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On the existence of solutions for fractional differential inclusions with anti-periodic boundary conditions

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Abstract The existence of solutions of an anti-periodic boundary value problem for fractional differential inclusions of order $\alpha \in (2, 3]$ is investigated. Several results are obtained by using suitable fixed point theorems when the right hand side has convex or non convex values.

Keywords Fractional differential inclusion · Anti-periodic boundary conditions · Fixed point

Mathematics Subject Classification (2000) 34A60 · 34A12 · 34A40

1 Introduction

Differential equations with fractional order have recently proved to be strong tools in the modelling of many physical phenomena. As a consequence there was an intensive development of the theory of differential equations of fractional order ([4, 21, 23, 26] etc.). The study of fractional differential inclusions was initiated by El-Sayed and Ibrahim [17]. Very recently several qualitative results for fractional differential inclusions were obtained in [5, 9–12, 19, 25] etc. Applied problems require definitions of fractional derivative allowing the utilization of physically interpretable initial or boundary conditions. Caputo's fractional derivative, originally introduced in [7] and afterwards adopted in the theory of linear visco elasticity, satisfies this demand. For a consistent bibliography on this topic, historical remarks and examples we refer to [3].

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In this paper we study the following fractional differential inclusion

$$D_c^{\alpha}x(t) \in F(t, x(t)) \quad \text{a.e. } ([0, T]), \tag{1.1}$$

$$x(0) = -x(T), \qquad x'(0) = -x'(T), \qquad x''(0) = -x''(T) \tag{1.2}$$

where $\alpha \in (2, 3]$, D_c^{α} is the Caputo fractional derivative and $F : I \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ is a set-valued map.

The present paper is motivated by a recent paper of Ahmad [1], where it is considered problem (1.1)–(1.2) with $F(.,.)$ single valued and several existence results are obtained by using fixed point techniques.

The aim of our paper is to extend the study in [1] to the set-valued framework and to present some existence results for problem (1.1)–(1.2). Our results are essentially based on a nonlinear alternative of Leray-Schauder type, on Bressan-Colombo selection theorem for lower semicontinuous set-valued maps with decomposable values and on Covitz and Nadler set-valued contraction principle. The methods used are standard, however their exposition in the framework of problem (1.1)–(1.2) is new.

We note that the methods based on Leray-Schauder alternative have been used intensively for the study of differential equations during the last few years (e.g., [13–15, 20] etc.).

The paper is organized as follows: in Sect. 2 we recall some preliminary facts that we need in the sequel and in Sect. 3 we prove our main results.

2 Preliminaries

In this section we sum up some basic facts that we are going to use later.

Let (X, d) be a metric space with the corresponding norm $|.|$ and let $I \subset \mathbf{R}$ be a compact interval. Denote by $\mathcal{L}(I)$ the σ-algebra of all Lebesgue measurable subsets of I, by $\mathcal{P}(X)$ the family of all nonempty subsets of X and by $\mathcal{B}(X)$ the family of all Borel subsets of X. If $A \subset I$ then $\chi_A(.) : I \to \{0, 1\}$ denotes the characteristic function of A. For any subset $A \subset X$ we denote by $\overline{A}$ the closure of A.

Recall that the Pompeiu-Hausdorff distance of the closed subsets $A, B \subset X$ is defined by

$$d_H(A, B) = \max\{d^*(A, B), d^*(B, A)\}, \qquad d^*(A, B) = \sup\{d(a, B); a \in A\},$$

where $d(x, B) = \inf_{y \in B} d(x, y)$.

As usual, we denote by $C(I, X)$ the Banach space of all continuous functions $x(.) : I \to X$ endowed with the norm $|x(.)|_C = \sup_{t \in I} |x(t)|$ and by $L^1(I, X)$ the Banach space of all (Bochner) integrable functions $x(.) : I \to X$ endowed with the norm $|x(.)|_1 = \int_I |x(t)| \mathrm{d}t$.

A subset $D \subset L^1(I, X)$ is said to be *decomposable* if for any $u(\cdot), v(\cdot) \in D$ and any subset $A \in \mathcal{L}(I)$ one has $u\chi_A + v\chi_B \in D$, where $B = I \setminus A$.

