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Global well-posedness for semilinear hyperbolic equations with dissipative term

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Abstract We study the initial boundary value problem of semilinear hyperbolic equations with dissipative term. By introducing a family of potential wells we derive the invariant sets and vacuum isolating of solutions. Then we prove the global existence, nonexistence and asymptotic behaviour of solutions. In particular we obtain some sharp conditions for global existence and nonexistence of solutions.

Keywords Semilinear hyperbolic equation · Potential wells · Global existence · Nonexistence · Asymptotic behaviour

Mathematics Subject Classification (2000) 35L25

1 Introduction

On the global wellposedness of solution of the initial boundary value problem (IBVP) for semilinear hyperbolic equations

$$\begin{cases} u_{tt} - \Delta u = f(u), \quad x \in \Omega, t > 0, \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \quad x \in \Omega, \\ u(x,t) = 0, \quad x \in \partial\Omega,\ t \geq 0, \end{cases} \tag{1}$$

there have been a lot of results [1–9]. On the global wellposedness of solutions to IBVP for semilinear hyperbolic equations with dissipative term

$$u_{tt} - \Delta u + \gamma u_t = f(u), \quad x \in \Omega, t > 0, \tag{2}$$

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$$u(x,0)=u_0(x), \qquad u_t(x,0)=u_1(x), \quad x\in\Omega, \tag{3}$$

$$u(x,t)=0, \quad x\in\partial\Omega,\ t\geq 0, \tag{4}$$

we can find a host of literature [10–24].

Recently in [9], problem (1) was studied. By introducing a family of potential wells W_δ and corresponding family V_δ the above mentioned problems were resolved. Moreover, some new results on invariant sets, vacuum isolating and global existence of solutions were obtained.

In this paper we study problem (2)–(4), where $\gamma\geq 0$, $\Omega\subset\mathbb{R}^n$ is a bounded domain, $f(u)$ satisfies the following conditions

(i) $f(u)\in C^1(\mathbb{R})$ and

$$\text{(H)} \quad u(uf'(u)-f(u))\geq 0, \quad \forall u\in\mathbb{R},$$

where the equality holds only for $u=0$.

(ii) There exists a $a>0$ and q such that

$$|f(u)|\leq a|u|^q, \quad \forall u\in\mathbb{R},$$

where $1<q<\infty$ if $n=1,2$; $1<q<\frac{n+2}{n-2}$ if $n\geq 3$.

(iii) $(p+1)F(u)\leq uf(u)$, $\forall u\in\mathbb{R}$, for some $1<p\leq q$ and $F(u)=\int_0^u f(s)\mathrm{d}s$.

Remark 1.1 Note that from the assumption (H) in [9] one can derive the assumption (H) in the present paper. Therefore the assumption (H) in the present paper is weaker than the assumption on $f(u)$ in [9].

Since (2) includes the damping term γu_t, the normal convexity method employed in [3] cannot be directly used to derive the global nonexistence of solutions. Therefore the main difficulty of the present paper is to improve the classical convexity method for proving the global nonexistence of solutions, as well as to obtain a sharp condition of global existence and nonexistence of solutions for problem (2)–(4). In addition, due to the fact that almost all of the relative works focus on the case $E(0)<d$, where d is the depth of the potential well defined for the problem, it is also a difficult open problem to prove the global existence or nonexistence of solutions for problem (2)–(4) with critical initial data $E(0)=d$.

The main purpose of this paper is to answer the following questions regarding problem (2)–(4):

(i) Under what conditions the solutions exist globally in time. And under what conditions the existence time of solutions is finite. Whether there exists a sharp condition for global existence of solutions.
(ii) How to prove the asymptotic behaviour of solutions.
(iii) For the critical initial data $E(0)=d$, where d is the depth of potential well, how to prove the global existence, nonexistence and asymptotic behaviour of solutions.

This paper is organized as follows.

- in Sect. 2 we recall some preliminary lemmas and introduce a family of potential wells, by which we not only obtain some new results on the global well-posedness of solutions but also derive a sharp condition for global existence and nonexistence of solutions in the rest sections;
- in Sect. 3 we discuss the invariant sets and vacuum isolating of solutions for problem (2)–(4) for $0 < E(0) < d$ and $E(0) \le 0$ respectively;
- in Sect. 4 we prove the global existence and nonexistence of solutions and give a sharp condition for global existence of solutions for problem (2)–(4) for $E(0) < d$;
- in Sect. 5 we prove the asymptotic behaviour of solutions for problem (2)–(4) for $0 < E(0) < d$;
- in the last section we prove the global existence, nonexistence and asymptotic behaviour of solutions for problem (2)–(4) with the critical data $E(0) = d$.

Throughout this paper, we set the notations: $\|\cdot\|_p = \|\cdot\|_{L^p(\Omega)}$, $\|\cdot\| = \|\cdot\|_2$. And$(u, v) = \int_\Omega uv\mathrm{d}x$ denotes the L^2-inner product.

In order to prove the main theorems we give the following Proposition 1.2.

Proposition 1.2 *Let q satisfy the condition in* (*H*), *then the embedding $H^1(\Omega) \hookrightarrow L^{q+1}(\Omega)$ is compact.*

2 Preliminary lemmas and introducing of $\{W_\delta\}$ and $\{V_\delta\}$

In this section, before introducing potential wells, we define some functionals. Then some preliminary lemmas are given to show some of their properties. Finally we introduce a family of potential wells W_δ and corresponding family V_δ.

As did in [9] for problem (2)–(4) we define

$$J(u) = \frac{1}{2}\|\nabla u\|^2 - \int_\Omega F(u)\mathrm{d}x,$$

$$I(u) = \|\nabla u\|^2 - \int_\Omega uf(u)\mathrm{d}x,$$

$$I_\delta(u) = \delta\|\nabla u\|^2 - \int_\Omega uf(u)\mathrm{d}x, \quad \delta > 0.$$

Lemma 2.1 *Let $f(u)$ satisfy* (*H*),

$$g(u) = \frac{f(u)}{u}, \quad u \neq 0.$$

Then

(i) $\lim_{u\to 0} g(u) = 0$;
(ii) $g(u)$ *is increasing on* $(0, \infty)$, *decreasing on* $(-\infty, 0)$;
(iii) $f(u)u \ge 0$ *for* $u \in \mathbb{R}$, *where the equality holds only for* $u = 0$;
(iv) $f(u)$ *is increasing on* $(-\infty, \infty)$;

(v)

$$0 \le F(u) \le \frac{a}{q+1}|u|^{q+1}.$$

Proof

(i) follows from (ii) in (H).
(ii) follows from (i) in (H) and

$$g'(u) = \frac{uf'(u) - f(u)}{u^2}.$$

(iii) follows from above (i) and (ii) of this lemma.
(iv) follows from (i) in (H) and (iii) in this lemma.
(v) follows from (iii) in this lemma and (ii) in (H). □

Lemma 2.2 [3] *Let $f(u)$ satisfy (H). Then $F(u) \ge B|u|^{p+1}$ for $|u| \ge 1$ and some $B > 0$.*

By the proof of Lemma 2.5 in [9] it is easy to see the following Lemma 2.3.

Lemma 2.3 *Let $f(u)$ satisfy (H),*

$$\varphi(\lambda) = \frac{1}{\lambda}\int_\Omega uf(\lambda u)\mathrm{d}x.$$

Then

(i) $\lim_{\lambda\to 0}\varphi(\lambda) = 0$, $\lim_{\lambda\to+\infty}\varphi(\lambda) = +\infty$;
(ii) $\varphi(\lambda)$ *is increasing on* $0 < \lambda < \infty$.

From Lemmas 2.1–2.3, by the argument in [9] we can obtain the following Lemmas 2.4–2.7.

