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Newton–Kantorovich approximations under weak continuity conditions

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Abstract The paper is concerned with the application of Kantorovich-type majorants for the convergence of Newton's method to a locally unique solution of a nonlinear equation in a Banach space setting. The Fréchet-derivative of the operator involved satisfies only a rather weak continuity condition. Using our new idea of recurrent functions, we obtain sufficient convergence conditions, as well as error estimates. The results compare favorably to earlier ones (Ezquerro, Hernández in IMA J. Numer. Anal. 22:187–205, 2002 and Proinov in J. Complex. 26:3–42, 2010).

Keywords Newton's method · Kantorovich-type majorants · Banach space · Recurrent functions · Semilocal convergence

Mathematics Subject Classification (2000) 65J15 · 65R20 · 47H17 · 49M15 · 45G10

1 Introduction

In this study we are concerned with the problem of approximating a locally unique solution $x^\star$ of equation

$$F(x) + G(x) = 0, \tag{1.1}$$

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where F is defined on an open convex subset $\mathcal{D}$ of a Banach space $\mathcal{X}$, with values in a Banach space $\mathcal{Y}$, and $G : \mathcal{D} \longrightarrow \mathcal{Y}$ is a continuous operator.

The field of computational sciences has seen a considerable development in mathematics, engineering sciences, and economic equilibrium theory. Researchers in this field are faced with the problem of solving a variety of equations. We note that in computational sciences, the practice of numerical analysis for finding such solutions is essentially connected to variants of Newton's method. For example, dynamic systems are mathematically modeled by difference or differential equations, and their solutions usually represent the states of the systems. For the sake of simplicity, assume that a time-invariant system is driven by the equation $\dot{x} = T(x)$, for some suitable operator T, where x is the state. Then the equilibrium states are determined by solving (1.1). Similar equations are used in the case of discrete systems. The unknowns of engineering equations can be functions (difference, differential, and integral equations), vectors (systems of linear or nonlinear algebraic equations), or real or complex numbers (single algebraic equations with single unknowns). Except in special cases, the most commonly used solution methods are iterative—when starting from one or several initial approximations a sequence is constructed that converges to a solution of the equation. Iteration methods are also applied for solving optimization problems. In such cases, the iteration sequences converge to an optimal solution of the problem at hand. Since all of these methods have the same recursive structure, they can be introduced and discussed in a general framework.

We shall use the Newton-type method (NTM):

$$x_{n+1} = x_n - F'(x_n)^{-1}(F(x_n) + G(x)), \quad x_0 \in \mathcal{D}\ (n \geq 0), \tag{1.2}$$

to generate a sequence $\{x_n\}$ approximating $x^\star$. Here, $F'(x) \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ $(x \in \mathcal{D})$ denotes th Fréchet-derivative of operator F [5, 18].

If $G(x) = 0$ $(x \in \mathcal{D})$, then we obtain the popular Newton's method (NM)

$$x_{n+1} = x_n - F'(x_n)^{-1} F(x_n), \quad x_0 \in \mathcal{D}\ (n \geq 0). \tag{1.3}$$

Zinčenko [27], Zabrejko–Nguen [26], Chen–Yamamoto [12], Appell et al. [1], Deuflhard [14, 15], Yamamoto [25], Rheinboldt [24], Dennis [13], Cătinaş [11], Hernández–Ezquerro [16], Potra [19–21], Proinov [22, 23], and Argyros [2–4, 6] have provided a convergence analysis of (NTM) under various conditions. A survey of such results can be found in [5, 7, 8], and the references there.

Let $x_0 \in \mathcal{D}$, and $R > 0$ be such that

$$U(x_0, R) = \{x \in \mathcal{X} : \|x - x_0\| < R\} \subseteq \mathcal{D}. \tag{1.4}$$

