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Boundary value problems for fractional differential equations involving Caputo derivative in Banach spaces

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Abstract In this paper, we study boundary value problems for fractional differential equations involving Caputo derivative in Banach spaces. A generalized singular type Gronwall inequality is given to obtain an important priori bounds. Some sufficient conditions for the existence of solutions are established by virtue of fractional calculus and fixed point method under some mild conditions. Two examples are given to illustrate the results.

Keywords Boundary value problems · Fractional differential equations · Generalized singular Gronwall inequality · Existence of solutions

Mathematics Subject Classification (2000) 26A33 · 34B05

1 Introduction

Fractional differential equations have recently been proved to be valuable tools in the modelling of many phenomena in various fields of engineering, physics and eco-

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nomics. We can find numerous applications in viscoelasticity, electrochemistry, control and electromagnetic. There has been a significant development in fractional differential equations. One can see the monographs of Kilbas et al. [10], Miller and Ross [11], Lakshmikantham et al. [13], Podlubny [14]. Particulary, Agarwal et al. [1] establish sufficient conditions for the existence and uniqueness of solutions for various classes of initial and boundary value problem for fractional differential equations and inclusions involving the Caputo fractional derivative in finite dimensional spaces. Recently, some fractional differential equations and optimal controls in Banach spaces are studied by Balachandran et al. [3, 4], Benchohra et al. [5], El-Borai [6], Henderson and Ouahab [8], Hernández et al. [9], Mophou and N'Guérékata [12], Wang et al. [17–22], Zhou et al. [23–27].

To our knowledge, boundary value problems for fractional differential equations involving the Caputo derivative in infinite dimensional spaces has not been studied extensively. In this paper, we extend the earlier work [5] on first order boundary value problem (BVP for short), for fractional differential equations in finite dimensional spaces to infinite dimensional spaces of the type

$$\begin{cases} {}^cD^\alpha y(t) = f(t, y(t)), & 0 < \alpha < 1,\ t \in J = [0, T], \\ ay(0) + by(T) = c, \end{cases} \tag{1}$$

where ${}^cD^\alpha$ is the Caputo fractional derivative of order α, $f : J \times X \to X$ where X is a Banach spaces and a, b, c are real constants with $a + b \neq 0$.

We present existence and uniqueness results for the fractional BVP (1) by virtue of fractional calculus and fixed point method. Compared with the results appeared in [5], there are at least three differences: (i) the work space is not R but the Banach spaces X; (ii) the assumptions are more general and easy to check; (iii) a priori bounds is established by a new singular type Gronwall inequality (Lemma 3.2) given by us.

The rest of this paper is organized as follows. In Sect. 2, we give some notations and recall some concepts and preparation results. In Sect. 3, we give a generalized singular type Gronwall inequality which can be used to establish the estimate of fixed point set $\{y = \lambda F y, \lambda \in (0, 1)\}$. In Sect. 4, we give three main results (Theorems 4.1–4.3), the first result based on Banach contraction principle, the second result based on Schaefer's fixed point theorem, the third result based on nonlinear alternative of Leray-Schauder type. Two examples are given in Sect. 5 to demonstrate the application of our main results. These results can be considered as a contribution to this emerging field.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. We denote $C(J, X)$ the Banach space of all continuous functions from J into X with the norm $\|y\|_\infty := \sup\{\|y(t)\| : t \in J\}$. For measurable functions $m : J \to R$, define the norm $\|m\|_{L^p(J,R)} = \left(\int_J |m(t)|^p dt\right)^{1/p}$, $1 \le p < \infty$. We denote $L^p(J, R)$ the Banach space of all Lebesgue measurable functions m with $\|m\|_{L^p(J,R)} < \infty$.

We need some basic definitions and properties of the fractional calculus theory which are used further in this paper. For more details, see [10].

Definition 2.1 The fractional order integral of the function $h \in L^1([a, b], R)$ of order $\alpha \in R_+$ is defined by

$$I_a^\alpha h(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds$$

where Γ is the Gamma function.

Definition 2.2 For a function h given on the interval $[a, b]$, the αth Riemann-Liouville fractional order derivative of h, is defined by

$$(D_{a+}^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} h(s)ds,$$

here $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

Definition 2.3 For a function h given on the interval $[a, b]$, the Caputo fractional order derivative of h, is defined by

$$({}^c D_{a+}^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} h^{(n)}(s)ds,$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

Lemma 2.4 *Let $\alpha > 0$, then the differential equation ${}^c D^\alpha h(t) = 0$ has solutions*

$$h(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},$$

where $c_i \in R$, $i = 0, 1, 2, \ldots, n$, $n = [\alpha] + 1$.

Lemma 2.5 *Let $\alpha > 0$, then*

$$I^\alpha ({}^c D^\alpha h)(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},$$

for some $c_i \in R$, $i = 0, 1, 2, \ldots, n$, $n = [\alpha] + 1$.

Now, let us recall the definition of a solution of the fractional BVP (1).

