J Appl Math Comput (2012) 39:235–251
DOI 10.1007/s12190-011-0520-1

JAMC

COMPUTATIONAL MATHEMATICS

Existence and global exponential stability of pseudo almost periodic solution for SICNNs with mixed delays

Farouk Chérif

Received: 16 August 2011 / Published online: 5 November 2011

Abstract Shunting Inhibitory Artificial Neural Networks are biologically inspired networks in which the synaptic interactions are mediated via a nonlinear mechanism called shunting inhibition, which allows neurons to operate as adaptive nonlinear filters. This paper considers the problem of existence and exponential stability of the pseudo almost periodic solution for shunting inhibitory cellular neural networks with mixed delays. The Banach fixed point theorem and the variant of a certain integral inequality with explicit estimate are used to establish the results. The results of this paper are new and they complement previously known results.

Keywords Globally exponential stability · Pseudo almost periodic solution · Neural network

Mathematics Subject Classification (2000) 34C27 · 37B25 · 92C20

1 Introduction

Cellular neural networks, introduced by Chua and Yang [12, 13], have been extensively investigated due to their important applications in such fields as image processing and pattern recognition [5–9]. Later, Bouzerdoum and Pinter and Bouzerdoum [3, 4] have introduced a new class of CNNs, namely the shunting inhibitory CNNs (SICNNs). SICNNs have been extensively applied in psychophysics, speech, perception, robotic, adaptive pattern recognition, vision, and image processing. It is well known that studies on SICNNs not only involve a discussion of stability properties, but also involve many dynamic behaviors such as periodic oscillatory behavior and

F. Chérif (✉)
Department of Mathematics, SCANN, Military Academy, Avenue du maréchal Tito, 4000 Sousse, Tunisia
e-mail: faroukcheriff@yahoo.fr

almost periodic oscillatory properties. This is modeled using the notion of pseudo almost periodic functions which can be represented as an almost periodic process plus an ergodic component. Hence, there have been extensive results on the problem of the existence and stability of the equilibrium point, periodic and almost periodic solutions of SICNNs with constant time delays and time-varying delays in the literature. We refer the reader to [10, 20] and the references cited therein. However, there exist few results on the existence and exponential stability of the almost periodic solutions of SICNNs with continuously distributed delays. On the other hand, it is well known, that time delays are inevitable in the interactions between neurons. This means, that there exist time delays in the information processing of neurons due to various reasons. For instance, time delays can be caused by the finite switching speed of amplifier circuits in neural networks or deliberately introduced to achieve tasks of dealing with motion related problems, such as moving image processing, pattern recognition, robotics, etc. Consequently, time delays in the neural networks make the dynamic behaviors become more complex, and may destabilize the stable equilibria and admit almost periodic oscillation, pseudo almost periodic motion, bifurcation and chaos.

Few years ago, Chen and Cao [10] have investigated the existence of almost periodic solution of the following system of SICNNs:

$$\frac{dx_{ij}}{dt} = -a_{ij}x_{ij} - \sum_{C_{ij}^{kl}\in N_r(i,j)} C_{ij}^{kl} f\big(x_{kl}(t-\tau)\big)x_{ij}(t) + L_{ij}(t) \tag{1}$$

However, to the author's best knowledge, there is no published paper considering the pseudo almost periodic solutions for SICNNs neural networks with mixed delays. Motivated by the above discussion, we consider the following more general SICNNs:

$$\begin{aligned}\frac{dx_{ij}}{dt} = -a_{ij}x_{ij} &- \sum_{B_{ij}^{kl}\in N_r(i,j)} B_{ij}^{kl}(t) g\big(x_{kl}(t-\tau)\big)x_{ij}(t)\\ &- \sum_{C_{ij}^{kl}\in N_r(i,j)} C_{ij}^{kl}(t)\left(\int_0^{\infty} K_{ij}(u) f\big(x_{kl}(t-u)\big)du\right)x_{ij}(t) + L_{ij}(t)\end{aligned} \tag{2}$$

The main purpose of this paper is to obtain some sufficient conditions for the existence, uniqueness and exponentially stability of the pseudo almost periodic solution for system (2). By applying fixed point theorem and differential inequality techniques, we derive some new sufficient conditions ensuring the existence, uniqueness and exponential stability of the pseudo almost periodic solution in the convex set, which are new and they complement previously known results. Moreover, an example is provided to illustrate the effectiveness of the new results.

2 Almost periodic and pseudo almost periodic functions

In this section, we would like to recall some basic notations and results of almost periodicity [2, 11, 15] and pseudo almost periodicity [17–19] which shall come into play later on. $BC(\mathbb{R},\mathbb{R}^n)$ denotes the set of bounded continued functions from $\mathbb{R}$

to $\mathbb{R}^n$. Note that $(BC(\mathbb{R}, \mathbb{R}^n), \|.\|_\infty)$ is a Banach space where $\|.\|_\infty$ denotes the sup norm $\|f\|_\infty := \sup_{t\in\mathbb{R}} \|f(t)\|$.

Definition 1 Let $k \in \mathbb{N}$ and $f \in C(\mathbb{R}, \mathbb{R}^k)$. We say that f is *almost periodic (Bohr a.p.) or uniformly almost periodic (u.a.p.),* when the following property is satisfied:

$$\forall \varepsilon > 0, \quad \exists l_\epsilon > 0, \quad \forall \alpha \in \mathbb{R}, \quad \exists \delta \in [\alpha, \alpha + l_\varepsilon], \quad \big\| f(\cdot + \delta) - f(\cdot)\big\|_\infty \le \varepsilon.$$

We denote by $AP(\mathbb{R}, \mathbb{R}^k)$ the set of the Bohr a.p. functions from $\mathbb{R}$ to $\mathbb{R}^k$. It is well-known that the set $AP(\mathbb{R}, \mathbb{R}^k)$ is a Banach space with the supremum norm.

Besides, the concept of pseudo almost periodicity was introduced by Zhang [17–19] in the early nineties. It is a natural generalization of the classical almost periodicity. Define the class of functions $PAP_0(\mathbb{R}, \mathbb{R}^k)$ as follows

$$PAP_0(\mathbb{R}, X) = \left\{ f \in BC\big(\mathbb{R}, \mathbb{R}^k\big) / \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} \|f(t)\| dt = 0 \right\}.$$

Definition 2 A function $f \in BC(\mathbb{R}, \mathbb{R}^k)$ is called pseudo almost periodic if it can be expressed as $f = h + \varphi$, where $h \in AP(\mathbb{R}, \mathbb{R}^k)$ and $\varphi \in PAP_0(\mathbb{R}, \mathbb{R}^k)$. The collection of such functions will be denoted by $PAP(\mathbb{R}, \mathbb{R}^k)$.

Remark 1 The functions h and φ in the above definition are respectively called the almost periodic component and the ergodic perturbation of the pseudo almost periodic function f. Further, the decomposition given in the definition above is unique. Observe that $(PAP(\mathbb{R}, \mathbb{R}^k), \|\cdot\|_\infty)$ is a Banach space and

$$AP\big(\mathbb{R}, \mathbb{R}^k\big) \subsetneq PAP\big(\mathbb{R}, \mathbb{R}^k\big) \subset BC\big(\mathbb{R}, \mathbb{R}^k\big)$$

since the function $\phi(t) = \sin^2 t + \sin^2 \sqrt{5}t + \exp(-t^2 \cos^2 t)$ is pseudo almost periodic function but not almost periodic.

