J Appl Math Comput (2012) 38:225–241
DOI 10.1007/s12190-011-0475-2

JAMC

Existence and uniqueness of positive solutions to m-point boundary value problem for nonlinear fractional differential equation

Sihua Liang · Jihui Zhang

Received: 22 October 2010 / Published online: 16 February 2011

Abstract In this paper, we consider the following nonlinear fractional m-point boundary value problem

$$D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, \quad 0 < t < 1,\ 2 < \alpha \leq 3,$$
$$u(0) = u'(0) = 0, \qquad u'(1) = \sum_{i=1}^{m-2} \beta_i u'(\xi_i),$$

where D_{0+}^{α} is the standard Riemann-Liouville fractional derivative. By the properties of the Green function, the lower and upper solution method and fixed-point theorem in partially ordered sets, some new existence and uniqueness of positive solutions to the above boundary value problem are established. As applications, examples are presented to illustrate the main results.

Keywords Fractional differential equation · Partially ordered sets · Fixed-point theorem · Lower and upper solution method · Positive solution

Mathematics Subject Classification (2000) 26A33 · 34B18 · 34B27

S. Liang (✉)
College of Mathematics, Changchun Normal University, Changchun 130032, Jilin, PR China
e-mail: liangsihua@163.com

J. Zhang
Institute of Mathematics, School of Mathematical Science, Nanjing Normal University, Nanjing, 210046 Jiangsu, PR China
e-mail: jihuiz@jlonline.com

1 Introduction

Recently, an increasing interest in studying the existence of solutions for boundary value problems of fractional order functional differential equations has been observed [4, 10, 21]. Fractional differential equations describe many phenomena in various fields of science and engineering such as physics, mechanics, chemistry, control, engineering, etc. For an extensive collection of such results, we refer the readers to the monographs by Samko et al. [19], Podlubny [17] and Kilbas et al. [9].

On the other hand, some basic theory for the initial value problems of fractional differential equations involving Riemann-Liouville differential operator has been discussed by Lakshmikantham [10–12], Z. Bai and H. Lü [3], A. M. A. El-Sayed et al. [6, 7] and C. Bai [1, 2], S. Zhang [20], etc.

El-Shahed [7] considered the following nonlinear fractional boundary value problem

$$\begin{aligned} &D_{0+}^{\alpha} u(t) + \lambda a(t) f(u(t)) = 0, \quad 0 < t < 1,\ 2 < \alpha \le 3, \\ &u(0) = u'(0) = u'(1) = 0, \end{aligned}$$

where D_{0+}^{α} is the standard Riemann-Liouville fractional derivative. They used the Krasnoselskii's fixed-point theorem on cone expansion and compression to show the existence and non-existence of positive solutions for the above fractional boundary value problem.

Liang and Zhang [13] considered the following nonlinear fractional boundary value problem

$$\begin{aligned} &D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, \quad 0 < t < 1,\ 3 < \alpha \le 4, \\ &u(0) = u'(0) = u''(0) = u''(1) = 0, \end{aligned}$$

where D_{0+}^{α} is the standard Riemann-Liouville fractional derivative. By means of lower and upper solution method and fixed-point theorems, some results on the existence of positive solutions to the above boundary value problems are obtained. But the uniqueness is not treated.

Li, Luo and Zhou [14] considered the following three point boundary value problems of fractional order differential equation

$$\begin{aligned} &D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, \quad 0 < t < 1,\ 1 < \alpha \le 2, \\ &u(0) = 0, \qquad D_{0_+}^{\beta} u(1) = a D_{0_+}^{\beta} u(\xi), \end{aligned}$$

where D_{0+}^{α} is the standard Riemann-Liouville fractional derivative. The existence and multiplicity results of positive solutions by using some fixed-point theorems. But the uniqueness is not treated.

In this paper, we deal with the following m-point boundary value problem

$$D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, \quad 0 < t < 1,\ 2 < \alpha \le 3, \tag{1.1}$$

$$u(0) = u'(0) = 0, \qquad u'(1) = \sum_{i=1}^{m-2} \beta_i u'(\xi_i), \tag{1.2}$$

where D_{0+}^{α} is the standard Riemann-Liouville fractional derivative. $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$ satisfies $0 < \sum_{i=1}^{m-2} \beta_i \xi_i^{\alpha-2} < 1$.

From the above works, we can see a fact, although the fractional boundary value problems have been investigated by some authors, we note that the results dealing with the existence of positive solutions of multi-point boundary value problems of fractional order differential equations are relatively scarce. Motivated by all the works above, in this paper we discuss the boundary value problem (1.1) and (1.2). Using lower and upper solution method and a fixed point theorem in partially ordered sets, we give some new existence and uniqueness criteria for boundary value problem (1.1) and (1.2). Finally, we present some examples to demonstrate our results. Existence of fixed point in partially ordered sets has been considered recently in [5, 8, 15, 16, 18]. This work is motivated by papers [5, 13].

2 Preliminaries

We need the following definitions and lemmas that will be used to prove our the main results.

Definition 2.1 Let $(E, \|\cdot\|)$ be a real Banach space. A nonempty, closed, convex set $P \subset E$ is said to be a cone provided the following are satisfied:

(a) if $y \in P$ and $\lambda \geq 0$, then $\lambda y \in P$;
(b) if $y \in P$ and $-y \in P$, then $y = 0$.

If $P \subset E$ is a cone, we denote the order induced by P on E by $\leq$, that is, $x \leq y$ if and only if $y - x \in P$.