Lemma 2.4 *Let $f(u)$ satisfy (H), $u \in H_0^1(\Omega)$ and $\|\nabla u\| \ne 0$. Then*

(i) $\lim_{\lambda\to 0} J(\lambda u) = 0$, $\lim_{\lambda\to+\infty} J(\lambda u) = -\infty$;
(ii) *On the interval* $0 < \lambda < \infty$ *there exists a unique* $\bar\lambda = \bar\lambda(u)$ *such that*

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}J(\lambda u)\bigg|_{\lambda=\bar\lambda} = 0;$$

(iii) $J(\lambda u)$ *is increasing on* $0 \le \lambda \le \bar\lambda$, *decreasing on* $\bar\lambda \le \lambda < \infty$ *and takes the maximum at* $\lambda = \bar\lambda$;
(iv) $I(\lambda u) = \lambda\frac{\mathrm{d}}{\mathrm{d}\lambda}J(\lambda u)$;
(v) $I(\lambda u) > 0$ *for* $0 < \lambda < \bar\lambda$, $I(\lambda u) < 0$ *for* $\bar\lambda < \lambda < \infty$, *and* $I(\bar\lambda u) = 0$.

Lemma 2.5 *Let $f(u)$ satisfy (H). Assume that $u \in H_0^1(\Omega)$ and $0 < \|\nabla u\| < r(\delta)$. Then $I_\delta(u) > 0$. In particular, if $0 < \|\nabla u\| < r(1)$, then $I(u) > 0$, where*

$$r(\delta) = \left(\frac{\delta}{aC_*^{q+1}}\right)^{\frac{1}{q-1}}, \qquad C_* = \sup_{u \in H_0^1(\Omega), u \neq 0} \frac{\|u\|_{q+1}}{\|\nabla u\|}.$$

Lemma 2.6 *Let $f(u)$ satisfy (H). Assume that $u \in H_0^1(\Omega)$ and $I_\delta(u) < 0$. Then $\|\nabla u\| > r(\delta)$. In particular, if $I(u) < 0$, then $\|\nabla u\| > r(1)$.*

Lemma 2.7 *Let $f(u)$ satisfy (H). Assume that $u \in H_0^1(\Omega)$, $I_\delta(u) = 0$ and $\|\nabla u\| \neq 0$. Then $\|\nabla u\| \geq r(\delta)$. In particular, if $I(u) = 0$ and $\|\nabla u\| \neq 0$, then $\|\nabla u\| \geq r(1)$.*

Definition 2.8 For problem (2)–(4) we define

$$d = \inf_{u \in \mathcal{N}} J(u), \quad \mathcal{N} = \{u \in H_0^1(\Omega) | I(u) = 0, \|\nabla u\| \neq 0\},$$

$$d(\delta) = \inf_{u \in \mathcal{N}_\delta} J(u), \quad \mathcal{N}_\delta = \{u \in H_0^1(\Omega) | I_\delta(u) = 0, \|\nabla u\| \neq 0\}, \quad \delta > 0.$$

By the same proof of Lemma 2.9 in [9] we can obtain the following Theorem 2.9.

Theorem 2.9 *Let p satisfy (H). Then*

(i) $d(\delta) \geq a(\delta) r^2(\delta)$ *for* $a(\delta) = \frac{1}{2} - \frac{\delta}{p+1}$, $0 < \delta < \frac{p+1}{2}$. *In particular, we have* $d \geq \frac{p-1}{2(p+1)} (\frac{1}{\alpha C_*^{q+1}})^{\frac{2}{q-1}}$.

(ii) $\lim_{\delta \to 0} d(\delta) = 0$, *there exists a* $\delta_0 \geq \frac{p+1}{2}$ *such that* $d(\delta_0) = 0$ *and* $d(\delta) > 0$ *for* $0 < \delta < \delta_0$.

(iii) $d(\delta)$ *is strictly increasing on* $0 < \delta \leq 1$, *decreasing on* $1 \leq \delta \leq \delta_0$ *and takes the maximum* $d = d(1)$ *at* $\delta = 1$.

Definition 2.10 Now for problem (2)–(4) we define

$$W = \{u \in H_0^1(\Omega) \mid I(u) > 0,\ J(u) < d\} \cup \{0\};$$

$$V = \{u \in H_0^1(\Omega) \mid I(u) < 0,\ J(u) < d\};$$

$$W_\delta = \{u \in H_0^1(\Omega) \mid I_\delta(u) > 0,\ J(u) < d(\delta)\} \cup \{0\}, \quad 0 < \delta < \delta_0;$$

$$V_\delta = \{u \in H_0^1(\Omega) \mid I_\delta(u) < 0,\ J(u) < d(\delta)\}, \quad 0 < \delta < \delta_0.$$

3 Invariant sets and vacuum isolating of solutions

In this section we discuss the invariant sets and vacuum isolating of solutions for problem (2)–(4) for $0 < E(0) < d$ and $E(0) \leq 0$ respectively.

Definition 3.1 $u = u(x,t) \in L^\infty(0, T; H_0^1(\Omega))$ with $u_t \in L^\infty(0, T; L^2(\Omega))$ is called a weak solution of problem (2)–(4) on $\Omega \times [0, T)$ if

(i)

$$(u_t, v) + \int_0^t (\nabla u, \nabla v)\mathrm{d}\tau = \int_0^t (f(u), v)\,\mathrm{d}\tau + (u_1, v),$$

$$\forall v \in H_0^1(\Omega),\ t \in (0, T); \tag{5}$$

(ii) $u(x, 0) = u_0(x)$ in $H_0^1(\Omega)$;

(iii)

$$E(t) + \gamma \int_0^t \|u_\tau\|^2 \mathrm{d}\tau \le E(0), \quad \forall t \in [0, T), \tag{6}$$

where

$$E(t) = \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\nabla u\|^2 - \int_\Omega F(u)\mathrm{d}x = \frac{1}{2}\|u_t\|^2 + J(u).$$

Remark 3.2 From $u \in L^\infty(0, T; H_0^1(\Omega))$, $u_t \in L^\infty(0, T; L^2(\Omega))$ and (2) we can obtain $u_{tt} \in L^\infty(0, T; H^{-1}(\Omega))$.

By (6) and the similar argument to that in [9] we can obtain the following Theorems 3.3–3.5 and Lemma 3.6.

At first we introduce the invariance of sets W_δ and V_δ.

Theorem 3.3 *Let $f(u)$ satisfy (H), $u_0(x) \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$. Assume that $0 < e < d$, (δ_1, δ_2) is the maximal interval including $\delta = 1$ such that $d(\delta) > e$ for $\delta \in (\delta_1, \delta_2)$. Then*

(i) *All weak solutions of problem (2)–(4) with $0 < E(0) \le e$ belong to W_δ for $\delta \in (\delta_1, \delta_2)$, provided $I(u_0) > 0$ or $\|\nabla u_0\| = 0$.*
(ii) *All weak solutions of problem (2)–(4) with $0 < E(0) \le e$ belong to V_δ for $\delta \in (\delta_1, \delta_2)$, provided $I(u_0) < 0$.*

By Theorem 3.3 we can obtain the theorem below.

Theorem 3.4 *Let $f(u)$, $u_i(x)$ $(i = 0, 1)$, e and (δ_1, δ_2) be the same as those in Theorem 3.3. Assume that $0 < E(0) \le e$. Then for any $\delta \in (\delta_1, \delta_2)$ both sets W_δ and V_δ are invariant, thereby both sets*

$$W_{\delta_1\delta_2} = \bigcup_{\delta_1<\delta<\delta_2} W_\delta \quad \text{and} \quad V_{\delta_1\delta_2} = \bigcup_{\delta_1<\delta<\delta_2} V_\delta$$

are invariant respectively under the flow of (2)–(4).

