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A study of higher-order nonlinear ordinary differential equations with four-point nonlocal integral boundary conditions

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Abstract This paper investigates some new existence results for an nth-order nonlinear differential equation with four-point nonlocal integral boundary conditions (strip/slit like conditions). Our results are based on some standard fixed point theorems and Leray-Schauder degree theory.

Keywords Nonlinear differential equations · Four-point integral boundary conditions · Fixed point theorem · Leray-Schauder degree

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1 Introduction

Nonlocal multi-point problems constitute an important class of boundary value problems and have been addressed by many authors, for instance, see [1, 3, 6, 7, 9, 11–16, 18–23]. The multi-point boundary conditions appear in certain problems of thermodynamics, elasticity and wave propagation, see [18] and the references therein. The multi-point boundary conditions may be understood in the sense that the controllers at the end points dissipate or add energy according to censors located at intermediate positions. However, much of the literature dealing with four-point boundary conditions is restricted to the involvement of nonlocal parameters in the solution or gradient

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of the solution of the problem. Recently, Ahmad and Ntouyas [5] studied a second order boundary value problem with four-point nonlocal integral boundary conditions.

The aim of the present paper is to develop some existence results for the following nth-order boundary value problem with four-point nonlocal integral boundary conditions

$$\begin{cases} x^{(n)}(t) = f(t, x(t)), \quad 0 < t < 1, \\ x(0) = \alpha \int_0^\xi x(s)ds, \qquad x'(0) = 0, \qquad x''(0) = 0, \dots, x^{(n-2)}(0) = 0, \\ x(1) = \beta \int_\eta^1 x(s)ds, \quad 0 < \xi < \eta < 1, \end{cases} \tag{1.1}$$

where f is a given continuous function, and α, β are real numbers.

Integral boundary conditions have various applications in applied fields such as blood flow problems, chemical engineering, thermoelasticity, underground water flow, population dynamics, etc. For a detailed description of the integral boundary conditions, we refer the reader to the papers [4, 10] and references therein. It has been observed that the limits of integration in the integral part of the boundary conditions are often taken to be fixed. In the present study, limits of integration in the integral boundary conditions involve the parameters $0 < \xi < \eta < 1$. The application of such conditions can be found in the slit/strip diffraction problems [2, 8].

The paper is organized as follows: in Sect. 1, we prove an auxiliary lemma that we need in the sequel. The main results based on the Banach contraction principle and Krasnoselskii's theorem are presented in Sect. 2, while the third result relying on Leray-Schauder degree theory, is obtained in Sect. 3. The methods used are standard, however their exposition in the framework of problem (1.1) is new.

Lemma 1.1 *For a given $y \in C[0, 1]$, a unique solution of the boundary value problem*

$$\begin{cases} x^{(n)}(t) = y(t), \quad 0 < t < 1, \\ x(0) = \alpha \int_0^\xi x(s)ds, \qquad x'(0) = 0, \qquad x''(0) = 0, \dots, x^{(n-2)}(0) = 0, \\ x(1) = \beta \int_\eta^1 x(s)ds, \quad 0 < \xi < \eta < 1, \end{cases} \tag{1.2}$$

is given by

$$\begin{aligned} x(t) = {} & \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} y(s)ds \\ & + \frac{1}{n\Delta}\Bigg[\alpha\big(n - \beta(1-\eta^n)\big) \int_0^\xi \left(\int_0^s \frac{(s-m)^{n-1}}{(n-1)!} y(m)dm \right) ds \\ & + \alpha\beta\xi^n \int_\eta^1 \left(\int_0^s \frac{(s-m)^{n-1}}{(n-1)!} y(m)dm \right) ds - \alpha\xi^n \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} y(s)ds \Bigg] \\ & + \frac{t^{n-1}}{\Delta}\Bigg[-\alpha\big(1 - \beta(1-\eta)\big) \int_0^\xi \left(\int_0^s \frac{(s-m)^{n-1}}{(n-1)!} y(m)dm \right) ds \end{aligned}$$

$$+\beta(1-\alpha\xi)\int_\eta^1\left(\int_0^s\frac{(s-m)^{n-1}}{(n-1)!}y(m)dm\right)ds$$
$$-(1-\alpha\xi)\int_0^1\frac{(1-s)^{n-1}}{(n-1)!}y(s)ds\Bigg], \tag{1.3}$$

where

$$\Delta=\frac{\alpha\xi^n}{n}\big(1-\beta(1-\eta)\big)+(1-\alpha\xi)\left(1-\frac{\beta(1-\eta^n)}{n}\right)\neq 0. \tag{1.4}$$

