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Parameter-uniform numerical methods for singularly perturbed mixed boundary value problems using grid equidistribution

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Abstract In this paper, we present the analysis of an upwind scheme for obtaining the solution of a convection-diffusion two-point boundary value problem with Robin boundary conditions. The solution is obtained on a suitable nonuniform mesh which is formed by equidistributing the arc-length monitor function. It is shown that the discrete solution obtained by the upwind scheme converges uniformly with respect to the perturbation parameter. Numerical results are presented that demonstrate the sharpness of the theoretical estimates.

Keywords Singularly perturbed boundary value problems · Robin BCs · Upwind scheme · Adaptive mesh · Uniform convergence

Mathematics Subject Classification (2000) 65L10 · 65L12

1 Introduction

In this article, we consider the following form of singularly perturbed mixed boundary value problem (SPMBVP):

$$\begin{cases} Lu(x) \equiv -\varepsilon u''(x) - a(x)u'(x) + b(x)u(x) = f(x), \quad x \in \Omega = (0,1), \\ B_0 u(0) \equiv \beta_1 u(0) - \beta_2 \varepsilon u'(0) = s_0, \qquad B_1 u(1) \equiv \lambda_1 u(1) + \lambda_2 u'(1) = s_1, \end{cases} \tag{1.1}$$

where $0 < \varepsilon \ll 1$ is a small singular perturbation parameter, the functions $a(x)$, $b(x)$, $f(x)$ are sufficiently smooth and s_0, s_1 are given constants. Further, we assume that the coefficients are such that $\alpha^* \geq a(x) > \alpha > 0$, $b(x) \geq 0$ and the constants satisfy

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$\beta_1, \beta_2 \geq 0$, $\beta_1 + \beta_2 > 0$, $\lambda_2 \geq 0$ and $\lambda_1 > 0$. Under these assumptions, the above problem (1.1) has a unique solution which exhibits a layer behavior near $x = 0$.

The literature of approximation of the solutions of convection-diffusion problems is large and many finite difference methods have been proposed to approximate the solutions ([2, 6, 11], the book [12]). But at the same time there are only a few articles that consider finite difference approximations of solutions of convection-diffusion problems with Robin type boundary conditions includes [1, 3, 10] and the recent book [5]. In [3], the conservative form of singularly perturbed ordinary differential equations with mixed boundary conditions is considered and a class of conservative difference schemes with uniform mesh are discussed while in [1], the authors used upwind scheme on special kind of nonuniform mesh namely piecewise-uniform Shishkin mesh to derive the approximation. The limitation of this kind of mesh is that it requires considerable amount of information about the exact solution before solving the problem which is not always available.

Here in this paper, we apply the upwind scheme on a more general nonuniform mesh known as *adaptive grids* for model problems of type (1.1) and find the optimal order of error estimates i.e., $\mathcal{O}(N^{-1})$ while in [1], the authors used Shishkin mesh to obtain first-order error estimate up to a logarithm factor i.e., of order $\mathcal{O}(N^{-1} \ln N)$. Also the adaptive grids approach has the advantage that it can be applied using little or no a priori information about the solution. For singular perturbation problems the aim is to cluster automatically the grid points within the boundary layer and an obvious choice of adaptivity criterion is therefore the solution gradient [2, 7, 11]. There are certain results for adaptive mesh approach to the solution of singular perturbation problems where the upwind finite difference scheme is applied on a nonuniform mesh formed by equidistributing the arc-length monitor function $M(x) = \sqrt{1 + |u'(x)|^2}$ [2, 4, 9, 11].

An outline of the paper is as follows. In the next section, we study the pertinent properties of the solution $u(x)$ of (1.1); in particular, a decomposition into regular and singular component is derived. In Sect. 3, we describe the upwind scheme, the generation of the adaptive grids and some results on equidistributing grid that are used later. The convergence analysis of the numerical solution obtained by the upwind scheme on the adaptive grid is discussed in Sect. 4. Finally, Sect. 5 gives a few numerical examples that confirm our theoretical bounds.

Throughout this paper C denotes a generic positive constant independent of the grid points x_j and the parameters ε and N (the number of mesh intervals) which can take different values at different places, even in the same argument. Whenever we write $\phi = \mathcal{O}(\psi)$, we mean that $|\phi| \leq C|\psi|$. To simplify the notation, we set $g_j = g(x_j)$ for any function g, while g_j^N denotes an approximation of g at x_j. Here $\|\cdot\|$ denotes the supremum norm, which is defined by

$$\|g\| = \sup_{\xi \in D} |g(\xi)|,$$

for a function g defined on some domain D.

2 Continuous problem

Lemma 2.1 (Maximum principle) *Let v be a smooth function satisfying $B_0v(0) \geq 0$, $B_1v(1) \geq 0$ and $Lv(x) \geq 0$, $\forall x \in \Omega$. Then $v(x) \geq 0$, $\forall x \in \overline{\Omega}$.*

Proof Let $x^* \in \overline{\Omega}$ be such that $v(x^*) = \min v(x), x \in \overline{\Omega}$ and assume that $v(x^*) < 0$. Clearly $x^* \neq 1$. Now consider the three cases: $\lambda_2 = 0, \lambda_1/\lambda_2 > \alpha/2\varepsilon$ and $\lambda_1/\lambda_2 < \alpha/\varepsilon$.

Case (i): If $\lambda_2 = 0$, form the boundary condition, we have $v(1) \geq 0$. Define the auxiliary function $w(x) = v(x)\exp(\alpha x/2\varepsilon)$. Now for $x^* \in \Omega$, then $w'(x^*) = 0$ and $w''(x^*) \geq 0$ and thus $Lv(x) < 0$ which contradicts our hypothesis. The only possibility is that $x^* = 0$, which means that $w(0) < 0$ and $w'(0) \geq 0$. Using the auxiliary function we can conclude that $v(0) < 0$ and $v'(0) \geq 0$, which is also a contradiction.

Case (ii): Let $\lambda_1/\lambda_2 > \alpha/2\varepsilon$. If $x^* = 1$, then the minimum of $w(x)$ also occurs at $x = 1$, and thus $w(1) < 0$, $w'(1) \leq 0$. From this it follows that $v'(1) \leq -(\alpha/2\varepsilon)v(1)$ and since $\lambda_1/\lambda_2 > \alpha/2\varepsilon$ we have $v'(1) < -(\lambda_1/\lambda_2)v(1)$ which violates our hypothesis. If $x^* \in \Omega$, then $Lv(x^*) < 0$ which again contradicts our hypothesis. So the only possibility left out is that $w(x)$ attains its minimum at $x = 0$ which can be discarded by same argument given in the first case.

Case (iii): Finally, if $\lambda_1/\lambda_2 \leq \alpha/\varepsilon$, define the auxiliary function $w(x) = v(x) \times \exp(\lambda_1 x/2\lambda_2)$. By giving similar argument we can conclude $x^* \neq 1$. If $x^* \in \Omega$, then $Lv(x^*) < 0$ which contradicts our hypothesis. Thus the only possibility left is that $x^* = 0$, which can be excluded as before.