3 Description system and preliminaries

In this paper, we study a class of shunting inhibitory cellular neural networks with distributed time delay. The dynamics of a cell C_{ij} are described by the following equation

$$\begin{aligned} \frac{dx_{ij}}{dt} = {} & -a_{ij}x_{ij} - \sum_{B_{ij}^{kl} \in N_r(i,j)} B_{ij}^{kl}(t) g\big(x_{kl}(t-\tau)\big) x_{ij}(t) \\ & - \sum_{C_{ij}^{kl} \in N_r(i,j)} C_{ij}^{kl}(t) \left(\int_0^\infty K_{ij}(u) f\big(x_{kl}(t-u)\big) du \right) x_{ij}(t) + L_{ij}(t) \end{aligned} \tag{2}$$

$i = 1, 2, \dots, m;\ j = 1, 2, \dots, n$, where C_{ij} denote the cell at the (i, j) position of the lattice at time t, the r-neighborhood $N_r(i, j)$ of C_{ij} is determined by

$$N_r(i, j) = \big\{ C_{kl} : \max\big(|k - i|, |l - j|\big) \le r, 1 \le k \le m, 1 \le l \le n \big\}.$$

Here $x_{ij}(\cdot)$ represents the activity of the cell $C_{ij}(\cdot)$, $L_{ij}(\cdot)$ is the external input to $C_{ij}(\cdot)$, the constant $a_{ij} > 0$ is the passive decay rate of the cell activity, $B_{ij}^{kl}\ (\cdot)$ and $C_{ij}^{kl}\ (\cdot)$ are the connection or coupling strength of postsynaptic activity of the cell transmitted to the cell $C_{ij}\ (\cdot)$ depending upon discrete delays and distributed delays, respectively; the activation functions f and g are continuous representing the output or firing rate of the cell $C_{ij}(t)$; τ represents axonal signal transmission delay. Note that the functions $B_{ij}^{kl}\ (\cdot), C_{ij}^{kl}\ (\cdot), L_{ij}(\cdot), K_{ij}(\cdot)$ are all pseudo almost periodic. Throughout this paper, we set

$$x(t) = \{x_{ij}(t)\} = \begin{pmatrix} x_{11}(t) \\ \vdots \\ x_{1n}(t) \\ \vdots \\ x_{m1}(t) \\ \vdots \\ x_{mn}(t) \end{pmatrix}$$

For all $x(t) = \{x_{ij}(t)\} \in \mathbb{R}^{n\times m}$, we define the norm

$$\|x(t)\| = \max_{ij}\{|x_{ij}(t)|\}$$

We also consider the following conditions (H_1), (H_2), (H_3), (H_4), (H_5) and (H_6).

(H_1) $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies the Lipschitz condition, i.e. there exists positive scalar $L_f > 0$ such that for all $x, y \in \mathbb{R}$

$$|f(x) - f(y)| \leq L_f|x - y| \tag{3}$$

(H_2) $g : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies the Lipschitz condition, i.e. there exists positive scalar $L_g > 0$ such that for all $x, y \in \mathbb{R}$

$$|g(x) - g(y)| \leq Lg|x - y| \tag{4}$$

(H_3) For all $1 \leq i \leq m$ and $1 \leq j \leq n$, the functions $B_{ij}^{kl}\ (\cdot), C_{ij}^{kl}\ (\cdot), L_{ij}(\cdot), K_{ij}(\cdot)$ are pseudo almost periodic,

(H_4) there exist non-negative constants M_f, M_g, L, r, ρ such that

$$0 < M_f = \sup_{x\in\mathbb{R}}|f(x)|, \qquad 0 < Mg = \sup_{x\in\mathbb{R}}|g(x)|, \qquad L = \max_{i,j}\left\{\frac{L_{ij}}{a_{ij}}\right\},$$
$$R = \sup_{t\in\mathbb{R}}\max_{i,j}\left|\left(\frac{M_g \sum_{B_{ij}^{kl}\in N_r(i,j)} |B_{ij}^{kl}(t)| + M_f \sum_{C_{ij}^{kl}\in N_r(i,j)} |C_{ij}^{kl}(t)|}{a_{ij}}\right)\right|,$$
$$\varrho = \frac{RI}{1-R}$$

and for all $t \in \mathbb{R}$, for all $1 \leq i \leq m$; $1 \leq j \leq n$

$$a = \left(\frac{\sum_{B_{ij}^{kl}\in N_r(i,j)}(M_g + L_g\rho)|B_{ij}^{kl}(t)| + \sum_{C_{ij}^{kl}\in N_r(i,j)}(L_f\rho + M_f)|C_{ij}^{kl}(t)|}{a_{ij}}\right) < 1.$$

(H_5) For all $1 \leq i \leq m$ and $1 \leq j \leq n$, the delay kernels $K_{ij} : [0, +\infty) \longrightarrow \mathbb{R}$ are continuous, integrable and satisfying

$$\int_0^\infty K_{ij}(u) = 1, \qquad \int_0^\infty K_{ij}(u) e^{\mu u} du \prec +\infty,$$

where μ is a sufficiently non negative small constant.

(H_6) For every sufficiently small $t > 0$

$$a_{ij} - \left(\sum_{B_{ij}^{kl} \in N_r(i,j)} \left| B_{ij}^{kl}(t_{ij}) \right| M_g + L_g \rho + \sum_{C_{ij}^{kl} \in N_r(i,j)} \left| C_{ij}^{kl}(t_{ij}) \right| M_f + L_f \rho \right)$$

is non-negative.

Definition 3 [14] Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function, then $\frac{D^+ f(t)}{dt}$ is defined as

$$\frac{D^+ f(t)}{dt} = \limsup_{h \to 0^+} \frac{f(t+h) - f(t)}{h}$$

Remark 2 The upper-right Dini derivative $\frac{D^+ V|f(t)|}{dt}$ of $|f(t)|$ is given by

$$\frac{D^+ V|f(t)|}{dt} = sign\big(f(t)\big) \frac{dz(t)}{dt}$$

where $sign(\cdot)$ is the signum function.

4 Existence of the pseudo almost periodic solution

In order to prove the first main result of this paper, we shall demonstrate some lemmas.

Lemma 1 *If* $\varphi \in PAP(\mathbb{R}, \mathbb{R}^k)$, *then* $\varphi(\cdot - h) \in PAP(\mathbb{R}, \mathbb{R}^k)$ *for all* $h \in \mathbb{R}$.

Proof By Definition, we can write $\varphi = \varphi_1 + \varphi_2$, where $\varphi_1 \in AP(\mathbb{R}, \mathbb{R}^k)$ and $\varphi_2 \in PAP_0(\mathbb{R}, \mathbb{R}^k)$. Obviously,

$$\varphi(\cdot - h) = \varphi_1(\cdot - h) + \varphi_2(\cdot - h).$$

Observe that $\varphi_1(\cdot - h) \in AP(\mathbb{R}, \mathbb{R}^k)$ and

$$\begin{aligned} 0 \leq \frac{1}{2T} \int_{-T}^{T} \left\| \varphi_2(t-h) \right\| dt &= \frac{1}{2T} \int_{-T-h}^{T-h} \left\| \varphi_2(t \right\| dt \\ &\leq \frac{2T + 2h}{2T(2T+2h)} \int_{-T-h}^{T+h} \left\| \varphi_2(t) \right\| dt, \end{aligned}$$

which implies that $\varphi_2(\cdot \; - h) \in PAP_0(\mathbb{R}, \mathbb{R}^k)$. So $\varphi(\cdot - h) \in PAP(\mathbb{R}, \mathbb{R}^k)$. □

Lemma 2 *Suppose that assumptions* (H_1)–(H_3) *hold. If* $\varphi \in PAP(\mathbb{R}, \mathbb{R})$, *then for all* i, j *the function* $\Psi_{ij} : t \longmapsto (\int_0^\infty K_{ij}(u)\varphi(t-u)du)$ *belongs to* $PAP(\mathbb{R}, \mathbb{R})$.