Theorem 3.5 *Under the assumptions of Theorem 3.4 for all weak solutions of problem (2)–(4) we have*

$$u(t) \notin \mathcal{N}_{\delta_1\delta_2} = \bigcup_{\delta_1<\delta<\delta_2} \mathcal{N}_\delta.$$

Next we consider the case $E(0) \le 0$.

To discuss the invariant sets for $E(0) \le 0$, we introduce a lemma here.

Lemma 3.6 *Let $f(u)$ satisfy (H), $u_0(x) \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$. Assume that $E(0) = 0$ and $\|\nabla u_0\| \ne 0$. Then all weak solutions of problem (2)–(4) satisfy*

$$\|\nabla u\| \ge r_0 = \left(\frac{q+1}{2aC_*^{q+1}}\right)^{\frac{1}{q-1}}.$$

Now we can obtain the invariance of V_δ for $E(0) \le 0$.

Theorem 3.7 *Let $f(u)$ satisfy (H), $u_0(x) \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$. Assume that $E(0) < 0$ or $E(0) = 0$, $\|\nabla u_0\| \ne 0$. Then all weak solutions of problem (2)–(4) belong to V_δ for $\delta \in (0, \frac{p+1}{2})$.*

Proof Let $u(t)$ be any weak solution of problem (2)–(4) with $E(0) < 0$ or $E(0) = 0$, $\|\nabla u_0\| \ne 0$, T be the existence time of $u(t)$. From (6) we can get

$$\begin{aligned}\frac{1}{2}\|u_t\|^2 &+ a(\delta)\|\nabla u\|^2 + \frac{1}{p+1} I_\delta(u) \\ &\le \frac{1}{2}\|u_t\|^2 + J(u) \le E(0), \quad \delta \in \left(0, \frac{p+1}{2}\right), t \in [0, T).\end{aligned} \tag{7}$$

If $E(0) < 0$, then (7) gives $I_\delta(u) < 0$ and $J(u) < 0 < d(\delta)$ for $\delta \in (0, \frac{p+1}{2})$, $t \in [0, T)$. If $E(0) = 0$ and $\|\nabla u_0\| \ne 0$, then Lemma 3.6 gives $\|\nabla u\| \ge r_0$ for $0 \le t < T$. Again by (7) we get $I_\delta(u) < 0$ and $J(u) < 0 < d(\delta)$ for $\delta \in (0, \frac{p+1}{2})$, $0 \le t < T$. Hence for above two cases we always have $u \in V_\delta$ for $\delta \in (0, \frac{p+1}{2})$, $0 \le t < T$. □

4 Global existence and nonexistence of solutions

In this section we prove the global existence and nonexistence of solutions and give a sharp condition for global existence of solutions for problem (2)–(4) with $E(0) < d$.

Firstly we consider the global existence of weak solution of problem (2)–(4).

Theorem 4.1 *Let $\gamma \ge 0$, $f(u)$ satisfy (H), $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$. Assume that $E(0) < d$, $I(u_0) > 0$ or $\|\nabla u_0\| = 0$. Then problem (2)–(4) admits a global weak solution $u(t) \in L^\infty(0, \infty; H_0^1(\Omega))$ with $u_t(t) \in L^\infty(0, \infty; L^2(\Omega))$ and $u(t) \in W$ for $0 \le t < \infty$.*

Proof Let $\{w_j(x)\}$ be a system of base functions in $H_0^1(\Omega)$. Construct the approximate solutions

$$u_m(x,t) = \sum_{j=1}^{m} g_{jm}(t) w_j(x), \quad m = 1, 2, \dots,$$

satisfying

$$(u_{mtt}, w_s) + (\nabla u_m, \nabla w_s) + \gamma(u_{mt}, w_s) = (f(u_m), w_s), \quad s = 1, 2, \ldots, m; \quad (8)$$

$$u_m(x,0) = \sum_{j=1}^{m} a_{jm} w_j(x) \to u_0(x) \quad \text{in } H_0^1(\Omega), \quad (9)$$

$$u_{mt}(x,0) = \sum_{j=1}^{m} b_{jm} w_j(x) \to u_1(x) \quad \text{in } L^2(\Omega). \quad (10)$$

Multiplying (8) by $g'_{sm}(t)$ and summing for s we get

$$\frac{\mathrm{d}E_m(t)}{\mathrm{d}t} + \gamma \|u_{mt}\|^2 = 0$$

$$E_m(t) + \gamma \int_0^t \|u_{m\tau}\|^2 \mathrm{d}\tau = E_m(0) < d, \quad 0 \le t < \infty \quad (11)$$

for sufficiently large m

$$E_m(t) = \frac{1}{2}\|u_{mt}\|^2 + \frac{1}{2}\|\nabla u_m\|^2 - \int_\Omega F(u_m)\mathrm{d}x.$$

From (11) by the argument in [9] it follows that there exists a u and subsequence $\{u_\nu\}$ of $\{u_m\}$ such that

$u_\nu \to u$ in $L^\infty(0,\infty; H_0^1(\Omega))$ weakly star and a.e. in $\Omega \times [0,\infty)$,

$u_\nu \to u$ in $L^{q+1}(\Omega)$ strongly for each $t > 0$,

$u_{\nu t} \to u_t$ in $L^\infty(0,\infty; L^2(\Omega))$ weakly star.

And u satisfies (i) and (ii) in Definition 3.1. Next we prove u satisfies (6). First we prove that

$$\lim_{\nu\to\infty} \int_\Omega F(u_\nu)\mathrm{d}x = \int_\Omega F(u)\mathrm{d}x.$$

In fact

$$\begin{aligned} &\left| \int_\Omega F(u_\nu)\mathrm{d}x - \int_\Omega F(u)\mathrm{d}x \right| \\ &\quad \le \int_\Omega |f(u + \theta_\nu(u_\nu - u))|\,|u_\nu - u|\mathrm{d}x \\ &\quad \le \|f(u + \theta_\nu(u_\nu - u)\|_r \|u_\nu - u\|_{q+1}, \quad 0 < \theta_\nu < 1,\ r = \frac{q+1}{q}. \end{aligned}$$

Since

$$\begin{aligned} \|f(u + \theta_\nu(u_\nu - u)\|_r^r &\le a^r \int_\Omega \left(|u + \theta_\nu(u_\nu - u)|^q\right)^r \mathrm{d}x \\ &= a^r \|u + \theta_\nu(u_\nu - u)\|_{q+1}^{q+1} \le C \end{aligned}$$

we get

$$\lim_{\nu\to\infty}\int_\Omega F(u_\nu)\mathrm{d}x=\int_\Omega F(u)\mathrm{d}x.$$