Proof We know that the general solution of the equation $x^{(n)}(t)=y(t)$ can be written as

$$x(t)=c_1+c_2t+c_3t^2+\cdots+c_nt^{n-1}+\int_0^t\frac{(t-s)^{n-1}}{(n-1)!}y(s)ds, \tag{1.5}$$

where $c_1,c_2,\ldots,c_n\in\mathbb{R}$ are arbitrary constants. Applying the boundary conditions for the problem (1.2), we find that $c_2=0,\ldots,c_{n-1}=0$,

$$c_1=\frac{\alpha}{\Delta}\left(1-\frac{\beta(1-\eta^n)}{n}\right)\int_0^\xi\left(\int_0^s\frac{(s-m)^{n-1}}{(n-1)!}y(m)dm\right)ds$$
$$+\frac{\alpha\xi^n}{n\Delta}\left\{\beta\int_\eta^1\left(\int_0^s\frac{(s-m)^{n-1}}{(n-1)!}y(m)dm\right)ds-\int_0^1\frac{(1-s)^{n-1}}{(n-1)!}y(s)ds\right\}$$

and

$$c_n=-\frac{\alpha}{\Delta}\big(1-\beta(1-\eta)\big)\int_0^\xi\left(\int_0^s\frac{(s-m)^{n-1}}{(n-1)!}y(m)dm\right)ds$$
$$+\frac{(1-\alpha\xi)}{\Delta}\left\{\beta\int_\eta^1\left(\int_0^s\frac{(s-m)^{n-1}}{(n-1)!}y(m)dm\right)ds-\int_0^1\frac{(1-s)^{n-1}}{(n-1)!}y(s)ds\right\},$$

where Δ is given by (1.4). Substituting the values of $c_1,\ldots,c_n$ in (1.5), we get (1.3). □

2 Existence results in Banach space

Let $(X,\|\cdot\|)$ be a Banach space and $\mathcal{C}=\mathrm{C}([0,1],X)$ denotes the Banach space of all continuous functions from $[0,1]\to X$ endowed with a topology of uniform convergence with the norm denoted by $\|\cdot\|$.

By Lemma 1.1, the problem (1.1) can be transformed to a fixed point problem as

$$x=\mathcal{F}(x), \tag{2.1}$$

where $\mathcal{F}:\mathcal{C}\to\mathcal{C}$ is given by

$$
\begin{aligned}
(\mathcal{F}x)(t) = & \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(s,x(s))ds \\
& + \frac{1}{n\Delta}\Bigg[\alpha\big(n-\beta(1-\eta^n)\big)\int_0^{\xi}\left(\int_0^s \frac{(s-m)^{n-1}}{(n-1)!} f(m,x(m))dm\right)ds \\
& + \alpha\beta\xi^n \int_\eta^1 \left(\int_0^s \frac{(s-m)^{n-1}}{(n-1)!} f(m,x(m))dm\right)ds \\
& - \alpha\xi^n \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} f(s,x(s))ds\Bigg] \\
& + \frac{t^{n-1}}{\Delta}\Bigg[-\alpha\big(1-\beta(1-\eta)\big)\int_0^{\xi}\left(\int_0^s \frac{(s-m)^{n-1}}{(n-1)!} f(m,x(m))dm\right)ds \\
& + \beta(1-\alpha\xi)\int_\eta^1 \left(\int_0^s \frac{(s-m)^{n-1}}{(n-1)!} f(m,x(m))dm\right)ds \\
& - (1-\alpha\xi)\int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} f(s,x(s))ds\Bigg], \quad t\in[0,1].
\end{aligned}
$$

For the forthcoming analysis, we need the following assumptions:

(A$_1$) $\|f(t,x)-f(t,y)\| \le L\|x-y\|, \quad \forall t\in[0,1],\ L>0,\ x,y\in X;$
(A$_2$) $\|f(t,x)\| \le \mu(t), \quad \forall (t,x)\in[0,1]\times X$, and $\mu\in C([0,1],\mathbb{R}^+)$.