□

Now we establish a priori bounds for the solution and its derivatives of the problem (1.1).

Lemma 2.2 *If u is the solution of the boundary value problem* (1.1), *then we have the following stability estimate*:

$$\|u\| \leq \alpha^{-1}\left(1 + \frac{\lambda_1}{\lambda_2}\right)\|f\| + C\left(\frac{|s_0|}{\beta_1 + \alpha\beta_2} + \frac{|s_1|}{\lambda_1}\right). \tag{2.1}$$

Proof Consider the following barrier function

$$\psi^{\pm}(x) = \frac{|s_0|}{(\beta_1 + \alpha\beta_2) - \beta_1(1 - \lambda_2\alpha/\lambda_1\varepsilon)e^{-\alpha/\varepsilon}}\left(e^{-\alpha x/\varepsilon} - \left(1 - \frac{\lambda_2\alpha}{\lambda_1\varepsilon}\right)e^{-\alpha/\varepsilon}\right)$$
$$+ \frac{|s_1|}{\lambda_1} + \alpha^{-1}\|f\|\left(1 + \frac{\lambda_2}{\lambda_1} - x\right) \pm u(x).$$

It is easy to check that $\beta_1\psi^{\pm}(0) - \varepsilon\beta_2(\psi^{\pm})'(0) \geq 0$ and $\lambda_1\psi^{\pm}(1) + \lambda_2(\psi^{\pm})'(1) \geq 0$. Now from (1.1) for $x \in \Omega$, we have $L\psi^{\pm}(x) \geq 0$. Hence, by applying the maximum principle given in Lemma 2.1, we can conclude that $\psi^{\pm}(x) \geq 0, \forall x \in \overline{\Omega}$ which is our desired result. □

Lemma 2.3 *The derivatives $u^{(k)}$ of the solution u of (1.1) satisfy the following bounds*:

$$\begin{aligned} &\|u^{(k)}\| \le C\varepsilon^{-k}\max\{\|f\|, \|u\|\}, \quad k=1,2,\\ &\|u^{(3)}\| \le C\varepsilon^{-3}\max\{\|f\|, \|f'\|, \|u\|\}, \end{aligned} \tag{2.2}$$

where C depend on the bounds of the coefficients and their derivatives.

Proof The proof is analogous to the one given in [5]. Note that

$$\int_0^x \big(f(t) - b(t)u(t) + a(t)u'(t)\big)\,dt \le \|f\| + C\|u\|, \tag{2.3}$$

where C depends on $\|a\|$, $\|b\|$, $\|a'\|$ and $\|b'\|$. Using the mean-value theorem, there exists a $z \in (0, \varepsilon)$ such that $|\varepsilon u'(z)| \le 2\|u\|$. Integrating the BVP (1.1), we have

$$\varepsilon(u'(x) - u'(0)) = \int_0^x \big(f(t) - b(t)u(t) + a(t)u'(t)\big)\,dt. \tag{2.4}$$

Combining (2.3) and (2.4), we have

$$|\varepsilon u'(x)| \le \|f\| + C\|u\|,$$

which gives the required result for $k = 1$. Again from the BVP (1.1), we have

$$\varepsilon u'' = bu - au' - f \quad \text{and} \quad \varepsilon u''' = (bu - au' - f)',$$

which gives successively the required bounds on the second and third derivatives. □

In order to derive the ε-uniform error estimate, we require sharper bounds on the derivatives of the solution, for this, we decompose the solution of (1.1) into regular and singular parts as follows:

$$u(x) = v(x) + w(x). \tag{2.5}$$

Now $v(x)$ can be written in an asymptotic expansion as

$$v(x) = v_0(x) + \varepsilon v_1(x) + \varepsilon^2 v_2(x),$$

where v_0, v_1 and v_2 are respectively the solutions of the following differential equations:

$$\begin{cases} a(x)v_0'(x) + b(x)v_0(x) = -f(x), & \lambda_1 v_0(1) + \lambda_2 v_0'(1) = s_1,\\ a(x)v_1'(x) + b(x)v_1(x) = -v_0''(x), & \lambda_1 v_1(1) + \lambda_2 v_1'(1) = 0,\\ Lv_2(x) = v_1''(x), \quad \beta_1 v_2(0) + \beta_2 v_2'(0) = 0, & \lambda_1 v_2(1) + \lambda_2 v_2'(1) = 0. \end{cases} \tag{2.6}$$

Hence, the regular component of the solution satisfies the BVP:

$$\begin{cases} Lv(x) = f(x), \\ \beta_1 v(0) - \varepsilon\beta_2 v'(0) = \beta_1 v_0(0) - \varepsilon\beta_2 v_0'(0) + \varepsilon(\beta_1 v_1(0) - \varepsilon\beta_2 v_1'(0)), \\ \lambda_1 v(1) + \lambda_2 v'(1) = s_1, \end{cases} \tag{2.7}$$

and the singular component satisfies:

$$\begin{cases} Lw(x) = 0, \\ \beta_1 w(0) - \varepsilon\beta_2 w'(0) = s_0 - (\beta_1 v(0) - \varepsilon\beta_2 v'(0)), \\ \lambda_1 w(1) + \lambda_2 w'(1) = 0. \end{cases} \tag{2.8}$$

In the following lemma, we obtain bounds for the components of the solution and their derivatives.

Lemma 2.4 *For sufficiently small* ε *and* $0 \le k \le 3$, *the derivatives of* v *and* w *satisfy the following bounds*:

$$\begin{aligned} &|v^{(k)}(x)| \le C(1 + \varepsilon^{2-k}), \\ &|w^{(k)}(x)| \le C\varepsilon^{-k} \exp(-\alpha x/\varepsilon), \quad \forall x \in \overline{\Omega}. \end{aligned} \tag{2.9}$$

Proof The proof is given in [1]. □

3 Numerical scheme and nonuniform grids

3.1 Discrete problem

Consider the difference approximation of (1.1) on a nonuniform grid $\overline{\Omega}^N = \{x_j\}_{j=0}^N$ and denote $h_j = x_j - x_{j-1}$. For a mesh function Z_j, we define the following difference operators:

$$\begin{aligned} &D^+ Z_j = \frac{Z_{j+1} - Z_j}{h_{j+1}}, \qquad D^- Z_j = \frac{Z_j - Z_{j-1}}{h_j}, \\ &D^+ D^- Z_j = \frac{2}{h_j + h_{j+1}} \Big(D^+ Z_j - D^- Z_j \Big). \end{aligned} \tag{3.1}$$

The upwind finite difference scheme for (1.1) takes the form

$$\begin{cases} L^N U_j^N \equiv -\varepsilon D^+ D^- U_j^N - a_j D^+ U_j^N + b_j U_j^N = f_j, \quad 1 \le j \le N-1, \\ B_0^N U_0^N \equiv \beta_1 U_0^N - \beta_2 \varepsilon D^+ U_0^N = s_0, \qquad B_1^N U_N^N \equiv \lambda_1 U_N^N + \lambda_2 D^- U_N^N = s_1. \end{cases} \tag{3.2}$$