Definition 2.6 (Definition 3.1, [1]) A function $y \in C^1(J, X)$ is said to be a solution of the fractional BVP (1) if y satisfies the equation ${}^c D^\alpha y(t) = f(t, y(t))$ a.e. on J, and the condition $ay(0) + by(T) = c$.

For the existence of solutions for the fractional BVP (1), we need the following auxiliary lemma.

Lemma 2.7 (Lemma 3.2, [1]) *A function $y \in C(J, X)$ is a solution of the fractional integral equation*

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \overline{f}(s)ds - \frac{1}{a+b}\left[\frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \overline{f}(s)ds - c\right],$$

if and only if y is a solution of the following fractional BVP

$$\begin{cases} {}^cD^\alpha y(t) = \overline{f}(t), \quad 0 < \alpha < 1, \ t \in J, \\ ay(0) + by(T) = c. \end{cases} \tag{2}$$

As a consequence of Lemma 2.7, we have the following result which is useful in what follows.

Lemma 2.8 *A function $y \in C(J, X)$ is a solution of the fractional integral equation*

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s)) ds$$
$$- \frac{1}{a+b} \left[\frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f(s, y(s)) ds - c \right],$$

if and only if y is a solution of the fractional BVP (1).

Lemma 2.9 (Bochner theorem, [2]) *A measurable function $f : J \to X$ is Bochner integrable if $\|f\|$ is Lebesbuge integrable.*

Lemma 2.10 (Mazur lemma, [2]) *If $\mathcal{K}$ is a compact subset of X, then its convex closure $\overline{conv}\mathcal{K}$ is compact.*

Lemma 2.11 (Ascoli-Arzela theorem, [15]) *Let $\mathcal{S} = \{s(t)\}$ is a function family of continuous mappings $s : [a, b] \to X$. If $\mathcal{S}$ is uniformly bounded and equicontinuous, and for any $t^* \in [a, b]$, the set $\{s(t^*)\}$ is relatively compact, then there exists a uniformly convergent function sequence $\{s_n(t)\}$ $(n = 1, 2, \ldots, t \in [a, b])$ in $\mathcal{S}$.*

Theorem 2.12 (Schaefer's fixed point theorem, [15]) *Let $F : X \to X$ completely continuous operator. If the set*

$$E(F) = \{x \in X : x = \lambda F x \ for \ some \ \lambda \in [0, 1]\}$$

is bounded, then F has fixed points.

Theorem 2.13 (Nonlinear alternative of Leray-Schauder type, [7]) *Let $\mathcal{C}$ a nonempty convex subset of X. Let U a nonempty open subset of C with $0 \in U$ and $F : \overline{U} \to \mathcal{C}$ compact and continuous operators. Then either*

(i) *F has fixed points.*
(ii) *There exist $y \in \partial U$ and $\lambda \in [0, 1]$ with $y = \lambda F(y)$.*

3 A generalized singular type Gronwall's inequality

In order to apply the Schaefer fixed point theorem to show the existence of solutions, we need a new generalized singular type Gronwall inequality with mixed type singular integral operator. It will play an essential role in the study of BVP for fractional differential equations.

We first collect a generalized Gronwall inequality which appeared in our earlier work [16].

Lemma 3.1 (Lemma 3.2, [16]) *Let* $y \in C(J, X)$ *satisfy the following inequality*:

$$\|y(t)\| \leq a + b\int_0^t \|y(\theta)\|^{\lambda_1} d\theta + c\int_0^T \|y(\theta)\|^{\lambda_2} d\theta + d\int_0^t \|y_\theta\|_B^{\lambda_3} d\theta + e\int_0^T \|y_\theta\|_B^{\lambda_4} d\theta, t \in J,$$

where $\lambda_1, \lambda_3 \in [0, 1], \lambda_2, \lambda_4 \in [0, 1), a, b, c, d, e \geq 0$ *are constants and* $\|y_\theta\|_B = \sup_{0\leq s\leq\theta} \|y(s)\|$. *Then there exists a constant* $M^* > 0$ *such that*

$$\|y(t)\| \leq M^*.$$

Using the above generalized Gronwall inequality, we can obtain the following new generalized singular type Gronwall inequality.

Lemma 3.2 *Let* $y \in C(J, X)$ *satisfy the following inequality*:

$$\|y(t)\| \leq a + b\int_0^t (t-s)^{\alpha-1}\|y(s)\|^\lambda ds + c\int_0^T (T-s)^{\alpha-1}\|y(s)\|^\lambda ds, \quad (3)$$

where $\alpha \in (0, 1)$, $\lambda \in [0, 1-\frac{1}{p})$ *for some* $1 < p < \frac{1}{1-\alpha}$, $a, b, c \geq 0$ *are constants. Then there exists a constant* $M^* > 0$ *such that*

$$\|y(t)\| \leq M^*.$$

Proof Let

$$x(t) = \begin{cases} 1, & \|y(t)\| \leq 1, \\ y(t), & \|y(t)\| > 1. \end{cases}$$