For convenience, let us set

$$
\Lambda = \frac{1}{(n+1)!}\left((n+1)+\frac{\delta_1+\delta_2}{|\Delta|}\right), \tag{2.2}
$$

where

$$
\delta_1 = \frac{|\alpha|\xi^n}{n}\Big(\xi|n-\beta(1-\eta^n)| + |\beta|(1-\eta^{n+1}) + (n+1)\Big),
$$

and

$$
\delta_2 = |\alpha(1-\beta(1-\eta))|\xi^{n+1} + [|\beta|(1-\eta^{n+1})+(n+1)]|1-\alpha\xi|.
$$

Theorem 2.1 *Assume that $f:[0,1]\times X\to X$ is a jointly continuous function and satisfies the assumption* (A$_1$) *with $L<1/\Lambda$, where Λ is given by* (2.2). *Then the boundary value problem* (1.1) *has a unique solution.*

Proof Setting $\sup_{t\in[0,1]}\|f(t,0)\| = M$ and choosing $r \ge \frac{\Lambda M}{1-L\Lambda}$, we show that $\mathcal{F}B_r \subset B_r$, where $B_r = \{x\in\mathcal{C} : \|x\|\le r\}$. For $x\in B_r$, we have:

$$
\begin{aligned}
\|(\mathcal{F}x)(t)\| \le & \left|\frac{\alpha}{n\Delta}\Big[(n-\beta(1-\eta^n)) - n(1-\beta(1-\eta))t^{n-1}\Big]\right| \\
& \times \int_0^{\xi}\left(\int_0^s \frac{(s-m)^{n-1}}{(n-1)!}\|f(m,x(m))\|dm\right)ds
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \left|\frac{\beta}{n\Delta}\left[\alpha\xi^n + n(1-\alpha\xi)t^{n-1}\right]\right| \\
&\quad \times \int_\eta^1 \left(\int_0^s \frac{(s-m)^{n-1}}{(n-1)!}\|f(m,x(m))\|dm\right)ds \\
&\quad + \left|\frac{1}{n\Delta}\left[\alpha\xi^n + n(1-\alpha\xi)t^{n-1}\right]\right| \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!}\|f(s,x(s))\|ds \\
&\quad + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}\|f(s,x(s))\|ds \\
&\le \left|\frac{\alpha}{n\Delta}\left[(n-\beta(1-\eta^n)) - n(1-\beta(1-\eta))t^{n-1}\right]\right| \\
&\quad \times \int_0^\xi \left(\int_0^s \frac{(s-m)^{n-1}}{(n-1)!}(\|f(m,x(m)) - f(m,0)\| + \|f(m,0)\|)dm\right)ds \\
&\quad + \left|\frac{\beta}{n\Delta}\left[\alpha\xi^n + n(1-\alpha\xi)t^{n-1}\right]\right| \\
&\quad \times \int_\eta^1 \left(\int_0^s \frac{(s-m)^{n-1}}{(n-1)!}(\|f(m,x(m)) - f(m,0)\| + \|f(m,0)\|)dm\right)ds \\
&\quad + \left|\frac{1}{n\Delta}\left[\alpha\xi^n + n(1-\alpha\xi)t^{n-1}\right]\right| \\
&\quad \times \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!}(\|f(s,x(s)) - f(s,0)\| + \|f(s,0)\|)ds \\
&\quad + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}(\|f(s,x(s)) - f(s,0)\| + \|f(s,0)\|)ds \\
&\le (Lr+M)\left\{\left|\frac{\alpha}{n\Delta}\left[(n-\beta(1-\eta^n)) - n(1-\beta(1-\eta))t^{n-1}\right]\right|\right. \\
&\quad \times \int_0^\xi \left(\int_0^s \frac{(s-m)^{n-1}}{(n-1)!}dm\right)ds \\
&\quad + \left|\frac{\beta}{n\Delta}\left[\alpha\xi^n + n(1-\alpha\xi)t^{n-1}\right]\right| \int_\eta^1 \left(\int_0^s \frac{(s-m)^{n-1}}{(n-1)!}dm\right)ds \\
&\quad + \left|\frac{1}{n\Delta}\left[\alpha\xi^n + n(1-\alpha\xi)t^{n-1}\right]\right| \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!}ds \\
&\quad \left. + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}ds\right\} \\
&\le \frac{Lr+M}{(n+1)!}\left((n+1) + \frac{\delta_1+\delta_2}{|\Delta|}\right) \\
&= (Lr+M)\,\Lambda \le r.
\end{aligned}
$$