Hence from (4.4) we get

$$\begin{aligned}
&\frac{1}{2}\|u_t\|^2+\frac{1}{2}\|\nabla u\|^2+\gamma\int_0^t\|u_\tau\|^2\mathrm{d}\tau\\
&\quad\le\liminf_{\nu\to\infty}\frac{1}{2}\|u_{\nu t}\|^2+\liminf_{\nu\to\infty}\frac{1}{2}\|\nabla u_\nu\|^2+\liminf_{\nu\to\infty}\gamma\int_0^t\|u_{\nu\tau}\|^2\mathrm{d}\tau\\
&\quad\le\liminf_{\nu\to\infty}\left(\frac{1}{2}\|u_{\nu t}\|^2+\frac{1}{2}\|\nabla u_\nu\|^2+\gamma\int_0^t\|u_{\nu\tau}\|^2\mathrm{d}\tau\right)\\
&\quad=\liminf_{\nu\to\infty}\left(E_\nu(0)+\int_\Omega F(u_\nu)\mathrm{d}x\right)\\
&\quad=\lim_{\nu\to\infty}\left(E_\nu(0)+\int_\Omega F(u_\nu)\mathrm{d}x\right)\\
&\quad=E(0)+\int_\Omega F(u)\mathrm{d}x,
\end{aligned}$$

which gives (6). Finally by Theorem 3.3 we have $u\in W$ for $0\le t<\infty$. □

Corollary 4.2 *If in Theorem 4.1 the assumption "$E(0)<d$, $I(u_0)>0$" is replaced by "$0<E(0)<d$, $I_{\delta_2}(u_0)>0$", where (δ_1,δ_2) is the maximal interval including $\delta=1$ such that $d(\delta)>E(0)$ for $\delta\in(\delta_1,\delta_2)$, then problem (2)–(4) admits a global weak solution $u(t)\in L^\infty(0,\infty;H_0^1(\Omega))$ with $u_t(t)\in L^\infty(0,\infty;L^2(\Omega))$ and $u(t)\in W_\delta$ for $\delta\in(\delta_1,\delta_2)$, $0\le t<\infty$.*

Theorem 4.3 *If in Corollary 4.2 the assumption "$I_{\delta_2}(u_0)>0$ or $\|\nabla u_0\|=0$" is replaced by "$\|\nabla u_0\|<r(\delta_2)$", then problem (2)–(4) admits a global weak solution $u(t)\in L^\infty(0,\infty;H_0^1(\Omega))$ with $u_t(t)\in L^\infty(0,\infty;L^2(\Omega))$ satisfying*

$$\|\nabla u\|^2\le\frac{E(0)}{a(\delta_1)},\qquad \|u_t\|^2\le 2E(0),\quad 0\le t<\infty. \tag{12}$$

Proof First $\|\nabla u_0\|<r(\delta_2)$ gives $I_{\delta_2}(u_0)>0$ or $\|\nabla u_0\|=0$. Hence from Corollary 4.2 it follows that problem (2)–(4) admits a global weak solution $u(t)\in L^\infty(0,\infty;H_0^1(\Omega))$ with $u_t(t)\in L^\infty(0,\infty;L^2(\Omega))$ and $u(t)\in W_\delta$ for $\delta\in(\delta_1,\delta_2)$, $0\le t<\infty$. Finally in

$$\begin{aligned}
&\frac{1}{2}\|u_t\|^2+a(\delta)\|\nabla u\|^2+\frac{1}{p+1}I_\delta(u)\le\frac{1}{2}\|u_t\|^2+J(u)\le E(0),\\
&\quad\delta\in(\delta_1,\delta_2),\ 0\le t<\infty,
\end{aligned}$$

letting $\delta\to\delta_1$ we get (12). □

Next we consider the existence of global strong solution for problem (2)–(4).

Theorem 4.4 *Let $\gamma \geq 0$, $f(u)$ satisfy*

(i) $f(u) \in C^1$, $f(0) = 0$ *and*

$$(H_1) \quad u\left(uf'(u) - f(u)\right) \geq 0,$$

where the equality holds only for $u = 0$.

(ii) $|f'(u)| \leq b|u|^{q_1}$ *for some $b > 0$ and some $0 < q_1 \leq \frac{2}{n-2}$ if $n \geq 3$; $0 < q_1 < \infty$ if $n = 1, 2$.*

(iii) $(p+1)F(u) \leq uf(u)$ *for some $1 < p \leq q_1 + 1$ and $F(u) = \int_0^u f(s)\mathrm{d}s$.*

$$u_0(x) \in H^2(\Omega) \cap H_0^1(\Omega), \qquad u_1(x) \in H_0^1(\Omega).$$

Assume that $E(0) < d$ and $I(u_0) > 0$ or $\|\nabla u_0\| = 0$. Then problem (2)–(4) admits a global strong solution $u(t) \in L^\infty(0, T; H^2(\Omega)) \cap L^\infty(0, \infty; H_0^1(\Omega))$ with $u_t(t) \in L^\infty(0, T; H_0^1(\Omega)) \cap L^\infty(0, \infty; L^2(\Omega))$ and $u_{tt}(t) \in L^\infty(0, T; L^2(\Omega))$ for any $T > 0$ and $u(t) \in W$ for $0 \leq t < \infty$.

Proof Clearly from (H_1) one can get (H), where $q = q_1 + 1$. Let $\{w_j\}$ be the eigenfunction system of problem

$$\Delta w + \lambda w = 0, \quad x \in \Omega, \; w|_{\partial\Omega} = 0.$$

Construct the approximate solutions as shown in the proof of Theorem 4.1. Then from (11) we get

$$\frac{1}{2}\|u_{mt}\|^2 + \frac{p-1}{2(p+1)}\|\nabla u_m\|^2 + \frac{1}{p+1}I(u_m) \leq E_m(t) \leq E_m(0) < d$$

for sufficiently large m. Hence from $u_m \in W$ for sufficiently large m, we have

$$\|\nabla u_m\|^2 < \frac{2(p+1)}{p-1}d, \qquad \|u_{mt}\|^2 < 2d, \quad 0 \leq t < \infty. \tag{13}$$

Let $\{u_\nu\}$ and $u \in L^\infty(0, \infty; H_0^1(\Omega))$ with $u_t \in L^\infty(0, \infty; L^2(\Omega))$ be the same as those in the proof of Theorem 4.1. Then u is a global weak solution of problem (2)–(4).

Next multiplying (8) by $\lambda_s g'_{sm}(t)$ and summing for s we get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|\nabla u_{mt}\|^2 + \frac{1}{2}\|\Delta u_m\|^2\right) + \gamma\|\nabla u_{mt}\|^2 \\
&= \left(f'(u_m)\nabla u_m, \nabla u_{mt}\right) \\
&\leq \|f'(u_m)\|_r \|\nabla u_m\|_s \|\nabla u_{mt}\| \\
&\leq C\|f'(u_m)\|_r \|\Delta u_m\| \|\nabla u_{mt}\|,
\end{aligned} \tag{14}$$

where $s = \frac{2n}{n-2}$, $r = n$ if $n \geq 3$; $s = r = 4$ if $n = 1, 2$.

On the other hand, from (ii) in (H_1) and (13) we can get $\|f'(u_m)\|_r \le C$ for $0 \le t < \infty$. Thus from (14) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|\nabla u_{mt}\|^2 + \frac{1}{2}\|\Delta u_m\|^2\right) \le C\left(\|\nabla u_{mt}\|^2 + \|\Delta u_m\|^2\right).$$

By use of Gronwall inequality we get

$$\|\nabla u_{mt}\|^2 + \|\Delta u_m\|^2 \le C(T), \quad 0 \le t \le T. \tag{15}$$

Multiplying (8) by $g''_{sm}(t)$ and summing for s yield

$$\begin{aligned}\|u_{mtt}\|^2 &= (\Delta u_m - \gamma u_{mt} + f(u_m), u_{mtt}) \\ &\le (\|\Delta u_m\| + \gamma\|u_{mt}\| + \|f(u_m)\|)\|u_{mtt}\| \\ &\le C(T)\|u_{mtt}\|\end{aligned}$$

and

$$\|u_{mtt}\| \le C(T), \quad 0 \le t \le T. \tag{16}$$

From (15) and (16), it follows that $u \in L^\infty(0, T; H^2(\Omega))$ with $u_t \in L^\infty(0.T; H_0^1(\Omega))$ and $u_{tt} \in L^\infty(0, T; L^2(\Omega))$, $\forall T > 0$ and u is a global strong solution of problem (2)–(4). □

Now we discuss the global nonexistence of solutions of problem (2)–(4).