Equation (3.2) can be expressed in the following form of system of algebraic equations

$$\begin{cases} -r_j^- U_{j-1}^N + r_j^c U_j^N - r_j^+ U_{j+1}^N = f_j, \quad j = 1, \ldots, N-1, \\ r_0^c U_0^N + r_0^+ U_1^N = s_0, \qquad r_N^- U_{N-1}^N + r_N^c U_N^N = s_1, \end{cases} \tag{3.3}$$

where

$$\begin{cases} r_j^- = \dfrac{2\varepsilon}{h_j(h_j+h_{j+1})}, \qquad r_j^c = \dfrac{2\varepsilon}{h_j h_{j+1}} + \dfrac{a_j}{h_{j+1}} + b_j, \\ r_j^+ = \dfrac{2\varepsilon}{h_{j+1}(h_j+h_{j+1})} + \dfrac{a_j}{h_{j+1}}, \\ r_0^c = \beta_1 + \dfrac{\varepsilon\beta_2}{h_1}, \qquad r_0^+ = -\dfrac{\varepsilon\beta_2}{h_1}, \qquad r_N^- = -\dfrac{\lambda_2}{h_N}, \qquad r_N^c = \lambda_1 + \dfrac{\lambda_2}{h_N}. \end{cases} \tag{3.4}$$

In the tridiagonal system (3.4), the off-diagonal entries have the following properties:

$$r_0^+ < 0, \quad r_N^- < 0 \quad \text{and} \quad r_j^- > 0, \quad r_j^+ > 0, \quad j = 1, \ldots, N-1,$$

and

$$r_j^c + r_j^- + r_j^+ \geq 0, \quad \text{for } j = 1, \ldots, N-1, \tag{3.5}$$

which imply that the stiffness matrix is an M-matrix.

Lemma 3.1 (Discrete maximum principle) *Let L^N be the upwind finite difference operator defined in* (3.2) *and let Ω^N be an arbitrary mesh of $N+1$ mesh points. If $\mathcal{V}_j$ be any mesh function satisfying $\beta_1\mathcal{V}_0 - \varepsilon\beta_2 D^+\mathcal{V}_0 \geq 0$, $\lambda_1\mathcal{V}_N + \lambda_2 D^-\mathcal{V}_N \geq 0$ and $L^N\mathcal{V}_j \geq 0$, $1 \leq j \leq N-1$, then $\mathcal{V}_j \geq 0$ for $0 \leq j \leq N$.*

Proof Here, we consider the two cases: $\beta_2 = 0$ and $\beta_2 \neq 0$.

Case (i): If $\beta_2 = 0$, then form the left boundary condition, we have $\mathcal{V}_0 \geq 0$. Define $\mathcal{V}_k = \min_j \mathcal{V}_j < 0$. Now $D^+\mathcal{V}_k \geq 0$ and $D^+D^-\mathcal{V}_k \geq 0$, which results $L^N\mathcal{V}_k \leq 0$. To avoid such contradiction, we must take $\mathcal{V}_k = \mathcal{V}_{k-1} = \mathcal{V}_{k+1} < 0$. By repeating this argument we reach at $\mathcal{V}_0 < 0$ which is a contradiction.

Case (ii): When $\beta_2 \neq 0$. Define a mesh function as $\mathcal{W}_j = (1 + (\beta_1 h_j)/(\beta_2\varepsilon))\mathcal{V}_j$. Consider $\mathcal{W}_k = \min_j\{\mathcal{W}_j\}$. Assuming $\mathcal{V}_k < 0$ implies $\mathcal{W}_k < 0$. Note that if $k = N$, then $D^-\mathcal{V}_N \leq 0$ and also $\mathcal{V}_N < 0$ which violates the right hand side boundary condition and hence $k \neq N$. We thus have two remaining possibilities.

First, suppose $k = 0$, so that $\mathcal{W}_0 = \min_j\{\mathcal{W}_j\}$. Using the left boundary conditions, we get $\mathcal{W}_1 \leq \mathcal{W}_0$ and since $\mathcal{W}_0$ is the minimum, we have $\mathcal{W}_1 = \mathcal{W}_0$. Repeating the same argument, we can reach at $\mathcal{W}_N = \mathcal{W}_0$ which means $D^+\mathcal{V}_{N-1} = 0$ and $D^-\mathcal{V}_N = 0$. So we have the right boundary condition as $\lambda_1\mathcal{V}_N + \lambda_2 D^-\mathcal{V}_N \leq 0$ which contradicts the hypothesis.

Finally, suppose $0 < k < N$ for which $\mathcal{V}_k = \min_j\{\mathcal{V}_j\} < 0$ be the minimum, then $D^+\mathcal{V}_k \geq 0$ and $D^+D^-\mathcal{V}_k \geq 0$ which gives $L^N\mathcal{V}_k \leq 0$. To avoid this contradiction, we can take $\mathcal{V}_k = \mathcal{V}_{k-1} = \mathcal{V}_{k+1} < 0$ which is again a contradiction to the right hand side boundary condition. □

3.2 Grid equidistribution

A commonly-used technique in adaptive grid generation is based on the idea of equidistribution. A grid Ω^N is said to be equidistributing, if

$$\int_{x_{j-1}}^{x_j} M(u(s),s)ds = \int_{x_j}^{x_{j+1}} M(u(s),s)ds, \quad j=1,\dots,N-1, \tag{3.6}$$

where $M(u(x),x) > 0$ is called the monitor function. Equivalently, (3.6) can be expressed as

$$\int_{x_{j-1}}^{x_j} M(u(s),s)ds = \frac{1}{N}\int_0^1 M(u(s),s)ds, \quad j=1,\dots,N. \tag{3.7}$$

For practical purposes, it is common to use monitor functions which are bounded away from zero to maintain a sensible distribution of mesh points throughout the domain. Here, we consider the scaled arc-length monitor function

$$M(u(x),x) = \sqrt{1+(u'(x))^2}, \tag{3.8}$$

which is bounded below by unity. The optimal choice of the monitor function depends on the problem being solved, the numerical discretization being used and the norm of the error that is to be minimized. In practice, the monitor function is often based on a simple function of the derivatives of the unknown solution. For more details about the equidistribution principle and the choice of monitoring functions, one can refer the articles [2, 7].

Remark 3.2 One desirable property of the monitoring function is that there exist some positive constants $C_1 \le C_2$ such that

$$C_1 \le M(u(x),x) \quad \text{and} \quad \int_0^1 M(u(x),x)dx \le C_2.$$

Now combining the above with (3.7), we have $h_j \le (C_2/C_1)N^{-1}$, $j=1,\dots,N-1$.