In this study, we are motivated by the elegant works by Hernández–Ezquerro [16], Proinov [22, 23], and optimization considerations. They provided a semilocal convergence analysis using the following conditions:

$$
\begin{aligned}
(\mathcal{C}):\quad & F'(x_0)^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X}), \text{ and for any } x, y \in \overline{U}(x_0, r) \quad (0 < r \leq R):\\
& \|F'(x_0)^{-1}(F'(x) - F'(y))\| \leq \omega(\|x - y\|),\\
& \omega(tr) \leq h(t)\omega(r), \quad t \in [0, 1],\ r \in [0, R],\\
& \|F'(x_0)^{-1}(G(x) - G(y))\| \leq \omega_1(\|x - y\|)\|x - y\|,\\
& \|F'(x_0)^{-1}(F(x_0) + G(x_0))\| \leq \eta,
\end{aligned} \tag{1.5}
$$

and scalar iteration $\{v_n\}$ ($n \geq 0$) given by

$$
v_0 = 0, \qquad v_1 = \eta,
$$

$$
v_{n+1} = v_n + \frac{H\omega(v_n - v_{n-1}) + \omega_1(v_n - v_{n-1})}{1 - \omega(v_n)}(v_n - v_{n-1}) \quad (n \geq 1),
$$

as the majorizing sequence for $\{x_n\}$.

Here, ω, ω_1 are non-decreasing, non-negative function defined on interval $[0, R]$. h is a function on $[0, 1]$, and

$$
H = \int_0^1 h(t)\,dt.
$$

Condition (1.5) has been successfully used to sharpen the error bounds obtained for particular expressions [16] (see also [23, Sect. 7]). Note that such a function h always exists. Indeed, if ω is a nonzero function on $\mathcal{J} = [0, +\infty)$, then one can define function $h : [0, 1] \longrightarrow \mathbb{R}$ by

$$
h(s) = \sup\left\{\frac{\omega(s\,t)}{\omega(t)} : t \in [0, \infty),\ \text{with } \omega(t) > 0\right\}.
$$

Clearly, function h so defined satisfies (1.5), and has the following properties [23]:

- $h(0) = 0$, $h(1) = 1$ provided that $\omega(0) = 0$;
- h is nondecreasing on $[0, 1]$ provided that ω is nondecreasing on $\mathcal{J}$;
- h is continuous on $[0, 1]$ provided that ω is nondecreasing on $\mathcal{J}$;
- h is identical to 1 on $[0, 1]$ if ω is non-decreasing on $\mathcal{J}$ and $\omega(0) > 0$.

Several choices of function h can be found in [23].

Recently [10], we provided a finer convergence than in [16, 23], under the same computational cost using the following set of conditions:

$$
\begin{aligned}
(\mathcal{H}):\quad & F'(x_0)^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X}), \text{and for any } x, y \in \overline{U}(x_0, r) \quad (0 < r \leq R):\\
& \|F'(x_0)^{-1}(F'(x) - F'(y))\| \leq \omega(\|x - y\|),\\
& \omega(tr) \leq h(t)\omega(r), \quad t \in [0, 1],\ r \in [0, R],\\
& \|F'(x_0)^{-1}(F'(x) - F'(x_0))\| \leq \omega_0(\|x - x_0\|),\\
& \|F'(x_0)^{-1}(G(x) - G(y))\| \leq \omega_1(\|x - y\|)\|x - y\|,\\
& \|F'(x_0)^{-1}(F(x_0) + G(x_0))\| \leq \eta,
\end{aligned} \tag{1.6}
$$

and scalar iteration $\{t_n\}$ ($n \geq 0$) given by

$$t_0 = 0, \qquad t_1 = \eta,$$
$$t_{n+1} = t_n + \frac{H\omega(t_n - t_{n-1}) + \omega_1(t_n - t_{n-1})}{1 - \omega_0(t_n)}\,(t_n - t_{n-1}) \quad (n \geq 1), \tag{1.7}$$

as the majorizing sequence for $\{x_n\}$.

Here, ω_0 is a non-decreasing, non-negative function on $[0, R]$.

Condition (1.6) is not an additional hypothesis, since in practice, the computation of function ω requires that of ω_0.

Note also that:

$$\omega_0(r) \leq \omega(r), \quad r \in [0, R] \tag{1.8}$$

holds in general, and $\frac{\omega}{\omega_0}$ can be arbitrarily large [5].

In this study, we still use the set of hypotheses $(\mathcal{H})$, but we generate new convergence conditions for majorizing sequence $\{t_n\}$. This way we provide an even finer analysis than in the previously stated approaches.