Theorem 4.5 *Let* $0 \le \gamma < (p-1)\lambda_1$, $f(u)$ *satisfy* (*H*), $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$. *Assume that* $E(0) < d$ *and* $I(u_0) < 0$. *Then the existence time of solution for problem* (2)–(4) *is finite, where*

$$\lambda_1 = \inf_{u \in H_0^1(\Omega), \|\nabla u\| \ne 0} \frac{\|\nabla u\|}{\|u\|}.$$

Proof Let $u(t)$ be any weak solution of problem (2)–(4), T be the maximal existence time of $u(t)$. Let us prove $T < \infty$. If it is false, then $T = +\infty$. Let

$$M(t) = \|u\|^2,$$

then

$$\dot{M}(t) = 2(u_t, u),$$

$$\ddot{M}(t) = 2\|u_t\|^2 + 2(u_{tt}, u) = 2\|u_t\|^2 - 2\gamma(u_t, u) - 2I(u), \tag{17}$$

which together with

$$\frac{1}{2}\|u_t\|^2 + \frac{p-1}{2(p+1)}\|\nabla u\|^2 + \frac{1}{p+1}I(u) \le E(t) \le E(0)$$

gives

$$\ddot{M}(t) \geq (p+3)\|u_t\|^2 - 2\gamma(u_t, u) + (p-1)\|\nabla u\|^2 - 2(p+1)E(0). \tag{18}$$

(i) If $E(0) \leq 0$, then

$$\ddot{M}(t) \geq (p+3)\|u_t\|^2 - 2\gamma(u_t, u) + (p-1)\lambda_1^2\|u\|^2.$$

From $\gamma < (p-1)\lambda_1$ it follows that there exists a $\varepsilon \in (0, p-1)$ such that

$$\gamma^2 < (p-1-\varepsilon)(p-1)\lambda_1^2.$$

Then

$$\ddot{M}(t) \geq (4+\varepsilon)\|u_t\|^2 + (p-1-\varepsilon)\|u_t\|^2 - 2\gamma(u_t, u) + (p-1)\lambda_1^2\|u\|^2.$$

From this and

$$\begin{aligned} 2\gamma(u_t, u) &\leq (p-1-\varepsilon)\|u_t\|^2 + \frac{\gamma^2}{p-1-\varepsilon}\|u\|^2 \\ &\leq (p-1-\varepsilon)\|u_t\|^2 + (p-1)\lambda_1^2\|u\|^2, \end{aligned}$$

we get

$$\ddot{M}(t) \geq (4+\varepsilon)\|u_t\|^2. \tag{19}$$

Hence by Schwartz inequality we get

$$\begin{aligned} &\ddot{M}(t)M(t) - \frac{4+\varepsilon}{4}\dot{M}^2(t) \geq (4+\varepsilon)(\|u_t\|^2\|u\|^2 - (u_t, u)^2) \geq 0, \\ &(M^{-\alpha}(t))'' = \frac{-\alpha}{M^{\alpha+2}(t)}\left(\ddot{M}(t)M(t) - (\alpha+1)\dot{M}^2(t)\right) \leq 0, \\ &\alpha = \frac{\varepsilon}{4},\ 0 \leq t < \infty. \end{aligned}$$

Hence there exists a $T_1 > 0$ such that

$$\lim_{t \to T_1} M^{-\alpha}(t) = 0$$

and

$$\lim_{t \to T_1} M(t) = +\infty,$$

which contradicts $T = +\infty$.

(ii) $0 < E(0) < d$.

In this case from Theorem 3.3 we have $u \in V_\delta$ for $1 < \delta < \delta_2$, $0 \leq t < \infty$, where (δ_1, δ_2) is the maximal interval including $\delta = 1$ such that $d(\delta) > E(0)$ for

$\delta \in (\delta_1, \delta_2)$. Hence $I_\delta(u) < 0$ and $\|\nabla u\| > r(\delta)$ for $1 < \delta < \delta_2$, $0 \le t < \infty$. So we have $I_{\delta_2}(u) \le 0$ and $\|\nabla u\| \ge r(\delta_2)$ for $0 \le t < \infty$. From (17) we get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(e^{\gamma t}\dot{M}(t)) &= 2e^{\gamma t}(\|u_t\|^2 - I(u)) \\
&= 2e^{\gamma t}(\|u_t\|^2 + (\delta_2 - 1)\|\nabla u\|^2 - I_{\delta_2}(u)) \\
&\ge 2e^{\gamma t}(\delta_2 - 1)r^2(\delta_2) = C(\delta_2)e^{\gamma t}, \\
e^{\gamma t}\dot{M}(t) &\ge C(\delta_2)\int_0^t e^{\gamma\tau}\mathrm{d}\tau + \dot{M}(0) = \frac{C(\delta_2)}{\gamma}(e^{\gamma t} - 1) + \dot{M}(0), \\
\dot{M}(t) &\ge \frac{C(\delta_2)}{\gamma}(1 - e^{-\gamma t}) + e^{-\gamma t}\dot{M}(0).
\end{aligned}$$

Hence there exists a $t_0 > 0$ such that

$$\dot{M}(t) \ge \frac{C(\delta_2)}{2\gamma} \quad \text{for } t \ge t_0$$

and

$$M(t) \ge \frac{C(\delta_2)}{2\gamma}(t - t_0) + M(t_0) \ge \frac{C(\delta_2)}{2\gamma}(t - t_0), \quad t \ge t_0. \tag{20}$$

On the other hand, from $\gamma < (p-1)\lambda_1$ it follows that there exists a $\varepsilon \in (0, p-1)$ such that

$$\gamma^2 < (p - 1 - \varepsilon)((p-1)\lambda_1^2 - \varepsilon).$$

From (18) we have

$$\begin{aligned}
\ddot{M}(t) &\ge (p+3)\|u_t\|^2 - 2\gamma(u_t, u) + (p-1)\lambda_1^2\|u\|^2 - 2(p+1)E(0) \\
&= (4+\varepsilon)\|u_t\|^2 + (p - 1 - \varepsilon)\|u_t\|^2 - 2\gamma(u_t, u) \\
&\quad + ((p-1)\lambda_1^2 - \varepsilon)\|u\|^2 + \varepsilon M(t) - 2(p+1)E(0).
\end{aligned} \tag{21}$$

From (21) and

$$\begin{aligned}
2\gamma(u_t, u) &\le (p - 1 - \varepsilon)\|u_t\|^2 + \frac{\gamma^2}{p - 1 - \varepsilon}\|u\|^2 \\
&\le (p - 1 - \varepsilon)\|u_t\|^2 + ((p-1)\lambda_1^2 - \varepsilon)\|u\|^2,
\end{aligned}$$

we get

$$\ddot{M}(t) \ge (4+\varepsilon)\|u_t\|^2 + \varepsilon M(t) - 2(p+1)E(0).$$

From (20) it follows that there exists a $t_1 > 0$ such that

$$\varepsilon M(t) > 2(p+1)E(0) \quad \text{for } t > t_1$$

and

$$\ddot{M}(t) > (4+\varepsilon)\|u_t\|^2, \quad t > t_1.$$

The remainder of this proof is same with that in the proof of case (i).

Therefore for above two cases we always have $T < \infty$. □

From Theorem 4.1 and Theorem 4.5 we can obtain the following sharp condition for global existence of solution for problem (2)–(4):

Theorem 4.6 *Let* $0 \le \gamma < (p-1)\lambda_1$, $f(u)$ *satisfy* (*H*), $u_0(x) \in H_0^1(\Omega)$, $u_1(x) \in L^2(\Omega)$. *Assume* $E(0) < d$. *Then when* $I(u_0) > 0$, *problem* (2)–(4) *admits a global weak solution*; *and when* $I(u_0) < 0$, *the problem does not admits any global weak solution*.