It follows from condition (3) and Hölder inequality that

$$\begin{aligned} \|x(t)\|^\lambda &\leq \|x(t)\| \\ &\leq (a+1) + b\int_0^t (t-s)^{\alpha-1}\|x(s)\|^\lambda ds + c\int_0^T (T-s)^{\alpha-1}\|x(s)\|^\lambda ds \\ &\leq (a+1) + b\left(\int_0^t (t-s)^{p(\alpha-1)} ds\right)^{\frac{1}{p}} \left(\int_0^t \|x(s)\|^{\frac{\lambda p}{p-1}} ds\right)^{\frac{p-1}{p}} \\ &\quad + c\left(\int_0^T (T-s)^{p(\alpha-1)} ds\right)^{\frac{1}{p}} \left(\int_0^T \|x(s)\|^{\frac{\lambda p}{p-1}} ds\right)^{\frac{p-1}{p}} \\ &\leq (a+1) + b\left[\frac{T^{p(\alpha-1)+1}}{p(\alpha-1)+1}\right]^{\frac{1}{p}} \int_0^t \|x(s)\|^{\frac{\lambda p}{p-1}} ds \end{aligned}$$

$$+ c\left[\frac{T^{p(\alpha-1)+1}}{p(\alpha-1)+1}\right]^{\frac{1}{p}}\int_0^T \|x(s)\|^{\frac{\lambda p}{p-1}}ds.$$

This implies that

$$\|x(t)\| \le (a+1) + b\left[\frac{T^{p(\alpha-1)+1}}{p(\alpha-1)+1}\right]^{\frac{1}{p}}\int_0^t \|x(s)\|^{\frac{\lambda p}{p-1}}ds$$
$$+ c\left[\frac{T^{p(\alpha-1)+1}}{p(\alpha-1)+1}\right]^{\frac{1}{p}}\int_0^T \|x(s)\|^{\frac{\lambda p}{p-1}}ds,$$

where $0 < \frac{\lambda p}{p-1} < 1$.

By Lemma 3.1, one can complete the rest proof immediately. □

4 Main results

Before stating and proving the main results, we introduce the following hypotheses.

(H1) The function $f : J \times X \to X$ is strongly measurable with respect to t on J.

(H2) There exists a constant $\alpha_1 \in (0, \alpha)$ and real-valued function $m(t) \in L^{\frac{1}{\alpha_1}}(J, R)$ such that

$$\|f(t,u_1) - f(t,u_2)\| \le m(t)\|u_1 - u_2\|, \quad \text{for each } t \in J, \quad \text{and all } u_1, u_2 \in X.$$

(H3) There exists a constant $\alpha_2 \in (0, \alpha)$ and real-valued function $h(t) \in L^{\frac{1}{\alpha_2}}(J, R)$ such that

$$\|f(t,y)\| \le h(t), \quad \text{for each } t \in J, \quad \text{and all } y \in X.$$

For brevity, let $M = \|m\|_{L^{\frac{1}{\alpha_1}}(J,R)}$, $H = \|h\|_{L^{\frac{1}{\alpha_2}}(J,R)}$.

Our first result is based on Banach contraction principle.

Theorem 4.1 *Assume that* (H1)–(H3) *hold. If*

$$\Omega_{\alpha,T} = \frac{MT^{\alpha-\alpha_1}}{\Gamma(\alpha)(\frac{\alpha-\alpha_1}{1-\alpha_1})^{1-\alpha_1}}\left(1 + \frac{|b|}{|a+b|}\right) < 1, \tag{4}$$

then the system (1) *has a unique solution on* J.

Proof For each $t \in J$, we have

$$\int_0^t \left\|(t-s)^{\alpha-1}f(s,y(s))\right\|ds \le \left(\int_0^t (t-s)^{\frac{\alpha-1}{1-\alpha_2}}ds\right)^{1-\alpha_2}\left(\int_0^t (h(s))^{\frac{1}{\alpha_2}}ds\right)^{\alpha_2}$$
$$\le \frac{T^{\alpha-\alpha_2}H}{(\frac{\alpha-\alpha_2}{1-\alpha_2})^{1-\alpha_2}}.$$

Thus, $\|(t-s)^{\alpha-1}f(s,y(s))\|$ is Lebesgue integrable with respect to $s\in[0,t]$ for all $t\in J$ and $y\in C(J,X)$. Then $(t-s)^{\alpha-1}f(s,y(s))$ is Bochner integrable with respect to $s\in[0,t]$ for all $t\in J$ due to Lemma 2.9.