The paper is organized as follows: Sect. 2 contains the semilocal convergence analysis of (NTM), whereas the applications, numerical examples, and comparisons with earlier results can be found in Sect. 3.

2 Semilocal convergence analysis of (NTM)

We need to define some sequences and functions.

Definition 2.1 Let constants η, H, and functions h_0, ω_0, ω, and ω_1 be as in the introduction of this study. Let scalar sequence $\{t_n\}$ given by (1.7). Define functions f_n, g_n, p_n on $[0, 1)$ and q on $I_q = [0, 1) \times [0, \eta] \times [\eta, \frac{\eta}{1-s}]^3$ by

$$f_n(s) = H\omega(s^{n-1}\eta) + \omega_1(s^{n-1}\eta) + s(\omega_0((1 + s + \cdots + s^{n-1})\eta) - 1), \tag{2.1}$$

$$\begin{aligned} g_n(s) = {} & H(\omega(s^n\eta) - \omega(s^{n-1}\eta)) + \omega_1(s^n\eta) - \omega_1(s^{n-1}\eta) \\ & + s(\omega_0((1 + s + \cdots + s^n)\eta) - \omega_0((1 + s + \cdots + s^{n-1})\eta)), \end{aligned} \tag{2.2}$$

$$\begin{aligned} p_n(s) = {} & H(\omega(s^{n+1}\eta) + \omega(s^{n-1}\eta) - 2\omega(s^n\eta)) \\ & + \omega_1(s^{n+1}\eta) + \omega_1(s^{n-1}\eta) - 2\omega_1(s^n\eta) \\ & + s(\omega_0((1 + s + \cdots + s^{n+1})\eta) + \omega_0((1 + s + \cdots + s^{n-1})\eta) \\ & - 2\omega_0((1 + s + \cdots + s^n)\eta)), \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} q(s,\lambda,\beta_0,\beta,\gamma_0) &= H(\omega(s^2\lambda)+\omega(\lambda)-2\omega(s,\lambda)) \\ &\quad +\omega_1(s^2\lambda)+\omega_1(\lambda)-2\omega_1(s\,\lambda) \\ &\quad +s(\omega_0(\beta_0+\beta+\gamma_0)+\omega_0(\beta)-2\omega_0(\beta_0+\beta)). \end{aligned} \tag{2.4}$$

Define function f_∞ on [0, 1) by

$$f_\infty(s) = \lim_{n\longrightarrow\infty} f_n(s). \tag{2.5}$$

It then follows from (2.1), and (2.5) that

$$f_\infty(s) = \omega_0\left(\frac{\eta}{1-s}\right) - 1. \tag{2.6}$$

It follows from (2.1)–(2.4) that:

$$f_{n+1}(s) = f_n(s) + g_n(s), \tag{2.7}$$

$$h_{n+1}(s) = g_n(s) + p_n(s), \tag{2.8}$$

and for

$$\lambda = s^{n-1}\eta, \qquad \beta_0 = s^{n+1}\eta, \qquad \beta = \eta\sum_{i=0}^{n-1} s^i, \qquad \gamma_0 = s^n\eta, \tag{2.9}$$

we have

$$q(s,\lambda,\beta_0,\beta,\gamma_0) = p_n(s). \tag{2.10}$$

We need the following result on majorizing sequences for (NTM).

Lemma 2.2 *Let iteration* $\{t_n\}$ *given by* (1.7), *and functions* f_n, g_n, p_n, q *be as given in Defintion* 2.1.

Assume there exists $\alpha \in (0,1)$, *such that*:

$$\omega_0(\eta) < 1, \tag{2.11}$$

$$\frac{H\omega(\eta)+\omega_1(\eta)}{1-\omega_0(\eta)} \le \alpha, \tag{2.12}$$

$$q(s,\lambda,\beta_0,\beta,\gamma_0) \ge 0 \quad \text{on } I_q, \tag{2.13}$$

$$g_1(\alpha) \ge 0, \tag{2.14}$$

and

$$f_\infty(\alpha) \le 0. \tag{2.15}$$

Then, scalar iteration $\{t_n\}$ $(n \geq 0)$, *is well defined, non-decreasing, bounded from above by*:

$$t^{\star\star} = \frac{\eta}{1-\alpha}, \tag{2.16}$$

and converges to its unique least upper bound $t^\star$ *satisfying*

$$t^\star \in [0, t^{\star\star}]. \tag{2.17}$$

Moreover, the following estimates hold for all $n \geq 0$:

$$0 \leq t_{n+1} - t_n \leq \alpha(t_n - t_{n-1}) \leq \alpha^n \eta, \tag{2.18}$$

and

$$0 \leq t^\star - t_n \leq \frac{\eta}{1-\alpha}\alpha^n. \tag{2.19}$$

Proof Estimate (2.18) is true if:

$$0 \leq \frac{H\omega(t_n - t_{n-1}) + \omega_1(t_n - t_{n-1})}{1 - \omega_0(t_n)} \leq \alpha \tag{2.20}$$

hold for all $n \geq 1$.

In view of (1.7), (2.11), and (2.12), estimate (2.20) holds for $n = 1$. We also have by (2.20) that $0 \leq t_2 - t_1 \leq \alpha\ (t_1 - t_0)$. That is (2.18) holds for $n = 1$. Let us assume that (2.18), and (2.20) hold for all $k \leq n$. Then, we have by (2.18) that

$$t_n \leq \frac{1-\alpha^n}{1-\alpha}\eta. \tag{2.21}$$

Using the induction hypotheses, and (2.21), we see that (2.20) certainly holds, if

$$H\omega(\alpha^{n-1}\eta) + \omega_1(\alpha^{n-1}\eta) \leq \alpha\left(1 - \omega_0\left(\frac{1-\alpha^n}{1-\alpha}\eta\right)\right) \tag{2.22}$$

or

$$H\omega(\alpha^{n-1}\eta) + \omega_1(\alpha^{n-1}\eta) + \alpha\omega_0\left(\frac{1-\alpha^n}{1-\alpha}\eta\right) - \alpha \leq 0.$$

Let $s = \alpha$. Estimates (2.22) motivates us to define function f_n given by (2.1), and show instead of (2.22):

$$f_n(\alpha) \leq 0. \tag{2.23}$$

We have by (2.7)–(2.10) (for $s = \alpha$), and (2.14) that

$$f_{n+1}(\alpha) \geq f_n(\alpha) \quad (n \geq 1). \tag{2.24}$$

In view of (2.5), (2.6), and (2.24), estimate (2.23) holds, if (2.15) is true, since

$$f_n(\alpha) \leq f_\infty(\alpha) \quad (n \geq 1). \tag{2.25}$$

The induction is completed.

It follows that iteration $\{t_n\}$ is non-decreasing, bounded from above by $t^{\star\star}$ (given by (2.16)), and as such it converges to its unique least upper bound $t^\star$ satisfying (2.17). Finally, estimate (2.19) follows from (2.18) by using standard majorization techniques [5, 7, 8].

That completes the proof of Lemma 2.2. □

Hypotheses $(\mathcal{H})$, and those of Lemma 2.1 will be called from now $(\mathcal{A})$. We provide the main semilocal convergence result for (NTM).

Theorem 2.3 *Assume hypotheses* $(\mathcal{A})$ *hold. Then, sequence* $\{x_n\}$ $(n \geq 0)$ *generated by* (*NTM*) *is well defined, remains in* $\overline{U}(x_0, t^\star)$ *for all* $n \geq 0$, *and converges to a solution* $x^\star \in \overline{U}(x_0, t^\star)$ *of equation* $F(x) + G(x) = 0$.

Moreover, the following estimates hold for all $n \geq 0$:

$$\|x_n - x^\star\| \leq t^\star - t_n. \tag{2.26}$$

Furthermore, if there exists

$$R_0 \in [t^\star, R], \tag{2.27}$$

such that

$$H\omega(R_0) + \omega_1(R_0) + \omega_0(t^\star) < 1, \tag{2.28}$$

then $x^\star$ *is the unique solution of equation* $F(x) + G(x) = 0$ *in* $\overline{U}(x_0, R_0)$.