To simplify the treatment, we construct the monitor function (3.8) in terms of the exact solution of the BVP (1.1). Now equidistribution can also be thought of as giving rise to a mapping $x = x(\xi)$ relating a computational coordinate $\xi \in [0,1]$ to the physical coordinate $x \in [0,1]$ defined by

$$\int_0^{x(\xi)} M(u(s),s)ds = \xi \int_0^1 M(u(s),s)ds = \xi\ell, \tag{3.9}$$

where ℓ is the length of u over $\overline{\Omega}$. Now

$$\frac{dx}{d\xi} = \frac{\ell}{\sqrt{1+(u'(x))^2}}.$$

More precisely, we have

$$x_j = \int_0^{\xi_j} \frac{\ell}{\sqrt{1+u'(s)^2}} ds, \qquad \xi_j = \frac{j}{N}, \quad j = 0, \ldots, N. \tag{3.10}$$

Hence, the step sizes of the mesh are given by

$$h_j = x_j - x_{j-1} = \int_{\xi_{j-1}}^{\xi_j} \frac{\ell}{\sqrt{1+(u'(s))^2}} ds. \tag{3.11}$$

For truly adaptive algorithm, the monitoring function has to be approximated from the numerical solution. Let U_j^N be the piecewise linear interpolant of knots $(x_j, u(x_j))$. From equidistribution principle (3.7), we have

$$\left[1 + (D^- U_j^N)^2\right] dx^2 = (\ell d\xi)^2.$$

In other words, we can construct the mesh from (3.7) as the solution of the following nonlinear system of equations:

$$\begin{cases} (x_{j+1} - x_j)^2 + (U_{j+1}^N - U_j^N)^2 = (x_j - x_{j-1})^2 + (U_j^N - U_{j-1}^N)^2, \\ \quad j = 1, \ldots, N-1, \\ x_0 = 0, \qquad x_N = 1. \end{cases} \tag{3.12}$$

The system of (3.2) and (3.12) are solved simultaneously to obtain the solution U_j^N and the grids x_j. Note that although (3.2) represents a linear set of equations for U_j^N, the fact that for the grid we require to equidistribute the monitor function based on U_j^N in (3.7) is nonlinearly linked to the solution.

Lemma 3.3 *If the mesh $\overline{\Omega}^N$ is generated by* (3.12), *then*

- *There are $\mathcal{O}(N)$ grid points inside the boundary layer $(0, x_K)$. Moreover, $h_j \le C\varepsilon$ for $j \le K$.*
- *There are $\mathcal{O}(1)$ grid points inside the transition region (x_K, x_J), where $\mathcal{O}(1)$ is independent of ε and N.*
- *There are $\mathcal{O}(N)$ grid points inside the regular region $(x_J, 1)$ and $h_j \le CN^{-1}$ for $j \ge J+1$, where $|u'(x)| \gg 1$ if $x < x_J$ and $|u'(x)| = \mathcal{O}(1)$ if $x > x_J$.*

Proof The proof can be found in [11]. □

4 Error analysis

4.1 Local truncation error

The local truncation error of the difference scheme (3.2) at the grid x_j is given by

$$\tau_j = L^N U_j^N - Lu(x_j), \tag{4.1}$$

where u and U_j^N denote the solutions of (1.1) and (3.2) respectively.

In order to obtain the bound of the local truncation error, we require the following lemma.

Lemma 4.1 *For any $\psi \in \mathcal{C}^3(\overline{\Omega})$, we have*

$$\left|\left(D^+ - \frac{d}{dx}\right)\psi(x_j)\right| \le \frac{1}{x_{j+1}-x_j}\int_{x_j}^{x_{j+1}}(x_{j+1}-s)\psi''(s)ds,$$

$$\left|\left(D^+D^- - \frac{d^2}{dx^2}\right)\psi(x_j)\right| \le \frac{1}{x_{j+1}-x_{j-1}}\left[\frac{1}{h_{j+1}}\int_{x_j}^{x_{j+1}}(x_{j+1}-s)^2\psi'''(s)ds \right.$$
$$\left. - \frac{1}{h_j}\int_{x_{j-1}}^{x_j}(s-x_{j-1})^2\psi'''(s)ds\right].$$

Proof The complete proof can be found in Lemma 4.1 of [8]. □

Lemma 4.2 *The truncation error defined in* (4.1) *has the following bound*:

$$|\tau_j| \le \frac{C}{\varepsilon N}\exp\left(\frac{-\alpha x_{j-1}}{\varepsilon}\right). \tag{4.2}$$

Proof Using the Taylor series expansion and Lemma 4.1, the truncation error (4.1) can be expressed as

$$\tau_j = \frac{-\varepsilon}{h_j+h_{j+1}}\left[\frac{1}{h_{j+1}}\int_{x_j}^{x_{j+1}}(x_{j+1}-s)^2u'''(s)ds - \frac{1}{h_j}\int_{x_{j-1}}^{x_j}(s-x_{j-1})^2u'''(s)ds\right]$$
$$+ \frac{a_j}{h_{j+1}}\int_{x_j}^{x_{j+1}}(x_{j+1}-s)u''(s)ds, \tag{4.3}$$

from which we obtain the following bound

$$|\tau_j| < \varepsilon\int_{x_{j-1}}^{x_{j+1}}|u'''(s)|ds + C\int_{x_{j-1}}^{x_{j+1}}|u''(s)|ds. \tag{4.4}$$

If we invoke the derivative bounds of the continuous solution given in Lemma 2.3 in the first term, the above expression becomes

$$|\tau_j| < C\int_{x_{j-1}}^{x_{j+1}}|u''(s)|ds. \tag{4.5}$$

From (3.10), we have

$$|\tau_j| \le C\ell\int_{\xi_{j-1}}^{\xi_{j+1}}\frac{|u''(x)|}{\sqrt{1+u'(x)^2}}d\xi \le \frac{C}{\varepsilon}\int_{\xi_{j-1}}^{\xi_{j+1}}\frac{|u'(x)|}{\sqrt{1+u'(x)^2}}d\xi. \tag{4.6}$$

From Lemma 2.3, we know that $|u'(x)| = \mathcal{O}(1/\varepsilon)$. Using this bound of the solution, we can get constants C_3 and C_4 such that

$$\frac{C_3}{\varepsilon}\exp\left(\frac{-\alpha^* x}{\varepsilon}\right) \le u'(x) \le \frac{C_4}{\varepsilon}\exp\left(\frac{-\alpha x}{\varepsilon}\right),$$

holds. Now (4.6) can be written as

$$\begin{aligned} |\tau_j| &\le \frac{C}{\varepsilon}\int_{\xi_{j-1}}^{\xi_{j+1}} \frac{\frac{C_4}{\varepsilon}\exp(\frac{-\alpha x}{\varepsilon})}{\sqrt{1+(\frac{C_3}{\varepsilon})^2\exp(\frac{-2\alpha^* x}{\varepsilon})}}d\xi \le \frac{C}{\varepsilon N}\frac{\frac{C_4}{\varepsilon}\exp(\frac{-\alpha x_{j-1}}{\varepsilon})}{\sqrt{1+(\frac{C_3}{\varepsilon})^2\exp(\frac{-2\alpha^* x_{j-1}}{\varepsilon})}} \\ &\le R_{j-1}\exp\left(\frac{-\omega x_{j-1}}{\varepsilon}\right), \end{aligned} \tag{4.7}$$

where $0 < \omega < 1$ is independent of ε, N and $R_{j-1} = \frac{C}{\varepsilon N}\frac{(C_4/\varepsilon)\exp(-(\alpha-\omega)x_{j-1}/\varepsilon)}{\sqrt{1+(C_3/\varepsilon)^2\exp(-2\alpha^* x_{j-1}/\varepsilon)}}$.