Now, for $x, y \in \mathcal{C}$ and for each $t \in [0, 1]$, we obtain

$$\begin{aligned}
&\|(\mathcal{F}x)(t) - (\mathcal{F}y)(t)\| \\
&\quad \le \left|\frac{\alpha}{n\Delta}\left[(n - \beta(1-\eta^n)) - n(1-\beta(1-\eta))t^{n-1}\right]\right| \\
&\qquad \times \int_0^\xi \left(\int_0^s \frac{(s-m)^{n-1}}{(n-1)!}\|f(m, x(m)) - f(m, y(m))\|dm\right)ds \\
&\qquad + \left|\frac{\beta}{n\Delta}\left[\alpha\xi^n + n(1-\alpha\xi)t^{n-1}\right]\right| \\
&\qquad \times \int_\eta^1 \left(\int_0^s \frac{(s-m)^{n-1}}{(n-1)!}\|f(m, x(m)) - f(m, y(m))\|dm\right)ds \\
&\qquad + \left|\frac{1}{n\Delta}\left[\alpha\xi^n + n(1-\alpha\xi)t^{n-1}\right]\right|\int_0^1 \frac{(1-s)^{n-1}}{(n-1)!}\|f(s, x(s)) - f(s, y(s))\|ds \\
&\qquad + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}\|f(s, x(s)) - f(s, y(s))\|ds \\
&\quad \le L\|x - y\|\left\{\left|\frac{\alpha}{n\Delta}\left[(n - \beta(1-\eta^n)) - n(1-\beta(1-\eta))t^{n-1}\right]\right|\right. \\
&\qquad \times \int_0^\xi \left(\int_0^s \frac{(s-m)^{n-1}}{(n-1)!}dm\right)ds \\
&\qquad + \left|\frac{\beta}{n\Delta}\left[\alpha\xi^n + n(1-\alpha\xi)t^{n-1}\right]\right|\int_\eta^1 \left(\int_0^s \frac{(s-m)^{n-1}}{(n-1)!}dm\right)ds \\
&\qquad \left. + \left|\frac{1}{n\Delta}\left[\alpha\xi^n + n(1-\alpha\xi)t^{n-1}\right]\right|\int_0^1 \frac{(1-s)^{n-1}}{(n-1)!}ds + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}ds\right\} \\
&\quad \le \frac{L}{(n+1)!}\left((n+1) + \frac{\delta_1 + \delta_2}{|\Delta|}\right)\|x - y\| \\
&\quad = L\Lambda\|x - y\|,
\end{aligned}$$

where Λ is given by (2.2). Observe that Λ depends only on the parameters involved in the problem. As $L < 1/\Lambda$, therefore $\mathcal{F}$ is a contraction. Thus, the conclusion of the theorem follows by the contraction mapping principle (Banach fixed point theorem). □

Our next existence result is based on Krasnoselskii's fixed point theorem [17].

Theorem 2.2 (Krasnoselskii's fixed point theorem). *Let M be a closed convex and nonempty subset of a Banach space X. Let A, B be the operators such that* (i) $Ax + By \in M$ *whenever* $x, y \in M$; (ii) A *is compact and continuous*; (iii) B *is a contraction mapping*. *Then there exists $z \in M$ such that $z = Az + Bz$.*

Theorem 2.3 *Let $f : [0, 1] \times X \to X$ be a jointly continuous function mapping bounded subsets of $[0, 1] \times X$ into relatively compact subsets of X, and the assumptions (A_1) and (A_2) hold, with*

$$\frac{L(\delta_1 + \delta_2)}{|\Delta|(n+1)!} < 1. \tag{2.3}$$

Then the boundary value problem (1.1) has at least one solution on $[0, 1]$.