Proof We shall show using induction on k:

$$\|x_k - x_{k-1}\| \leq t_k - t_{k-1}, \tag{2.29}$$

and

$$\|x_k - x_0\| \leq t_k. \tag{2.30}$$

Estimate (2.29) holds by (1.7), whereas (2.30) is true for $k = 1$ as equality. Let us assume (2.29), and (2.30) hold for all $m \leq k$. Using hypotheses (1.6), and (2.11), we get:

$$\|F'(x_0)^{-1}(F'(x_1) - F'(x_0))\| \leq \omega_0(\|x_1 - x_0\|) \leq \omega_0(\eta) < 1. \tag{2.31}$$

It follows from (2.31), and the Banach lemma on invertible operators [5, 7, 8] that $F'(x_0)^{-1}$ exists, and

$$\|F'(x_1)^{-1}F'(x_0)\| \leq \frac{1}{1 - \omega_0(\|x_1 - x_0\|)} \leq \frac{1}{1 - \omega_0(\eta)}. \tag{2.32}$$

We also showed in Lemma 2.2 that

$$\omega_0(t_k) < 1. \tag{2.33}$$

It then follows as in (2.31) with t_k, x_k replacing t_1, x_1 that $F'(x_k)^{-1}$ exists, and

$$\|F'(x_k)^{-1}F'(x_0)\| \le \frac{1}{1-\omega_0(\|x_k-x_0\|)} \le \frac{1}{1-\omega_0(t_k)}. \tag{2.34}$$

Using (1.2), hypotheses $(\mathcal{H})$, and (2.34)

$$\begin{aligned}
\|x_{k+1}-x_k\| &= \|F'(x_k)^{-1}(F(x_k)+G(x_k))\| \\
&\le \|F'(x_k)^{-1}F'(x_0)\| \bigg\| F'(x_0)\Big(F(x_k)+G(x_k) \\
&\quad - F'(x_{k-1})(x_k-x_{k-1}) - F(x_{k-1}) - G(x_{k-1})\Big)\bigg\| \\
&\le \frac{1}{1-\omega_0(t_k)}\bigg(\int_0^1 \|F'(x_0)^{-1}(F'(x_{k-1}+t(x_k-x_{k-1})) \\
&\quad - F'(x_{k-1}))\|\,\|x_k-x_{k-1}\|\,dt \\
&\quad + \|F'(x_0)^{-1}(G(x_k)-G(x_{k-1}))\|\bigg) \\
&\le \frac{1}{1-\omega_0(t_k)}\bigg(\int_0^1 \omega(t\|x_k-x_{k-1}\|)\,dt \\
&\quad + \omega_1(\|x_k-x_{k-1}\|)\bigg)\|x_k-x_{k-1}\| \\
&\le \frac{H\omega(t_k-t_{k-1})+\omega_1(t_k-t_{k-1})}{1-\omega_0(t_k)}(t_k-t_{k-1}) \\
&= t_{k+1}-t_k,
\end{aligned} \tag{2.35}$$

which shows (2.29) for all $k \ge 1$.

Moreover, we have:

$$\begin{aligned}
\|x_{k+1}-x_0\| &\le \|x_{k+1}-x_k\| + \|x_k-x_0\| \\
&\le t_{k+1}-t_k+t_k = t_{k+1} \le t^\star.
\end{aligned} \tag{2.36}$$

The induction for (2.29), and (2.30) is completed.

In view of Lemma 2.2, (2.29), and (2.30), sequence $\{x_n\}$ is Cauchy in a Banach space $\mathcal{X}$, and as such it converges to some $x^\star \in \overline{U}(x_0,t^\star)$ (since $\overline{U}(x_0,t^\star)$ is a closed set).

Using (2.35), we get

$$\|F'(x_0)^{-1}(F(x_k)+G(x_k))\| \le (H\omega(t_k-t_{k-1})+\omega_1(t_k-t_{k-1}))\omega(t_k-t_{k-1}). \tag{2.37}$$

By letting $k \longrightarrow \infty$ in (2.37), we obtain $F(x^\star)+G(x^\star)=0$.

Estimate (2.26) follows from (2.29) by using standard majorization techniques [5, 7, 8].

Finally, to show uniqueness of $x^\star$ in $\overline{U}(x_0, R_0)$, let us assume $y^\star$ is a solution in $\overline{U}(x_0, R_0)$.