Definition 2.2 [17] The integral

$$I_{0+}^{s} f(x) = \frac{1}{\Gamma(s)} \int_0^x \frac{f(t)}{(x-t)^{1-s}} dt, \quad x > 0,$$

where $s > 0$, is called Riemann-Liouville fractional integral of order s and $\Gamma(s)$ is the Euler gamma function defined by

$$\Gamma(s) = \int_0^{+\infty} t^{s-1} e^{-t} dt, \quad s > 0.$$

Definition 2.3 [9] For a function $f(x)$ given in the interval $[0, \infty)$, the expression

$$D_{0+}^{s} f(x) = \frac{1}{\Gamma(n-s)} \left(\frac{d}{dx}\right)^n \int_0^x \frac{f(t)}{(x-t)^{s-n+1}} dt,$$

where $n = [s] + 1$, $[s]$ denotes the integer part of number s, is called the Riemann-Liouville fractional derivative of order s.

The following two lemmas can be found in [3, 9] which are crucial in finding an integral representation of fractional boundary value problem (1.1) and (1.2).

Lemma 2.1 [3, 9] *Let* $\alpha > 0$ *and* $u \in C(0,1) \cap L(0,1)$. *Then fractional differential equation*

$$D_{0+}^{\alpha} u(t) = 0$$

has

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}, \quad c_i \in \mathbb{R},\ i = 0, 1, \ldots, n,\ n = [\alpha] + 1$$

as unique solutions.

Lemma 2.2 [3, 9] *Assume that* $u \in C(0,1) \cap L(0,1)$ *with a fractional derivative of order* $\alpha > 0$ *that belongs to* $C(0,1) \cap L(0,1)$. *Then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, \ldots, n$, $n = [\alpha] + 1$.

The following fixed-point theorems in partially ordered sets are fundamental and important to the proofs of our main results.

Theorem 2.1 [8] *Let* $(E, \le)$ *be a partially ordered set and suppose that there exists a metric* d *in* E *such that* (E, d) *is a complete metric space. Assume that* E *satisfies the following condition*

$$\begin{aligned} &\text{if } \{x_n\} \text{ is a nondecreasing sequence in } E \text{ such that } x_n \to x, \\ &\quad \text{then } x_n \le x,\ \forall\, n \in \mathbb{N}. \end{aligned} \tag{2.1}$$

Let $T : E \to E$ *be nondecreasing mapping such that*

$$d(Tx, Ty) \le d(x, y) - \psi(d(x, y)), \quad \text{for } x \ge y,$$

where $\psi : [0, +\infty) \to [0, +\infty)$ *is a continuous and nondecreasing function such that* ψ *is positive in* $(0, +\infty)$, $\psi(0) = 0$ *and* $\lim_{t\to\infty} \psi(t) = \infty$. *If there exists* $x_0 \in E$ *with* $x_0 \le T(x_0)$, *then* T *has a fixed point.*

If we consider that $(E, \le)$ satisfies the following condition

$$\text{for } x, y \in E \text{ there exists } z \in E \text{ which is comparable to } x \text{ and } y, \tag{2.2}$$

then we have the following result.

Theorem 2.2 [15] *Adding condition* (2.2) *to the hypotheses of Theorem* 2.1, *we obtain uniqueness of the fixed point.*

3 Related lemmas

Lemma 3.1 *Let* $\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-2}\neq 1$. *If* $h\in C[0,1]$, *then the boundary value problem*

$$D_{0+}^{\alpha}u(t)+h(t)=0,\qquad 0<t<1,\ 2<\alpha\le 3, \tag{3.1}$$

$$u(0)=u'(0)=0,\qquad u'(1)=\sum_{i=1}^{m-2}\beta_i u'(\xi_i), \tag{3.2}$$

has a unique solution

$$u(t)=\int_0^1 G(t,s)h(s)ds+\frac{t^{\alpha-1}\sum_{i=1}^{m-2}\beta_i}{(\alpha-1)\big(1-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-2}\big)}\int_0^1 H(\xi_i,s)h(s)ds, \tag{3.3}$$

where

$$G(t,s)=\frac{1}{\Gamma(\alpha)}\begin{cases} t^{\alpha-1}(1-s)^{\alpha-2}-(t-s)^{\alpha-1}, & 0\le s\le t\le 1,\\ t^{\alpha-1}(1-s)^{\alpha-2}, & 0\le t\le s\le 1,\end{cases} \tag{3.4}$$

$$H(t,s)=\frac{\partial G(t,s)}{\partial t}=\frac{\alpha-1}{\Gamma(\alpha)}\begin{cases} t^{\alpha-2}(1-s)^{\alpha-2}-(t-s)^{\alpha-2}, & 0\le s\le t\le 1,\\ t^{\alpha-2}(1-s)^{\alpha-2}, & 0\le t\le s\le 1.\end{cases} \tag{3.5}$$

Proof By Lemma 2.2, the solution of (3.1) can be written as

$$u(t)=c_1t^{\alpha-1}+c_2t^{\alpha-2}+c_3t^{\alpha-3}-\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds.$$

From (3.2), we know that $c_2=c_3=0$ and

$$c_1=\frac{1}{\Gamma(\alpha)\big(1-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-2}\big)}\bigg[\int_0^1(1-s)^{\alpha-2}h(s)ds - \sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i}(\xi_i-s)^{\alpha-2}h(s)ds\bigg].$$