Remark 4.7 Note that the proof of Theorem 4.5 for the case $0 < E(0) < d$ strongly depend on the fact that $u \in V_\delta$ for $1 < \delta < \delta_2$, where u is the solution of problem (2)–(4) with $0 < E(0) < d$, $I(u_0) < 0$. Therefore the introducing of the family $\{V_\delta\}$ is necessary for the proof of Theorem 4.5.

5 Asymptotic behaviour of solution

In this section we prove the asymptotic behaviour of solution for problem (2)–(4) with $0 < E(0) < d$.

Lemma 5.1 *Let* $\gamma \ge 0$, $f(u)$ *satisfy* (*H*) *and* u *be the weak solution of problem* (2)–(4) *with* $0 < E(0) < d$, $I(u_0) > 0$ *or* $\|\nabla u_0\| = 0$. *Then*

(i) $I(u) = \|u_t\|^2 - \frac{\mathrm{d}}{\mathrm{d}t}(u_t, u) - \frac{\gamma}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2$;
(ii) $I(u) \ge (1-\delta_1)\|\nabla u\|^2$,

where (δ_1, δ_2) *is the maximal interval including* $\delta = 1$ *such that* $d(\delta) > E(0)$ *for* $\delta \in (\delta_1, \delta_2)$.

Proof Let $u(t)$ be a weak solution of problem (2)–(4) with $0 < E(0) < d$, $I(u_0) > 0$ or $\|\nabla u_0\| = 0$, T be the existence time of $u(t)$.

(i) Multiply (2) by u and integrate on Ω, we can derive the conclusion.
(ii) From Theorem 3.3 we have $u(t) \in W_\delta$ for $\delta_1 < \delta < 1$, $0 \le t < T$. Hence $I_\delta(u) \ge 0$ for $\delta_1 < \delta < 1$, $0 \le t < T$ and $I_{\delta_1}(u) \ge 0$ for $0 \le t < T$. Thus we get

$$I(u) = \|\nabla u\|^2 - \int_\Omega u f(u)\mathrm{d}x = (1-\delta_1)\|\nabla u\|^2 + I_{\delta_1}(u) \ge (1-\delta_1)\|\nabla u\|^2. \quad \square$$

Then we have the following theorem on the asymptotic behaviour of the strong solutions of problem (2)–(4) for $0 < E(0) < d$.

Theorem 5.2 *Let $\gamma > 0$, $f(u)$ satisfy (H_1), $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $u_1 \in H_0^1(\Omega)$. Assume that $0 < E(0) < d$, $I(u_0) > 0$ or $\|\nabla u_0\| = 0$. Then for the global strong solution of problem (2)–(4) given in Theorem 4.4 we have*

$$E(t) \le Ce^{-\lambda t}, \quad 0 \le t < \infty \tag{22}$$

and

$$\|u_t\|^2 + \|\nabla u\|^2 \le C_1 e^{-\lambda t}, \quad 0 \le t < \infty \tag{23}$$

for some positive constants C, C_1 and λ.

Proof Let $u(t)$ be a global strong solution of problem (2)–(4) given by Theorem 4.4. Then by Theorem 4.4 and Theorem 3.3 we have $u(t) \in L^\infty(0, T; H^2(\Omega)) \cap L^\infty(0, \infty; H_0^1(\Omega))$ with $u_t(t) \in L^\infty(0, T; H_0^1(\Omega)) \cap L^\infty(0, \infty; L^2(\Omega))$, $u_{tt}(t) \in L^\infty(0, \infty; L^2(\Omega))$, $\forall T > 0$ and $u(t) \in W_\delta$ for $\delta_1 < \delta < \delta_2$, $0 \le t < \infty$, where (δ_1, δ_2) is the maximal interval including $\delta = 1$ such that $d(\delta) > E(0)$ for $\delta \in (\delta_1, \delta_2)$.

Multiplying (2) by u_t, integrating on Ω and multiplying the obtained equality by $e^{\alpha t}$ $(\alpha > 0)$ leads

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\alpha t}E(t)\right) + \gamma e^{\alpha t}\|u_t\|^2 = \alpha e^{\alpha t}E(t), \quad 0 \le t < T, \ \forall T > 0. \tag{24}$$

Integrating (24) with respect to t we get

$$e^{\alpha t}E(t) + \gamma \int_0^t e^{\alpha\tau}\|u_\tau\|^2 \mathrm{d}\tau \le E(0) + \alpha \int_0^t e^{\alpha\tau}E(\tau)\mathrm{d}\tau, \quad 0 \le t < \infty. \tag{25}$$

From $u(t) \in W$ and

$$E(t) \ge \frac{1}{2}\|u_t\|^2 + \frac{p-1}{2(p+1)}\|\nabla u\|^2 + \frac{1}{p+1}I(u),$$

we get

$$E(t) \ge \frac{1}{2}\|u_t\|^2 + \frac{p-1}{2(p+1)}\|\nabla u\|^2 \ge 0, \quad 0 \le t < \infty. \tag{26}$$

Moreover form Lemma 5.1 it follows that

$$\begin{aligned}\int_0^t e^{\alpha\tau}E(\tau)\mathrm{d}\tau &\le \frac{1}{2}\int_0^t e^{\alpha\tau}\|u_\tau\|^2\mathrm{d}\tau + \frac{1}{2}\int_0^t e^{\alpha\tau}\|\nabla u\|^2\mathrm{d}\tau \\ &\le \frac{1}{2}\int_0^t e^{\alpha\tau}\|u_\tau\|^2\mathrm{d}\tau + \frac{1}{2(1-\delta_1)}\int_0^t e^{\alpha\tau}I(u)\mathrm{d}\tau \\ &= \frac{1}{2}\left(1 + \frac{1}{1-\delta_1}\right)\int_0^t e^{\alpha\tau}\|u_\tau\|^2\mathrm{d}\tau \\ &\quad - \frac{1}{2(1-\delta_1)}\int_0^t e^{\alpha\tau}\frac{\mathrm{d}}{\mathrm{d}\tau}\left((u_\tau, u) + \frac{\gamma}{2}\|u\|^2\right)\mathrm{d}\tau \end{aligned} \tag{27}$$

$$
\begin{aligned}
&-\int_0^t e^{\alpha\tau}\frac{\mathrm{d}}{\mathrm{d}\tau}\left((u_\tau,u)+\frac{\gamma}{2}\|u\|^2\right)\mathrm{d}\tau\\
&=(u_1,u_0)+\frac{\gamma}{2}\|u_0\|^2-e^{\alpha t}\left((u_t,u)+\frac{\gamma}{2}\|u\|^2\right)\\
&\quad+\alpha\int_0^t e^{\alpha\tau}\left((u_\tau,u)+\frac{\gamma}{2}\|u\|^2\right)\mathrm{d}\tau\\
&\le\frac{1}{2}\left(\|u_1\|^2+(1+\gamma)\|u_0\|^2\right)+\frac{1}{2}e^{\alpha t}\left(\|u_t\|^2+(1+\gamma)\|u\|^2\right)\\
&\quad+\frac{\alpha}{2}\int_0^t e^{\alpha\tau}\left(\|u_\tau\|^2+(1+\gamma)\|u\|^2\right)\mathrm{d}\tau. \qquad (28)
\end{aligned}
$$

From (26)–(28) it follows that

$$
\begin{aligned}
&e^{\alpha t}E(t)+\gamma\int_0^t e^{\alpha\tau}\|u_\tau\|^2\mathrm{d}\tau\\
&\quad\le C_0E(0)+\frac{\alpha}{2}\left(1+\frac{1}{1-\delta_1}\right)\int_0^t e^{\alpha\tau}\|u_\tau\|^2\mathrm{d}\tau+\alpha C_1e^{\alpha t}E(t)\\
&\quad+\alpha^2C_1\int_0^t e^{\alpha\tau}E(\tau)\mathrm{d}\tau, \qquad (29)
\end{aligned}
$$

where C_0 and C_1 are positive constants. Take α such that

$$0<\alpha<\min\left\{\frac{1}{2C_1},\frac{2\gamma}{1+\frac{1}{1-\delta_1}}\right\}.$$

Then (29) gives

$$e^{\alpha t}E(t)\le 2C_0E(0)+2\alpha^2C_1\int_0^t e^{\alpha\tau}E(\tau)\mathrm{d}\tau,\quad 0\le t<\infty$$

and by Gronwall inequality we obtain

$$e^{\alpha t}E(t)\le 2C_0E(0)e^{2C_1\alpha^2 t}$$

and (22), where $C=2C_0E(0)>0$, $\lambda=\alpha(1-2C_1\alpha)>0$.