Using the approximation

$$x^\star - x_{k+1} = x^\star - x_k + F'(x_k)^{-1}(F(x_k) + G(x_k)) - F'(x_k)^{-1}(F(y^\star) + G(y^\star)), \tag{2.38}$$

as in (2.35), we obtain in turn:

$$\begin{aligned} \|y^\star - x_{k+1}\| &\le \frac{1}{1-\omega_0(t_k)}\Bigg(\int_0^1 \|F'(x_0)^{-1}(F'(x_{k-1} + t(x^\star - x_k)) \\ &\quad - F'(x_k))\|\|x^\star - x_k\|\,dt + \|F'(x_0)^{-1}(G(y^\star) - G(x_k))\|\Bigg) \\ &\le \frac{1}{1-\omega_0(t_k)}\Bigg(\int_0^1 \omega(t\|y^\star - x_k\|)\,dt + \omega_1(\|y^\star - x_k\|)\Bigg)\|y^\star - x_k\| \\ &\le \frac{H\omega(R_0) + \omega_1(R_0)}{1-\omega_0(t^\star)}\|y^\star - x_k\| \\ &< \|y^\star - x_k\| \quad \text{(by (2.28))}, \end{aligned} \tag{2.39}$$

which implies $\lim_{k\to\infty} x_k = y^\star$. But we showed $\lim_{k\to\infty} x_k = x^\star$. Hence, we deduce

$$x^\star = y^\star.$$

That completes the proof of Theorem 2.3. □

Remark 2.4

(a) The limit point $t^\star$ can be replaced in Theorem 2.3 by $t^{\star\star}$ given in the closed form by (2.16).
(b) Note that more general conditions than the ones given in [23], and ours (introduced in this study) were provided in [4] to show the local (and semilocal) convergence of two-point Newton-like methods (see also [5, 7, 17]).

3 Special Cases and Applications

Let us consider the case of Newton's method. That is: $G(x) = 0$ for all $x \in \mathcal{D}$, and $\omega_1(s) = 0$ for all $s \in [0, 1]$.

In this case, we note that:

$$\omega_0(s) \le \omega(s), \quad s \in [0,1], \tag{3.1}$$

holds, and $\frac{\omega}{\omega_0}$ can be arbitrarily large [5, 7, 8].

Comparison with a result by Ezquerro, and Hernández [16] (see also Proinov [23, Theorem 7.3]): If one reproduces these results in affine invariant form, then the corresponding to $\{t_n\}$ majorizing sequence is essentially given by:

$$\begin{aligned} &v_0 = 0, \qquad v_1 = \eta, \\ &v_{n+1} = v_n + \frac{H\omega(v_n - v_{n-1}) + \omega_2(v_{n-1})}{1-\omega(v_n)}\,(v_n - v_{n-1}) \quad (n \ge 1). \end{aligned} \tag{3.2}$$

Then, under the hypotheses of our Theorem 2.3 and the corresponding ones in [16, 23], we note the following advantages:

(i) Majorizing sequence $\{t_n\}$ is tighter than $\{v_n\}$, since an inductive argument shows:

$$t_n < v_n \quad (n \ge 2) \tag{3.3}$$

and

$$t_{n+1} - t_n < v_{n+1} - v_n \quad (n \ge 2). \tag{3.4}$$

(ii) The information on the location of the solution is at least as precise, since:

$$t^\star = \lim_{n\to\infty} t_n \le \lim_{n\to\infty} v_n = v^\star. \tag{3.5}$$

It turns out our sufficient convergence conditions are weaker. Indeed, for simplicity, let us consider the case, when

$$\omega(s) = Ls, \qquad \omega_0(s) = L_0 s, \quad \text{and} \quad h(t) = \frac{1}{2}. \tag{3.6}$$

Then, the iterations $\{t_n\}$, $\{v_n\}$ become:

$$t_0 = 0, \qquad t_1 = \eta, \qquad t_{n+1} = t_n + \frac{L(t_n - t_{n-1})^2}{2(1 - L_0\, t_n)} \quad (n \ge 1), \tag{3.7}$$

$$v_0 = 0, \qquad v_1 = \eta, \qquad v_{n+1} = v_n + \frac{L(v_n - v_{n-1})^2}{2(1 - L v_n)} \quad (n \ge 1). \tag{3.8}$$

Note that iteration (3.8) converges if the famous for its simplicity and clarity Newton–Kantorovich hypothesis

$$K = L\eta \le \frac{1}{2} \tag{3.9}$$

holds [17].