Therefore, the unique solution of boundary value problem (3.1), (3.2) is

$$\begin{aligned} u(t)={}&-\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds\\ &+\frac{t^{\alpha-1}}{\Gamma(\alpha)\big(1-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-2}\big)}\bigg[\int_0^1(1-s)^{\alpha-2}h(s)ds\\ &-\sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i}(\xi_i-s)^{\alpha-2}h(s)ds\bigg]\end{aligned}$$

$$
\begin{aligned}
&= -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds \\
&\quad + \left(\frac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^{\alpha-1}\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-2}}{\Gamma(\alpha)\left(1-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-2}\right)}\right)\int_0^1 (1-s)^{\alpha-2}h(s)ds \\
&\quad - \frac{t^{\alpha-1}\sum_{i=1}^{m-2}\beta_i}{\Gamma(\alpha)\left(1-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-2}\right)}\int_0^{\xi_i}(\xi_i-s)^{\alpha-2}h(s)ds \\
&= \frac{1}{\Gamma(\alpha)}\int_0^t \left(t^{\alpha-1}(1-s)^{\alpha-2}-(t-s)^{\alpha-1}\right)h(s)ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_t^1 t^{\alpha-1}(1-s)^{\alpha-2}h(s)ds \\
&\quad + \frac{t^{\alpha-1}\sum_{i=1}^{m-2}\beta_i}{\Gamma(\alpha)\left(1-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-2}\right)}\left[\int_0^{\xi_i}\xi_i^{\alpha-2}(1-s)^{\alpha-2}h(s)ds\right. \\
&\quad \left. - \int_0^{\xi_i}(\xi_i-s)^{\alpha-2}h(s)ds\right] \\
&\quad + \frac{t^{\alpha-1}\sum_{i=1}^{m-2}\beta_i}{\Gamma(\alpha)\left(1-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-2}\right)}\int_{\xi_i}^1 \xi_i^{\alpha-2}(1-s)^{\alpha-2}h(s)ds \\
&= \int_0^1 G(t,s)h(s)ds + \frac{t^{\alpha-1}\sum_{i=1}^{m-2}\beta_i}{(\alpha-1)\left(1-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-2}\right)}\int_0^1 H(\xi_i,s)h(s)ds.
\end{aligned}
$$

The proof is complete. □

Lemma 3.2 *G is a continuous function and $G(t,s)\geq 0$.*

Proof The continuity of G is easily checked. On the other hand, for $0\leq t\leq s\leq 1$ it is obvious that

$$G(t,s)=\frac{t^{\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)}\geq 0.$$

In the case $0\leq s\leq t\leq 1$ $(s\neq 1)$, we have

$$
\begin{aligned}
G(t,s) &= \frac{1}{\Gamma(\alpha)}\left[\frac{t^{\alpha-1}(1-s)^{\alpha-1}}{1-s}-(t-s)^{\alpha-1}\right] \\
&\geq \frac{1}{\Gamma(\alpha)}\left[t^{\alpha-1}(1-s)^{\alpha-1}-(t-s)^{\alpha-1}\right] \\
&= \frac{1}{\Gamma(\alpha)}\left[(t-ts)^{\alpha-1}-(t-s)^{\alpha-1}\right] \\
&\geq 0.
\end{aligned}
$$

Moreover, as $G(t, 1) = 0$, then we conclude that $G(t, s) \geq 0$ for all $(t, s) \in [0, 1] \times [0, 1]$. The proof is complete. □

Remark 3.1 Obviously, by Lemmas 3.1 and 3.2, we have $u(t) \geq 0$ if $h(t) \geq 0$ on $t \in [0, 1]$.

Lemma 3.3

$$\sup_{t\in[0,1]} \int_0^1 G(t, s)ds = \frac{1}{(\alpha - 1)\Gamma(\alpha + 1)}, \qquad \int_0^1 H(\eta, s)ds = \frac{\eta^{\alpha-2} - \eta^{\alpha-1}}{\Gamma(\alpha)}.$$

Proof Since

$$\begin{aligned}
\int_0^1 G(t, s)ds &= \int_0^t G(t, s)ds + \int_t^1 G(t, s)ds \\
&= \frac{1}{\Gamma(\alpha)} \int_0^t (t^{\alpha-1}(1 - s)^{\alpha-2} - (t - s)^{\alpha-1})ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_t^1 t^{\alpha-1}(1 - s)^{\alpha-2}ds \\
&= \frac{1}{\Gamma(\alpha)} \left(\frac{t^{\alpha-1}}{\alpha - 1} - \frac{t^\alpha}{\alpha} \right).
\end{aligned}$$

On the other hand, let

$$\phi(t) = \int_0^1 G(t, s)ds = \frac{1}{\Gamma(\alpha)} \left(\frac{t^{\alpha-1}}{\alpha - 1} - \frac{t^\alpha}{\alpha} \right),$$

then, as

$$\phi'(t) = \frac{1}{\Gamma(\alpha)} \left(t^{\alpha-2} - t^{\alpha-1} \right) > 0, \quad \text{for } t > 0,$$

the function $\phi(t)$ is strictly increasing and, consequently,

$$\begin{aligned}
\sup_{t\in[0,1]} \phi(t) &= \sup_{t\in[0,1]} \int_0^1 G(t, s)ds = \phi(1) = \frac{1}{\Gamma(\alpha)} \left(\frac{1}{\alpha - 1} - \frac{1}{\alpha} \right) \\
&= \frac{1}{\alpha(\alpha - 1)\Gamma(\alpha)} = \frac{1}{(\alpha - 1)\Gamma(\alpha + 1)}.
\end{aligned}$$

By direct computation, we have

$$\int_0^1 H(\eta, s)ds = \frac{\eta^{\alpha-2} - \eta^{\alpha-1}}{\Gamma(\alpha)}.$$

The proof is complete. □

Remark 3.2 From Lemma 3.3, we have

$$L = \sup_{t\in[0,1]} \int_0^1 G(t,s)ds + \frac{\sum_{i=1}^{m-2} \beta_i}{(\alpha-1)\big(1-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-2}\big)} \int_0^1 H(\xi_i,s)ds$$
$$= \frac{1}{\alpha(\alpha-1)\Gamma(\alpha)} + \frac{\sum_{i=1}^{m-2}\beta_i(\xi_i^{\alpha-2}-\xi_i^{\alpha-1})}{\Gamma(\alpha)(\alpha-1)\big(1-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-2}\big)}. \quad (3.6)$$