Let us denote $y_{j-1} = (C/\varepsilon)\exp(-\alpha x_{j-1}/\varepsilon)$, $g(y) = y/\sqrt{1+y^2}$ which is an increasing function in $[0, y^*]$ where $y^* = \sqrt{(1-\omega)/\omega}$. Since $\omega = \mathcal{O}(1)$, we have $y^* = \mathcal{O}(\omega)$ and hence $g(y^*) = \mathcal{O}(1)$. Therefore, we can express

$$R_{j-1} \le \frac{C}{\varepsilon N}g(y_{j-1}) \le \frac{C}{\varepsilon N}g(y^*) \le \frac{C}{\varepsilon N},$$

and hence,

$$|\tau_j| \le \frac{C}{\varepsilon N}\exp\left(\frac{-\alpha x_{j-1}}{\varepsilon}\right),$$

which is the required result. □

Lemma 4.3 *The solution of the constant coefficient problem given by*

$$\begin{cases} -\varepsilon D^+D^-\Phi_j - \alpha D^+\Phi_j = 0, & 1 \le j \le N-1, \\ \beta_1\Phi_0 - \beta_2\varepsilon D^+\Phi_0 = 1, & \lambda_1\Phi_N + \lambda_2 D^-\Phi_N = 0, \end{cases} \tag{4.8}$$

on a uniform mesh or on the nonuniform mesh Ω^N satisfies $|\Phi_j| \le C$ and $D^+\Phi_j \le 0$, $\forall 1 \le j \le N-1$.

Proof Assume that the mesh is uniform, then $h_j = h = N^{-1}$. The solution of the difference equation (4.8) is of the form

$$\Phi_j = \frac{\mu^{N-j} + (\alpha\lambda_2/\lambda_1\varepsilon) - 1}{[\beta_1(\mu^N + (\alpha\lambda_2/\lambda_1\varepsilon) - 1) - \beta_2\mu^{N-1}]}, \quad \text{where } \mu = 1 + \frac{\alpha h}{\varepsilon},$$

and therefore

$$D^+\Phi_j = \frac{\mu^{N-j-1}}{\varepsilon[\beta_1(\mu^N + (\alpha\lambda_2/\lambda_1\varepsilon) - 1) - \beta_2\mu^{N-1}]} \le 0. \tag{4.9}$$

Now, consider the case, when the mesh is nonuniform, then the solution is of the form

$$\Phi_j = \frac{\mu_j^{N-j} + (\lambda_2\alpha/\lambda_1\varepsilon) - 1}{\mu_j^N + (\lambda_2\alpha/\lambda_1\varepsilon) - 1}, \quad \text{where } \mu_j = 1 + \frac{\alpha h_j}{\varepsilon}.$$

Applying the forward difference operator, we obtain

$$D^+\Phi_j = \frac{\alpha\mu^{N-j-1}}{\varepsilon[\mu^N + (\alpha\lambda_2/\lambda_1\varepsilon) - 1]} \leq 0. \tag{4.10}$$

Combining (4.9) and (4.10), we have the desired result. □

Analogous to the continuous case, the discrete solution U_j^N can also be decomposed into the sum

$$U_j^N = V_j^N + W_j^N, \tag{4.11}$$

where V_j^N and W_j^N are, respectively, the solutions of the problems:

$$\begin{cases} L^N V_j^N = f_j, \qquad x_j \in \Omega^N, \\ \beta_1 V_0^N - \varepsilon\beta_2 D^+ V_0^N = \beta_1 v(0) - \varepsilon\beta_2 v'(0), \\ \lambda_1 V_N^N + \lambda_2 D^- V_N^N = \lambda_1 v(1) + \lambda_2 v'(1), \end{cases} \tag{4.12}$$

and

$$\begin{cases} L^N W_j^N = 0, \qquad x_j \in \Omega^N, \\ \beta_1 W_0^N - \varepsilon\beta_2 D^+ W_0^N = \beta_1 w(0) - \varepsilon\beta_2 w'(0), \\ \lambda_1 W_N^N + \lambda_2 D^- W_N^N = 0. \end{cases} \tag{4.13}$$

Now we will obtain the error estimates for each of these components separately.

Lemma 4.4 *The error in the regular component of the numerical solution is obtained as*

$$|V_j^N - v(x_j)| \leq CN^{-1}, \quad \forall x_j \in \Omega^N,$$

where V_j^N is the solution of (4.12) *and v is the solution of* (2.7).

Proof Consider the local truncation error

$$L^N V_j^N - Lv(x_j) = -\varepsilon\left(\frac{d^2}{dx^2} - D^+D^-\right)v(x_j) - a_j\left(\frac{d}{dx} - D^+\right)v(x_j).$$

Then, by using the bounds given in (4.1) and in (2.9), we have

$$|L^N V_j^N - Lv(x_j)| \leq \frac{\varepsilon}{3}(x_{j+1} - x_{j-1})\|v^{(3)}\| + \frac{a_j}{2}(x_{j+1} - x_j)\|v^{(2)}\| \leq CN^{-1}.$$

Let us use the mesh function

$$\Psi^{\pm}(x_j) = CN^{-1}\left(\phi_j + \frac{1}{\lambda_1}\right) \pm (V_j^N - v)(x_j),$$

where ϕ_j is the solution of the constant coefficient problem

$$\begin{cases} -\varepsilon D^+D^-\phi_j - \alpha D^+\phi_j + b\phi_j = f_j, & 1 \le j \le N-1, \\ \beta_1\phi_0 - \beta_2\varepsilon D^+\phi_0 = 1, & \lambda_1\phi_N + \lambda_2 D^-\phi_N = 0. \end{cases} \tag{4.14}$$

Using the bounds given in Lemmas 2.4, 4.1 and 4.3, the following inequalities hold:

$$|\beta_1(V_0^N - v(0)) - \varepsilon\beta_2(D^+V_0^N - v'(0))| \le CN^{-1},$$

$$|\lambda_1(V_0^N - v(1)) + \lambda_2(D^-V_N^N - v'(1))| \le CN^{-1}.$$

Now by choosing C large enough, we can show that $\beta_1\Psi_0^{\pm} - \varepsilon\beta_2 D^+\Psi_0^{\pm} \ge 0$ and $\lambda_1\Psi_N^{\pm} + \lambda_2 D^-\Psi_N^{\pm} \ge 0$ and $L^N\Psi^{\pm}(x_j) \ge 0$. Hence, by applying the discrete maximum principle given in Lemma 3.1, the result follows. □

Lemma 4.5 *The error in the singular component of the numerical solution is obtained as*

$$|W_j^N - w(x_j)| \le CN^{-1}, \quad \forall x_j \in \Omega^N,$$

where W_j^N is the solution of (4.13) *and w is the solution of* (2.8).