Proof Letting $\sup_{t\in[0,1]} |\mu(t)| = \|\mu\|$, we fix

$$\overline{r} \geq \frac{\|\mu\|}{(n+1)!}\left((n+1) + \frac{\delta_1 + \delta_2}{|\Delta|}\right),$$

and consider $B_{\overline{r}} = \{x \in \mathcal{C} : \|x\| \leq \overline{r}\}$. We define the operators $\mathcal{P}$ and $\mathcal{Q}$ on $B_{\overline{r}}$ as

$$(\mathcal{P}x)(t) = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(s, x(s))ds,$$

$$\begin{aligned}
(\mathcal{Q}x)(t) = \frac{1}{n\Delta}\Bigg[&\alpha\Big(n - \beta(1-\eta^n)\Big)\int_0^{\xi}\left(\int_0^s \frac{(s-m)^{n-1}}{(n-1)!} f(m, x(m))dm\right)ds \\
&+ \alpha\beta\xi^n \int_\eta^1 \left(\int_0^s \frac{(s-m)^{n-1}}{(n-1)!} f(m, x(m))dm\right)ds \\
&- \alpha\xi^n \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} f(s, x(s))ds\Bigg] \\
&+ \frac{t^{n-1}}{\Delta}\Bigg[-\alpha\Big(1 - \beta(1-\eta)\Big)\int_0^{\xi}\left(\int_0^s \frac{(s-m)^{n-1}}{(n-1)!} f(m, x(m))dm\right)ds \\
&+ \beta(1-\alpha\xi)\int_\eta^1 \left(\int_0^s \frac{(s-m)^{n-1}}{(n-1)!} f(m, x(m))dm\right)ds \\
&- (1-\alpha\xi)\int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} f(s, x(s))ds\Bigg], \quad t \in [0, 1].
\end{aligned}$$

For $x, y \in B_{\overline{r}}$, we find that

$$\|\mathcal{P}x + \mathcal{Q}y\| \leq \frac{\|\mu\|}{(n+1)!}\left((n+1) + \frac{\delta_1 + \delta_2}{|\Delta|}\right) \leq \overline{r}.$$

Thus, $\mathcal{P}x + \mathcal{Q}y \in B_{\overline{r}}$. It follows from the assumption (A_1) together with (2.3) that $\mathcal{Q}$ is a contraction mapping. Continuity of f implies that the operator $\mathcal{P}$ is continuous. Also, $\mathcal{P}$ is uniformly bounded on $B_{\overline{r}}$ as

$$\|\mathcal{P}x\| \leq \frac{\|\mu\|}{n!}.$$

Now we prove the compactness of the operator $\mathcal{P}$.

In view of (A_1), we define $\sup_{(t,x)\in[0,1]\times B_{\overline{r}}} \|f(t,x)\| = \overline{f} < \infty$, and consequently we have

$$
\begin{aligned}
&\|(\mathcal{P}x)(t_1) - (\mathcal{P}x)(t_2)\| \\
&= \left\| \int_0^{t_1} \frac{[(t_1-s)^{n-1} - (t_2-s)^{n-1}]}{(n-1)!} f(s,x(s))ds - \int_{t_1}^{t_2} \frac{(t_2-s)^{n-1}}{(n-1)!} f(s,x(s))ds \right\| \\
&\le \frac{\overline{f}}{n!} |t_1^n - t_2^n|,
\end{aligned}
$$

which is independent of x. Thus, $\mathcal{P}$ is equicontinuous. Using the fact that f maps bounded subsets into relatively compact subsets, we have that $\mathcal{P}(\mathcal{A})(t)$ is relatively compact in X for every t, where $\mathcal{A}$ is a bounded subset of $\mathcal{C}$. So $\mathcal{P}$ is relatively compact on $B_{\overline{r}}$. Hence, by the Arzelá-Ascoli Theorem, $\mathcal{P}$ is compact on $B_{\overline{r}}$. Thus all the assumptions of Theorem 2.2 are satisfied. So the conclusion of Theorem 2.2 implies that the boundary value problem (1.1) has at least one solution on [0, 1]. □

3 Existence of solution via Leray–Schauder degree theory

Theorem 3.1 *Let $f : [0,1] \times X \to X$. Assume that there exist constants $0 \le \kappa < \frac{1}{\Lambda}$, where Λ is given by (2.2) and $M > 0$ such that $\|f(t,x)\| \le \kappa\|x\| + M$ for all $t \in [0,1]$, $x \in X$. Then the boundary value problem (1.1) has at least one solution.*