Consider $T : X \to \mathcal{P}(X)$ a set-valued map. A point $x \in X$ is called a fixed point for $T(.)$ if $x \in T(x)$. $T(.)$ is said to be bounded on bounded sets if $T(B) := \bigcup_{x \in B} T(x)$ is a bounded subset of X for all bounded sets B in X. $T(.)$ is said to be compact if

$T(B)$ is relatively compact for any bounded sets B in X. $T(.)$ is said to be totally compact if $\overline{T(X)}$ is a compact subset of X. $T(.)$ is said to be upper semicontinuous if for any open set $D \subset X$, the set $\{x \in X; T(x) \subset D\}$ is open in X. $T(.)$ is called completely continuous if it is upper semicontinuous and totally bounded on X.

It is well known that a compact set-valued map $T(.)$ with nonempty compact values is upper semicontinuous if and only if $T(.)$ has a closed graph.

We recall the following nonlinear alternative of Leray-Schauder type and its consequences.

Theorem 2.1 [24] *Let D and $\overline{D}$ be the open and closed subsets in a normed linear space X such that $0 \in D$ and let $T : \overline{D} \to \mathcal{P}(X)$ be a completely continuous set-valued map with compact convex values. Then either*

(i) *the inclusion $x \in T(x)$ has a solution, or*
(ii) *there exists $x \in \partial D$ (the boundary of D) such that $\lambda x \in T(x)$ for some $\lambda > 1$.*

Corollary 2.2 *Let $B_r(0)$ and $\overline{B_r(0)}$ be the open and closed balls in a normed linear space X centered at the origin and of radius r and let $T : \overline{B_r(0)} \to \mathcal{P}(X)$ be a completely continuous set-valued map with compact convex values. Then either*

(i) *the inclusion $x \in T(x)$ has a solution, or*
(ii) *there exists $x \in X$ with $|x| = r$ and $\lambda x \in T(x)$ for some $\lambda > 1$.*

Corollary 2.3 *Let $B_r(0)$ and $\overline{B_r(0)}$ be the open and closed balls in a normed linear space X centered at the origin and of radius r and let $T : \overline{B_r(0)} \to X$ be a completely continuous single valued map with compact convex values. Then either*

(i) *the equation $x = T(x)$ has a solution, or*
(ii) *there exists $x \in X$ with $|x| = r$ and $x = \lambda T(x)$ for some $\lambda < 1$.*

We recall that a multifunction $T(.) : X \to \mathcal{P}(X)$ is said to be lower semicontinuous if for any closed subset $C \subset X$, the subset $\{s \in X; G(s) \subset C\}$ is closed.

If $F(.,.) : I \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ is a set-valued map with compact values and $x(.) \in C(I, \mathbf{R})$ we define

$$S_F(x) := \{f \in L^1(I, \mathbf{R});\ f(t) \in F(t, x(t)) \quad \text{a.e. } (I)\}.$$

We say that $F(.,.)$ is of *lower semicontinuous type* if $S_F(.)$ is lower semicontinuous with closed and decomposable values.

Theorem 2.4 [6] *Let S be a separable metric space and $G(.) : S \to \mathcal{P}(L^1(I, \mathbf{R}))$ be a lower semicontinuous set-valued map with closed decomposable values.*

Then $G(.)$ has a continuous selection (i.e., there exists a continuous mapping $g(.) : S \to L^1(I, \mathbf{R})$ such that $g(s) \in G(s)\ \forall s \in S$).

A set-valued map $G : I \to \mathcal{P}(\mathbf{R})$ with nonempty compact convex values is said to be *measurable* if for any $x \in \mathbf{R}$ the function $t \to d(x, G(t))$ is measurable.

A set-valued map $F(.,.) : I \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ is said to be *Carathéodory* if $t \to F(t,x)$ is measurable for any $x \in \mathbf{R}$ and $x \to F(t,x)$ is upper semicontinuous for almost all $t \in I$.

$F(.,.)$ is said to be L^1-*Carathéodory* if for any $l > 0$ there exists $h_l(.) \in L^1(I,\mathbf{R})$ such that $\sup\{|v|; v \in F(t,x)\} \le h_l(t)$ a.e. (I), $\forall x \in \overline{B_l(0)}$.

Theorem 2.5 [22] *Let X be a Banach space, let $F(.,.) : I \times X \to \mathcal{P}(X)$ be a L^1-Carathéodory set-valued map with $S_F \neq \emptyset$ and let $\Gamma : L^1(I,X) \to C(I,X)$ be a linear continuous mapping.*

Then the set-valued map $\Gamma \circ S_F : C(I,X) \to \mathcal{P}(C(I,X))$ defined by

$$(\Gamma \circ S_F)(x) = \Gamma(S_F(x))$$

has compact convex values and has a closed graph in $C(I,X) \times C(I,X)$.

