\left(s, d_1 \xi^{\sigma}(s)\right)\Delta s \quad \text{for } t \geq t_0.$$

Letting $t = t_0$ in the above inequality, we obtain a contradiction to condition (3.1).

A similar argument holds if we assume that x is an eventually negative solution of Eq. (1.1). This completes the proof. □

Next, we present a sufficient condition for the oscillation of strongly sublinear equation (1.1).

Theorem 3.2 *Suppose that Eq.* (1.1) *is strongly sublinear. If*

$$\int_{t_0}^{\infty} \left(\frac{s}{\sigma(s)}\right)^{\gamma} \xi(s) \left| f\left(s, c\sigma(s)\right)\right| \Delta s = \infty \quad \textit{for all } c \neq 0, \tag{3.18}$$

then Eq. (1.1) *is oscillatory.*

Proof Let x be a nonoscillatory solution of Eq. (1.1) and assume that $x(t) > 0$ for $t \geq t_0 \in \mathbb{T}$, where to is chosen so large that the four cases of Lemma 2.2 and (2.7) in Lemma 2.3 hold with $\xi(t) < 1$ for all $t \geq t_0$.

Next, we consider the following four cases:

Case (I). There exist positive constants k_1 and k_2 such that

$$k_1 \sigma(t) \leq x^{\sigma}(t) \leq k_2 \sigma(t) \quad \text{for all } t \geq t_0. \tag{3.19}$$

From (3.3) and (3.19) and using the strong sublinearity of f we get

$$\begin{aligned} \infty > \int_{t_0}^{\infty} f\left(s, x(s)\right)\Delta s &= \int_{t_0}^{\infty} \frac{f(s, x^{\sigma}(s))}{(x^{\sigma}(s))^{\gamma}} \left(x^{\sigma}(s)\right)^{\gamma} \Delta s \\ &\geq \int_{t_0}^{\infty} \frac{f(s, k_2\sigma(s))}{(k_2\sigma(s))^{\gamma}} \left(k_1\sigma(s)\right)^{\gamma} \Delta s \\ &\geq \left(\frac{k_1}{k_2}\right)^{\gamma} \int_{t_0}^{\infty} \xi(s) f\left(s, k_2\sigma(s)\right)\Delta s, \end{aligned}$$

which contradicts (3.18).

Case (II). There exist positive constants b_1 and b_2 such that

$$b_1 \le x^\sigma(t) \le b_2\sigma(t) \quad \text{for } t \ge t_0. \tag{3.20}$$

From (3.6), (3.20) and strong sublinearity of f, we obtain

$$\begin{aligned}\infty > \int_{t_0}^\infty \sigma(s) f\big(s, x(s)\big)\Delta s &= \int_{t_0}^\infty \sigma(s)\frac{f(s, x^\sigma(s))}{(x^\sigma(s))^\gamma}\big(x^\sigma(s)\big)^\gamma \Delta s \\ &\ge \left(\frac{b_1}{b_2}\right)^\gamma \int_{t_0}^\infty s^{1-\gamma}\left(\frac{s}{\sigma(s)}\right)^\gamma f\big(s, b_2\sigma(s)\big)\Delta s \\ &\ge c\left(\frac{b_1}{b_2}\right)^\gamma \int_{t_0}^\infty \left(\frac{s}{\sigma(s)}\right)^\gamma \xi(s) f\big(s, b_2\sigma(s)\big)\Delta s,\end{aligned}$$

where c is a positive constant. This contradicts condition (3.18).

Next we consider:

Case (III). Integrating Eq. (1.1) from u to $t \ge u \ge t_0$ twice, we get

$$\begin{aligned}-x^{\Delta^2}(t) &\ge \int_u^t \int_u^s \frac{1}{a(t)} f\big(\tau, x^\sigma(\tau)\big)\Delta\tau\,\Delta s \\ &= \int_u^t \left(\frac{t-\sigma(\tau)}{a(t)}\right) f\big(\tau, x^\sigma(\tau)\big)\Delta\tau.\end{aligned}$$

Integrating this inequality from u to $t \ge u \ge t_0$ we find

$$x^\Delta(u) \ge \int_u^t \left(\int_u^s \left(\frac{s-\sigma(\tau)}{a(s)}\right)\Delta s\right) f\big(\tau, x^\sigma(\tau)\big)\Delta\tau.$$

There exists a constant $c_1 > 0$ such that

$$x^\Delta(t) \ge \int_t^\infty c_1\xi(\tau) f\big(\tau, x^\sigma(\tau)\big)\Delta\tau. \tag{3.21}$$

Since $x^\Delta(t) > 0$ and $x^{\Delta^2}(t) < 0$ for $t \ge t_0$, there exists a constant $c_2 > 0$ such that

$$x(t) \ge c_2 t x^\Delta(t) \quad \text{for } t \ge t_0. \tag{3.22}$$

Using (3.21) in (3.22) we have

$$x(t) \ge ct\int_t^\infty \xi(s) f\big(s, x^\sigma(s)\big)\Delta s,$$

where $c = c_1c_2$. Next, we let

$$y(t) = \int_t^\infty \xi(s) f\big(s, x^\sigma(s)\big)\Delta s.$$

Then

$$-y^\Delta(t) = \xi(t) f\big(t, x^\sigma(t)\big) = \xi(t)\frac{f(t, x^\sigma(t))}{(x^\sigma(t))^\gamma}\big(x^\sigma(t)\big)^\gamma \quad \text{for } t \ge t_0. \tag{3.23}$$