Proof By definition, we can write $\varphi = \varphi_1 + \varphi_2$, where $\varphi_1 \in AP(\mathbb{R}, \mathbb{R})$ and $\varphi_2 \in PAP_0(\mathbb{R}, \mathbb{R})$. Then

$$\begin{aligned}\Psi_{ij}(t) &= \int_0^\infty K_{ij}(u)\big(\varphi_1(t-u) + \varphi_2(t-u)\big)du \\ &= \int_0^\infty K_{ij}(u)\varphi_1(t-u)du + \int_0^\infty K_{ij}(u)\varphi_2(t-u)du \\ &= \Psi_{ij}^1(t) + \Psi_{ij}^2(t).\end{aligned}$$

First, let us prove that $\Psi_{ij}^1 \in AP(\mathbb{R}, \mathbb{R})$. For all $\varepsilon > 0$ one has

$$\exists l_\varepsilon > 0, \quad \forall \alpha \in \mathbb{R}, \quad \exists \delta \in [\alpha, \alpha + l_\varepsilon], \quad \big|\varphi_1(\cdot + \delta) - \varphi_1(\cdot)\big|_\infty \le \varepsilon,$$

and consequently

$$\begin{aligned}\big|\Psi_{ij}^1(t+\delta) - \Psi_{ij}^1(t)\big| &= \left|\int_0^\infty K_{ij}(u)\varphi_1(t+\delta-u)du - \int_0^\infty K_{ij}(u)\varphi_1(t-u)du\right| \\ &= \left|\int_0^\infty K_{ij}(u)\varphi_1(t+\delta-u)du - \int_0^\infty K_{ij}(u)\varphi_1(t-u)du\right| \\ &= \left|\int_0^\infty K_{ij}(u)\big[\varphi_1(t+\delta-u) - \varphi_1(t-u)\big]du\right| \\ &= \left|\int_{-\infty}^t K_{ij}(t-u)\big[\varphi_1(u+\delta) - \varphi_1(u)\big]du\right| \\ &\le \int_{-\infty}^t \big|K_{ij}(t-u)\big|\big|\varphi_1(u+\delta) - \varphi_1(u)\big|du \\ &\le \varepsilon,\end{aligned}$$

which implies that $\Psi_{ij}^1 \in AP(\mathbb{R}, \mathbb{R})$. Now we turn our attention to the function Ψ_{ij}^2. One has

$$\begin{aligned}&\lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^T \big|\Psi_{ij}^2(t)(t)\big|dt \\ &\quad= \lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^T \left|\int_0^\infty K_{ij}(u)\varphi_2(t-u)du\right|dt \\ &\quad\le \lim_{T\to+\infty} \frac{1}{2T}\int_0^\infty \big|K_{ij}(u)\big|\left(\int_{-T}^T \big|\varphi_2(t-u)\big|dt\right)du \\ &\quad= \lim_{T\to+\infty} \frac{1}{2T}\int_0^\infty \big|K_{ij}(u)\big|\left(\int_{-T-u}^{T-u} \big|\varphi_2(z)\big|dz\right)du \\ &\quad\le \lim_{T\to+\infty} \frac{1}{2T}\int_0^\infty \big|K_{ij}(u)\big|\left(\int_{-T-u}^{T+u} \big|\varphi_2(z)\big|dz\right)du \\ &\quad= \lim_{T\to+\infty} \int_0^\infty \big|K_{ij}(u)\big|\frac{T+u}{2T(T+u)}\left(\int_{-T-u}^{T+u} \big|\varphi_2(z)\big|dz\right)du \\ &\quad= 0,\end{aligned}$$

which implies that $\Psi_{ij}^2(t) \in PAP_0(\mathbb{R}, \mathbb{R})$. Consequently $\Psi_{ij}(t) \in PAP(\mathbb{R}, \mathbb{R})$. □

Lemma 3 *Suppose that assumptions* (H_1)–(H_4) *hold. Define the nonlinear operator* Γ *by*: *for each* $\varphi = (\varphi_{11}, \dots, \varphi_{1n}, \dots, \varphi_{m1}, \dots, \varphi_{mn})^t \in PAP(\mathbb{R}, \mathbb{R}^{n\times m})$

$$x_\varphi(t) = \begin{pmatrix} \int_{-\infty}^t e^{-a_{11}(t-s)} F_{11}(s, \varphi_{11}(s))ds \\ \vdots \\ \int_{-\infty}^t e^{-a_{1n}(t-s)} F_{1n}(s, \varphi_{1n}(s))ds \\ \vdots \\ \int_{-\infty}^t e^{-a_{m1}(t-s)} F_{m1}(s, \varphi_{m1}(s))ds \\ \vdots \\ \int_{-\infty}^t e^{-a_{nm}(t-s)} F_{nm}(s, \varphi_{nm}(s))ds \end{pmatrix}$$

where

$$F_{ij}\big(s, \varphi_{ij}(s)\big) = -\sum_{B_{ij}^{kl} \in N_r(i,j)} B_{ij}^{kl}(s) g\big(\varphi_{kl}(s-\tau)\big)\varphi_{ij}(s) - \sum_{C_{ij}^{kl} \in N_r(i,j)} C_{ij}^{kl}(s)\left(\int_0^\infty K_{ij}(u) f\big(\varphi_{kl}(s-u)\big)du\right)\varphi_{ij}(s) + L_{ij}(s). \tag{5}$$

Then Γ *maps* $PAP(\mathbb{R}, \mathbb{R}^{n\times m})$ *into itself.*

Proof Let $\varphi \in PAP(\mathbb{R}, \mathbb{R}^{n\times m})$. Immediately by Lemma 2 the function

$$t \longmapsto \left(\int_0^\infty K_{ij}(u)\varphi_{kl}(t-u)du\right)$$

belongs to $PAP(\mathbb{R}, \mathbb{R}^{n\times m})$. Next, by using (H_1), (H_2), (H_3) and the composition theorem of [1] it follows that the functions $\Phi_{ij} : s \longmapsto F_{ij}(s, \varphi_{ij}(s))$ belong to $PAP(\mathbb{R}, \mathbb{R})$. Consequently, one can write $\Phi_{ij} = \Phi_{ij}^1 + \Phi_{ij}^2$ where $\Phi_{ij}^1 \in AP(\mathbb{R}, \mathbb{R})$ and $\Phi_{ij}^2 \in PAP_0(\mathbb{R}, \mathbb{R})$. So, for all $1 \le i \le m$ and $1 \le j \le n$

$$(\Gamma_{ij}\varphi_{ij})(t) = \int_{-\infty}^t e^{-a_{ij}(t-s)}\Phi_{ij}^1(s)ds + \int_{-\infty}^t e^{-a_{ij}(t-s)}\Phi_{ij}^2(s)ds = \Theta_1(t) + \Theta_2(t).$$