Lemma 3.4 *$G(t,s)$ is strictly increasing in the first variable.*

Proof For s fixed, we let

$$g_1(t) = \frac{1}{\Gamma(\alpha)}\Big(t^{\alpha-1}(1-s)^{\alpha-2} - (t-s)^{\alpha-1}\Big), \quad \text{for } s \le t,$$
$$g_2(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1}(1-s)^{\alpha-1}, \quad \text{for } t \le s.$$

It is easy to check that $g_1(t)$ is strictly increasing on $[s,1]$ and $g_2(t)$ is strictly increasing on $[0,s]$. Then we have the following cases:

Case 1: $t_1, t_2 \le s$ and $t_1 < t_2$. In this case, we have $g_2(t_1) < g_2(t_2)$, i.e. $G(t_1,s) < G(t_2,s)$.
Case 2: $s \le t_1, t_2$ and $t_1 < t_2$. In this case, we have $g_1(t_1) < g_1(t_2)$, i.e. $G(t_1,s) < G(t_2,s)$.
Case 3: $t_1 \le s \le t_2$ and $t_1 < t_2$. In this case, we have $g_2(t_1) \le g_2(s) = g_1(s) \le g_1(t_2)$.

We claim that $g_2(t_1) < g_1(t_2)$. In fact, if $g_2(t_1) = g_1(t_2)$, then $g_2(t_1) = g_2(s) = g_1(s) = g_1(t_2)$, from the monotone of g_1 and g_2, we have $t_1 = s = t_2$, which contradicts with $t_1 < t_2$. This fact implies that $G(t_1,s) < G(t_2,s)$. The proof is complete. □

4 Uniqueness of a positive and nondecreasing solution for the boundary value problems (1.1)–(1.2)

In this section, we establish the existence and uniqueness of a positive and nondecreasing solution for the boundary value problems (1.1)–(1.2) by using a fixed point theorem in partially ordered sets. The basic space used in this section is $E = C[0,1]$. Then E is a real Banach space with the norm $\|u\| = \max_{0\le t\le 1}|u(t)|$. Note that this space can be equipped with a partial order given by

$$x, y \in C[0,1], \quad x \le y \quad \Leftrightarrow \quad x(t) \le y(t), \quad t \in [0,1].$$

In [15] it is proved that $(C[0,1], \le)$ with the classic metric given by

$$d(x,y) = \sup_{0\le t\le 1}\{|x(t)-y(t)|\}$$

satisfied condition (2.1) of Theorem 2.1. Moreover, for $x, y \in C[0,1]$ as the function $\max\{x,y\} \in C[0,1]$, $(C[0,1], \le)$, satisfies condition (2.2).

The main result of this paper is the following.

Theorem 4.1 *The boundary value problem* (1.1)–(1.2) *has a unique positive and strictly increasing solution* $u(t)$ *if the following conditions are satisfied*:

(i) $f : [0, 1] \times [0, +\infty) \to [0, +\infty)$ *is continuous and nondecreasing respect to the second variable and* $f(t, u(t)) \not\equiv 0$ *for* $t \in Z \subset [0, 1]$ *with* $\mu(Z) > 0$ (μ *denotes the Lebesgue measure*);

(ii) *There exists* $0 < \lambda < L^{-1}$ *such that for* $u, v \in [0, +\infty)$ *with* $u \geq v$ *and* $t \in [0, 1]$

$$f(t, u) - f(t, v) \leq \lambda \cdot \ln(u - v + 1).$$

Proof Consider the cone

$$K = \{u \in C[0, 1] : u(t) \geq 0\}.$$

As K is a closed set of $C[0, 1]$, K is a complete metric space with the distance given by $d(u, v) = \sup_{t\in[0,1]} |u(t) - v(t)|$.

Now, we consider the operator T defined by

$$Tu(t) = \int_0^1 G(t, s)f(s, u(s))ds + \frac{t^{\alpha-1}\sum_{i=1}^{m-2}\beta_i}{(\alpha-1)\big(1-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-2}\big)}\int_0^1 H(\xi_i, s)f(s, u(s))ds,$$

by Lemma 3.2 and condition (i), we have that $T(K) \subset K$.

We now show that all the conditions of Theorems 2.1 and 2.2 are satisfied.

Firstly, by condition (i), for $u, v \in K$ and $u \geq v$, we have

$$\begin{aligned} Tu(t) &= \int_0^1 G(t, s)f(s, u(s))ds \\ &\quad + \frac{t^{\alpha-1}\sum_{i=1}^{m-2}\beta_i}{(\alpha-1)\big(1-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-2}\big)}\int_0^1 H(\xi_i, s)f(s, u(s))ds \\ &\geq \int_0^1 G(t, s)f(s, v(s))ds \\ &\quad + \frac{t^{\alpha-1}\sum_{i=1}^{m-2}\beta_i}{(\alpha-1)\big(1-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-2}\big)}\int_0^1 H(\xi_i, s)f(s, v(s))ds \\ &= Tv(t). \end{aligned}$$

This proves that T is a nondecreasing operator.