Note that if $dim\, X < \infty$, and $F(.,.)$ is as in Theorem 2.5, then $S_F(x) \neq \emptyset$ for any $x(.) \in C(I,X)$ (e.g., [22]).

Let (X,d) be a metric space and consider a set valued map T on X with nonempty values in X. T is said to be a λ-contraction if there exists $0 < \lambda < 1$ such that:

$$d_H(T(x),T(y)) \le \lambda d(x,y) \quad \forall x,y \in X,$$

where $d_H(.,.)$ denotes the Pompeiu-Hausdorff distance.

The set-valued contraction principle [16] states that if X is complete, and $T : X \to \mathcal{P}(X)$ is a set valued contraction with nonempty closed values, then $T(.)$ has a fixed point, i.e. a point $z \in X$ such that $z \in T(z)$.

Definition 2.6 [21] (a) *The fractional integral of order* $\alpha > 0$ of a Lebesgue integrable function $f(.) : (0,\infty) \to \mathbf{R}$ is defined by

$$I^\alpha f(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds,$$

provided the right-hand side is pointwise defined on $(0,\infty)$ and $\Gamma(.)$ is the (Euler's) Gamma function defined by $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt$.

(b) *The Caputo fractional derivative of order* $\alpha > 0$ of a function $f(.) : [0,\infty) \to \mathbf{R}$ is defined by

$$D_c^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{-\alpha+n-1} f^{(n)}(s)ds,$$

where $n = [\alpha] + 1$. It is assumed implicitly that $f(.)$ is n times differentiable whose n-th derivative is absolutely continuous.

In what follows $I = [0,T]$. We recall (e.g., [20]) that if $\alpha > 0$ and $f(.) \in C(I,\mathbf{R})$ or $f(.) \in L^\infty(I,\mathbf{R})$ then $(D_c^\alpha I^\alpha f)(t) \equiv f(t)$.

Lemma 2.7 [1] *For any* $f(.) \in C(I, \mathbf{R})$ *the unique solution of the boundary value problem*

$$D_c^\alpha x(t) = f(t) \quad \text{a.e. } ([0,T]), \quad \alpha \in (2,3],$$
$$x(0) = -x(T), \qquad x'(0) = -x'(T), \qquad x''(0) = -x''(T)$$

is given by

$$x(t) = \int_0^T G(t,s) f(s) ds,$$

where $G(.,.) : I \times I \to \mathbf{R}$ *is the Green function defined by*

$$G(t,s) := \begin{cases} \frac{(t-s)^{\alpha-1} - \frac{1}{2}(T-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(T-2t)(T-s)^{\alpha-2}}{4\Gamma(\alpha-1)} + \frac{t(T-t)(T-s)^{\alpha-3}}{4\Gamma(\alpha-2)}, \\ \quad \text{if } 0 \le s < t \le T, \\ -\frac{1}{2}\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(T-2t)(T-s)^{\alpha-2}}{4\Gamma(\alpha-1)} + \frac{t(T-t)(T-s)^{\alpha-3}}{4\Gamma(\alpha-2)}, \quad \text{if } 0 \le t < s \le T. \end{cases}$$

Note that $|G(t,s)| \le \frac{T^{\alpha-1}}{2}(\frac{1}{\Gamma(\alpha)} + \frac{1}{2\Gamma(\alpha-1)} + \frac{1}{2\Gamma(\alpha-2)})$ $\forall t, s \in I$.
Let $G_0 := \sup_{t,s \in I} |G(t,s)|$.

3 The main results

We are able now to present the existence results for problem (1.1)–(1.2). We consider first the case when $F(.,.)$ is convex valued.

Hypothesis 3.1 (i) $F(.,.) : I \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ has nonempty compact convex values and is Carathéodory.

(ii) There exist $\varphi(.) \in L^1(I, \mathbf{R})$ with $\varphi(t) > 0$ a.e. (I) and there exists a nondecreasing function $\psi : [0, \infty) \to (0, \infty)$ such that

$$\sup\{|v|;\ v \in F(t,x)\} \le \varphi(t)\psi(|x|) \quad \text{a.e. } (I), \quad \forall x \in \mathbf{R}.$$

Theorem 3.2 *Assume that Hypothesis* 3.1 *is satisfied and there exists* $r > 0$ *such that*

$$r > G_0 |\varphi|_1 \psi(r). \tag{3.1}$$

Then problem (1.1)–(1.2) *has at least one solution* $x(.)$ *such that* $|x(.)|_C < r$.