Let us prove the almost periodicity of the function

$$\Theta_1 : t \longmapsto \int_{-\infty}^t e^{-a_{ij}(t-s)}\Phi_{ij}^1(s)ds.$$

For $\varepsilon > 0$, we consider in view of the almost periodicity of Φ_{ij}^1, a number L such that in any interval $[\alpha, \alpha + L[$ one finds a number δ, with property that

$$\sup_{t\in\mathbb{R}}\left|\Phi_{ij}^1(t+\delta) - \Phi_{ij}^1(t)\right| < \epsilon,$$

where for all $1 \le i \le m$ and $1 \le j \le n$. Afterwards, we can write

$$\begin{aligned}\left|\Theta_1(t+\delta)-\Theta_1(t)\right| &= \left|\int_{-\infty}^{t+\delta} e^{-a_{ij}(t-s)}\Phi_{ij}^1(s)ds - \int_{-\infty}^{t} e^{-a_{ij}(t-s)}\Phi_{ij}^1(s)ds\right| \\ &= \left|\int_{-\infty}^{t} e^{-a_{ij}(t-s)}\Phi_{ij}^1(s+\delta)ds - \int_{-\infty}^{t} e^{-a_{ij}(t-s)}\Phi_{ij}^1(s)ds\right| \\ &\le \int_{-\infty}^{t} e^{-a_{ij}(t-s)}\left|\Phi_{ij}^1(s+\delta)-\Phi_{ij}^1(s)\right|ds \\ &\le \frac{\varepsilon}{a_{ij}},\end{aligned}$$

which implies that $\Theta_1 \in AP(\mathbb{R},\mathbb{R})$. Now, we turn our attention to Θ_2. We have to prove that for all $1 \le i \le m$ and $1 \le j \le n$

$$\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\left|\int_{-\infty}^{t} e^{-a_{ij}(t-s)}\Phi_{ij}^2(s)ds\right|dt = 0.$$

Clearly,

$$\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\left|\int_{-\infty}^{t} e^{-a_{ij}(t-s)}\Phi_{ij}^2(s)ds\right|dt \le I_1 + I_2$$

where

$$I_1 = \lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\left(\int_{-T}^{t}\left|e^{-a_{ij}(t-s)}\Phi_{ij}^2(s)\right|ds\right)dt$$

and

$$I_2 = \lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}dt\left(\int_{-\infty}^{-T}\left|e^{-a_{ij}(t-s)}\Phi_{ij}^2(s)\right|ds\right).$$

Now, we shall prove that $I_1 = I_2 = 0$

$$\begin{aligned}\frac{1}{2T}\int_{-T}^{T}\left(\int_{-T}^{t}\left|e^{-(t-s)a_{ij}}\Phi_{ij}^2(s)\right|ds\right)dt &= \frac{1}{2T}\int_{-T}^{T}\left(\int_{-T}^{t} e^{-(t-s)a_{ij}}\left|\Phi_{ij}^2(s)\right|ds\right)dt \\ &= \frac{1}{2T}\int_{-T}^{T}\left(\int_{0}^{t+T} e^{-\xi a_{ij}}\left|\Phi_{ij}^2(t-\xi)\right|d\xi\right)dt \\ &\le \frac{1}{2T}\int_{-T}^{T}\left(\int_{0}^{+\infty} e^{-\xi a_{ij}}\left|\Phi_{ij}^2(t-\xi)\right|d\xi\right)dt \\ &= \int_{0}^{+\infty} e^{-\xi a_{ij}}\left(\frac{1}{2T}\int_{-T}^{T}\left|\Phi_{ij}^2(t-\xi)\right|dt\right)d\xi \\ &\le \int_{0}^{+\infty} e^{-\xi a_{ij}}\left(\frac{1}{2T}\int_{-T-\xi}^{T-\xi}\left|\Phi_{ij}^2(u)\right|du\right)d\xi \\ &\le \int_{0}^{+\infty} e^{-\xi a_{ij}}\left(\frac{1}{2T}\int_{-T-\xi}^{T+\xi}\left|\Phi_{ij}^2(u)\right|du\right)d\xi\end{aligned}$$

Since the function $\Phi_{ij}^2(\cdot) \in PAP_0(\mathbb{R},\mathbb{R})$ then the function Θ_T defined by

$$\Theta_T(\xi) = \frac{T+\xi}{T}\frac{1}{2(T+\xi)}\int_{-T-\xi}^{T+\xi}\left|\Phi_{ij}^2(u)\right|du$$

is bounded and satisfy $\lim_{T\to+\infty}\Theta_T(\xi)=0$. Consequently, by the Lebesgue dominated convergence theorem, we obtain

$$I_1=\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\left(\int_{-T}^{t}\left|e^{-(t-s)a_{ij}}\Phi_{ij}^2(s)\right|ds\right)dt=0.$$

On the other hand, notice that $|\Phi_{ij}^2|_\infty=\sup_{t\in\mathbb{R}}|\Phi_{ij}^2(t)|<\infty$ then

$$\begin{aligned}
I_2&=\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\left(\int_{-\infty}^{-T}\left|e^{-(t-s)a_{ij}}\Phi_{ij}^2(s)\right|ds\right)dt\\
&=\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\left(\int_{-\infty}^{-T}e^{-(t-s)a_{ij}}\left|\Phi_{ij}^2(s)\right|ds\right)dt\\
&\le\lim_{T\to+\infty}\frac{\sup_{t\in\mathbb{R}}|\Phi_{ij}^2(t)|}{2T}\int_{-T}^{T}\left(\int_{t+T}^{+\infty}e^{-\xi a_{ij}}d\xi\right)dt\\
&\le\lim_{T\to+\infty}\frac{\sup_{t\in\mathbb{R}}|\Phi_{ij}^2(t)|}{2T}\int_{-T}^{T}\left(\int_{2T}^{+\infty}e^{-\xi a_{ij}}d\xi\right)dt\\
&=\lim_{T\to+\infty}\frac{\sup_{t\in\mathbb{R}}|\Phi_{ij}^2(t)|}{a_{ij}}e^{-2a_{ij}T}=0
\end{aligned}$$

Consequently, the function Θ_2 belongs to $PAP_0(\mathbb{R},\mathbb{R})$. So for all $1\le i\le m$ and $1\le j\le n$, $(\Gamma_{ij}\varphi_{ij})$ belongs to $PAP(\mathbb{R},\mathbb{R})$ and consequently $(\Gamma\varphi)$ belongs to $PAP(\mathbb{R},\mathbb{R}^{n\times m})$. □

Theorem 1 *Suppose that assumptions* (H_1)–(H_4) *hold. Then* (2) *has a unique pseudo almost periodic solution in the convex set*

$$\mathcal{B}=B(\varphi_0,\rho)=\left\{\varphi\in PAP\big(\mathbb{R},\mathbb{R}^{n\times m}\big),\ \|\varphi-\varphi_0\|\le\frac{RL}{1-R}\right\}\tag{6}$$

where

$$\varphi_0(t)=\begin{pmatrix}\int_{-\infty}^{t}e^{-a_{11}(t-s)}L_{11}(s)ds\\ \vdots\\ \int_{-\infty}^{t}e^{-a_{1n}(t-s)}L_{1n}(s)ds\\ \vdots\\ \int_{-\infty}^{t}e^{-a_{m1}(t-s)}L_{m1}(s)ds\\ \vdots\\ \int_{-\infty}^{t}e^{-a_{nm}(t-s)}L_{nm}(s)ds\end{pmatrix}$$