Proof Using the bounds given in Lemma 2.4, the following inequalities holds:

$$\begin{aligned} &|\beta_1(W_0^N - w(0)) - \varepsilon\beta_2(D^+W_0^N - w'(0))| \\ &\quad = |\varepsilon\beta_2(D^+W_0^N - w'(0))| \le \left|\frac{\varepsilon}{h}\int_0^h (s-h)w''(s)ds\right| \le CN^{-1}. \end{aligned}$$

Similarly, we can show that

$$|\lambda_1(W_N^N - w(1)) + \lambda_2(D^-W_N^N - w'(1))| \le CN^{-1}.$$

First, we consider the case of uniform mesh. Using Lemmas 2.4 and 4.1, the truncation error corresponding to the singular part is

$$|L^NW_j^N - Lw(x_j)| \le C\varepsilon^{-1}(x_{j+1} - x_{j-1})e^{-\alpha x_{j-1}/\varepsilon} \le C\varepsilon^{-1}N^{-1}e^{-\alpha x_{j-1}/\varepsilon}. \tag{4.15}$$

We employ the barrier function

$$\chi^{\pm}(x_j) = \frac{Ce^{2\tau h/\varepsilon}}{\tau(\alpha - \tau)}\varepsilon^{-1}N^{-1}\left(\gamma_j + \frac{1}{\lambda_1}\right) \pm \left(W_j^N - w(x_j)\right),$$

where τ is a constant with $0 < \tau < \alpha$ and γ_j is the solution of the problem (4.14). By choosing C large enough such that $\chi^{\pm}(x_j)$ satisfy the discrete maximum principle given in Lemma 3.1 and using the fact $\gamma_j \le C$, we can conclude

$$\left(W_j^N - w(x_j)\right) \le \frac{Ce^{2\tau h/\varepsilon}}{\tau(\alpha-\tau)}\varepsilon^{-1}N^{-1}\gamma_j \le CN^{-1}. \tag{4.16}$$

Now consider the mesh to be nonuniform. Using Lemma 3.3, the domain Ω^N is broadly divided into two subdomains $[0, x_K)$ and $[x_K, 1]$. We wish to find the error in both the subdomains separately.

First, consider the points inside the regular region i.e., $x_j \in [x_K, 1]$. Using the triangle inequality, we have

$$|W_j^N - w(x_j)| \le |W_j^N| + |w(x_j)|.$$

We know from Lemma 2.4 that $|w(x_j)| \le Ce^{-\alpha x_j/\varepsilon} \le CN^{-1}$. It remains to find the bound for $|W_j^N|$. Let us consider the constant coefficient problem (assuming $b(x) \equiv 0$)

$$\begin{aligned} &-\varepsilon D^+D^-Y_j - \alpha D^+Y_j = 0, \quad 1 \le j \le N-1,\\ &Y_0 = 1, \qquad \lambda_1 Y_N + \lambda_2 D^- Y_N = 0. \end{aligned}$$

Using Lemma 4.3, the solution of the above problem is of the form

$$Y_j = \frac{\mu_j^{N-j} + (\lambda_2\alpha/\lambda_1\varepsilon) - 1}{\mu_j^N + (\lambda_2\alpha/\lambda_1\varepsilon) - 1}, \quad \text{where } \mu_j = 1 + \alpha h_j/\varepsilon. \tag{4.17}$$

From Lemma 3.3, we know that if $h_j \in [x_K, 1]$, then $h_j \le CN^{-1}$. Now using the fact that $(1 + (2\ln N)/N)^{-N/2} \le 2N^{-1}$, we obtain

$$\frac{\alpha}{\varepsilon}\mu_j^{-N/2} \le 2. \tag{4.18}$$

So

$$-\frac{\varepsilon}{\alpha}D^+Y_K = \frac{\mu_j^{K-1}}{\mu_j^K + (\lambda_2\alpha/\lambda_1\varepsilon) - 1} \ge \frac{1}{\mu_j}\left[\frac{1}{1 + (\lambda_2\alpha/\lambda_1\varepsilon)\mu_j^{-N/2}}\right].$$

Combining the above inequality with (4.18), we get

$$-\frac{\varepsilon}{\alpha}D^+Y_K = \frac{1}{\mu_j}\left[\frac{\lambda_1}{\lambda_1 + 2\lambda_2}\right]. \tag{4.19}$$

Using (4.18) and (4.19) in (4.17), we obtain

$$Y_j \le CN^{-1}\left(1 + \frac{2\lambda_2}{\lambda_1}\right),$$

and note that $D^+Y_j \le 0$. Consider the barrier function

$$\xi^{\pm}(x_j) = |\beta_1 W_0^N - \varepsilon\beta_2 D^+ W_0^N| Y_j \pm W_j^N,$$

then $\beta_1\xi_0^{\pm} - \varepsilon\beta_2 D^+\xi_0^{\pm} \ge 0$, $\lambda_1\xi_N^{\pm} + \lambda_2 D^-\xi_N^{\pm} \ge 0$ and in addition, $L^N\xi_j^{\pm} \ge 0$. Thus applying the discrete maximum principle, we have

$$|W_j^N| \le |\beta_1 W_0^N - \varepsilon\beta_2 D^+ W_0^N| Y_j \le CN^{-1}.$$

Finally, it remain to show the bound of the error for $x_j \in [0, x_K)$. The proof is similar to the case of uniform mesh except we use the discrete maximum principle on $[0, x_K)$. Analogously, let us take the mesh function

$$\eta^{\pm}(x_j) = \frac{Ce^{2\tau h/\varepsilon}}{\tau(\alpha-\tau)}\varepsilon^{-1}N^{-1}\left(Z_j + \frac{1}{\lambda_1}\right) \pm (W_j^N - w)(x_j),$$

where τ is a constant with $0 < \tau < \alpha$ and Z_j is the solution of the following problem:

$$\begin{cases} -\varepsilon D^+D^-Z_j - \tau D^+Z_j = 0, & 1 \le j \le K, \\ \beta_1 Z_0 - \beta_2\varepsilon D^+ Z_0 = 1, & \lambda_1 Z_K + \lambda_2 D^- Z_K = 0. \end{cases} \tag{4.20}$$

Now the solution is

$$Z_j = \frac{\mu^{K-j} + (\lambda_2\tau/\lambda_1\varepsilon) - 1}{\mu^K + (\lambda_2\tau/\lambda_1\varepsilon) - 1}, \quad \text{where } \mu = 1 + \tau h_j/\varepsilon,$$

and we can see that

$$D^+Z_j = -\frac{\mu^{K-j-1}}{\varepsilon[\mu^K + (\lambda_2\tau/\lambda_1\varepsilon) - 1]} \le 0.$$

Note that $\beta_1\eta_0^{\pm} - \beta_2\varepsilon D^+\eta_0^{\pm} \ge 0$, $\lambda_1\eta_K^{\pm} + \lambda_2 D^-\eta_K^{\pm} \ge 0$ and $L^N\eta_j^{\pm} \ge 0$; therefore applying the discrete maximum principle given in Lemma 3.1, we can have $\eta_j^{\pm} \ge 0$. Hence, for $x_j \in [0, x_K)$, we have

$$|W_j^N - w(x_j)| \le CN^{-1}, \tag{4.21}$$

and hence, this completes the proof of the lemma. □

The above error estimates for the regular and the singular component of the numerical solution now lead to the following error estimate.