Proof Let $X = C(I, \mathbf{R})$ and consider $r > 0$ as in (3.1). It is obvious that the existence of solutions to problem (1.1)–(1.2) reduces to the existence of the solutions of the integral inclusion

$$x(t) \in \int_0^T G(t,s) F(s, x(s)) ds, \quad t \in I. \tag{3.2}$$

Consider the set-valued map $T : \overline{B_r(0)} \to \mathcal{P}(C(I, \mathbf{R}))$ defined by

$$T(x) := \{v(.) \in C(I, \mathbf{R});\ v(t) := \int_0^T G(t,s) f(s) ds,\ f \in \overline{S_F(x)}\}. \tag{3.3}$$

We show that $T(.)$ satisfies the hypotheses of Corollary 2.2.

First, we show that $T(x) \subset C(I, \mathbf{R})$ is convex for any $x \in C(I, \mathbf{R})$.

If $v_1, v_2 \in T(x)$ then there exist $f_1, f_2 \in S_F(x)$ such that for any $t \in I$ one has

$$v_i(t) = \int_0^T G(t,s) f_i(s) ds, \quad i = 1, 2.$$

Let $0 \leq \alpha \leq 1$. Then for any $t \in I$ we have

$$(\alpha v_1 + (1-\alpha) v_2)(t) = \int_0^T G(t,s)[\alpha f_1(s) + (1-\alpha) f_2(s)] ds.$$

The values of $F(.,.)$ are convex, thus $S_F(x)$ is a convex set and hence $\alpha v_1 + (1 - \alpha) v_2 \in T(x)$.

Secondly, we show that $T(.)$ is bounded on bounded sets of $C(I, \mathbf{R})$.

Let $B \subset C(I, \mathbf{R})$ be a bounded set. Then there exist $m > 0$ such that $|x|_C \leq m$ $\forall x \in B$.

If $v \in T(x)$ there exists $f \in S_F(x)$ such that $v(t) = \int_0^T G(t,s) f(s) ds$. One may write for any $t \in I$

$$|v(t)| \leq \int_0^T |G(t,s)|.|f(s)| ds \leq \int_0^T |G(t,s)| \varphi(s) \psi(|x(t)|) ds$$

and therefore

$$|v|_C \leq G_0 |\varphi|_1 \psi(m) \quad \forall v \in T(x),$$

i.e., $T(B)$ is bounded.

We show next that $T(.)$ maps bounded sets into equi-continuous sets.

Let $B \subset C(I, \mathbf{R})$ be a bounded set as before and $v \in T(x)$ for some $x \in B$. There exists $f \in S_F(x)$ such that $v(t) = \int_0^T G(t,s) f(s) ds$. Then for any $t, \tau \in I$ we have

$$\begin{aligned} |v(t) - v(\tau)| &\leq \left| \int_0^T G(t,s) f(s) ds - \int_0^T G(\tau,s) f(s) ds \right| \\ &\leq \int_0^T |G(t,s) - G(\tau,s)|.|f(s)| ds \\ &\leq \int_0^T |G(t,s) - G(\tau,s)| \varphi(s) \psi(m) ds. \end{aligned}$$

It follows that $|v(t) - v(\tau)| \to 0$ as $t \to \tau$. Therefore, $T(B)$ is an equi-continuous set in $C(I, \mathbf{R})$.

We apply now Arzela-Ascoli's theorem we deduce that $T(.)$ is completely continuous on $C(I, \mathbf{R})$.

In the next step of the proof we prove that $T(.)$ has a closed graph.

Let $x_n \in C(I, \mathbf{R})$ be a sequence such that $x_n \to x^*$ and $v_n \in T(x_n)$ $\forall n \in \mathbf{N}$ such that $v_n \to v^*$. We prove that $v^* \in T(x^*)$.

Since $v_n \in T(x_n)$, there exists $f_n \in S_F(x_n)$ such that $v_n(t) = \int_0^T G(t,s) f_n(s) ds$.