Proof Set $\mathcal{B}=B(\varphi_0,\rho)=\{\varphi\in PAP(\mathbb{R},\mathbb{R}^{n\times m}),\ \|\varphi-\varphi_0\|\le\frac{RI}{1-R}\}$. Clearly, $\mathcal{B}$ is a closed convex subset of $PAP(\mathbb{R},\mathbb{R}^{n\times m})$ and

$$\left\|\varphi_0(t)\right\|=\sup_{t\in\mathbb{R}}\max_{i,j}\left\{\int_{-\infty}^{t}e^{-(t-s)a_{ij}}L_{ij}(s)ds\right\}\le\sup_{t\in\mathbb{R}}\max_{i,j}\left\{\frac{L_{ij}}{a_{ij}}\right\}=\max_{i,j}\left\{\frac{L_{ij}}{a_{ij}}\right\}=L.$$

Therefore, for any $\varphi \in \mathcal{B}$ by using the estimate just obtained, we see that

$$\|\varphi\| \leq \|\varphi - \varphi_0\| + \|\varphi_0\| \leq \frac{RL}{1-r} + L = \frac{L}{1-r}$$

Let us prove that the operator Γ is a self-mapping from $\mathcal{B}$ to $\mathcal{B}$. For short, we denote in the rest of the proof $N_r(i, j) = N_r$. Let us consider $\varphi \in \mathcal{B}$ then we have

$$\begin{aligned}
\big\|(\Gamma\varphi)(t) - \varphi_0(t)\big\| &= \sup_{t\in\mathbb{R}} \max_{i,j} \bigg| \int_{-\infty}^{t} e^{-a_{ij}(t-s)} \bigg(\sum_{B_{ij}^{kl}\in N_r} B_{ij}^{kl}(s) g\big(\varphi_{kl}(s-\tau)\big)\varphi_{ij}(s) \\
&\quad + \sum_{C_{ij}^{kl}\in N_r} C_{ij}^{kl}(s) \bigg(\int_0^{\infty} K_{ij}(s) f\big(\varphi_{kl}(s-u)\big) du \bigg) \varphi_{ij}(s) ds \bigg) \bigg| \\
&\leq \sup_{t\in\mathbb{R}} \max_{i,j} \bigg\{ \bigg| \int_{-\infty}^{t} e^{-a_{ij}(t-s)} \bigg(M_g \sum_{B_{ij}^{kl}\in N_r} \big|B_{ij}^{kl}(s)\big| \\
&\quad + M_f \sum_{C_{ij}^{kl}\in N_r} \big|C_{ij}^{kl}(s)\big| ds \bigg) \bigg| \bigg\} \|\varphi\| \\
&\leq \sup_{t\in\mathbb{R}} \max_{i,j} \left\{ \frac{M_g \sum_{B_{ij}^{kl}\in N_r} |B_{ij}^{kl}(t)| + M_f \sum_{C_{ij}^{kl}\in N_r} |C_{ij}^{kl}(t)|}{a_{ij}} \right\} \|\varphi\| \\
&\leq \frac{RL}{1-L},
\end{aligned}$$

which implies that $(\Gamma\varphi) \in \mathcal{B}$. Next, we shall prove that the operator Γ is a contraction mapping of $\mathcal{B}$. In fact, in view of (H_2), (H_3) and (H_4) for any $\varphi, \psi \in \mathcal{B}$, we have

$$\begin{aligned}
&\big\|(\Gamma\varphi) - (\Gamma\psi)\big\| \\
&= \sup_{t\in\mathbb{R}} \max_{i,j} \bigg\{ \bigg| \int_{-\infty}^{t} e^{-(t-s)a_{ij}} \bigg(\sum_{B_{ij}^{kl}\in N_r} B_{ij}^{kl}(s) g\big(\varphi_{kl}(s-\tau)\big)\varphi_{ij}(s) \\
&\quad - \sum_{B_{ij}^{kl}\in N_r} B_{ij}^{kl}(s) g\big(\varphi_{kl}(s-\tau)\big)\psi_{ij}(s) \\
&\quad + \sum_{B_{ij}^{kl}\in N_r} B_{ij}^{kl}(t) \big[g\big(\varphi_{kl}(s-\tau)\big) - g\big(\psi_{kl}(s-\tau)\big) \big] \psi_{ij}(s) \\
&\quad + \sum_{C_{ij}^{kl}\in N_r} C_{ij}^{kl}(s) \bigg(\int_0^{\infty} K_{ij}(u) f\big(\varphi_{kl}(s-u)\big) du \bigg) \varphi_{ij}(s) ds \\
&\quad - \sum_{C_{ij}^{kl}\in N_r} C_{ij}^{kl}(t) \bigg(\int_0^{\infty} K_{ij}(u) f\big(\psi_{kl}(t-u)\big) du \bigg) \psi_{ij}(t) \bigg) ds \bigg| \bigg\} \\
&\leq \sup_{t\in\mathbb{R}} \max_{i,j} \bigg\{ \int_{-\infty}^{t} e^{-(t-s)a_{ij}} M_g \bigg(\sum_{B_{ij}^{kl}\in N_r} \big|B_{ij}^{kl}(s)\big| \big|\varphi_{ij}(s) - \psi_{ij}(s)\big| \bigg)
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{B_{ij}^{kl}\in N_r} \left|B_{ij}^{kl}(s)\right|g\big(\varphi_{kl}(s-\tau)\big)-g\big(\psi_{kl}(s-\tau)\big)\big|\big|\psi_{ij}(s)\big| \\
&+ \sum_{C_{ij}^{kl}\in N_r} \left|C_{ij}^{kl}(s)\right|\left|\varphi_{ij}(s)\right| \\
&\times \left(\int_0^{\infty}\left|K_{ij}(u)\right|\left|f\big(\varphi_{kl}(s-u)\big)-f\big(\psi_{kl}(s-u)\big)\right|du\right) \\
&+ \sum_{C_{ij}^{kl}\in N_r} \left|C_{ij}^{kl}(s)\right|\left|\varphi_{ij}(s)-\psi_{ij}(s)\right|\int_0^{\infty}K_{ij}(u)f\big(\psi_{kl}(s-u)\big)duds\Bigg\} \\
\leq &\sup_{t\in\mathbb{R}}\max_{i,j}\Bigg\{\int_{-\infty}^{t}e^{-(t-s)a_{ij}}\Bigg(M_g\sum_{B_{ij}^{kl}\in N_r}\left|B_{ij}^{kl}(s)\right|\left|\varphi_{ij}(s)-\psi_{ij}(s)\right| \\
&+ L_g\sum_{B_{ij}^{kl}\in N_r(i,j)}\left|B_{ij}^{kl}(s)\right|\left|\varphi_{kl}(s-\tau)-\psi_{kl}(s-\tau)\right|\left|\psi_{ij}(s)\right| \\
&+ \sum_{B_{ij}^{kl}\in N_r} \left|B_{ij}^{kl}(s)\right|g\big(\varphi_{kl}(s-\tau)\big)-g\big(\psi_{kl}(s-\tau)\big)\big|\big|\psi_{ij}(s)\big| \\
&+ \sum_{C_{ij}^{kl}\in N_r} \left|C_{ij}^{kl}(s)\right|\left|\varphi_{ij}(s)\right| \\
&\times \left(\int_0^{\infty}\left|K_{ij}(u)\right|\left|f\big(\varphi_{kl}(s-u)\big)-f\big(\psi_{kl}(s-u)\big)\right|du\right) \\
&+ M_f\sum_{C_{ij}^{kl}\in N_r}\left|C_{ij}^{kl}(s)\right|\left|\varphi_{ij}(s)-\psi_{ij}(s)\right|\Bigg)ds\Bigg\} \\
\leq &\sup_{t\in\mathbb{R}}\max_{i,j}\Bigg\{\int_{-\infty}^{t}e^{-(t-s)a_{ij}}\Bigg(M_g\sum_{B_{ij}^{kl}\in N_r}\left|B_{ij}^{kl}(s)\right|\left|\varphi_{ij}(s)-\psi_{ij}(s)\right| \\
&+ L_g\sum_{B_{ij}^{kl}\in N_r}\left|B_{ij}^{kl}(s)\right|\left|\varphi_{kl}(s-\tau)-\psi_{kl}(s-\tau)\right|\left|\psi_{ij}(s)\right| \\
&+ \sum_{C_{ij}^{kl}\in N_r}\left|C_{ij}^{kl}(s)\right|L_f\int_0^{\infty}K_{ij}(u)\left|\varphi_{kl}(s-u)-\psi_{kl}(s-u)\right|du\left|\varphi_{ij}(s)\right| \\
&+ M_f\sum_{C_{ij}^{kl}\in N_r}\left|C_{ij}^{kl}(s)\right|\left|\varphi_{ij}(s)-\psi_{ij}(s)\right|\Bigg)ds\Bigg\} \\
\leq &\, a\|\varphi-\psi\|
\end{aligned}
$$