Hence, the fractional BVP (1) is equivalent to the following fractional integral equation

$$y(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,y(s))ds\\ -\frac{1}{a+b}\left[\frac{b}{\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}f(s,y(s))ds-c\right],\quad t\in J.$$

Let

$$r\geq\frac{T^{\alpha-\alpha_2}H}{\Gamma(\alpha)(\frac{\alpha-\alpha_2}{1-\alpha_2})^{1-\alpha_2}}+\frac{|b|}{|a+b|}\times\frac{T^{\alpha-\alpha_2}H}{\Gamma(\alpha)(\frac{\alpha-\alpha_2}{1-\alpha_2})^{1-\alpha_2}}+\frac{|c|}{|a+b|}.$$

Now we define the operator F on $B_r:=\{y\in C(J,X):\|y\|\leq r\}$ as follows

$$(Fy)(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,y(s))ds\\ -\frac{1}{a+b}\left[\frac{b}{\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}f(s,y(s))ds-c\right],\quad t\in J. \tag{5}$$

Therefore, the existence of a solution of the fractional BVP (1) is equivalent to that the operator F has a fixed point on B_r. We shall use the Banach contraction principle to prove that F has a fixed point. The proof is divided into two steps.

Step 1. $Fy\in B_r$ for every $y\in B_r$

For every $y\in B_r$ and any $\delta>0$, by (H3) and Hölder inequality, we get

$$\begin{aligned}&\|(Fy)(t+\delta)-(Fy)(t)\|\\ &\leq\left\|\frac{1}{\Gamma(\alpha)}\int_0^t[(t+\delta-s)^{\alpha-1}-(t-s)^{\alpha-1}]f(s,y(s))ds\right\|\\ &\quad+\left\|\frac{1}{\Gamma(\alpha)}\int_t^{t+\delta}(t+\delta-s)^{\alpha-1}f(s,y(s))ds\right\|\\ &\leq\frac{1}{\Gamma(\alpha)}\int_0^t[(t-s)^{\alpha-1}-(t+\delta-s)^{\alpha-1}]\|f(s,y(s))\|ds\\ &\quad+\frac{1}{\Gamma(\alpha)}\int_t^{t+\delta}(t+\delta-s)^{\alpha-1}\|f(s,y(s))\|ds\\ &\leq\frac{1}{\Gamma(\alpha)}\int_0^t[(t-s)^{\alpha-1}-(t+\delta-s)^{\alpha-1}]h(s)ds\\ &\quad+\frac{1}{\Gamma(\alpha)}\int_t^{t+\delta}(t+\delta-s)^{\alpha-1}h(s)ds\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\Gamma(\alpha)}\left(\int_0^t [(t-s)^{\frac{\alpha-1}{1-\alpha_2}} - (t+\delta-s)^{\frac{\alpha-1}{1-\alpha_2}}]ds\right)^{1-\alpha_2}\left(\int_0^t (h(s))^{\frac{1}{\alpha_2}}ds\right)^{\alpha_2}\\
&\quad + \frac{1}{\Gamma(\alpha)}\left(\int_t^{t+\delta} (t+\delta-s)^{\frac{\alpha-1}{1-\alpha_2}}ds\right)^{1-\alpha_2}\left(\int_t^{t+\delta} (h(s))^{\frac{1}{\alpha_2}}ds\right)^{\alpha_2}\\
&\leq \frac{H}{\Gamma(\alpha)}\left(\frac{t^{\frac{\alpha-\alpha_2}{1-\alpha_2}}}{\frac{\alpha-\alpha_2}{1-\alpha_2}} + \frac{\delta^{\frac{\alpha-\alpha_2}{1-\alpha_2}}}{\frac{\alpha-\alpha_2}{1-\alpha_2}} - \frac{(t+\delta)^{\frac{\alpha-\alpha_2}{1-\alpha_2}}}{\frac{\alpha-\alpha_2}{1-\alpha_2}}\right)^{1-\alpha_2} + \frac{H}{\Gamma(\alpha)}\left(\frac{\delta^{\frac{\alpha-\alpha_2}{1-\alpha_2}}}{\frac{\alpha-\alpha_2}{1-\alpha_2}}\right)^{1-\alpha_2}\\
&\leq \frac{H}{\Gamma(\alpha)(\frac{\alpha-\alpha_2}{1-\alpha_2})^{1-\alpha_2}}\left[\left(t^{\frac{\alpha-\alpha_2}{1-\alpha_2}} - (t+\delta)^{\frac{\alpha-\alpha_2}{1-\alpha_2}} + \delta^{\frac{\alpha-\alpha_2}{1-\alpha_2}}\right)^{1-\alpha_2} + \delta^{\alpha-\alpha_2}\right]\\
&\leq \frac{2H\delta^{\alpha-\alpha_2}}{\Gamma(\alpha)(\frac{\alpha-\alpha_2}{1-\alpha_2})^{1-\alpha_2}}.
\end{aligned}
$$

As $\delta \to 0$, the right-hand side of the above inequality tends to zero. Therefore, F is continuous on J, i.e., $Fy \in C(J, X)$.