Define $\Gamma : L^1(I, \mathbf{R}) \to C(I, \mathbf{R})$ by $(\Gamma(f))(t) := \int_0^T G(t,s) f(s) ds$. One has $\max_{t \in I} |v_n(t) - v^*(t)| = |v_n(.) - v^*(.)|_C \to 0$ as $n \to \infty$.

We apply Theorem 2.5 to find that $\Gamma \circ S_F$ has closed graph and from the definition of Γ we get $v_n \in \Gamma \circ S_F(x_n)$. Since $x_n \to x^*$, $v_n \to v^*$ it follows the existence of $f^* \in S_F(x^*)$ such that $v^*(t) = \int_0^T G(t,s) f^*(s) ds$.

Therefore, $T(.)$ is upper semicontinuous and compact on $\overline{B_r(0)}$. We apply Corollary 2.2 to deduce that either (i) the inclusion $x \in T(x)$ has a solution in $\overline{B_r(0)}$, or (ii) there exists $x \in X$ with $|x|_C = r$ and $\lambda x \in T(x)$ for some $\lambda > 1$.

Assume that (ii) is true. With the same arguments as in the second step of our proof we get $r = |x(.)|_C \le G_0 |\varphi|_1 \psi(r)$ which contradicts (3.1). Hence only (i) is valid and theorem is proved. □

Example 3.3 Consider the set-valued map defined by $F(t,x) = \{v \in \mathbf{R};\ f_1(t,x) \le v \le f_2(t,x)\}$ with $f_1, f_2 : I \times \mathbf{R} \to \mathbf{R}$ given measurable functions. We assume that for any $t \in I$, $f_1(t,.)$ is lower semicontinuous (i.e., the set $\{x \in \mathbf{R};\ f_1(t,x) > a\}$ is open for all $a \in \mathbf{R}$) and we assume that for any $t \in I$, $f_2(t,.)$ is upper semicontinuous (i.e., the set $\{x \in \mathbf{R};\ f_2(t,x) < a\}$ is open for all $a \in \mathbf{R}$). It is well known that $F(.,.)$ is compact convex valued and upper semicontinuous.

We take $T = 2\pi$, $\alpha = \frac{5}{2}$ and note that, since $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, in this case $|G(t,s)| \le G_0 = \frac{17\pi}{3\sqrt{2}}$ $\forall s, t \in [0, 2\pi]$. We choose f_1, f_2 such that $\max\{|f_1(t,x)|, |f_2(t,x)|;\ (t,x) \in I \times \mathbf{R}\} \le \frac{t}{2G_0\pi^2}|x|$. Obviously, the hypotheses of Theorem 3.2 are satisfied with $\psi(x) = x$, $\varphi(t) = \frac{t}{2G_0\pi^2}$ and any $r > 0$.

We consider now the case when $F(.,.)$ is not necessarily convex valued. Our first existence result in this case is based on the Leray-Schauder alternative for single valued maps and on Bressan Colombo selection theorem.

Hypothesis 3.4 (i) $F(.,.) : I \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ has compact values, $F(.,.)$ is $\mathcal{L}(I) \otimes \mathcal{B}(\mathbf{R})$ measurable and $x \to F(t,x)$ is lower semicontinuous for almost all $t \in I$.

(ii) There exist $\varphi(.) \in L^1(I, \mathbf{R})$ with $\varphi(t) > 0$ a.e. (I) and there exists a nondecreasing function $\psi : [0, \infty) \to (0, \infty)$ such that

$$\sup\{|v|;\ v \in F(t,x)\} \le \varphi(t)\psi(|x|) \quad \text{a.e. } (I), \quad \forall x \in \mathbf{R}.$$

Theorem 3.5 *Assume that Hypothesis 3.4 is satisfied and there exists $r > 0$ such that Condition (3.1) is satisfied.*

Then problem (1.1)–(1.2) has at least one solution on I.

Proof We note first that if Hypothesis 3.4 is satisfied then $F(.,.)$ is of lower semicontinuous type (e.g., [18]). Therefore, we apply Theorem 2.4 to deduce that there exists $f(.) : C(I, \mathbf{R}) \to L^1(I, \mathbf{R})$ such that $f(x) \in S_F(x)$ $\forall x \in C(I, \mathbf{R})$.

We consider the corresponding problem

$$x(t) = \int_0^T G(t,s)f(x(s))ds, \quad t \in I \tag{3.4}$$

in the space $X = C(I, \mathbf{R})$. It is clear that if $x(.) \in C(I, \mathbf{R})$ is a solution of the problem (3.4) then $x(.)$ is a solution to problem (1.1).