By (H_4), the operator Γ is a contraction mapping. Consequently, Γ possess a unique fixed point $\varphi_* \in \mathcal{B} \subset PAP(\mathbb{R}, \mathbb{R}^{n\times m})$ that is $\Gamma(\varphi_*) = \varphi_*$. Hence, φ_* is the unique pseudo almost periodic of the model (2). □

5 Stability of the pseudo almost periodic solution

In this section, we study the exponential stability of the unique almost periodic solution of the model (2).

Theorem 2 *Suppose that assumptions* (H_1)–(H_6) *hold. Let* $\varphi = (\varphi_{11}, \ldots, \varphi_{1n}, \ldots, \varphi_{m1}, \ldots, \varphi_{mn})^t$ *the unique pseudo almost periodic solution of* (2) *in* $\mathcal{B}$. *Then there exists a constant* $\mu > 0$ *such that for any solution* x *of* (2) *in* $\mathcal{B}$ *and for all* $t > 0$ *we have*

$$\left\|x(t) - \varphi(t)\right\| \leq Me^{-\mu t}$$

where $M = \sup_{t \leq 0} \|x(t) - \varphi(t)\|$.

Proof For all $1 \leq i \leq m, 1 \leq j \leq n$, set

$$\chi_{ij}(t) = (t - a_{ij}) + \sum_{B_{ij}^{kl} \in N_r(i,j)} |B_{ij}^{kl}(t_{ij})| M_g + L_g e^{t\tau} \rho$$

$$+ \sum_{C_{ij}^{kl} \in N_r(i,j)} |C_{ij}^{kl}(t_{ij})| M_f + L_f \rho \int_0^{+\infty} K_{ij}(s) e^{ts} ds \tag{7}$$

By hypothesis (H_6), $\psi_{ij}(0) < 0$. Since the functions $t \longmapsto \chi_{ij}(t)$ are continuous functions on $\mathbb{R}^+$ there exists a sufficiently small constant μ such that

$$\chi_{ij}(\mu) < 0, \quad 1 \leq i \leq m, 1 \leq i \leq n.$$

Take an arbitrary $\epsilon > 0$. For all $1 \leq i \leq m, 1 \leq i \leq n$, set

$$z_{ij}(t) = \left|x_{ij}(t) - \varphi_{ij}(t)\right| e^{\mu t}.$$

Then for all $1 \leq i \leq m, 1 \leq j \leq n$, and for all $t < 0$, one has

$$z_{ij}(t) \leq M < M + \epsilon.$$

In the following, we shall prove that for all $t > 0$ and for all $1 \leq i \leq m, 1 \leq j \leq n$

$$z_{ij}(t) \leq M + \epsilon.$$

Suppose the contrary. Then there exists $1 \leq i_0 \leq m, 1 \leq j_0 \leq n$ such that

$$A_{i_0 j_0} = \left\{t > 0, z_{i_0 j_0}(t) > M + \epsilon\right\} \neq \emptyset$$

Let

$$t_{ij} = \begin{cases} \inf\{t > 0, z_{ij}(t) > M + \epsilon\} \neq \emptyset & \{t > 0, z_{ij}(t) > M + \epsilon\} \neq \emptyset \\ +\infty & \{t > 0, z_{ij}(t) > M + \epsilon\} = \emptyset \end{cases}$$

It follows that $t_{ij} > 0$ and $z_{ij}(t) \leq M + \epsilon$, for all $t < t_{ij}$ Let us denote $t_{pq} = \min_{ij} t_{ij}$, where $1 \leq p \leq m, 1 \leq q \leq n$. It follows that $0 < t_{pq} < +\infty$ and for all $t \leq t_{pq}$, one has

$$z_{ij}(t) \leq M + \epsilon, \quad 1 \leq i \leq m, 1 \leq j \leq n.$$

In addition, noting that $t_{pq} = \inf\{t > 0, z_{pq}(t) > M + \epsilon\}$, we obtain immediately