On the other hand, for $u \geq v$ and by condition (ii) we have

$$\begin{aligned} d(Tu, Tv) &= \sup_{0\leq t\leq 1} |(Tu)(t) - (Tv)(t)| = \sup_{0\leq t\leq 1} ((Tu)(t) - (Tv)(t)) \\ &\leq \sup_{0\leq t\leq 1} \int_0^1 G(t, s)(f(s, u(s)) - f(s, v(s)))ds \end{aligned}$$

$$+\frac{\sum_{i=1}^{m-2}\beta_i}{(\alpha-1)\big(1-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-2}\big)}$$
$$\times\int_0^1 H(\xi_i,s)(f(s,u(s))-f(s,v(s)))ds$$
$$\leq \sup_{0\leq t\leq 1}\int_0^1 G(t,s)\lambda\cdot\ln(u(s)-v(s)+1)ds$$
$$+\frac{\sum_{i=1}^{m-2}\beta_i}{(\alpha-1)\big(1-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-2}\big)}$$
$$\times\int_0^1 H(\xi_i,s)\lambda\cdot\ln(u(s)-v(s)+1)ds.$$

Since the function $h(x)=\ln(x+1)$ is nondecreasing, by Lemma 3.3 and condition (ii), then we have

$$\begin{aligned}
&d(Tu,Tv)\\
&\quad\leq \lambda\ln(\|u-v\|+1)\left(\sup_{0\leq t\leq 1}\int_0^1 G(t,s)ds\right.\\
&\qquad\left.+\frac{\sum_{i=1}^{m-2}\beta_i}{(\alpha-1)\big(1-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-2}\big)}\int_0^1 H(\xi_i,s)ds\right)\\
&\quad=\lambda\ln(\|u-v\|+1)\cdot L\\
&\quad\leq \|u-v\|-(\|u-v\|-\ln(\|u-v\|+1)).
\end{aligned}$$

Let $\psi(x)=x-\ln(x+1)$. Obviously $\psi:[0,+\infty)\to[0,+\infty)$ is continuous, nondecreasing, positive in $(0,+\infty)$, $\psi(0)=0$ and $\lim_{x\to+\infty}\psi(x)=+\infty$. Thus, for $u\geq v$, we have

$$d(Tu,Tv)\leq d(u,v)-\psi(d(u,v)).$$

As $G(t,s)\geq 0$ and $f\geq 0$,

$$\begin{aligned}
(T0)(t)&=\int_0^1 G(t,s)f(s,0)ds\\
&\quad+\frac{t^{\alpha-1}\sum_{i=1}^{m-2}\beta_i}{(\alpha-1)\big(1-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-2}\big)}\int_0^1 H(\xi_i,s)f(s,0)ds\geq 0
\end{aligned}$$

and by Theorem 2.1 we know that problem (1.1)–(1.2) has at least one nonnegative solution. As $(K,\leq)$ satisfies condition (2.2), thus, Theorem 2.2 implies that uniqueness of the solution.

Finally, we will prove that this solution $u(t)$ is strictly increasing function. As $u(0)=\int_0^1 G(0,s)f(s,u(s))ds$ and $G(0,s)=0$ we have $u(0)=0$.

Moreover, if we take $t_1, t_2 \in [0, 1]$ with $t_1 < t_2$, we can consider the following cases.

Case 1: $t_1 = 0$. In this case, $u(t_1) = 0$ and, as $u(t) \geq 0$, suppose that $u(t_2) = 0$. Then

$$0 = u(t_2) = \int_0^1 G(t_2, s) f(s, u(s)) ds + \frac{t_2^{\alpha-1} \sum_{i=1}^{m-2} \beta_i}{(\alpha - 1)\big(1 - \sum_{i=1}^{m-2} \beta_i \xi_i^{\alpha-2}\big)} \int_0^1 H(\xi_i, s) f(s, u) ds.$$

This implies that

$$G(t_2, s) \cdot f(s, u(s)) = 0, \quad \text{a.e.} \quad (s)$$

and as $G(t_2, s) \neq 0$ a.e. (s) we get $f(s, u(s)) = 0$ a.e. (s).

On the other hand, f is nondecreasing respect to the second variable, then we have

$$f(s, 0) \leq f(s, u(s)) = 0, \quad \text{a.e.} \quad (s)$$

which contradicts the condition (i) $f(t, 0) \neq 0$ for $t \in Z \subset [0, 1](\mu(Z) \neq 0)$. Thus $u(t_1) = 0 < u(t_2)$.

Case 2: $0 < t_1$. In this case, let us take $t_2, t_1 \in [0, 1]$ with $t_1 < t_2$, then

$$\begin{aligned} u(t_2) - u(t_1) &= (Tu)(t_2) - (Tu)(t_1) \\ &= \int_0^1 (G(t_2, s) - G(t_1, s)) f(s, u(s)) ds \\ &\quad + \frac{(t_2^{\alpha-1} - t_1^{\alpha-1}) \sum_{i=1}^{m-2} \beta_i}{(\alpha - 1)\big(1 - \sum_{i=1}^{m-2} \beta_i \xi_i^{\alpha-2}\big)} \int_0^1 H(\xi_i, s) f(s, u(s)) ds. \end{aligned}$$

Taking into account Lemma 3.4 and the fact that $f \geq 0$, we get $u(t_2) - u(t_1) \geq 0$.

Suppose that $u(t_2) = u(t_1)$ then

$$\int_0^1 (G(t_2, s) - G(t_1, s)) f(s, u(s)) ds = 0$$

and this implies

$$(G(t_2, s) - G(t_1, s)) f(s, u(s)) = 0 \quad \text{a.e.} \quad (s).$$

Again, Lemma 3.4 gives us

$$f(s, u(s)) = 0 \quad \text{a.e.} \quad (s)$$

and using the same reasoning as above we have that this contradicts condition (i) $f(t, 0) \neq 0$ for $t \in Z \subset [0, 1]$ $(\mu(Z) \neq 0)$. Thus $u(t_1) = 0 < u(t_2)$. The proof is complete. □

5 Single positive solution of the boundary value problems (1.1)–(1.2)

In this section, we establish the existence of single positive solution for boundary value problem (1.1) and (1.2) by lower and upper solution method. We assume that $f:[0,1]\times[0,+\infty)\to[0,+\infty)$ is continuous in this section.