Let $r > 0$ that satisfies condition (3.1) and define the set-valued map $T : \overline{B_r(0)} \to \mathcal{P}(C(I, \mathbf{R}))$ by

$$(T(x))(t) := \int_0^T G(t,s)f(x(s))ds.$$

Obviously, the integral equation (3.4) is equivalent with the operator equation

$$x(t) = (T(x))(t), \quad t \in I. \tag{3.5}$$

It remains to show that $T(.)$ satisfies the hypotheses of Corollary 2.3.

We show that $T(.)$ is continuous on $\overline{B_r(0)}$. From Hypotheses 3.4(ii) we have

$$|f(x(t))| \leq \varphi(t)\psi(|x(t)|) \quad \text{a.e. } (I)$$

for all $x(.) \in C(I, \mathbf{R})$. Let $x_n, x \in \overline{B_r(0)}$ such that $x_n \to x$. Then

$$|f(x_n(t))| \leq \varphi(t)\psi(r) \quad \text{a.e. } (I).$$

From Lebesgue's dominated convergence theorem and the continuity of $f(.)$ we obtain, for all $t \in I$

$$\begin{aligned} \lim_{n\to\infty} (T(x_n))(t) &= \lim_{n\to\infty} \int_0^T G(t,s)f(x_n(s))ds \\ &= \int_0^T G(t,s)f(x(s))ds = (T(x))(t), \end{aligned}$$

i.e., $T(.)$ is continuous on $\overline{B_r(0)}$.

Repeating the arguments in the proof of Theorem 3.2 with corresponding modifications it follows that $T(.)$ is compact on $\overline{B_r(0)}$. We apply Corollary 2.3 and we find that either (i) the equation $x = T(x)$ has a solution in $\overline{B_r(0)}$, or (ii) there exists $x \in X$ with $|x|_C = r$ and $x = \lambda T(x)$ for some $\lambda < 1$. □

As in the proof of Theorem 3.2 if the statement (ii) holds true, then we obtain a contradiction to (3.1). Thus only the statement (i) is true and problem (1.1) has a solution $x(.) \in C(I, \mathbf{R})$ with $|x(.)|_C < r$.

In order to obtain an existence result for problem (1.1)–(1.2) by using the set-valued contraction principle we introduce the following hypothesis on F.

Hypothesis 3.6 (i) $F(.,.) : I \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ has nonempty compact values and, for every $x \in \mathbf{R}$, $F(., x)$ is measurable.

(ii) There exists $L(.) \in L^1(I, \mathbf{R}_+)$ such that for almost all $t \in I$, $F(t, \cdot)$ is $L(t)$-Lipschitz in the sense that

$$d_H(F(t,x), F(t,y)) \leq L(t)|x-y| \quad \forall x, y \in \mathbf{R}$$

and $d(0, F(t,0)) \leq L(t)$ a.e. (I).

Denote $L_0 := \int_0^T L(s)ds$.

Theorem 3.7 *Assume that Hypothesis 3.6. is satisfied and $G_0 L_0 < 1$. Then the problem (1.1)–(1.2) has a solution.*

Proof We transform the problem (1.1)–(1.2) into a fixed point problem. Consider the set-valued map $T : C(I, \mathbf{R}) \to \mathcal{P}(C(I, \mathbf{R}))$ defined by

$$T(x) := \left\{ v(.) \in C(I, \mathbf{R});\ v(t) := \int_0^T G(t,s) f(s) ds,\ f \in S_{F,x} \right\}.$$

Note that since the set-valued map $F(., x(.))$ is measurable with the measurable selection theorem (e.g., Theorem III.6 in [8]) it admits a measurable selection $f(.) : I \to \mathbf{R}$. Moreover, from Hypothesis 3.6

$$|f(t)| \leq L(t) + L(t)|x(t)|,$$

i.e., $f(.) \in L^1(I, \mathbf{R})$. Therefore, $S_{F,x} \neq \emptyset$.

It is clear that the fixed points of $T(.)$ are solutions of problem (1.1)–(1.2). We shall prove that $T(.)$ fulfills the assumptions of Covitz Nadler contraction principle.

First, we note that since $S_{F,x} \neq \emptyset$, $T(x) \neq \emptyset$ for any $x(.) \in C(I, \mathbf{R})$.