$$z_{pq}(t_{pq}) = M + \epsilon, D^+ z_{pq}(t_{pq}) \geq 0.$$

Now since $x(\cdot)$ and $\varphi(\cdot)$ are solutions of (2), we get

$$\begin{aligned}
0 &\leq D^+ z_{pq}(t_{pq}) \\
&= D^+\big[\big|x_{pq}(t) - \varphi_{pq}(t)\big|e^{\mu t}\big]_{|t=t_{pq}} \\
&= e^{\mu t_s}\left[\mu\big|x_s(t) - \varphi_s(t)\big| + \frac{D^+|x_i(t) - \varphi_i(t)|}{dt}\bigg|_{t=t_{pq}}\right] \\
&= \big|x_{pq}(t_{pq}) - \varphi_{pq}(t_{pq})\big|\mu e^{\mu t_{pq}} + e^{\mu t_{pq}} sgn\big(x_{pq}(t_{pq}) - \varphi_{pq}(t_{pq})\big) \\
&\quad \times \Bigg\{-a_{pq}\big(x_{pq}(t_{pq}) - \varphi_{pq}(t_{pq})\big) \\
&\quad + \sum_{B_{ij}^{kl} \in N_r(p,q)} B_{pq}^{kl}(t_{pq})\big|g\big(x_{kl}(t_{pq} - \tau)\big)x_{pq}(t_{pq}) \\
&\quad - g\big(\varphi_{kl}(t_{pq} - \tau)\big)\varphi_{pq}(t_{pq})\big|e^{\mu t_{pq}} \\
&\quad + \sum_{C_{ij}^{kl} \in N_r(p,q)} C_{pq}^{kl}(t_{pq})\bigg|\int_0^{+\infty} k_{ij}(u) f\big(x_{kl}(t_{pq} - u)\big)x_{pq}(t_{pq}) \\
&\quad - f\big(\varphi_{kl}(t_{pq} - u)\big)du\varphi_{pq}(t_{pq})\bigg|e^{\mu t_{pq}}\Bigg\} \\
&< \big|x_{pq}(t_{pq}) - \varphi_{pq}(t_{pq})\big|\mu e^{\mu t_{pq}} - a_{pq}e^{\mu t_{pq}}\big|x_{pq}(t_{pq}) - \varphi_{pq}(t_{pq})\big| \\
&\quad + \sum_{B_{ij}^{kl} \in N_r(p,q)} \Big(\big|B_{pq}^{kl}(t_{pq})\big|g\big(x_{kl}(t_{pq} - \tau)\big)\big|x_{pq}(t_{pq}) - \varphi_{pq}(t_{pq})\big| \\
&\quad + \big|g\big(x_{kl}(t_{pq} - \tau)\big) - g\big(\varphi_{kl}(t_{pq} - \tau)\big)\big|\big|\varphi_{pq}(t_{pq})\big|e^{\mu t_{pq}} \\
&\quad + \big|g\big(x_{kl}(t_{pq} - \tau)\big) - g\big(\varphi_{kl}(t_{pq} - \tau)\big)\big|\big|\varphi_{pq}(t_{pq})\big|\Big) \\
&\quad + \sum_{C_{ij}^{kl} \in N_r(p,q)} C_{pq}^{kl}(t_{pq})e^{\mu t_{pq}}\bigg|\int_0^{+\infty} k_{ij}(u) f\big(x_{kl}(t_{pq} - u)\big)du \\
&\quad \times \big|x_{pq}(t_{pq}) - \varphi_{pq}(t_{pq})\big| \\
&\quad + \int_0^{+\infty} k_{ij}(u)\big|f\big(x_{kl}(t_{pq} - u)\big) - f\big(\varphi_{kl}(t_{pq} - u)\big)\big|du\varphi_{pq}(t_{pq})\bigg| \\
&\leq (M + \epsilon)(\mu - a_{pq}) + \sum_{B_{ij}^{kl} \in N_r(p,q)} \big|B_{pq}^{kl}(t_{pq})\big|M_g z_{pq}(t_{pq}) + L_g\big|z_{kl}(t_{pq} - \tau)\big|e^{\mu\tau}\rho \\
&\quad + \sum_{C_{ij}^{kl} \in N_r(p,q)} \big|C_{pq}^{kl}(t_{pq})\big|z_{pq}(t_{pq})M_f \int_0^{+\infty} k_{ij}(u)du \\
&\quad + \int_0^{+\infty} |\varphi_{pq}(t_{pq})|L_f K_{ij}(u)e^{\mu u} z_{kl}(t_{pq} - u)du
\end{aligned}$$

$$\leq (M+\epsilon)\Bigg((\mu - a_{pq}) + \sum_{B_{ij}^{kl}\in N_r(p,q)} \left|B_{pq}^{kl}(t_{pq})\right| M_g + L_g e^{\mu\tau}\rho$$
$$+ \sum_{C_{ij}^{kl}\in N_r(p,q)} \left|C_{pq}^{kl}(t_{pq})\right| M_f + L_f \rho \int_0^{+\infty} k_{ij}(u) e^{\mu u} du\Bigg)$$

It follows that $\chi_{ij}(\mu) \geq 0$ which contradicts the fact that $\chi_{ij}(\mu) < 0$. Thus we obtain that for all $t > 0$,

$$z_{ij}(t) = \left|x_{ij}(t) - \varphi_{ij}(t)\right| \leq (M+\epsilon)e^{-\mu t}.$$

Note that $\|x(t) - \varphi(t)\| = \sup_{t\in\mathbb{R}} \max_{ij} |x_i(t) - \varphi_i(t)|$, then passing to limit when $\epsilon \to 0^+$ we obtain for all $t > 0$

$$\left\|x(t) - \varphi(t)\right\| \leq (M+\epsilon)e^{-\mu t}.$$

The proof of this theorem is completed. □

Remark 3 Our results and the method used in the proof of Theorem 1 are essentially new. Moreover, it should be pointed out that the proof of the exponential stability is similar than Theorem 5 in [8]. Notice that the pseudo almost periodicity is without importance in the proof of the above theorem and we have to replace the varying-time delay $\tau(t)$ by a constant delay τ. Further, if $B_{ij}^{kl}(t) = 0$ then this paper will be the natural continuation of [21] since the authors studied the almost periodic solutions. Besides, our model is more general than [20]. On the other hand if $a_{ij}(t) = a_{ij}$ and $C_{ij}^{kl}(t) = 0$ then this paper improve and generalize [16].

To our best knowledge, there is no published paper considering the pseudo almost periodic solutions for shunting inhibitory cellular neural networks with impulses and mixed delays.

6 An example

In order to illustrate some features of our main results, in this section, we will apply our main results to some special three-dimensional systems and demonstrate the efficiencies of our criteria.

Let us consider

$$\frac{dx_{ij}}{dt} = -a_{ij}x_{ij} - \sum_{B_{ij}^{kl}\in N_r(i,j)} B_{ij}^{kl}(t) g(x_{kl}(t-\tau)) x_{ij}(t)$$
$$- \sum_{C_{ij}^{kl}\in N_r(i,j)} C_{ij}^{kl}(t) \left(\int_0^{\infty} K_{ij}(u) f(x_{kl}(t-u)) du\right) x_{ij}(t) + L_{ij}(t) \quad (8)$$

where $i, j \in \{1, 2, 3\}$.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 5 & 8 & 8 \\ 7 & 5 & 6 \\ 4 & 3 & 8 \end{pmatrix}$$

$$\begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix} = \begin{pmatrix} 0.1\cos\sqrt{2}t & 0.2\cos\sqrt{5}t & 0.3\cos\sqrt{2}t \\ 0.6\cos\sqrt{2}t & 0 & 0.5\cos\sqrt{5}t \\ 0 & 0.4\cos\sqrt{2}t & 0.2\cos\sqrt{5}t \end{pmatrix}$$

$$\begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix} = \begin{pmatrix} 0.2\cos\sqrt{2}t & 0.4\cos\sqrt{5}t & 0 \\ 0.3\cos\sqrt{2}t & 0 & 0.3\cos\sqrt{5}t \\ 0.5\cos\sqrt{5}t & 0.6\cos\sqrt{2}t & 0.5\cos\sqrt{5}t \end{pmatrix}$$

$$\begin{pmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{pmatrix} = \begin{pmatrix} \frac{\cos t + e^{-t^2\cos^2 t}}{20} & \frac{\sin t + e^{-t^2\cos^2 t}}{20} & \frac{\cos t + e^{-t^2\cos^2 t}}{20} \\ \frac{\sin\sqrt{2}t + e^{-t^2\cos^2 t}}{20} & \frac{\cos t + e^{-t^2\cos^2 t}}{20} & \frac{\cos + e^{-t^2\cos^2 t}}{20} \\ \frac{\sin t + e^{-t^2\cos^2 t}}{20} & \frac{\sin t + e^{-t^2\cos^2 t}}{20} & \frac{\sin t + \sin\sqrt{2}t + e^{-t^2\cos^2 t}}{20} \end{pmatrix}$$