Proof Lets us define an operator $\mathcal{F} : \mathcal{C} \to \mathcal{C}$ as

$$x = \mathcal{F}(x), \tag{3.1}$$

where

$$
\begin{aligned}
(\mathcal{F}x)(t) &= \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(s,x(s))ds \\
&\quad + \frac{1}{n\Delta}\Bigg[\alpha\Big(n - \beta(1-\eta^n)\Big) \int_0^{\xi} \left(\int_0^s \frac{(s-m)^{n-1}}{(n-1)!} f(m,x(m))dm \right) ds \\
&\quad + \alpha\beta\xi^n \int_{\eta}^1 \left(\int_0^s \frac{(s-m)^{n-1}}{(n-1)!} f(m,x(m))dm \right) ds \\
&\quad - \alpha\xi^n \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} f(s,x(s))ds \Bigg] \\
&\quad + \frac{t^{n-1}}{\Delta}\Bigg[-\alpha\Big(1 - \beta(1-\eta)\Big) \int_0^{\xi} \left(\int_0^s \frac{(s-m)^{n-1}}{(n-1)!} f(m,x(m))dm \right) ds \\
&\quad + \beta(1-\alpha\xi) \int_{\eta}^1 \left(\int_0^s \frac{(s-m)^{n-1}}{(n-1)!} f(m,x(m))dm \right) ds \\
&\quad - (1-\alpha\xi) \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} f(s,x(s))ds \Bigg], \quad t \in [0,1].
\end{aligned}
$$

In view of the fixed point problem (3.1), we just need to prove the existence of at least one solution $x \in \mathcal{C}$ satisfying (3.1). Define a suitable ball B_R with radius $R > 0$ as

$$B_R = \{x \in \mathcal{C} : \|x\| < R\},$$

where R will be fixed later. Then, it is sufficient to show that $\mathcal{F} : \overline{B}_R \to \mathcal{C}$ satisfies

$$x \neq \lambda \mathcal{F} x, \quad \forall x \in \partial B_R \quad \text{and} \quad \forall \lambda \in [0, 1]. \tag{3.2}$$

Let us set

$$H(\lambda, x) = \lambda \mathcal{F} x, \quad x \in X, \ \lambda \in [0, 1].$$

Then, by the Arzelá-Ascoli Theorem, $h_\lambda(x) = x - H(\lambda, x) = x - \lambda \mathcal{F} x$ is completely continuous. If (3.2) is true, then the following Leray-Schauder degrees are well defined and by the homotopy invariance of topological degree, it follows that

$$\begin{aligned} \deg(h_\lambda, B_R, 0) &= \deg(I - \lambda \mathcal{F}, B_R, 0) = \deg(h_1, B_R, 0) \\ &= \deg(h_0, B_R, 0) = \deg(I, B_R, 0) = 1 \neq 0, \quad 0 \in B_r, \end{aligned}$$

where I denotes the identity operator. By the nonzero property of Leray-Schauder degree, $h_1(t) = x - \lambda \mathcal{F} x = 0$ for at least one $x \in B_R$. In order to prove (3.2), we assume that $x = \lambda \mathcal{F} x$ for some $\lambda \in [0, 1]$ and for all $t \in [0, 1]$ so that

$$\begin{aligned} \|x(t)\| &= \|\lambda(\mathcal{F} x)(t)\| \\ &\le \left|\frac{\alpha}{n\Delta}\left[(n - \beta(1 - \eta^n)) - n(1 - \beta(1 - \eta))t^{n-1}\right]\right| \\ &\quad \times \int_0^\xi \left(\int_0^s \frac{(s - m)^{n-1}}{(n-1)!}\|f(m, x(m))\|dm\right)ds \\ &\quad + \left|\frac{\beta}{n\Delta}\left[\alpha\xi^n + n(1 - \alpha\xi)t^{n-1}\right]\right| \int_\eta^1 \left(\int_0^s \frac{(s - m)^{n-1}}{(n-1)!}\|f(m, x(m))\|dm\right)ds \\ &\quad + \left|\frac{1}{n\Delta}\left[\alpha\xi^n + n(1 - \alpha\xi)t^{n-1}\right]\right| \int_0^1 \frac{(1 - s)^{n-1}}{(n-1)!}\|f(s, x(s))\|ds \\ &\quad + \int_0^t \frac{(t - s)^{n-1}}{(n-1)!}\|f(s, x(s))\|ds \\ &\le (\kappa\|x\| + M)\left\{\left|\frac{\alpha}{n\Delta}\left[(n - \beta(1 - \eta^n)) - n(1 - \beta(1 - \eta))t^{n-1}\right]\right|\right. \\ &\quad \times \int_0^\xi \left(\int_0^s \frac{(s - m)^{n-1}}{(n-1)!}dm\right)ds \\ &\quad + \left|\frac{\beta}{n\Delta}\left[\alpha\xi^n + n(1 - \alpha\xi)t^{n-1}\right]\right| \int_\eta^1 \left(\int_0^s \frac{(s - m)^{n-1}}{(n-1)!}dm\right)ds \\ &\quad \left. + \left|\frac{1}{n\Delta}\left[\alpha\xi^n + n(1 - \alpha\xi)t^{n-1}\right]\right| \int_0^1 \frac{(1 - s)^{n-1}}{(n-1)!}ds + \int_0^t \frac{(t - s)^{n-1}}{(n-1)!}ds\right\} \end{aligned}$$