Secondly, we prove that $T(x)$ is closed for any $x(.) \in C(I, \mathbf{R})$.

Let $\{x_n\}_{n \geq 0} \in T(x)$ such that $x_n(.) \to x^*(.)$ in $C(I, \mathbf{R})$. Then $x^*(.) \in C(I, \mathbf{R})$ and there exists $f_n \in S_{F,x}$ such that

$$x_n(t) = \int_0^T G(t,s) f_n(s) ds.$$

Since $F(.,.)$ has compact values and Hypothesis 3.6 is satisfied we may pass to a subsequence (if necessary) to get that $f_n(.)$ converges to $f(.) \in L^1(I, \mathbf{R})$ in $L^1(I, \mathbf{R})$. In particular, $f \in S_{F,x}$ and for any $t \in I$ we have

$$x_n(t) \to x^*(t) = \int_0^T G(t,s) f(s) ds,$$

i.e., $x^* \in T(x)$ and $T(x)$ is closed.

Finally, we show that $T(.)$ is a contraction on $C(I, \mathbf{R})$.

Let $x_1(.), x_2(.) \in C(I, \mathbf{R})$ and $v_1 \in T(x_1)$. Then there exist $f_1 \in S_{F,x_1}$ such that

$$v_1(t) = \int_0^T G(t,s) f_1(s) ds, \quad t \in I.$$

Consider the set-valued map

$$H(t) := F(t, x_2(t)) \cap \{x \in \mathbf{R};\ |f_1(t) - x| \le L(t)|x_1(t) - x_2(t)|\}, \quad t \in I.$$

From Hypothesis 3.6 one has

$$d_H(F(t, x_1(t)), F(t, x_2(t))) \le L(t)|x_1(t) - x_2(t)|,$$

hence $H(.)$ has nonempty closed values. Moreover, since $H(.)$ is measurable, there exists $f_2(.)$ a measurable selection of $H(.)$. It follows that $f_2 \in S_{F,x_2}$ and for any $t \in I$

$$|f_1(t) - f_2(t)| \le L(t)|x_1(t) - x_2(t)|.$$

Define

$$v_2(t) = \int_0^T G(t, s) f_2(s) ds, \quad t \in I.$$

and we have

$$\begin{aligned} |v_1(t) - v_2(t)| &\le \int_0^T |G(t, s)|.|f_1(s) - f_2(s)|ds \le G_0 \int_0^T |f_1(s) - f_2(s)|ds \\ &\le G_0 \int_0^T L(s)|x_1(s) - x_2(s)|ds \le G_0 L_0 |x_1 - x_2|_C. \end{aligned}$$

So, $|v_1 - v_2|_C \le G_0 L_0 |x_1 - x_2|_C$.
From an analogous reasoning by interchanging the roles of x_1 and x_2 it follows

$$d_H(T(x_1), T(x_2)) \le G_0 L_0 |x_1 - x_2|_C.$$

Therefore, $T(.)$ admits a fixed point which is a solution to problem (1.1)–(1.2). □

Example 3.8 Consider the following anti-periodic boundary value problem

$$D_c^{\frac{5}{2}} x(t) \in \left[0, \frac{1}{G_0(t+3)^2} \cdot \frac{|x|}{1+|x|}\right], \quad t \in [0, 2\pi], \tag{3.6}$$

$$x(0) = -x(2\pi), \qquad x'(0) = -x'(2\pi), \qquad x''(0) = -x''(2\pi) \tag{3.7}$$

Here, $T = 2\pi$, $G_0 = \frac{17\pi}{3\sqrt{2}}$, $L(t) = \frac{1}{9G_0}$, $L_0 = \frac{2\pi}{9G_0}$. Obviously, $G_0 L_0 = \frac{2\pi}{9} < 1$ and thus, by Theorem 3.7, problem (3.6)–(3.7) has a solution on $[0, 2\pi]$.

Remark 3.9 If $\alpha \in (1, 2]$, then problem (1.1)–(1.2) reduces to problem

$$D_c^{\alpha} x(t) \in F(t, x(t)) \quad \text{a.e. } ([0, T]), \tag{3.8}$$

$$x(0) = -x(T), \qquad x'(0) = -x'(T). \tag{3.9}$$

A result similar to Theorem 3.2 is obtained in [2] for problem (3.8)–(3.9). Obviously, similar results as in Theorem 3.5 and Theorem 3.7 may be obtained for problem (3.8)–(3.9).

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