However, iteration (3.7) converges under weaker conditions provided that $L_0 < L$. We need the following result for the convergence of majorizing sequence $\{t_n\}$.

Lemma 3.1 (Argyros and Hilout [9, 10]) *Assume there exist constants* $L_0 \geq 0$, $L \geq 0$ *with* $L_0 \leq L$, *and* $\eta \geq 0$, *such that*:

$$q_0 = \overline{L}\eta \leq \frac{1}{2}, \tag{3.10}$$

where,

$$\overline{L} = \frac{1}{8}\left(L + 4L_0 + \sqrt{L^2 + 8L_0 L}\right). \tag{3.11}$$

The inequality in (3.10) *is strict, if* $L_0 = 0$.

Then, sequence $\{t_k\}$ $(k \geq 0)$ *given by* (3.7) *is well defined, nondecreasing, bounded above by* $t^{\star\star}$, *and converges to its unique least upper bound* $t^\star \in [0, t^{\star\star}]$, *where*

$$t^{\star\star} = \frac{2\eta}{2-\delta}, \tag{3.12}$$

$$1 \leq \delta = \frac{4L}{L + \sqrt{L^2 + 8L_0 L}} < 2 \quad \text{for } L_0 \neq 0. \tag{3.13}$$

Moreover, the following estimates hold:

$$L_0 t^\star \leq 1, \tag{3.14}$$

$$0 \leq t_{k+1} - t_k \leq \frac{\delta}{2}(t_k - t_{k-1}) \leq \cdots \leq \left(\frac{\delta}{2}\right)^k \eta \quad (k \geq 1), \tag{3.15}$$

$$t_{k+1} - t_k \leq \left(\frac{\delta}{2}\right)^k (2q_0)^{2^k - 1}\eta \quad (k \geq 0), \tag{3.16}$$

$$0 \leq t^\star - t_k \leq \left(\frac{\delta}{2}\right)^k \frac{(2q_0)^{2^k-1}\eta}{1 - (2q_0)^{2^k}} \quad (2q_0 < 1)\ (k \geq 0). \tag{3.17}$$

Remark 3.2 If $L_0 = L$, Lemma 3.1 provides the usual error bounds appearing essentially in the Newton–Kantorovich theorem [17].

However, if $L_0 < L$, then our sufficient convergence condition (3.10) is weaker than (3.9). Finally, our ratio $2q_0$ is also smaller than $2K$.

Hence, the ω-function approach [16, 23] does not necessarily produce the weakest sufficient convergence conditions not even in the simplest case of Newton's method under the simple Lipschitz conditions (3.6).

Moreover, in the general case, our approach takes advantage of estimate (3.1). Note that the more precise than ω, function ω_0 does not appear in the convergence analysis of (NTM) [16, 23]. Furthemore, the ration α of linear convergence is given for iteration $\{t_n\}$ but not for $\{v_n\}$.

Let us provide examples where (3.9) is violated, but (3.10) holds, and $L_0 < L$.

Example 3.3 Let $\mathcal{X} = \mathcal{Y} = \mathbb{R}^2$, equipped with the max-norm, and

$$x_0 = (1, 1)^T, \qquad \mathcal{D}_0 = \{x : \|x - x_0\| \le 1 - \beta\}, \quad \beta \in \left[0, \frac{1}{2}\right).$$

Define function F on $\mathcal{D}_0$ by

$$F(x) = (\xi_1^3 - \beta, \xi_2^3 - \beta)^T, \quad x = (\xi_1, \xi_2)^T. \tag{3.18}$$

The Fréchet-derivative of operator F is given by

$$F'(x) = \begin{bmatrix} 3\xi_1^2 & 0 \\ 0 & 3\xi_2^2 \end{bmatrix}.$$

Using hypotheses of Theorem 2.3, we get:

$$\eta = \frac{1}{3}(1 - \beta), \qquad L_0 = 3 - \beta, \quad \text{and} \quad L = 2(2 - \beta).$$

The Newton–Kantorovich condition (3.5) is violated, since

$$\frac{4}{3}(1 - \beta)(2 - \beta) > 1 \quad \text{for all } \beta \in \left[0, \frac{1}{2}\right).$$

Hence, there is no guarantee that (NTM) converges to $x^\star = (\sqrt[3]{\beta}, \sqrt[3]{\beta})^T$, starting at x_0. However, our condition (3.10) is true for all $\beta \in I = [0.450339002, 0.5)$. Hence, the conclusions of our Theorem 2.3 can apply to solve (3.18) for all $\beta \in I$.