Let the r-neighborhood $N_r(i, j)$ $(i, j \in \{1, 2, 3\})$ of Cij be

$$\begin{cases} N_r(1,1) = \{C_{11}, C_{12}, C_{21}, C_{22}\} \qquad N_r(1,2) = \{C_{12}, C_{11}, C_{22}, C_{13}\} \\ N_r(2,1) = \{C_{21}, C_{11}, C_{22}, C_{31}\} \qquad N_r(3,1) = \{C_{31}, C_{21}, C_{22}, C_{32}\} \\ N_r(2,2) = \{C_{22}, C_{21}, C_{12}, C_{23}, C_{32}\} \qquad N_r(3,2) = \{C_{32}, C_{22}, C_{31}, C_{33}\} \\ N_r(1,3) = \{C_{12}, C_{13}, C_{23}\} \qquad N_r(2,3) = \{C_{23}, C_{13}, C_{22}, C_{33}\} \\ N_r(3,3) = \{C_{33}, C_{32}, C_{23}, C_{22}\} \end{cases} \tag{9}$$

It follows

$$\begin{cases} \sum\limits_{C_{kl}\in N_r(1,1)} |C_{11}^{kl}(t)| = 0.5|\cos\sqrt{2}t| + 0.4|\cos\sqrt{5}t| \\ \sum\limits_{C_{kl}\in N_r(1,2)} C_{12}^{kl}(t) = 0.2|\cos\sqrt{2}t| + 0.4|\cos\sqrt{5}t| \\ \sum\limits_{C_{kl}\in N_r(1,3)} C_{13}^{kl}(t) = 0.7|\cos\sqrt{5}t| \\ \sum\limits_{C_{kl}\in N_r(2,1)} C_{21}^{kl}(t) = 0.5|\cos\sqrt{2}t| + 0.5|\cos\sqrt{5}t| \\ \sum\limits_{C_{kl}\in N_r(2,2)} C_{22}^{kl}(t) = 0.7|\cos\sqrt{2}t| + 0.5|\cos\sqrt{5}t| \\ \sum\limits_{C_{kl}\in N_r(2,3)} C_{23}^{kl}(t) = 0.8|\cos\sqrt{5}t| \\ \sum\limits_{C_{kl}\in N_r(3,1)} C_{31}^{kl}(t) = 0.3|\cos\sqrt{2}t| + 0.8|\cos\sqrt{5}t| \\ \sum\limits_{C_{kl}\in N_r(3,2)} C_{32}^{kl}(t) = 0.6|\cos\sqrt{2}t| + |\cos\sqrt{5}t| \\ \sum\limits_{C_{kl}\in N_r(3,3)} C_{33}^{kl}(t) = 0.6|\cos\sqrt{2}t| + 0.8|\cos\sqrt{5}t| \end{cases} \tag{10}$$

Similarly,

$$\begin{cases} \sum_{B_{kl}\in N_r(1,1)} |B_{11}^{kl}(t)| = 0.7|\cos\sqrt{2}t| + 0.2|\cos\sqrt{5}t| \\ \sum_{B_{kl}\in N_r(1,2)} |B_{12}^{kl}(t)| = 0.4|\cos\sqrt{2}t| + 0.2|\cos\sqrt{5}t| \\ \sum_{B_{kl}\in N_r(1,3)} |B_{13}^{kl}(t)| = 0.7|\cos\sqrt{2}t| + 0.3|\cos\sqrt{5}t| \\ \sum_{B_{kl}\in N_r(2,1)} |B_{21}^{kl}(t)| = 0.4|\cos\sqrt{2}t| + 0.2|\cos\sqrt{5}t| \\ \sum_{B_{kl}\in N_r(2,2)} |B_{22}^{kl}(t)| = |\cos\sqrt{2}t| + 0.7|\cos\sqrt{5}t| \\ \sum_{B_{kl}\in N_r(2,3)} |B_{23}^{kl}(t)| = 0.3|\cos\sqrt{2}t| + 0.7|\cos\sqrt{5}t| \\ \sum_{B_{kl}\in N_r(3,1)} |B_{31}^{kl}(t)| = |\cos\sqrt{2}t| \\ \sum_{B_{kl}\in N_r(3,2)} |B_{32}^{kl}(t)| = 0.4|\cos\sqrt{2}t| + 0.2|\cos\sqrt{5}t| \\ \sum_{B_{kl}\in N_r(3,3)} |B_{33}^{kl}(t)| = 0.4|\cos\sqrt{2}t| + 0.7|\cos\sqrt{5}t| \end{cases} \tag{11}$$

Pose $f(x) = g(x) = \frac{1}{20}(|x+1| - |x-1|)$. Clearly $M_f = M_g = 0.1$

$$\begin{aligned} R &= \sup_{t\in\mathbb{R}} \max_{i,j} \left| \left(\frac{M_g \sum_{B_{ij}^{kl}\in N_r(i,j)} |B_{ij}^{kl}(t)| + M_f \sum_{C_{ij}^{kl}\in N_r(i,j)} |C_{ij}^{kl}(t)|}{a_{ij}} \right) \right| \\ &= \sup_{t\in\mathbb{R}} \max_{i,j} \left\{ \left| \left(\frac{M_g 1.7 + M_f 1.6}{a_{ij}} \right) \right| \right\} = 0.11 \end{aligned} \tag{12}$$

$$L = \sup_{t\in\mathbb{R}} \max_{i,j} \left\{ \frac{L_{ij}}{a_{ij}} \right\} = \max_{i,j} \left\{ \frac{L_{ij}}{a_{ij}} \right\} = \frac{1}{30}.$$

Now, since

$$a = \sup_{t\in\mathbb{R}} \max_{i,j} \left(\frac{\sum_{B_{ij}^{kl}\in N_r} (M_g + L_g\rho)|B_{ij}^{kl}(t)| + \sum_{C_{ij}^{kl}\in N_r} (L_f\rho + M_f)|C_{ij}^{kl}(t)|}{a_{ij}} \right) < 1$$

and for all $i, j \in \{1, 2, 3\}$

$$\sum_{B_{ij}^{kl}\in N_r(i,j)} \left|B_{ij}^{kl}(t_{ij})\right| M_g + L_g\rho + \sum_{C_{ij}^{kl}\in N_r(i,j)} \left|C_{ij}^{kl}(t_{ij})\right| M_f + L_f\rho < 3 \tag{13}$$

then all the conditions of Theorems 1 and 2 are satisfied and consequently by Theorem 2 it follows that the unique pseudo almost periodic solution of (8) is exponential stable.

7 Conclusion

In this paper, some novel sufficient conditions are presented ensuring the existence and uniqueness of the pseudo almost periodic solution for shunting inhibitory cellular neural networks with mixed distributed delays. Note that we just require that activation function is globally Lipschitz continuous, which is less conservative and less restrictive than the monotonic assumption in previous results. The method is very concise and the obtained results are new and they complement previously known results [20]. Finally, an illustrative example is given to demonstrate the effectiveness of the obtained results.

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