Moreover, for $y \in B_r$ and all $t \in J$, we get

$$
\begin{aligned}
&\|(Fy)(t)\|\\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,y(s))\|ds\\
&\quad + \frac{|b|}{|a+b|\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\|f(s,y(s))\|ds + \frac{|c|}{|a+b|}\\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds\\
&\quad + \frac{|b|}{|a+b|\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}h(s)ds + \frac{|c|}{|a+b|}\\
&\leq \frac{1}{\Gamma(\alpha)}\left(\int_0^t (t-s)^{\frac{\alpha-1}{1-\alpha_2}}ds\right)^{1-\alpha_2}\left(\int_0^t (h(s))^{\frac{1}{\alpha_2}}ds\right)^{\alpha_2}\\
&\quad + \frac{|b|}{|a+b|\Gamma(\alpha)}\left(\int_0^T (T-s)^{\frac{\alpha-1}{1-\alpha_2}}ds\right)^{1-\alpha_2}\left(\int_0^T (h(s))^{\frac{1}{\alpha_2}}ds\right)^{\alpha_2} + \frac{|c|}{|a+b|}\\
&\leq \frac{T^{\alpha-\alpha_2}H}{\Gamma(\alpha)(\frac{\alpha-\alpha_2}{1-\alpha_2})^{1-\alpha_2}} + \frac{|b|}{|a+b|} \times \frac{T^{\alpha-\alpha_2}H}{\Gamma(\alpha)(\frac{\alpha-\alpha_2}{1-\alpha_2})^{1-\alpha_2}} + \frac{|c|}{|a+b|}\\
&\leq r,
\end{aligned}
$$

which implies that $\|Fy\|_\infty \leq r$. Thus, we can conclude that for all $y \in B_r$, $Fy \in B_r$. i.e., $F : B_r \to B_r$.

Step 2. F is a contraction mapping on B_r.

For $x, y \in B_r$ and any $t \in J$, using (H2) and Hölder inequality, we get

$$
\begin{aligned}
&\|(Fx)(t)-(Fy)(t)\| \\
&\quad \leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,x(s))-f(s,y(s))\|ds \\
&\qquad + \frac{|b|}{|a+b|\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\|f(s,x(s))-f(s,y(s))\|ds \\
&\quad \leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}m(s)\|x(s)-y(s)\|ds \\
&\qquad + \frac{|b|}{|a+b|\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}m(s)\|x(s)-y(s)\|ds \\
&\quad \leq \frac{\|x-y\|_\infty}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}m(s)ds \\
&\qquad + \frac{|b|\|x-y\|_\infty}{|a+b|\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}m(s)ds \\
&\quad \leq \frac{\|x-y\|_\infty}{\Gamma(\alpha)}\left(\int_0^t (t-s)^{\frac{\alpha-1}{1-\alpha_1}}ds\right)^{1-\alpha_1}\left(\int_0^t (m(s))^{\frac{1}{\alpha_1}}ds\right)^{\alpha_1} \\
&\qquad + \frac{|b|\ \|x-y\|_\infty}{|a+b|\Gamma(\alpha)}\left(\int_0^T (T-s)^{\frac{\alpha-1}{1-\alpha_1}}ds\right)^{1-\alpha_1}\left(\int_0^T (m(s))^{\frac{1}{\alpha_1}}ds\right)^{\alpha_1} \\
&\quad \leq \frac{\|x-y\|_\infty}{\Gamma(\alpha)}\frac{T^{\alpha-\alpha_1}}{(\frac{\alpha-\alpha_1}{1-\alpha_1})^{1-\alpha_1}}\|m\|_{L^{\frac{1}{\alpha_1}}(J,R^+)} \\
&\qquad + \frac{|b|\|x-y\|_\infty}{|a+b|\Gamma(\alpha)}\frac{T^{\alpha-\alpha_1}}{(\frac{\alpha-\alpha_1}{1-\alpha_1})^{1-\alpha_1}}\|m\|_{L^{\frac{1}{\alpha_1}}(J,R^+)} \\
&\quad \leq \left[\frac{MT^{\alpha-\alpha_1}}{\Gamma(\alpha)(\frac{\alpha-\alpha_1}{1-\alpha_1})^{1-\alpha_1}}\left(1+\frac{|b|}{|a+b|}\right)\right]\|x-y\|_\infty.
\end{aligned}
$$

So we obtain

$$\|Fx-Fy\|_\infty \leq \Omega_{\alpha,T}\|x-y\|_\infty.$$

Thus, F is a contraction due to the condition (4).

By Banach contraction principle, we can deduce that F has an unique fixed point which is just the unique solution of the fractional BVP (1). □

Our second result is based on the well known Schaefer's fixed point theorem.

We make the following assumptions:

(H4) The function $f : J \times X \to X$ is continuous.

(H5) There exist constants $\lambda \in [0, 1-\frac{1}{p})$ for some $1 < p < \frac{1}{1-\alpha}$ and $N > 0$ such that

$$\|f(t,u)\| \leq N(1+\|u\|^\lambda) \quad \text{for each } t \in J \quad \text{and all } u \in X.$$

(H6) For every $t \in J$, the set $K = \{(t-s)^{\alpha-1} f(s, y(s)) : y \in C(J, X), s \in [0, t]\}$ is relatively compact.

Theorem 4.2 *Assume that* (H4)–(H6) *hold. Then the fractional BVP* (1) *has at least one solution on* J.

Proof Transform the fractional BVP (1) into a fixed point problem. Consider the operator $F : C(J, X) \to C(J, X)$ defined as (5). It is obvious that F is well defined due to (H4).