Example 3.4 Consider the following nonlinear boundary value problem [5]

$$\begin{cases} u'' = -u^3 - \gamma u^2, \\ u(0) = 0, \qquad u(1) = 1. \end{cases}$$

It is well known that this problem can be formulated as the integral equation

$$u(s) = s + \int_0^1 Q(s, t)(u^3(t) + \gamma u^2(t))\, dt \tag{3.19}$$

where, Q is the Green function:

$$Q(s, t) = \begin{cases} t(1 - s), & t \le s, \\ s(1 - t), & s < t. \end{cases}$$

We observe that

$$\max_{0 \le s \le 1} \int_0^1 |Q(s, t)| = \frac{1}{8}.$$

Let $\mathcal{X} = \mathcal{Y} = \mathcal{C}[0, 1]$, with norm

$$\|x\| = \max_{0\le s\le 1} |x(s)|.$$

Then problem (3.19) is in the form (1.1), where, $F : \mathcal{D} \longrightarrow \mathcal{Y}$ is defined as

$$[F(x)](s) = x(s) - s - \int_0^1 Q(s,t)(x^3(t) + \gamma x^2(t))\,dt,$$

and

$$G(x)(s) = 0.$$

It is easy to verify that the Fréchet derivative of F is defined in the form

$$[F'(x)v](s) = v(s) - \int_0^1 Q(s,t)(3x^2(t) + 2\gamma x(t))v(t)\,dt.$$

If we set $u_0(s) = s$, and $\mathcal{D} = U(u_0, R)$, then since $\|u_0\| = 1$, it is easy to verify that $U(u_0, R) \subset U(0, R+1)$. It follows that $2\gamma < 5$, then

$$\|I - F'(u_0)\| \le \frac{3\|u_0\|^2 + 2\gamma\|u_0\|}{8} = \frac{3+2\gamma}{8},$$

$$\|F'(u_0)^{-1}\| \le \frac{1}{1 - \frac{3+2\gamma}{8}} = \frac{8}{5-2\gamma},$$

$$\|F(u_0)\| \le \frac{\|u_0\|^3 + \gamma\|u_0\|^2}{8} = \frac{1+\gamma}{8},$$

$$\|F(u_0)^{-1}F(u_0)\| \le \frac{1+\gamma}{5-2\gamma}.$$

On the other hand, for $x, y \in \mathcal{D}$, we have

$$[(F'(x) - F'(y))v](s) = -\int_0^1 Q(s,t)(3x^2(t) - 3y^2(t) + 2\gamma(x(t) - y(t)))v(t)\,dt.$$

Consequently (see [5]),

$$\|F'(x) - F'(y)\| \le \frac{\gamma + 6R + 3}{4}\|x - y\|,$$

$$\|F'(x) - F'(u_0)\| \le \frac{2\gamma + 3R + 6}{8}\|x - u_0\|.$$

Therefore, conditions of Theorem 2.3 hold with

$$\eta = \frac{1+\gamma}{5-2\gamma}, \qquad L = \frac{\gamma + 6R + 3}{4}, \qquad L_0 = \frac{2\gamma + 3R + 6}{8}.$$

Note also that $L_0 < L$.

Finally, we note that relevant results have been given by us for more general Newton-type method in [7] under different sufficient convergence conditions. In the special case of (NTM), our conditions are different from [16, 23]. Moreover, we provide the ratio of linear convergence, not given with the approach in [16, 23].

4 Conclusion

Using ω-type conditions, we provided a semilocal convergence analysis for (NTM) in order to approximate a locally unique solution of an nonlinear equation in a Banach space.

Using a combination of ω-condition and center-ω condition, instead of only ω-condition, and our new idea of recurrent functions, we presented an analysis with the following advantages over the works in [16, 23]: weaker sufficient convergence conditions, tighter error bounds and larger convergence domain in some interesting cases.

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