Theorem 4.6 *If u is the solution of problem* (1.1) *and U_j^N is the solution of* (3.2) *on the mesh defined by* (3.12), *then we have*

$$\max_{0\le j\le N} |u(x_j) - U_j^N| \le CN^{-1}.$$

Proof From (2.5) and (4.11), we have

$$|u(x_j) - U_j^N| \leq |v(x_j) - V_j^N| + |w(x_j) - W_j^N|.$$

Using the error bounds for the regular and singular components given in Lemmas 4.4 and 4.5 respectively, we obtain the desired bound. □

5 Numerical results

In this section to validate the theoretical results, we apply the proposed numerical scheme to several test problems with constant and variable coefficients. For comparison purposes, we use the upwind difference scheme on the piecewise-uniform Shishkin mesh.

Example 5.1 Consider the test problem

$$\begin{cases} -\varepsilon u''(x) - u'(x) + u(x) = 0, & x \in (0,1), \\ u(0) - \varepsilon u'(0) = 1, & u(1) + u'(0) = 1. \end{cases} \tag{5.1}$$

The exact solution is given by

$$u(x) = C_1 \exp(m_1 x) + C_2 \exp(m_2 x), \quad \text{where } m_{1,2} = \frac{-1 \pm \sqrt{1+4\varepsilon}}{2\varepsilon},$$

and

$$C_1 = \frac{\varepsilon m_2 - 1 - (1+m_2)\exp(m_2)}{(1+m_1)(1-\varepsilon m_2)\exp(m_1) - (1+m_2)(1-\varepsilon m_1)\exp(m_2)},$$
$$C_2 = \frac{\varepsilon m_1 - 1 - (1+m_1)\exp(m_1)}{(1+m_1)(1-\varepsilon m_2)\exp(m_1) - (1+m_2)(1-\varepsilon m_1)\exp(m_2)}.$$

This BVP has a boundary layer in the left end at $x = 0$.

Example 5.2 Consider the variable coefficient problem

$$\begin{cases} \varepsilon u''(x) + \frac{1}{1+x} u'(x) = x + 1, & x \in (0,1), \\ u(0) - \varepsilon u'(0) = 1, & u(1) + u'(0) = 1. \end{cases} \tag{5.2}$$

The solution $u(x)$ is of the form

$$u(x) = \frac{(x+1)^3}{6\varepsilon+3} + C_3 \left[\frac{(x+1)^{1-1/\varepsilon}}{\varepsilon - 1} - \frac{2^{1-1/\varepsilon}}{\varepsilon - 1} - \frac{2^{-1/\varepsilon}}{\varepsilon} \right] + \left[1 - \frac{20}{6\varepsilon+3} \right],$$

where

$$C_3 = \frac{(19+3\varepsilon)/(6\varepsilon+3)}{((1-2^{1-1/\varepsilon})(\varepsilon-1) - 2^{-1/\varepsilon}/\varepsilon) - 1}.$$

The above problem has a boundary layer at the left side of the domain near $x = 0$.

For boundary layer on the left, the piecewise-uniform Shishkin mesh $\overline{\Omega}_\varepsilon^N$ is constructed by partitioning the domain $[0, 1]$ into two subdomains $[0, \sigma)$ and $[\sigma, 1]$. σ is the point of transition from a fine mesh to the coarse mesh and is defined as $\sigma = \min\{1/2, \sigma_0 \varepsilon \ln N\}$, $\sigma_0 = 1/\alpha$, where α defined earlier. The definition of σ guarantees the existence of some points inside the layer region. Without loss of generality, assume that N is even. Now we will place $N/2$ numbers of subintervals in each of the subdomains.

For any value of N and ε, we calculate the maximum pointwise errors E_ε^N and the ε-uniform errors E^N by

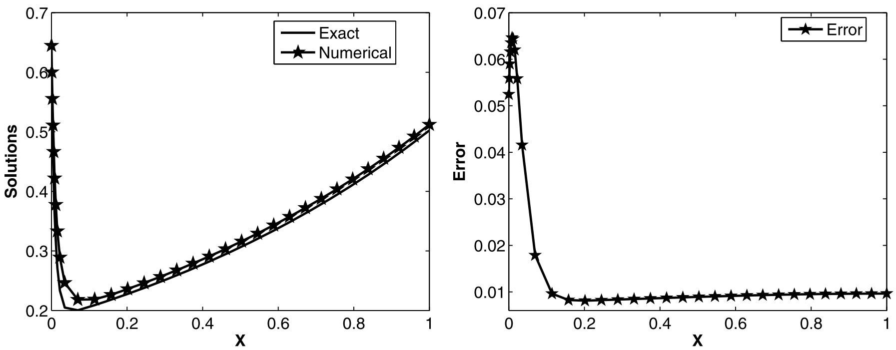
$$E_\varepsilon^N = \max_{0 \le j \le N} |u(x_j) - U_j^N| \quad \text{and} \quad E^N = \max_{0 < \varepsilon \le 1} E_\varepsilon^N, \tag{5.3}$$

where u is the exact solution and U_j is the numerical solution obtained by using N mesh intervals in the domain $\overline{\Omega}^N$. The corresponding rate of convergence are calculated by

$$r_\varepsilon^N = \log_2 \left(\frac{E_\varepsilon^N}{E_\varepsilon^{2N}} \right) \quad \text{and} \quad r^N = \log_2 \left(\frac{E^N}{E^{2N}} \right). \tag{5.4}$$

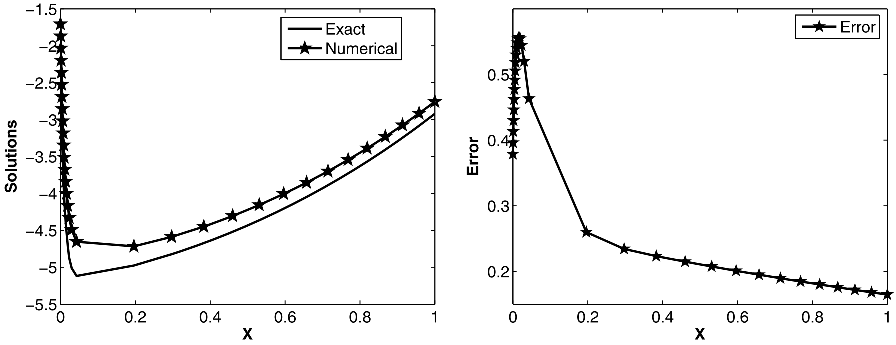
Figures 1(a) and 1(b) represent the numerical solution along with the exact solution and the corresponding error obtained on the adaptive grid for Example 5.1 respectively. Also we have plotted similar graphs for Example 5.2 in Figs. 2(a) and 2(b). In Tables 1 and 2, we present the maximum pointwise error and the corresponding order of convergence of the solution for Examples 5.1 and 5.2 respectively, which clearly shows that the proposed method is ε-uniform convergent of order one.