For the sake of convenience, we subdivide the proof into several steps.

Step 1. F is continuous.

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $C(J, X)$. Then for each $t \in J$, we have

$$\begin{aligned}
&\|(Fy_n)(t) - (Fy)(t)\| \\
&\quad \le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(s, y_n(s)) - f(s, y(s))\| ds \\
&\qquad + \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \|f(s, y_n(s)) - f(s, y(s))\| ds \\
&\quad \le \frac{\|f(\cdot, y_n(\cdot)) - f(\cdot, y(\cdot))\|_\infty}{\Gamma(\alpha)} \left[\int_0^t (t-s)^{\alpha-1} ds + \frac{|b|}{|a+b|} \int_0^T (T-s)^{\alpha-1} ds \right] \\
&\quad \le \frac{T^\alpha}{\Gamma(\alpha+1)} \left(1 + \frac{|b|}{|a+b|}\right) \|f(\cdot, y_n(\cdot)) - f(\cdot, y(\cdot))\|_\infty.
\end{aligned}$$

Since f is continuous, we have

$$\begin{aligned}
&\|Fy_n - Fy\|_\infty \\
&\quad \le \frac{T^\alpha}{\Gamma(\alpha+1)} \left(1 + \frac{|b|}{|a+b|}\right) \|f(\cdot, y_n(\cdot)) - f(\cdot, y(\cdot))\|_\infty \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Step 2. F maps bounded sets into bounded sets in $C(J, X)$.

Indeed, it is enough to show that for any $\eta^* > 0$, there exists a $\ell > 0$ such that for each $y \in B_{\eta^*} = \{y \in C(J, X) : \|y\|_\infty \le \eta^*\}$, we have $\|Fy\|_\infty \le \ell$.

For each $t \in J$, we get

$$\begin{aligned}
&\|(Fy)(t)\| \\
&\quad \le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(s, y(s))\| ds \\
&\qquad + \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \|f(s, y(s))\| ds + \frac{|c|}{|a+b|} \\
&\quad \le \frac{N}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (1 + \|y(s)\|^\lambda) ds \\
&\qquad + \frac{|b|N}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} (1 + \|y(s)\|^\lambda) ds + \frac{|c|}{|a+b|}
\end{aligned}$$

$$\leq \frac{N}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds + \frac{|b|N}{|a+b|\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}ds + \frac{|c|}{|a+b|}$$
$$+ \frac{N}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|y(s)\|^{\lambda}ds + \frac{|b|N}{|a+b|\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\|y(s)\|^{\lambda}ds$$
$$\leq \frac{NT^{\alpha}}{\Gamma(\alpha+1)} + \frac{|b|NT^{\alpha}}{|a+b|\Gamma(\alpha+1)} + \frac{|c|}{|a+b|} + \frac{NT^{\alpha}(\eta^*)^{\lambda}}{\Gamma(\alpha+1)} + \frac{|b|NT^{\alpha}(\eta^*)^{\lambda}}{|a+b|\Gamma(\alpha+1)},$$

which implies that

$$\|Fy\|_{\infty} \leq \frac{NT^{\alpha}}{\Gamma(\alpha+1)}\left(1+\frac{|b|}{|a+b|}\right) + \frac{|c|}{|a+b|} + \frac{NT^{\alpha}(\eta^*)^{\lambda}}{\Gamma(\alpha+1)}\left(1+\frac{|b|}{|a+b|}\right) := \ell.$$

Step 3. F maps bounded sets into equicontinuous sets of $C(J, X)$.
Let $0 \leq t_1 < t_2 \leq T$, $y \in B_{\eta^*}$. Using (H5), we have

$$\begin{aligned}
&\|(Fy)(t_2) - (Fy)(t_1)\| \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^{t_1} [(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}]\|f(s, y(s))\|ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2} (t_2-s)^{\alpha-1}\|f(s, y(s))\|ds \\
&\leq \frac{N}{\Gamma(\alpha)}\int_0^{t_1} [(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}](1+\|y(s)\|^{\lambda})ds \\
&\quad + \frac{N}{\Gamma(\alpha)}\int_{t_1}^{t_2} (t_2-s)^{\alpha-1}(1+\|y(s)\|^{\lambda})ds \\
&\leq \frac{N(1+(\eta^*)^{\lambda})}{\Gamma(\alpha)}\int_0^{t_1} [(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}]ds \\
&\quad + \frac{N(1+(\eta^*)^{\lambda})}{\Gamma(\alpha)}\int_{t_1}^{t_2} (t_2-s)^{\alpha-1}ds \\
&\leq \frac{N(1+(\eta^*)^{\lambda})}{\Gamma(\alpha+1)}(|t_1^{\alpha} - t_2^{\alpha}| + 2(t_2-t_1)^{\alpha}) \\
&\leq \frac{3N(1+(\eta^*)^{\lambda})(t_2-t_1)^{\alpha}}{\Gamma(\alpha+1)}.
\end{aligned}$$

As $t_2 \to t_1$, the right-hand side of the above inequality tends to zero, therefore F is equicontinuous.