Lemma 5.1 *If* $u(t)\in C[0,1]$ *and is a positive solution of* (1.1) *and* (1.2), *then* $m\rho(t)\le u(t)\le M\rho(t)$, *where*

$$\rho(t)=\int_0^1 G(t,s)ds+\frac{t^{\alpha-1}\sum_{i=1}^{m-2}\beta_i}{(\alpha-1)\big(1-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-2}\big)}\int_0^1 H(\xi_i,s)ds,$$

and m, M *are two constants.*

Proof Since $u(t)\in C[0,1]$, there exists $M'>0$ so that $|u(t)|\le M'$ for $t\in[0,1]$. Taking

$$m:=\min_{(t,u)\in[0,1]\times[0,M']} f(t,u(t)),\qquad M:=\max_{(t,u)\in[0,1]\times[0,M']} f(t,u(t)).$$

By view of Lemma 3.1, we have

$$m\rho(t)\le u(t)\le M\rho(t).$$

Thus we finished the proof of Lemma 5.1. □

Now we introduce the following two definitions about the upper and lower solutions of fractional boundary value problem (1.1) and (1.2).

Definition 5.1 A function $\theta(t)$ is called a lower solution of fractional boundary value problem (1.1) and (1.2) if $\theta(t)\in C[0,1]$ and $\theta(t)$ satisfies

$$-D_{0+}^{\alpha}\theta(t)\le f(t,\theta(t)),\quad 0<t<1,\ 2<\alpha\le 3,$$

$$\theta(0)\le 0,\qquad \theta'(0)\le 0,\qquad \theta'(1)\le\sum_{i=1}^{m-2}\beta_i\theta'(\xi_i).$$

Definition 5.2 A function $\gamma(t)$ is called an upper solution of fractional boundary value problem (1.1) and (1.2) if $\gamma(t)\in C[0,1]$ and $\gamma(t)$ satisfies

$$-D_{0+}^{\alpha}\gamma(t)\ge f(t,\gamma(t)),\quad 0<t<1,\ 2<\alpha\le 3,$$

$$\gamma(0)\ge 0,\qquad \gamma'(0)\ge 0,\qquad \gamma'(1)\ge\sum_{i=1}^{m-2}\beta_i\gamma'(\xi_i).$$

The main result of this paper is the following.

Theorem 5.1 *The fractional boundary value problem* (1.1) *and* (1.2) *has a positive solution* $u(t)$ *if the following conditions are satisfied*:
(H_f) $f(t,u) \in C([0,1] \times [0,+\infty), \mathbb{R}^+)$ *is nondecreasing relative to* u, $f(t,\rho(t)) \not\equiv 0$ *for* $t \in (0,1)$ *and there exists a positive constant* $\mu < 1$ *such that*

$$k^{\mu} f(t,u) \leq f(t,ku), \quad \forall\, 0 \leq k \leq 1.$$

Proof At first, we will prove that the functions $\theta(t) = k_1 g(t)$, $\gamma(t) = k_2 g(t)$ are lower and upper solutions of (1.1) and (1.2), respectively, where $0 < k_1 \leq \min\{\frac{1}{a_2}, (a_1)^{\frac{\mu}{1-\mu}}\}$, $k_2 \geq \max\{\frac{1}{a_1}, (a_2)^{\frac{\mu}{1-\mu}}\}$ and

$$a_1 = \min\left\{1, \inf_{t\in[0,1]} f(t,\rho(t))\right\} > 0, \qquad a_2 = \max\left\{1, \sup_{t\in[0,1]} f(t,\rho(t))\right\}$$

and

$$g(t) = \int_0^1 G(t,s) f(s,\rho(s))ds + \frac{t^{\alpha-1}\sum_{i=1}^{m-2}\beta_i}{(\alpha-1)\left(1-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-2}\right)} \int_0^1 H(\xi_i,s) f(s,\rho(s))ds.$$

By view of Lemma 3.1, we know that $g(t)$ is a positive solution of the following equations

$$\begin{aligned} &D_{0+}^{\alpha} u(t) = f(t,\rho(t)), \quad 0<t<1,\ 2<\alpha\leq 3,\\ &u(0) = u'(0) = 0, \qquad u'(1) = \sum_{i=1}^{m-2}\beta_i u'(\xi_i). \end{aligned} \tag{5.1}$$

From the conclusion of Lemma 5.1, we know that

$$a_1\rho(t) \leq g(t) \leq a_2\rho(t), \quad \forall\, t\in[0,1]. \tag{5.2}$$

Thus, by virtue of the assumption of the Theorem 5.1, shows that

$$\begin{aligned} &k_1 a_1 \leq \frac{\theta(t)}{\rho(t)} \leq k_1 a_2 \leq 1, \qquad \frac{1}{k_2 a_2} \leq \frac{\rho(t)}{\gamma(t)} \leq \frac{1}{k_2 a_1} \leq 1,\\ &(k_1 a_1)^{\mu} \geq k_1, \qquad (k_2 a_2)^{\mu} \leq k_2. \end{aligned}$$