In this case, there exists a constant $b > 0$ such that

$$x(t) \le bt \quad \text{for } t \ge t_0. \tag{3.24}$$

Thus,

$$-y^{\Delta}(t) \geq \xi(t)\frac{f(t, b\sigma(t))}{(b\sigma(t))^{\gamma}}\big(c\sigma(t)y(t)\big)^{\gamma}$$

or

$$-\frac{y^{\Delta}(t)}{y^{\gamma}(t)} \geq \left(\frac{c}{b}\right)^{\gamma}\xi(t)f\big(t, b\sigma(t)\big) \quad \text{for } t \geq t_0.$$

Integrating this inequality from t_0 to t and using Lemma 2.1 we get

$$\int_{t_0}^{t}\frac{-(y^{1-\gamma}(s))^{\Delta}}{1-\gamma}\Delta s \geq -\int_{t_0}^{t}\frac{y^{\Delta}(s)}{y^{\gamma}(s)}\Delta s \geq \left(\frac{c}{b}\right)^{\gamma}\int_{t_0}^{t}\xi(s)f\big(s, b\sigma(s)\big)\Delta s,$$

or

$$\infty > \frac{y^{1-\gamma}(t_0)}{1-\gamma} \geq \left(\frac{c}{b}\right)^{\gamma}\int_{t_0}^{t}\xi(s)f\big(s, b\sigma(s)\big)\Delta s,$$

which contradicts condition (3.18).

Case (IV). As in the proof of Theorem 3.1, Case (III), we obtain (3.13) and (3.14) for $t \geq t_0$. Since $(ax^{\Delta^2})(t)$ is increasing, we see that

$$x^{\sigma}(t) \geq \xi^{\sigma}(t)\big((ax^{\Delta^2})(t)\big)^{\sigma} \geq \xi^{\sigma}(t)\big(ax^{\Delta^2}\big)(t) \quad \text{for } t \geq t_0. \tag{3.25}$$

Using (3.14) in (3.25), we get

$$x^{\sigma}(t) \geq b_1 t\xi^{\sigma}(t)\big(ax^{\Delta^2}\big)^{\Delta}(t) \quad \text{for } t \geq t_0. \tag{3.26}$$

Using (3.24) and the strong sublinearity of f in Eq. (1.1), we get

$$-Z^{\Delta}(t) = f\big(t, x^{\sigma}(t)\big) = \frac{f(t, x^{\sigma}(t))}{(x^{\sigma}(t))^{\gamma}}\big(x^{\sigma}(t)\big)^{\gamma} \geq \frac{f(t, c\sigma(t))}{(c\sigma(t))^{\gamma}}\big(b_1 t\xi^{\sigma}(t)Z(t)\big)^{\gamma},$$

where $Z(t) = (ax^{\Delta^2})^{\Delta}(t)$ for $t \geq t_0$. Thus

$$-\frac{Z^{\Delta}(t)}{Z^{\gamma}(t)} \geq \left(\frac{b_1}{c}\right)^{\gamma}\left(\frac{t}{\sigma(t)}\right)\xi^{\sigma}(t)f\big(t, c\sigma(t)\big) \quad \text{for } t \geq t_0.$$

The rest of the proof is similar to that of Case (III) and hence is omitted.

A similar argument holds if we assume that x is an eventually negative solution of Eq. (1.1). This completes the proof. □

Remarks

1. The results of this paper are presented in a form which is essentially new even for the special cases of Eq. (1.1) when $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$, i.e., for the corresponding differential equation

$$\big(a(t)x''(t)\big)'' + f\big(t, x(t)\big) = 0 \tag{3.27}$$

and the difference equation

$$\Delta^2\big(a(t)\Delta^2 x(t)\big) + f\big(t, x(t+1)\big) = 0. \tag{3.28}$$

For Eq. (3.27), we give the following oscillation result.

Theorem 3.3 *Assume*

$$\int_{t_0}^{\infty} \frac{\sigma(s)}{a(s)} ds < \infty.$$

(i) *If Eq.* (3.27) *is strongly superlinear and*

$$\int_{t_0}^{\infty} \sigma(s) \left| f\left(s, c\xi(s)\right) \right| ds = \infty \quad \text{for all } c \neq 0,$$

where

$$\xi(t) = \int_t^{\infty} \int_s^{\infty} \frac{1}{a(u)} du ds,$$

then Eq. (3.27) *is oscillatory.*

(ii) *If Eq.* (3.27) *is strongly sublinear and*

$$\int_{t_0}^{\infty} \xi(s) \left| f(s, cs) \right| ds = \infty \quad \text{for all constants } c \neq 0,$$

then Eq. (3.27) *is oscillatory.*

Also for Eq. (3.28) we obtain the following new oscillation result:

Theorem 3.4 *Suppose*

$$\sum_{s=t_0}^{\infty} \frac{\sigma(s)}{a(s)} < \infty.$$

(i) *If Eq.* (3.28) *is strongly superlinear and*

$$\sum_{s=t_0}^{\infty} (s+1) \left| f\left(s, c\xi(s+1)\right) \right| = \infty \quad \text{for every constant } c \neq 0,$$

where

$$\xi(t) = \sum_{s=t}^{\infty} \sum_{u=s}^{\infty} \frac{1}{a(u)},$$

then Eq. (3.28) *is oscillatory.*

(ii) *If Eq.* (3.28) *is strongly sublinear and*

$$\sum_{s=t_0}^{\infty} \left(\frac{s}{s+1} \right)^{\gamma} \xi(s) \left| f\left(s, c(s+1)\right) \right| = \infty \quad \text{for every constant } c \neq 0,$$

then Eq. (3.28) *is oscillatory.*

2. When either

$$\int_{t_0}^{\infty} \frac{s}{a(s)} \Delta s = \infty, \quad \text{or} \quad \int_{t_0}^{\infty} \frac{1}{a(s)} \Delta s = \infty$$

holds, we see from the proofs of our results that Cases (I) and (II) of Lemma 2.2 are the only cases to be considered and so easier oscillation criteria for Eq. (1.1) can be obtained. Here, we omit the details.

3. The results of this paper can be easily extended to delay dynamic equation of the form

$$\left(ax^{\Delta^2}\right)^{\Delta^2}(t) + f\left(t, x^{\sigma}\left(g(t)\right)\right) = 0,$$

where $g : \mathbb{T} \to \mathbb{T}$, $g(t) \leq t$ for $t \in \mathbb{T}$, g is non-decreasing and $\lim_{t\to\infty} g(t) = \infty$. The details are left to the reader.

4. We may employ other types of time scales, e.g., $\mathbb{T} = h\mathbb{Z}$ with $h > 0$, $\mathbb{T} = q^{\mathbb{N}_0}$ with $q > 1$, $\mathbb{T} = \mathbb{N}_0^2 \ldots$ etc., see [8, 9]. The details are left to the reader.

4 Examples

Example 4.1 Consider the differential equation

$$\left(t\sigma(t)x^{\Delta^2}\right)^{\Delta^2}(t) + \left(\sigma(t)x^{\sigma}(t)\right)^3 = 0, \quad t \geq 1. \tag{4.1}$$

Denote $a(t) = t\sigma(t)$, $f(t, x^{\sigma}(t)) = (\sigma(t)x^{\sigma}(t))^3$. It's easy to see that a is a positive real-valued rd-continuous function, $\int_1^{\infty} \frac{\sigma(s)}{a(s)} \Delta s = \int_1^{\infty} \frac{1}{t} \Delta t < \infty$, $\operatorname{sgn} f(t, x) = \operatorname{sgn} x$ and $f(t, x) \leq f(t, y)$, $x \leq y$, $t \geq 1$. Furthermore $f(t, x)$ is strongly superlinear with

$$\frac{|\sigma(t)x^{\sigma}(t)|^3}{|x^{\sigma}(t)|^2} \leq \frac{|\sigma(t)y^{\sigma}(t)|^3}{|y^{\sigma}(t)|^2} \quad \text{for } |x| \leq |y|,\ xy > 0,\ t > 1.$$

From Lemma 2.2 we get that

$$\xi(t) = \int_t^{\infty} \int_s^{\infty} \frac{1}{u\sigma(u)} \Delta u \Delta s = \int_t^{\infty} \frac{1}{s} \Delta s \geq \frac{1}{t}, \quad t > 1.$$

Then we obtain

$$\int_1^{\infty} \sigma(s)\left|f\left(s, c\xi^{\sigma}(s)\right)\right|\Delta s \geq \int_1^{\infty} |c|^3 \sigma(s) \Delta s = \infty,$$

for all $c \neq 0$, which shows that all conditions in Theorem 3.1 holds. So by Theorem 3.1 Eq. (4.1) is oscillatory.

Example 4.2 Consider the difference equations

$$\Delta^2\left(2^t \Delta^2 x_t\right) + e^t x_{t+1}^3 = 0 \tag{4.2}$$

here $f(t, x_{t+1}) = e^t x_{t+1}^3$, $a(t) = 2^n$. From Theorem 3.4 we get that

$$\xi(t) = \sum_{s=t}^{\infty} \sum_{u=s}^{\infty} \frac{1}{2^u} = \sum_{s=t}^{\infty} \frac{1}{2^{s-1}} = \frac{1}{2^{t-2}}.$$

Then we obtain

$$\sum_{s=1}^{\infty} (s+1)\left|f\left(s, c\xi^{\sigma}(s)\right)\right| = \sum_{s=1}^{\infty} \frac{c^3 e^s (s+1)}{2^{3s-6}} = \infty,$$

for all $c \neq 0$, which shows that all conditions in Theorem 3.4 holds. So by Theorem 3.4 Eq. (4.2) is oscillatory.

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