Now, let $\{y_n\}$, $n = 1, 2, \ldots$ be a sequence on B_{η^*}, and

$$(Fy_n)(t) = (F_1y_n)(t) + (F_2y_n)(T), \quad t \in J.$$

where

$$(F_1 y_n)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y_n(s)) ds, \quad t \in J,$$

$$(F_2 y_n)(T) = -\frac{1}{a+b} \left[\frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f(s, y_n(s)) ds - c \right].$$

In view of the condition (H6) and Lemma 2.10, we know that $\overline{conv}\, K$ is compact. For any $t^* \in J$,

$$\begin{aligned}
(F_1 y_n)(t^*) &= \frac{1}{\Gamma(\alpha)} \int_0^{t^*} (t^* - s)^{\alpha-1} f(s, y_n(s)) ds \\
&= \frac{1}{\Gamma(\alpha)} \lim_{k \to \infty} \sum_{i=1}^{k} \frac{t^*}{k} \left(t^* - \frac{it^*}{k} \right)^{\alpha-1} f\left(\frac{it^*}{k}, y_n\left(\frac{it^*}{k} \right) \right) \\
&= \frac{t^*}{\Gamma(\alpha)} \widetilde{\xi_n},
\end{aligned}$$

where

$$\widetilde{\xi_n} = \lim_{k \to \infty} \sum_{i=1}^{k} \frac{1}{k} \left(t^* - \frac{it^*}{k} \right)^{\alpha-1} f\left(\frac{it^*}{k}, y_n\left(\frac{it^*}{k} \right) \right).$$

Since $\overline{conv}\, K$ is convex and compact, we know that $\widetilde{\xi_n} \in \overline{conv}\, K$. Hence, for any $t^* \in J$, the set $\{(F_1 y_n)(t^*)\}$ is relatively compact. From Lemma 2.11, every $\{(F_1 y_n)(t)\}$ contains a uniformly convergent subsequence $\{(F_1 y_{n_k})(t)\}$, $k = 1, 2, \ldots$ on J. Thus, the set $\{F_1 y : y \in B_{\eta^*}\}$ is relatively compact. Similarly, one can obtain $\{(F_2 y_n)(T)\}$ contains a uniformly convergent subsequence $\{(F_2 y_{n_k})(T)\}$, $k = 1, 2, \ldots$. Thus, the set $\{F_2 y : y \in B_{\eta^*}\}$ is relatively compact. As a result, the set $\{Fy, y \in B_{\eta^*}\}$ is relatively compact.

As a consequence of Steps 1–3, we can conclude that F is continuous and completely continuous.

Step 4. A priori bounds.

Now it remains to show that the set

$$E(F) = \{ y \in C(J, X) : y = \lambda F y, \text{ for some } \lambda \in (0, 1) \}$$

is bounded.

Let $y \in E(F)$, then $y = \lambda F y$ for some $\lambda \in (0, 1)$. Thus, for each $t \in J$, we have

$$\begin{aligned}
y(t) = \lambda \Bigg(&\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s)) ds \\
&- \frac{1}{a+b} \left[\frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f(s, y(s)) ds - c \right] \Bigg).
\end{aligned}$$

For each $t \in J$, we have

$$\|y(t)\| \leq \frac{NT^\alpha}{\Gamma(\alpha+1)} + \frac{|b|NT^\alpha}{|a+b|\Gamma(\alpha+1)} + \frac{|c|}{|a+b|} + \frac{N}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|y(s)\|^\lambda ds + \frac{|b|N}{|a+b|\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\|y(s)\|^\lambda ds.$$

By Lemma 3.2, there exists a $M^* > 0$ such that

$$\|y(t)\| \leq M^*, \quad t \in J.$$

Thus for every $t \in J$, we have

$$\|y\|_\infty \leq M^*.$$

This shows that the set $E(F)$ is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that F has a fixed point which is a solution of the fractional BVP (1). □

In the following theorem we apply the nonlinear alternative of Leray-Schauder type in which the condition (H5) is weakened.

(H5′) There exist a $\lambda \in [0, 1 - \frac{1}{p})$ for some $1 < p < \frac{1}{1-\alpha}$ and a function $N(t) \in L^{\frac{1}{\alpha_3}}(J, R)$, $\alpha_3 \in (0, \alpha)$ such that

$$\|f(t,u)\| \leq N(t)(1 + \|u\|^\lambda) \quad \text{for each } t \in J \quad \text{and all } u \in X.$$

Theorem 4.3 *Assume that* (H4), (H5′), (H6) *hold. Then the fractional BVP* (1) *has at least one solution on* J.

Proof Consider the operator F defined in Theorem 4.2. It can be easily shown that F is continuous and completely continuous. Repeating the same process in Step 4 in Theorem 4.2, using (H5′) and Hölder inequality again, we have for each $t \in J$, there exists a $M^* > 0$ such that $\|y\|_\infty \leq M^*$.