Therefore, we have

$$\begin{aligned} f(t,\theta(t)) &= f\left(t, \frac{\theta(t)}{\rho(t)}\rho(t)\right) \geq \left(\frac{\theta(t)}{\rho(t)}\right)^{\mu} f(t,\rho(t))\\ &\geq (k_1 a_1)^{\mu} f(t,\rho(t)) \geq k_1 f(t,\rho(t)), \end{aligned}$$

$$k_2 f(t,\rho(t)) = k_2 f\left(t, \frac{\rho(t)}{\gamma(t)}\gamma(t)\right) \geq k_2 \left(\frac{\rho(t)}{\gamma(t)}\right)^{\mu} f(t,\gamma(t))$$
$$\geq k_2(k_2 a_2)^{-\mu} f(t,\gamma(t)) \geq f(t,\gamma(t)).$$

It implies that

$$\begin{aligned} -D_{0+}^{\alpha}\theta(t) &= k_1 f(t,\rho(t)) \leq f(t,\theta(t)), \quad 0<t<1,\ 2<\alpha\leq 3, \\ -D_{0+}^{\alpha}\gamma(t) &= k_2 f(t,\rho(t)) \geq f(t,\gamma(t)), \quad 0<t<1,\ 2<\alpha\leq 3. \end{aligned} \tag{5.3}$$

Obviously, $\theta(t) = k_1 g(t)$, $\gamma(t) = k_2 g(t)$ satisfies the boundary conditions (1.2). So, $\alpha(t) = k_1 g(t)$, $\beta(t) = k_2 g(t)$ are lower and upper solutions of (1.1) and (1.2) respectively.

Next, we will prove fractional boundary value problem

$$\begin{aligned} &-D_{0+}^{\alpha}u(t) = g(t,u(t)), \quad 0<t<1,\ 2<\alpha\leq 3, \\ &u(0)=u'(0)=0, \qquad u'(1) = \textstyle\sum_{i=1}^{m-2}\beta_i u'(\xi_i) \end{aligned} \tag{5.4}$$

has a solution, where

$$g(t,u(t)) = \begin{cases} f(t,\theta(t)), & \text{if } u(t)\leq\theta(t), \\ f(t,u(t)), & \text{if } \theta(t)\leq u(t)\leq\gamma(t), \\ f(t,\gamma(t)), & \text{if } \gamma(t)\leq u(t). \end{cases}$$

Thus, we consider the operator $A: C[0,1]\to C[0,1]$ define as follows

$$Au(t) = \int_0^1 G(t,s)g(s,u(s))ds + \frac{t^{\alpha-1}\sum_{i=1}^{m-2}\beta_i}{(\alpha-1)\left(1-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-2}\right)}\int_0^1 H(\xi_i,s)g(s,u(s))ds,$$

where $G(t,s)$ and $H(t,s)$ are defined in Lemma 3.1. It is clear that A is continuous in $C[0,1]$. Since the function $f(t,u)$ in nondecreasing in u, this shows that, for any $u\in C[0,1]$,

$$f(t,\theta(t)) \leq g(t,u(t)) \leq f(t,\gamma(t)) \quad \text{for } t\in[0,1].$$

The operator $A: C[0,1]\to C[0,1]$ is continuous in view of continuity of $G(t,s)$ and $g(t,u(t))$. By means of Arzela-Ascoli theorem, A is a compact operator. Therefore, from Leray-Schauder fixed point theorem, the operator A has a fixed point, i.e., fractional boundary value problem (5.4) has a solution.

Finally, we will prove that fractional boundary value problem (1.1) and (1.2) has a positive solution.

Suppose $u^*(t)$ is a solution of fractional boundary value problem (5.4). Since the function $f(t,u)$ is nondecreasing in u, we know that

$$f(t,\theta(t)) \leq g(t,u^*(t)) \leq f(t,\gamma(t)) \quad \text{for } t\in[0,1].$$

Thus

$$
\begin{aligned}
&-D_{0+}^{\alpha} z(t) \geq f(t, \gamma(t)) - g(t, u^*(t)) \geq 0, \\
&z(0) = z'(0) = 0, \qquad z'(1) = \sum_{i=1}^{m-2} \beta_i z'(\xi_i),
\end{aligned}
\tag{5.5}
$$

where $z(t) = \gamma(t) - u^*(t)$. By virtue of Remark 3.1, $z(t) \geq 0$, i.e., $u^*(t) \leq \gamma(t)$ for $t \in [0, 1]$. Similarly, $\theta(t) \leq u^*(t)$ for $t \in [0, 1]$. Therefore, $u^*(t)$ is a positive solution of fractional boundary value problem (1.1) and (1.2). We have finished the proof of Theorem 3.1. □

6 Example

Example 6.1 *The fractional boundary value problem*

$$
\begin{cases}
D_{0+}^{\frac{5}{2}} u(t) + (t^2+1)\ln(2+u(t)) = 0, \qquad 0 < t < 1, \\
u(0) = u'(0) = 0, \qquad u'(1) = \frac{1}{2} u'(\frac{1}{4})
\end{cases}
\tag{6.1}
$$

has a unique and strictly increasing solution.