$$\le \frac{\kappa\|x\| + M}{(n+1)!}\left((n+1) + \frac{\delta_1 + \delta_2}{|\Delta|}\right)$$
$$= (\kappa\|x\| + M)\Lambda,$$

which, on taking norm and solving for $\|x\|$, yields

$$\|x\| \le \frac{M\Lambda}{1 - \kappa\Lambda}.$$

Letting $R = \frac{M\Lambda}{1-\kappa\Lambda} + 1$, (3.2) holds. This completes the proof. □

Remark 3.1 By fixing the values of α and β in results of this paper, we obtain some new results. For instance, the existence results for a three-point nth-order boundary value problem with nonlocal integral boundary conditions of the form

$$x(0) = 0, \qquad x'(0) = 0, \dots, x^{(n-2)}(0) = 0, \qquad x(1) = \beta \int_\eta^1 x(s)ds, \quad 0 < \eta < 1,$$

can be obtained by taking $\alpha = 0$ while the results for a three-point nth-order boundary value problem with nonlocal integral boundary conditions

$$x(0) = \alpha \int_0^\xi x(s)ds, \qquad x'(0) = 0, \dots, x^{(n-2)}(0) = 0, \qquad x(1) = 0, \quad 0 < \xi < 1,$$

follow by taking $\beta = 0$ in the results of this paper.

4 Examples

Example 4.1 Consider the following fourth-order nonlinear ordinary differential equation with four-point nonlocal integral boundary conditions

$$\begin{cases} x^{(4)}(t) = \frac{L}{(t+1)^2}\frac{\|x\|}{1+\|x\|}, \quad L > 0,\ t \in [0, 1], \\ x(0) = \int_0^{1/3} x(s)ds, \quad x'(0) = 0, \quad x''(0) = 0, \quad x(1) = \int_{2/3}^1 x(s)ds. \end{cases} \tag{4.1}$$

Here, $\alpha = \beta = 1$, $\xi = 1/3$, $\eta = 2/3$, and $f(t,x) = \frac{L}{(t+1)^2}\frac{\|x\|}{1+\|x\|}$. Clearly $\|f(t,x) - f(t,y)\| \le L\|x - y\|$. Further, $\Delta = 130/243$, and

$$\Lambda = \frac{1}{(n+1)!}\left((n+1) + \frac{\delta_1 + \delta_2}{|\Delta|}\right) = 0.10298295.$$

For $L < 1/\Lambda = 9.71034526$, the conclusion of Theorem 2.1 holds. Therefore, the boundary value problem (4.1) has a unique solution on [0, 1].

Example 4.2 Consider the following fourth-order boundary value problem

$$\begin{cases} x^{(4)}(t) = \frac{1}{(4\pi)} \sin(2\pi \|x\|) + \frac{\|x\|}{1+\|x\|}, \quad t \in [0, 1], \\ x(0) = \int_0^{1/3} x(s)ds, \quad x'(0) = 0, \quad x''(0) = 0, \quad x(1) = \int_{2/3}^1 x(s)ds. \end{cases} \tag{4.2}$$

Here, $\alpha = \beta = 1$, $\xi = 1/3$, $\eta = 2/3$, and

$$\left| f(t, x) \right| = \left| \frac{1}{(4\pi)} \sin(2\pi \|x\|) + \frac{\|x\|}{1 + \|x\|} \right| \leq \frac{1}{2}\|x\| + 1.$$

Clearly $M = 1$ and

$$\kappa = \frac{1}{2} < \frac{1}{\Lambda} = 9.71034526.$$

Thus, all the conditions of Theorem 3.1 are satisfied and consequently the problem (4.2) has at least one solution.

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