Let

$$U = \{y \in C(J, X) : \|y\|_\infty < M^* + 1\}.$$

The operator $F : \overline{U} \to C(J, X)$ is continuous and completely continuous. From the choice of U, there is no $y \in \partial U$ such that $y = \lambda F(y)$, $\lambda \in (0, 1)$. As a consequence of the nonlinear alternative of Leray-Schauder type, we deduce that F has a fixed point $y \in \overline{U}$, which implies that the fractional BVP (1) has at least one solution $y \in C(J, X)$. □

5 Examples

In this section we give two examples to illustrate the usefulness of our main results.

Example 5.1 Let us consider the following fractional boundary value problem,

$$\begin{cases} {}^cD^\alpha y(t) = \frac{e^{-vt}|y(t)|}{(1+e^t)(1+|y(t)|)}, & \alpha \in (0,1),\ t \in J_1 := [0,1], \\ y(0) + y(1) = 0, \end{cases} \tag{6}$$

where $v > 0$ is a constant.

Set

$$f(t,y) = \frac{e^{-vt}y}{(1+e^t)(1+y)}, \quad (t,y) \in J_1 \times [0,\infty).$$

Let $y_1, y_2 \in [0,\infty)$ and $t \in J_1$. Then we have

$$\begin{aligned} |f(t,y_1) - f(t,y_2)| &= \frac{e^{-vt}}{(1+e^t)}\left|\frac{y_1}{1+y_1} - \frac{y_2}{1+y_2}\right| \\ &= \frac{e^{-vt}|y_1-y_2|}{(1+e^t)(1+y_1)(1+y_2)} \\ &\le \frac{e^{-vt}}{1+e^t}|y_1-y_2| \\ &\le \frac{e^{-vt}}{2}|y_1-y_2|. \end{aligned}$$

Obviously, for all $y \in [0,\infty)$ and each $t \in J_1$,

$$\begin{aligned} |f(t,y)| &= \frac{e^{-vt}}{1+e^t}\left|\frac{y}{1+y}\right| \\ &\le \frac{e^{-vt}}{1+e^t} \\ &\le \frac{e^{-vt}}{2}. \end{aligned}$$

For $t \in J_1$, $\beta \in (0,\alpha)$, let $m(t) = h(t) = \frac{e^{-vt}}{2} \in L^{\frac{1}{\beta}}(J_1,R)$, $M = \|\frac{e^{-vt}}{2}\|_{L^{\frac{1}{\beta}}(J_1,R)}$. Choosing some $v > 0$ large enough and suitable $\beta \in (0,\alpha)$, one can arrive at the following inequality

$$\Omega_{\alpha,1} = \frac{M1^{\alpha-\beta}}{\Gamma(\alpha)(\frac{\alpha-\beta}{1-\beta})^{1-\beta}} \times \frac{3}{2} < 1.$$

Thus all the assumptions in Theorem 4.1 are satisfied, our results can be applied to the problem (6).

Example 5.2 Let us consider another fractional boundary value problem,

$$\begin{cases} {}^cD^\alpha y(t) = \frac{t^\vartheta|y(t)|^\lambda}{(1+e^t)(1+|y(t)|)}, & \alpha \in (0,1),\ \vartheta > -\alpha,\ t \in J_1, \\ y(0) + y(1) = 0. \end{cases} \tag{7}$$

Set

$$f_1(t,x) = \frac{t^{\vartheta} x^{\lambda}}{(1+e^t)(1+x)}, \quad (t,x) \in J_1 \times [1,2].$$

Obviously, for all $y \in [1,2]$ and each $t \in J_1$,

$$\begin{aligned} |f_1(t,y)| &= \frac{t^{\vartheta}}{(1+e^t)}\left|\frac{y^{\lambda}}{1+y}\right| \le \frac{1}{(1+e^t)} \times \frac{|y|^{\lambda}}{2} \\ &\le \frac{1}{4}|y|^{\lambda}. \end{aligned}$$

Since $\vartheta > -\alpha$, it is not difficult to see

$$\begin{aligned} \int_0^t (t-s)^{\alpha-1}\frac{s^{\vartheta}|y(s)|^{\lambda}}{(1+e^s)(1+|y(s)|)}ds &\le \int_0^t (t-s)^{\alpha-1}s^{\vartheta}ds \\ &\le \frac{\Gamma(\alpha)\Gamma(\vartheta+1)}{\Gamma(\alpha+\vartheta+1)}t^{\alpha+\vartheta} \\ &\le \frac{\Gamma(\alpha)\Gamma(\vartheta+1)}{\Gamma(\alpha+\vartheta+1)}. \end{aligned}$$

As a result, the set $K_1 = \{(t-s)^{\alpha-1}\frac{s^{\vartheta}|y(s)|^{\lambda}}{(1+e^s)(1+|y(s)|)} : y \in C(J_1,[1,2]), s \in [0,t]\}$ is bounded and closed which implies that K_1 is compact. Thus, all the assumptions in Theorem 4.2 are satisfied, our results can be applied to the problem (7).

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