Furthermore from (22) and (26) we get (23). □

Lemma 5.3 *Let $\gamma>0$, $f(u)$ satisfy (H), $u_0(x)\in H_0^1(\Omega)$, $u_1(x)\in L^2(\Omega)$. Assume that $E(0)<d$, $I(u_0)>0$ or $\|\nabla u_0\|=0$. Then for the approximate solutions given in the proof of Theorem 4.1 we have*

(i)

$$I(u_m)=\|u_{mt}\|^2-\frac{\mathrm{d}}{\mathrm{d}t}(u_{mt},u_m)-\frac{\gamma}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_m\|^2,\quad\forall m;$$

(ii) $I(u_m)\ge(1-\delta_1)\|\nabla u_m\|^2$ *for sufficiently large m,* *where* (δ_1, δ_2) *is the maximal interval including* $\delta = 1$ *such that* $(\delta_1, \delta_2) \subset (0, \delta_0)$ *and* $E(0) < d(\delta)$ *for* $\delta \in (\delta_1, \delta_2)$.

Proof

(i) Multiplying (8) by $g_{sm}(t)$ and summing for s we can get (i) in this lemma.
(ii) First $E(0) < d(\delta)$ for $\delta \in (\delta_1, \delta_2)$ gives $E_m(0) < d(\delta)$ for $\delta \in (\delta_1, \delta_2)$ and sufficiently large m. Thus from (11) we have

$$\frac{1}{2}\|u_{mt}\|^2 + J(u_m) + \gamma \int_0^t \|u_{m\tau}\|^2 \mathrm{d}\tau \le E_m(0) < d(\delta),$$
$$\delta \in (\delta_1, \delta_2),\ 0 \le t < \infty. \tag{30}$$

From (30) by a similar argument to the proof of Theorem 3.1 in [9] we can obtain that $u_m \in W_\delta$ for $\delta \in (\delta_1, \delta_2)$, $0 \le t < \infty$ and sufficiently large m. Hence we have $I_\delta(u_m) \ge 0$ for $\delta \in (\delta_1, \delta_2)$, $0 \le t < \infty$ and sufficiently large m, which gives $I_{\delta_1}(u_m) \ge 0$ for $0 \le t < \infty$ and sufficiently large m. Hence we have

$$I(u_m) = (1 - \delta_1)\|\nabla u_m\|^2 + I_{\delta_1}(u_m) \ge (1 - \delta_1)\|\nabla u_m\|^2$$

for sufficiently large m. □

By the lemma above, we also prove the asymptotic behaviour of weak solution of problem (2)–(4) for $E(0) < d$.

Theorem 5.4 *Let* $\gamma > 0$, $f(u)$ *satisfy* (*H*), $u_0(x) \in H_0^1(\Omega)$, $u_1(x) \in L^2(\Omega)$. *Assume that* $E(0) < d$, $I(u_0) > 0$ *or* $\|\nabla u_0\| = 0$. *Then for the global weak solution* $u(t)$ *given in Theorem* 4.1, (22) *and* (23) *also hold.*

Proof Let $\{u_m\}$ be the approximate solutions give in the proof of Theorem 4.1. Then from the proof of Theorem 4.1 we have

$$\frac{\mathrm{d}E_m(t)}{\mathrm{d}t} + \gamma\|u_{mt}\|^2 = 0. \tag{31}$$

Multiplying (31) by $e^{\alpha t}(\alpha > 0)$ we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\alpha t}E_m(t)\right) + \gamma e^{\alpha t}\|u_{mt}\|^2 = \alpha e^{\alpha t}E_m(t), \quad 0 \le t < \infty, \tag{32}$$

where

$$E_m(t) = \frac{1}{2}\|u_{mt}\|^2 + \frac{1}{2}\|\nabla u_m\|^2 - \int_\Omega F(u_m)\mathrm{d}x.$$

Note that for u_m, both (i) and (ii) of Lemma 5.3 hold. Hence from (32) by a similar argument to the proof of Theorem 5.2 we can obtain

$$E_m(t) \le CE_m(0)e^{-\lambda t}, \quad 0 \le t < \infty$$

for some positive constants C and λ independent of m, which gives

$$\frac{1}{2}\|u_{mt}\|^2 + \frac{1}{2}\|\nabla u_m\|^2 \le CE_m(0)e^{-\lambda t} + \int_\Omega F(u_m)\mathrm{d}x, \quad 0 \le t < \infty. \tag{33}$$

Let $\{u_\nu\}$ be the subsequence of $\{u_m\}$ given in the proof of Theorem 4.1. Then we have

$$\lim_{\nu\to\infty}\int_\Omega F(u_\nu)\mathrm{d}x = \int_\Omega F(u)\mathrm{d}x.$$

Hence from (33) we get

$$\begin{aligned}\frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\nabla u\|^2 &\le \liminf_{\nu\to\infty}\frac{1}{2}\|u_{\nu t}\|^2 + \liminf_{\nu\to\infty}\frac{1}{2}\|\nabla u_\nu\|^2 \\ &\le \liminf_{\nu\to\infty}\left(\frac{1}{2}\|u_{\nu t}\|^2 + \frac{1}{2}\|\nabla u_\nu\|^2\right) \\ &\le \liminf_{\nu\to\infty}\left(CE_\nu(0)e^{-\lambda t} + \int_\Omega F(u_\nu)\mathrm{d}x\right) \\ &= \lim_{\nu\to\infty}\left(CE_\nu(0)e^{-\lambda t} + \int_\Omega F(u_\nu)\mathrm{d}x\right) \\ &= CE(0)e^{-\lambda t} + \int_\Omega F(u)\mathrm{d}x,\end{aligned}$$

which gives (22) and (23). □

Remark 5.5 Note that the proof of Theorem 5.2 and Theorem 5.4 strongly depend on the fact that $u \in W_\delta$ for $\delta_1 < \delta < 1$, where u is the solution of problem (2)–(4) with $0 < E(0) < d$, $I(u_0) > 0$ or $\|\nabla u_0\| = 0$, (δ_1, δ_2) is the maximal interval including $\delta = 1$ such that $d(\delta) > E(0)$ for $\delta \in (\delta_1, \delta_2)$. Therefore the introducing of potential wells $\{W_\delta\}$ is necessary for the proof of Theorem 5.2.

6 Global existence and asymptotic behaviour of solutions for problem (2)–(4) with $E(0) = d$

In this section we prove the global existence, nonexistence and asymptotic behaviour of solutions for problem (2)–(4) with the critical data $E(0) = d$.

We firstly prove the invariance of sets W' and V'.