Proof In this case, $f(t, u) = (t^2+1)\ln(2+u)$ for $(t, u) \in [0, 1] \times [0, \infty)$. Note that f is a continuous function and $f(t, u) \not\equiv 0$ for $t \in [0, 1]$. Moreover, f is nondecreasing respect to the second variable since $\frac{\partial f}{\partial u} = \frac{1}{u+2}(t^2+1) > 0$. On the other hand, for $u \geq v$ and $t \in [0, 1]$, we have

$$
\begin{aligned}
f(t, u) - f(t, v) &= (t^2+1)\ln(2+u) - (t^2+1)\ln(2+v) = (t^2+1)\ln\left(\frac{2+u}{2+v}\right) \\
&= (t^2+1)\ln\left(\frac{2+v+u-v}{2+v}\right) = (t^2+1)\ln\left(1+\frac{u-v}{2+v}\right) \\
&\leq (t^2+1)\ln(1+(u-v)) \leq 2\ln(1+u-v).
\end{aligned}
$$

In this case, $\lambda = 2, \xi_1 = \frac{1}{4}, \beta_1 = \frac{1}{2}, \alpha = \frac{5}{2}$ because

$$
L = \frac{1}{\alpha(\alpha-1)\Gamma(\alpha)} + \frac{\beta_1(\xi_1^{\alpha-2} - \xi_1^{\alpha-1})}{\Gamma(\alpha)(\alpha-1)(1-\beta_1\xi_1^{\alpha-2})} < \frac{1}{2} = \frac{1}{\lambda}.
$$

Thus Theorem 4.1 implies that boundary value problem (1.1)–(1.2) has a unique and strictly increasing solution. □

Example 6.2 *As an example we mention the following fractional boundary value problem*

$$\begin{cases} D_{0+}^{\frac{5}{2}} u(t) + f(t, u(t)) = 0, & 0 < t < 1, \\ u(0) = u'(0) = 0, & u'(1) = \sum_{i=1}^{m-2} \beta_i u'(\xi_i) \end{cases} \tag{6.2}$$

where D_{0+}^{α} is the standard Riemann-Liouville fractional derivative and

$$f(t, u) = t + u^{\mu}, \quad 0 < \mu < 1.$$

Proof Since $k^{\mu} \leq 1$ for $0 < \mu < 1$ and $0 \leq k \leq 1$. It is easy to check that

$$k^{\mu} f(t, u) = k^{\mu} t + k^{\mu} u^{\mu} \leq t + (ku)^{\mu} = f(t, ku).$$

Thus, by Theorem 5.1 we know that the boundary value problem (6.2) has a positive solution $u(t)$. □

Acknowledgements The Project is supported by NSFC (10871096), Ji Jiao Ke He Zi [2011] No. 196, Natural Science Foundation of Changchun Normal University.

The authors thank the anonymous referee for valuable suggestions regarding the original manuscript.

References

1. Bai, C.: Positive solutions for nonlinear fractional differential equations with coefficient that changes sign. Nonlinear Anal. **64**, 677–685 (2006)
2. Bai, C.: Triple positive solutions for a boundary value problem of nonlinear fractional differential equation. Electron. J. Qual. Theory Differ. Equ. **24**, 1–10 (2008)
3. Bai, Z., Lü, H.: Positive solutions for boundary value problem of nonlinear fractional differential equation. J. Math. Anal. Appl. **311**, 495–505 (2005)
4. Benchohra, M., Henderson, J., Ntouyas, S.K., Ouahab, A.: Existence results for fractional order functional differential equations with infinite delay. J. Math. Anal. Appl. **338**, 1340–1350 (2008)
5. Caballero Mena, J., Harjani, J., Sadarangani, K.: Existence and uniqueness of positive and nondecreasing solutions for a class of singular fractional boundary value problems. Bound. Value Probl. **2009**, 421310 (2009). doi:10.1155/2009/421310. 10 pages
6. El-Sayed, A.M.A., El-Mesiry, A.E.M., El-Saka, H.A.A.: On the fractional-order logistic equation. Appl. Math. Lett. **20**, 817–823 (2007)
7. El-Shahed, M.: Positive solutions for boundary value problem of nonlinear fractional differential equation. Abstr. Appl. Anal. **2007**, 10368 (2007). doi:10.1155/2007/10368. 8 pages
8. Harjani, J., Sadarangani, K.: Fixed point theorems for weakly contractive mappings in partially ordered sets. Nonlinear Anal. **71**, 3403–3410 (2009)
9. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, vol. 204. Elsevier, Amsterdam (2006)
10. Lakshmikantham, V.: Theory of fractional functional differential equations. Nonlinear Anal. **69**, 3337–3343 (2008)
11. Lakshmikantham, V., Vatsala, A.S.: Basic theory of fractional differential equations. Nonlinear Anal. **69**, 2677–2682 (2008)
12. Lakshmikantham, V., Vatsala, A.S.: General uniqueness and monotone iterative technique for fractional differential equations. Appl. Math. Lett. **21**, 828–834 (2008)
13. Liang, S., Zhang, J.H.: Positive solutions for boundary value problems of nonlinear fractional differential equation. Nonlinear Anal. **71**, 5545–5550 (2009)

14. Li, C.F., Luo, X.N., Zhou, Y.: Existence of positive solutions of the boundary value problem for nonlinear fractional differential equations. Comput. Math. Appl. **59**, 1363–1375 (2010)
15. Nieto, J.J., Rodríguez-López, R.: Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations. Order **22**, 223–239 (2005)
16. Nieto, J.J., Rodríguez-López, R.: Fixed point theorems in ordered abstract spaces. Proc. Am. Math. Soc., **135**(8), 2505–2517 (2007)
17. Podlubny, I.: Fractional Differential Equations. Mathematics in Sciences and Engineering, vol. 198. Academic Press, San Diego (1999)
18. O'Regan, D., Petrusel, A.: Fixed point theorems for generalized contractions in ordered metric spaces. J. Math. Anal. Appl. **341**, 1241–1252 (2008)
19. Samko, S.G., Kilbas, A.A., Marichev, O.I.: Fractional Integrals and Derivatives. Theory and Applications. Gordon and Breach, Yverdon (1993)
20. Zhang, S.: Existence of solution for a boundary value problem of fractional order. Acta Math. Sci. **26**, 220–228 (2006)
21. Zhou, Y.: Existence and uniqueness of fractional functional differential equations with unbounded delay. Int. J. Dyn. Syst. Differ. Equ. **1**, 239–244 (2008)