We have also compared the computational results using adaptive mesh to the numerical results using the Shishkin mesh which are shown in Tables 3 and 4 with



(a) Solution. *(b) Error.*

Fig. 1 Numerical solution with the exact solution and the corresponding error for Example 5.1 with $\varepsilon = 10^{-2}$ and $N = 32$



(a) Solution. *(b) Error.*

Fig. 2 Numerical solution with the exact solution and the corresponding error for Example 5.2 with $\varepsilon = 10^{-2}$ and $N = 32$

Table 1 Maximum point-wise errors E_ε^N and the rate of convergence r_ε^N for Example 5.1

ε		Number of intervals N					
		32	64	128	256	512	1024
1	E_ε^N	1.0908e−2	5.4606e−3	2.7320e−3	1.3664e−3	6.8332e−4	3.4169e−4
	r_ε^N	0.9982	0.9991	0.9996	0.9998	0.9999	
10^{-2}	E_ε^N	6.4626e−2	3.5456e−2	1.8811e−2	9.7417e−3	4.9658e−3	2.5126e−3
	r_ε^N	0.9596	0.9786	0.9890	0.9944	0.9972	
10^{-4}	E_ε^N	8.0792e−2	4.7244e−2	2.6781e−2	1.4919e−2	8.1599e−3	4.4533e−3
	r_ε^N	0.7741	0.8189	0.8441	0.8705	0.8737	
10^{-6}	E_ε^N	8.1147e−2	4.7652e−2	2.7158e−2	1.5176e−2	8.3265e−3	4.6297e−3
	r_ε^N	0.7680	0.8112	0.8396	0.8660	0.8468	
10^{-8}	E_ε^N	8.1160e−2	4.7663e−2	2.7162e−2	1.5180e−2	8.3289e−3	4.6307e−3
	r_ε^N	0.7679	0.8113	0.8394	0.8660	0.8469	
	E^N	**8.1160e−2**	**4.7663e−2**	**2.7162e−2**	**1.5180e−2**	**8.3289e−3**	**4.6307e−3**
	r^N	**0.7679**	**0.8113**	**0.8394**	**0.8660**	**0.8469**	

$\varepsilon = 10^{-3}$ and $\varepsilon = 10^{-6}$ for Examples 5.1 and 5.2, respectively. From these results, one can observe that adaptive mesh produces better results than that produced by using the Shishkin mesh. The advantage of this approach is that without any prior knowledge of the location and the width of the boundary layer, we are able to generate an appropriate nonuniform mesh suitable for the layer type problems.

6 Conclusion

In this article, we presented the analysis of an upwind scheme for obtaining the solution for singularly perturbed mixed BVPs of the form (1.1) on a suitable nonuniform

Table 2 Maximum point-wise errors E_ε^N and the rate of convergence r_ε^N for Example 5.2

ε		Number of intervals N					
		32	64	128	256	512	1024
10^{-2}	E_ε^N	5.5687e−1	3.0267e−1	1.5959e−1	8.3796e−2	4.4184e−2	2.2633e−2
	r_ε^N	0.8796	0.9234	0.9294	0.9234	0.9651	
10^{-4}	E_ε^N	5.8701e−1	3.1851e−1	1.7697e−1	9.5507e−2	4.9720e−2	2.6464e−2
	r_ε^N	0.8820	0.8478	0.8899	0.9418	0.9098	
10^{-8}	E_ε^N	5.8750e−1	3.1914e−1	1.7698e−1	9.5516e−2	4.9982e−2	2.6729e−2
	r_ε^N	0.8804	0.8506	0.8898	0.9344	0.9030	
	E^N	**5.8750e−1**	**3.1914e−1**	**1.7698e−1**	**9.5516e−2**	**4.9982e−2**	**2.6729e−2**
	r^N	**0.8804**	**0.8506**	**0.8898**	**0.9344**	**0.9030**	

Table 3 Comparison of numerical results with Shishkin mesh for Example 5.1

N	$\varepsilon = 10^{-3}$		$\varepsilon = 10^{-6}$	
	Shishkin mesh	Adaptive mesh	Shishkin mesh	Adaptive mesh
32	7.8107e−2	7.8136e−2	7.8268e−2	8.1147e−2
	0.6516	0.8007	0.6524	0.7680
64	4.9720e−2	4.4855e−2	4.9795e−2	4.7652e−2
	0.7204	0.8438	0.7209	0.8112
128	3.0176e−2	2.4991e−2	3.0212e−2	2.7158e−2
	0.7732	0.8756	0.7734	0.8396
256	1.7657e−2	1.3621e−2	1.7675e−2	1.5176e−2
	0.8113	0.9099	0.8114	0.8660
512	1.0062e−2	7.2494e−3	1.0071e−2	8.3265e−3
	0.8385	0.9395	0.8386	0.8468

Table 4 Comparison of numerical results with Shishkin mesh for Example 5.2

N	$\varepsilon = 10^{-3}$		$\varepsilon = 10^{-6}$	
	Shishkin mesh	Adaptive mesh	Shishkin mesh	Adaptive mesh
32	7.5247e−1	5.8435e−1	7.5587e−1	5.8753e−1
	0.7257	0.8879	0.7260	0.8804
64	4.5503e−1	3.1578e−1	4.5697e−1	3.1915e−1
	0.7647	0.8466	0.7646	0.8534
128	2.6782e−1	1.7561e−1	2.6897e−1	1.7664e−1
	0.8010	0.8961	0.8013	0.8871
256	1.5372e−1	9.4357e−2	1.5435e−1	9.5509e−2
	0.8318	0.9478	0.8318	0.9349
512	8.6365e−2	4.8917e−2	8.6717e−2	4.9960e−2
	0.8541	0.8895	0.8541	0.9028

adaptive grids based on equidistribution principle. We carried out the error analysis for the numerical solution and it is shown that the error converges at the rate of first-order, independently of the singular perturbation parameter. Numerical results obtained for some examples show that the proposed scheme is of first-order accurate. Hence, the key result established here is that the computed solution on the adaptive grids are uniformly convergent with respect to the perturbation parameter.

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