Lemma 6.1 *Let $\gamma > 0$, $f(u)$ satisfy (H), $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$,*

$$W' = \{u \in H_0^1(\Omega) | I(u) > 0\} \cup \{0\},$$

$$V' = \{u \in H_0^1(\Omega) | I(u) < 0\}.$$

Assume that $E(0) = d$. Then W' and V' are invariant under the flow of (2)–(4) respectively.

Proof We prove this lemma by considering two cases (i) and (ii) for W' and V' respectively.

(i) Let $u(t)$ be any weak solution of problem (2)–(4) with $E(0) = d$, $I(u_0) > 0$ or $\|\nabla u_0\| = 0$, T be the existence time of $u(t)$. We prove that $u(t) \in W'$ for $0 < t < T$. If it is false, then there exists a $t_0 \in (0, T)$ such that $u(t_0) \in \partial W'$, i.e. $I(u(t_0)) = 0$, $\|\nabla u(t_0)\| \neq 0$. Then we have $J(u(t_0)) \geq d$. Hence by

$$\frac{1}{2}\|u_t\|^2 + J(u) + \gamma \int_0^t \|u_\tau\|^2 \mathrm{d}\tau \leq E(0) = d,$$

we get $\int_0^{t_0} \|u_t\|^2 \mathrm{d}t = 0$ and $\|u_t\| = 0$ for $0 \leq t \leq t_0$, which implies $\frac{\mathrm{d}u}{\mathrm{d}t} = 0$ for $x \in \Omega$, $0 \leq t \leq t_0$ and $u(x,t) = u_0(x)$. Hence we have $I(u(t_0)) = I(u_0) > 0$ which contradicts $I(u(t_0)) = 0$.

(ii) Let $u(t)$ be any weak solution of problem (2)–(4) with $E(0) = d$, $I(u_0) < 0$, T be the existence time of $u(t)$. We prove that $I(u) < 0$ for $0 < t < T$. If it is false, then there exists a $t_0 \in (0, T)$ such that $u(t_0) \in \partial V'$, i.e. $I(u(t_0)) = 0$. Let t_0 be the first time such that $I(u) = 0$. Then $I(u) < 0$ for $0 < t < t_0$. Hence by Lemma 2.6 we have $\|\nabla u\| > r(1)$ for $0 < t < t_0$ and $\|\nabla u(t_0)\| \geq r(1)$. So we have $J(u(t_0)) \geq d$. The remainder of this proof is similar to the proof of part (i). □

Lemma 6.2 *Let $\gamma > 0$, $f(u)$ satisfy (H), $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$. Assume that $E(0) = d$, $u(t)$ be a weak solution (not steady state solution) of problem (2)–(4), T be the existence time of $u(t)$. Then there exists a $t_0 \in (0, T)$ such that*

$$\int_0^{t_0} \|u_t\|^2 \mathrm{d}t > 0. \tag{34}$$

Proof Let $u(t)$ be any solution (but not steady state solution) of problem (2)–(4) with $E(0) = d$, T be the existence time of $u(t)$. We prove that there exists a $t_0 \in (0, T)$ such that (34) holds. If it is false, then $\int_0^t \|u_\tau\|^2 \mathrm{d}\tau \equiv 0$ for $0 \leq t < T$, which gives $\|u_t\| = 0$ for $0 \leq t < T$. Hence we have $\frac{\mathrm{d}u}{\mathrm{d}t} = 0$ for $x \in \Omega$, $t \in [0, T)$, which gives $u(t) \equiv u_0$, i.e. $u(t)$ is a steady state solution of problem (2), (4). □

By the argument in the proof of Theorem 22 in [9] we can obtain the following theorems.

The global existence of weak solution

Theorem 6.3 *Let $\gamma \geq 0$, $f(u)$ satisfy (H), $u_0(x) \in H_0^1(\Omega)$, $u_1(x) \in L^2(\Omega)$. Assume that $E(0) = d$ and $I(u_0) \geq 0$. Then problem (2)–(4) admits a global weak solution $u(t) \in L^\infty(0, \infty; H_0^1(\Omega))$ with $u_t(t) \in L^\infty(0, \infty; L^2(\Omega))$ and $u(t) \in \bar{W} = W \cap \partial W$ for $0 \leq t < \infty$.*

The asymptotic behaviour of weak solution

Theorem 6.4 *Let $\gamma > 0$, $f(u)$ satisfy (H), $u_0(x) \in H_0^1(\Omega)$, $u_1(x) \in L^2(\Omega)$. Assume that $E(0) = d$ and $I(u_0) \geq 0$. Then for the global weak solution given in Theorem 6.3, both (22) and (23) hold.*

Proof We only consider the case $\|\nabla u_0\| \neq 0$. Let us recall the proof of Theorem 6.3 (see Theorem 5.1 in [9]). Take $\lambda_m = 1 - \frac{1}{m}$, $m = 2, 3, \ldots$, $u_{0m}(x) = \lambda_m u_0(x)$. Consider the initial conditions

$$u(x,0) = u_{0m}(x), \qquad u_t(x,0) = u_1(x), \quad x \in \Omega \tag{35}$$

and corresponding problem (2), (35), (4). Then we have $0 < E_m(0) < d$, $I(u_{0m}) > 0$, where

$$E_m(0) = \frac{1}{2}\|u_1\|^2 + \frac{1}{2}\|\nabla u_{0m}\|^2 - \int_\Omega F(u_{0m})\mathrm{d}x.$$

Hence from Theorem 4.1 and Theorem 5.4 it follows that for each m problem (2), (35), (4) admits a global weak solution $u_m(t) \in L^\infty(0,\infty; H_0^1(\Omega))$ with $u_{mt}(t) \in L^\infty(0,\infty; L^2(\Omega))$ satisfying

$$E_m(t) \le C E_m(0) e^{-\lambda t}, \quad 0 \le t < \infty$$

for some positive constants C and λ independent of m. The remainder of this proof is the same as the proof of Theorem 5.4. □

The global nonexistence of solution

Theorem 6.5 *Let $0 \le \gamma < (p-1)\lambda_1$, λ_1 is defined in Theorem 4.5, $f(u)$ satisfy (H), $u_0(x) \in H_0^1(\Omega)$, $u_1(x) \in L^2(\Omega)$. Assume that $E(0) = d$ and $I(u_0) < 0$. Then the existence time of solution but not the steady state solution $u(t)$ of problem (2)–(4) is finite.*

Proof Let $u(t)$ be a weak solution of problem (2)–(4), T be the maximal existence time of $u(t)$. We prove that if $u(t)$ is not a steady state solution of problem (2)–(4), then $T < \infty$. In fact, from Lemma 6.2 it follows that there exists a $t_0 \in (0,T)$ such that

$$\int_0^{t_0} \|u_t\|^2 \mathrm{d}t > 0$$

and

$$E(t_0) = E(0) - \gamma \int_0^{t_0} \|u_t\|^2 \mathrm{d}t < d.$$

On the other hand, from Lemma 6.1 we have $I(u(t_0)) < 0$. Hence from Theorem 4.5 it follow that the existence time T of $u(t)$ if finite. □

From Theorem 6.3 and Theorem 6.5 we can obtain the following sharp condition of global existence and nonexistence of solutions for problem (2)–(4) with $E(0) = d$.

Corollary 6.6 *Let $f(u)$ satisfy (H), $u_0(x) \in H_0^1(\Omega)$, $u_1(x) \in L^2(\Omega)$. Assume that $0 < \gamma < (p-1)\lambda_1$ and $E(0) = d$. Then when $I(u_0) > 0$ the solution $u(t)$ of problem (2)–(4) exists globally in time, and when $I(u_0) < 0$ the existence time of solution but not the steady state solution $u(t)$ of problem (2)–(4) is finite.*

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